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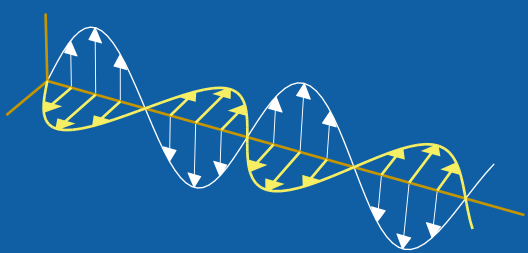

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Abstract

Modes of propagation of electromagnetic pulses in open circular waveguides are investigated systematically. Core and cladding both consist of simple (linear, homogeneous, isotropic), dispersive materials modeled by temporal convolution with physically sound susceptibility kernels. Under these circumstances, pulses cannot propagate along the guide unless the sum of the (first) initial derivatives of the electric and magnetic susceptibility kernels of the medium in the core is less than the corresponding sum for the medium in the cladding. Only a finite number of pulse modes can be excited, and relevant temporal Volterra integral equations of the second kind for these modes are derived. A theory of functions of integral operators is developed in order to obtain the results.

1 Introduction

Propagation of guided time-harmonic electromagnetic waves in closed and open dielectric structures is a technically important subject, see, *e.g.*, [2, 3, 5, 6, 12, 15]. Pulse propagation in waveguides has been considerably less attended to: Kristensson [10] managed to analyze the modes of propagation in closed empty guides using wave splitting technique. The results in [10] were extended to closed guides with isotropic fillings in [1]. The modes of propagation of electromagnetic pulses in open slab waveguides were discussed in [13] using a wave splitting technique in the normal direction. In the present article, the results in [13] are modified to cover pulse propagation in open circular waveguides, *e.g.*, optical fibers. However, wave splitting is not referred to. Instead, the problem is solved using an Ansatz that gives the right radial dependence.

In section 2, the basic field equations for relevant for pulse propagation in simple (linear, homogeneous, isotropic), dispersive materials are presented. We make use of a complex time-dependent electromagnetic field, cf. Stratton [14]. Modes of propagation of pulses in the circular waveguide are analyzed in section 3. Specifically, conditions for propagation of pulse modes on the susceptibility kernels of the dielectric constituents are presented and dispersion equations derived. In appendix A, functions of causal convolution operators, several of which are referred to in section 3, are discussed.

2 Basic equations for simple, dispersive media

2.1 Notation

The following notation is used: position is denoted by $\boldsymbol{r} = (x, y, z)$, time by t, electric and magnetic field vectors by $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{H}(\boldsymbol{r}, t)$, respectively, and electric and magnetic flux densities by $\boldsymbol{D}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$. Each field vector is written in the form $\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{e}_x E_x(\boldsymbol{r}, t) + \boldsymbol{e}_y E_y(\boldsymbol{r}, t) + \boldsymbol{e}_z E_z(\boldsymbol{r}, t) = (E_x(\boldsymbol{r}, t), E_y(\boldsymbol{r}, t), E_z(\boldsymbol{r}, t))$, where $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, and $\boldsymbol{e}_z$ are the basis vectors in the Cartesian frame. The dynamics of the fields is modeled by the macroscopic Maxwell equations: $\nabla \times \boldsymbol{E}(\boldsymbol{r},t) = -\partial_t \boldsymbol{B}(\boldsymbol{r},t)$ and $\nabla \times \boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{J}(\boldsymbol{r},t) + \partial_t \boldsymbol{D}(\boldsymbol{r},t)$, where $\boldsymbol{J}(\boldsymbol{r},t)$ is the current density. For brevity, the independent variables $(\boldsymbol{r},t)$ are often suppressed. The speed of light in vacuum and the intrinsic impedance of vacuum are denoted by c_0 and η_0, respectively. The constitutive relations of a simple, causal, time-invariant, and continuous material is written in the form $c_0\eta_0\boldsymbol{D} = \varepsilon\boldsymbol{E}$ and $c_0\boldsymbol{B} = \mu\eta_0\boldsymbol{H}$, where the *relative* permittivity and permeability operators of the medium are $\varepsilon = 1+\chi^e(t)*$ and $\mu = 1+\chi^m(t)*$, respectively, and the asterisk $(*)$ denotes temporal convolution [7]: $[\varepsilon\boldsymbol{E}](\boldsymbol{r},t) = \boldsymbol{E}(\boldsymbol{r},t) + (\chi^e * \boldsymbol{E})(\boldsymbol{r},t) = \boldsymbol{E}(\boldsymbol{r},t) + \int_{-\infty}^{\infty} \chi^e(t-t')\boldsymbol{E}(\boldsymbol{r},t')\,dt'$, where the integral kernels $\chi^e(t)$ and $\chi^m(t)$ are the susceptibility kernels of the medium. Owing to causality, these functions vanish for $t<0$, and, for $t>0$, they are assumed to be twice continuously differentiable. At a few occasions, the temporal Heaviside step $H(t)$ appear. Finally, the positive square root is intended wherever the square-root operator $\surd$ appears.

