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# PHYSICAL LIMITATIONS ON $D/Q$ FOR ANTENNAS

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**Abstract:** In this paper, physical limitations on the directivity  $D$  and Q-factor  $Q$  are derived for antennas of arbitrary shape. The quotient  $D/Q$  is shown to be bounded from above by the antenna volume and certain shape coefficients in terms of the eigenvalues of the high-contrast polarizability dyadic. The theory is exemplified by numerical results for the half-wave antenna with astonishing agreement.

## INTRODUCTION

The limitations presented in this paper generalize the classical results by Chu[1]. The most important advantage of the new limitations is that they are not restricted to the smallest circumscribing sphere, but instead hold for arbitrary geometries. In fact, the sphere is far from optimal with respect to the volume for many antennas, *cf.*, the half-wave antenna. The new formulation also permits studies of polarization effects, with important applications to diversity in MIMO systems.

In contrast to the numerous papers that have been published on the subject, the present analysis is solely based on the fundamental principles of linearity, time-translational invariance and causality rather than the spherical vector waves. Details on the analysis is given in Gustafsson *et al.*[2] and Sohl *et al.*[3].

## LIMITATIONS ON THE DIRECTIVITY AND THE Q-FACTOR

Consider an arbitrary antenna  $V$  subject to an incident plane-wave  $\mathbf{E}_{\text{in}}$  impinging in the  $\hat{\mathbf{k}}$ -direction. Let  $\mathbf{E}_0$  denote the Fourier amplitude of  $\mathbf{E}_{\text{in}}$  and introduce the electric and magnetic polarizations  $\hat{\mathbf{p}}_e = \mathbf{E}_0/|\mathbf{E}_0|$  and  $\hat{\mathbf{p}}_m = \hat{\mathbf{k}} \times \hat{\mathbf{p}}_e$ , respectively. The scattering properties of  $V$  in the forward direction are then described by the far-field amplitude via a linear and causal time-translational invariant convolution of  $\mathbf{E}_{\text{in}}$  and a certain temporal dyadic  $\mathbf{S}_t$ . Let  $\mathbf{S}$  denote the Fourier transform of  $\mathbf{S}_t$  and introduce  $\varrho(k) = \hat{\mathbf{p}}_e^* \cdot \mathbf{S}(k, \hat{\mathbf{k}}) \cdot \hat{\mathbf{p}}_e/k^2$ . The function  $\varrho$  is then holomorphic in the upper half-plane  $\text{Im } k > 0$  for a large class of dyadics  $\mathbf{S}_t$ . Particular useful in the forthcoming analysis is the static limit  $\varrho(0) = (\hat{\mathbf{p}}_e^* \cdot \gamma_e \cdot \hat{\mathbf{p}}_e + \hat{\mathbf{p}}_m^* \cdot \gamma_m \cdot \hat{\mathbf{p}}_m)/4\pi$ , where  $\gamma_e$  and  $\gamma_m$  denote the electric and magnetic polarizability dyadics, respectively.

A resonance model for  $\varrho$  in the vicinity of the resonance at  $k = k_0$  is

$$\varrho(k) = \varrho(0) \frac{iQk_0/(2k)}{1 - iQ(k/k_0 - k_0/k)/2}, \quad (1)$$

where  $k$  is real-valued and  $Q$  coincides with the Q-factor of the antenna when the relative bandwidth  $B$  is based on the half-power threshold, see Figure 1. For the generalization of (1) to multiple resonances, see Gustafsson *et al.*[2].

The extinction cross section  $\sigma_{\text{ext}}$  is directly related to the imaginary part of  $\varrho$  via the optical theorem  $\sigma_{\text{ext}} = 4\pi k \text{Im } \varrho$ . For the resonance at  $k = k_0$ , (1) yields that the partial realized gain  $(1 - |\rho|^2)G$

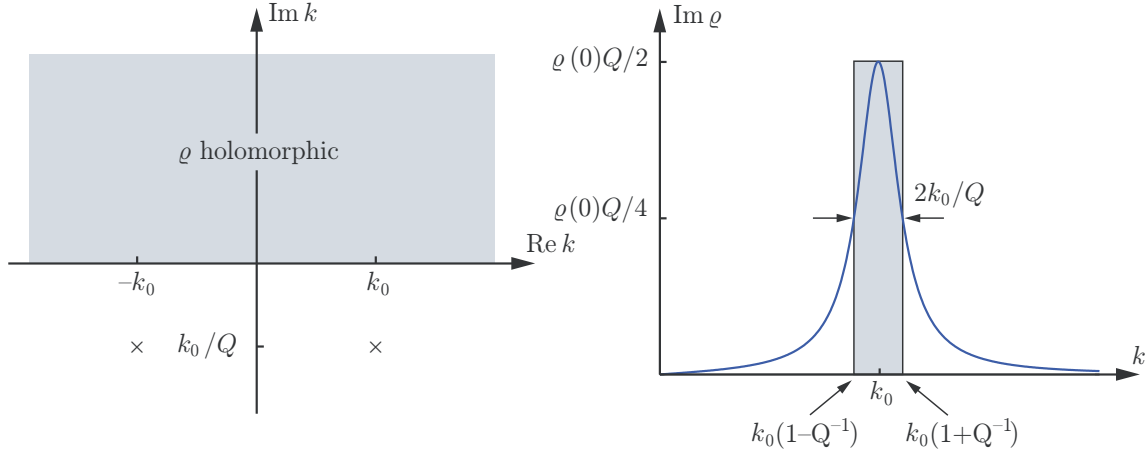


Figure 1: The symmetrically distributed poles  $\times$  of  $\varrho$  in the complex  $k$ -plane (left figure) and the corresponding single resonance model of  $\text{Im } \varrho$  when  $Q \gg 1$  (right figure).

satisfies

$$(1 - |\rho|^2)G = \frac{k^2 \sigma_a}{\pi} \leq \frac{k^2 \sigma_{\text{ext}}}{\pi} = \varrho(0) \frac{2k^2 Q k_0}{1 + Q^2 (k/k_0 - k_0/k)^2/4}, \quad (2)$$

where  $\rho$  and  $\sigma_a$  are the reflection coefficient and absorption cross section of the antenna, respectively. The upper bound in (2) is in general not isoperimetric but can be sharpened with a priori knowledge of the absorption efficiency  $\eta = \sigma_a/\sigma_{\text{ext}}$ . Recall that  $\sigma_{\text{ext}}$ ,  $\sigma_a$ , and  $G$  depend on the incident direction  $\hat{k}$  as well as the electric polarization  $\hat{p}_e$ .

For an antenna which is perfectly matched ( $\rho = 0$ ) at  $k = k_0$ , the partial realized gain  $(1 - |\rho|^2)G$  coincides with the partial directivity  $D$ . Under this assumption, (1) and (2) yields  $D/Q \leq 2k_0^3 \varrho(0)$ , or

$$\frac{D}{Q} \leq \frac{k_0^3}{2\pi} (\hat{p}_e^* \cdot \gamma_e \cdot \hat{p}_e + \hat{p}_m^* \cdot \gamma_m \cdot \hat{p}_m). \quad (3)$$

Inequality (3) is fundamental and holds for any linear, time-translational invariant and causal antenna. Recall that  $\gamma_e$  and  $\gamma_m$  scales with the volume of the antenna, *i.e.*, the right hand side of (3) is proportional to  $(k_0 a)^3$ , where  $a$  is the radius of, say, the volume-equivalent sphere.

From the analysis in Gustafsson *et al.*[2] it is clear that both  $\gamma_e$  and  $\gamma_m$  are bounded from above by the high-contrast polarizability dyadic  $\gamma_\infty$ . This fact can be used to estimate the quotient  $D/Q$  independently of the electric and magnetic polarizations, *viz.*,

$$\sup_{\hat{p}_e \cdot \hat{p}_m = 0} \frac{D}{Q} \leq \frac{k_0^3}{2\pi} (\gamma_1 + \gamma_2), \quad (4)$$

where  $\gamma_1$  and  $\gamma_2$  denote the largest and second largest eigenvalues of  $\gamma_\infty$ , respectively. A simple example of (4) is given by the sphere of radius  $a$ . In this case the right hand side of (4) is equal to  $4(k_0 a)^3$ , which is sharper than the corresponding result  $6(k_0 a)^3$  for the generalization of Chu[1] in the case of both TE- and TM-polarizations and  $k_0 a \ll 1$ .