2.2 Constitutive models

In this article, the continuity condition [4, 13]

$$\chi^e(+0) = \chi^m(+0) = 0. \tag{2.1}$$

is imposed on the susceptibility kernels of the isotropic medium. Condition (2.1) is met by the well-known Lorentz model (the resonance model),

$$\chi^e(t) = \frac{\omega_p^2}{\sqrt{\omega_0^2 - \left(\frac{\nu}{2}\right)^2}} \exp\left(-\frac{\nu t}{2}\right) \sin\left(\sqrt{\omega_0^2 - \left(\frac{\nu}{2}\right)^2}\, t\right) H(t),$$

which applies to bound electrons in insulators, and, by the Drude model,

$$\chi^e(t) = \frac{\omega_p^2}{\nu}\left(1 - \exp(-\nu t)\right) H(t),$$

which applies to free electrons in conductors (set $\omega_0 = 0$ in the Lorentz model), and by any linear combination of these models. On the other hand, the Debye model (the relaxation model) for polar liquids $\chi^e(t) = \alpha \exp(-\beta t) H(t)$ and Ohm's law for conductors (set $\beta = 0$ in the Debye model) violate the condition (2.1). Models that violate (2.1) have been described as "unphysical" in a major textbook on classical electrodynamics [6].

2.3 The complex electromagnetic field vector

Any time-dependent electromagnetic field $(\boldsymbol{E}, \boldsymbol{H})$ in a simple medium can be represented uniquely by the complex field vector [13, 14]

$$\boldsymbol{Q} = \frac{1}{2}\left(\boldsymbol{E} - i\mathcal{Z}\eta_0\boldsymbol{H}\right) = \frac{1}{2}\left(\boldsymbol{E} - i\mathcal{Y}^{-1}\eta_0\boldsymbol{H}\right), \tag{2.2}$$

which satisfies the first-order dispersive wave equation

$$\nabla \times \boldsymbol{Q} = -ic_0^{-1}\partial_t \mathcal{N}\, \boldsymbol{Q} - i\eta_0 \mathcal{Z}\boldsymbol{J}/2. \tag{2.3}$$

The real temporal integral operators

$$\begin{aligned}\mathcal{Z} &= 1 + Z(t)*,\\ \mathcal{Y} &= 1 + Y(t)*,\\ \mathcal{N} &= 1 + N(t)*\end{aligned}$$

are the relative intrinsic impedance, the relative intrinsic admittance, and the index of refraction of the medium, respectively. These operators are intrinsic operators of the medium related by $\mathcal{N} = \mu\mathcal{Y}$ and $\mathcal{N}\mathcal{Y} = \varepsilon$. In the non-magnetic case, $\mathcal{N} = \mathcal{Y}$.

Boundary conditions at an interface between two dispersive materials are that the tangential parts of $\mathrm{Re}\,(\boldsymbol{Q})$ and $\mathrm{Re}\,(i\mathcal{Y}\boldsymbol{Q})$ be continuous.

2.4 The intrinsic operators of the medium

Since $\mathcal{N}\mathcal{N} = \mu\varepsilon$, the refractive kernel $N(t)$ satisfies the Volterra integral equation of the second kind

$$2N(t) + (N * N)(t) = \chi^e(t) + \chi^m(t) + (\chi^e * \chi^m)(t).$$

Volterra integral equations of the second kind are uniquely solvable in the space of continuous functions in each compact time-interval and the solutions depend continuously on data [9]. Consequently, the refractive kernel inherits causality and smoothness properties from the susceptibility kernels.

The admittance and impedance kernels satisfy Volterra integral equation of the second kind

$$\begin{aligned}Y(t) + (Y * \chi^m)(t) &= N(t) - \chi^m(t),\\ Z(t) + (Z * N)(t) &= \chi^m(t) - N(t).\end{aligned}$$

and inherit causality and regularity from the susceptibility kernels. Observe that the continuity condition (2.1) implies that

$$N(+0) = Y(+0) = Z(+0) = 0. \tag{2.4}$$

2.5 Decomposition of the complex field

The complex electromagnetic field vector defined by (2.2) can be decomposed in its transverse and longitudinal components:

$$\boldsymbol{Q} = \boldsymbol{Q}_\perp + \boldsymbol{e}_z Q_z.$$

Similarly, the nabla operator is written as $\nabla = \nabla_\perp + \boldsymbol{e}_z\partial_z$ and the Laplacian as $\Delta = \Delta_\perp + \partial_z^2$. Decompositions as these are standard in waveguide theory.