The upper bounds in (3) and (4) differ only by a factor of  $\pi$  compared to the corresponding results in Gustafsson *et al.*[2] for the partial realized gain, *viz.*,  $G_\Lambda B \leq k_0^3 (\gamma_1 + \gamma_2)/2$ , where  $G_\Lambda = \min_{k \in K} (1 - |\rho|^2)G$  is the minimum realizable gain on the interval  $K = [k_0(1 - B/2), k_0(1 + B/2)]$ .

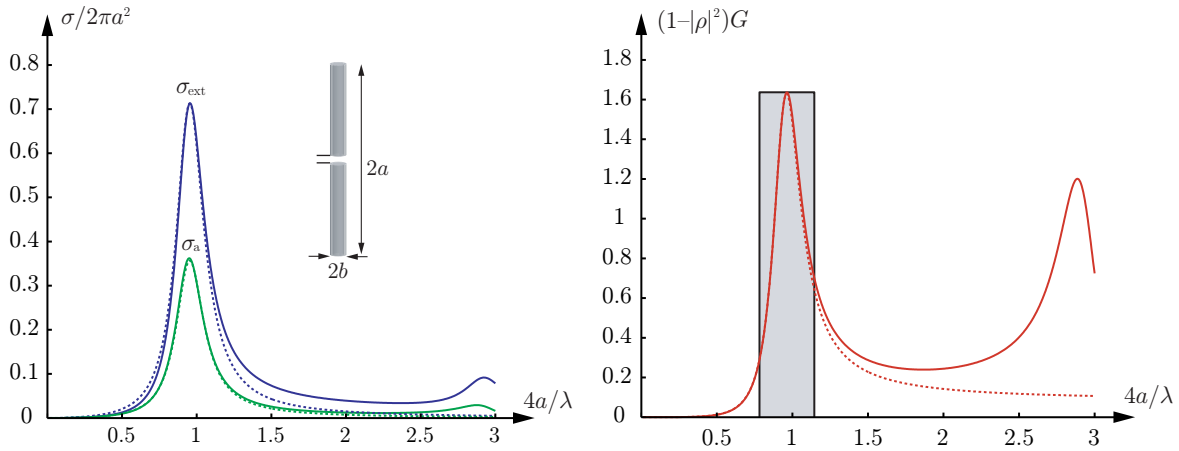


Figure 2: The extinction and absorption cross sections (left figure) and the partial realized gain (right figure) for the half-wave antenna. The different curves correspond to Hallén’s integral equation (solid), the resonance model for  $\varrho$  (dashed) and the physical limitation on  $G_{\Lambda}B$  (shaded).

### A NUMERICAL EXAMPLE: THE HALF-WAVE ANTENNA

The method of moments (MoM) solution of Hallén’s integral equation together with a gap feed model is used to determine the cross sections and partial realized gain for a cylindrical half-wave antenna with axial ratio  $b/a = 10^{-3}$ , see Figure 2. The antenna is resonant for  $2b \approx 0.47\lambda_0$  with the associated directivity  $D = 1.64$ . The relative bandwidth is about 8% at  $\text{SWR} = 2$  and the corresponding Q-factor based on a differentiation of the reflection coefficient is  $Q = 8.4$ . Note that the absorption efficiency at the resonance is  $\eta \approx 0.5$ , corresponding to a minimum scattering antenna.

The finite element method (FEM) is used to determine the eigenvalues  $\gamma_1$  och  $\gamma_2$  for the half-wave antenna. Under the assumptions of non-magnetic material parameters, the result is  $\gamma_1 = 0.71a^3$  with  $\gamma_2$  negligible. These eigenvalues inserted into (4) together with  $D = 1.64$  and  $\eta = 0.5$  yields  $Q \geq 8.4$  which coincides with the Q-factor above. In Figure 2, it is observed that the single resonance model with  $Q = 8.4$  is a good approximation of the cross sections and the realized gain. The corresponding bound on the product  $G_{\Lambda}B$  is illustrated by the rectangular region ( $G_{\Lambda} = 1.64$  and  $B = 38\%$ ) for an arbitrary minimum scattering antenna confined within the cylindrical region.

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