Recall that the complex field satisfies the first-order vector wave equation (2.3)

$$\nabla \times \boldsymbol{Q} = -ic_0^{-1}\partial_t \mathcal{N}\,\boldsymbol{Q} \tag{2.5}$$

in the absence of the source term. Consequently,

$$\nabla \cdot \boldsymbol{Q} = 0. \tag{2.6}$$

Taking the curl of both members of equation (2.5) and using (2.6) result in a second-order vector wave equation for the complex field:

$$-\Delta \boldsymbol{Q} = -c_0^{-2}\partial_t^2 \mathcal{N}^2\,\boldsymbol{Q}. \tag{2.7}$$

Decomposing the complex field in transverse and longitudinal components identifies the transverse and longitudinal parts of (2.5):

$$\begin{aligned}
&\partial_z(\boldsymbol{e}_z \times \boldsymbol{Q}_\perp) + \nabla_\perp Q_z \times \boldsymbol{e}_z = -ic_0^{-1}\partial_t \mathcal{N}\,\boldsymbol{Q}_\perp,\\
&\nabla_\perp \times \boldsymbol{Q}_\perp = -ic_0^{-1}\partial_t \mathcal{N} Q_z \boldsymbol{e}_z.
\end{aligned}$$

The transverse part can be written as

$$\partial_z \boldsymbol{Q}_\perp = \nabla_\perp Q_z + ic_0^{-1}\partial_t \mathcal{N}\,\boldsymbol{e}_z \times \boldsymbol{Q}_\perp, \tag{2.8}$$

where

$$Q_z = -ic_0 \partial_t^{-1} \mathcal{N}^{-1} \nabla_\perp \cdot \boldsymbol{e}_z \times \boldsymbol{Q}_\perp. \tag{2.9}$$

Equation (2.8) expresses the dynamics of the transverse field along the axis of the waveguide in terms of the transverse gradient of the longitudinal field component, which can be interpreted as a driving term. Equation (2.9) expresses the longitudinal field component in terms of the transverse rotation of the transverse field.

3 Modes in open circular waveguides

Consider a straight, cylindric open waveguide extended in the z-direction. The core is denoted by V_1 and the cladding by V_2. The cross-sections of the core and of the cladding are denoted by Ω_1 and Ω_2 and the boundary of the cross-section by $\partial\Omega$. The reference direction of the normal vector field $\boldsymbol{n} = \boldsymbol{n}(\boldsymbol{r}_\perp)$ along $\partial\Omega$ is outward with respect to the core. The tangential vector field $\boldsymbol{\tau} = \boldsymbol{\tau}(\boldsymbol{r}_\perp)$ along $\partial\Omega$ is defined by $\boldsymbol{\tau} = \boldsymbol{n} \times \boldsymbol{e}_z$. The binormal vector field along $\partial\Omega$ is thus $\boldsymbol{e}_z$.

As in [13], it is appropriate to introduce the intrinsic operators

$$\mathcal{N} = \begin{cases} \mathcal{N}_1 = 1 + N_1(t)\,* & (\boldsymbol{r}_\perp \in V_1),\\ \mathcal{N}_2 = 1 + N_2(t)\,* & (\boldsymbol{r}_\perp \in V_2),\end{cases}$$

$$\mathcal{Y} = \begin{cases} \mathcal{Y}_1 = 1 + Y_1(t)\,* & (\boldsymbol{r}_\perp \in V_1),\\ \mathcal{Y}_2 = 1 + Y_2(t)\,* & (\boldsymbol{r}_\perp \in V_2),\end{cases}$$

and

$$\mathscr{Z} = \begin{cases} \mathscr{Z}_1 = 1 + Z_1(t) * & (\boldsymbol{r}_\perp \in V_1), \\ \mathscr{Z}_2 = 1 + Z_2(t) * & (\boldsymbol{r}_\perp \in V_2). \end{cases}$$

for the two constituents and define wavenumber operators as

$$\mathcal{K} = c_0^{-1}\partial_t \mathcal{N} = \begin{cases} \mathcal{K}_1 = c_0^{-1}\partial_t \mathcal{N}_1 & (\boldsymbol{r}_\perp \in V_1), \\ \mathcal{K}_2 = c_0^{-1}\partial_t \mathcal{N}_2 & (\boldsymbol{r}_\perp \in V_2), \end{cases}$$

The general idea of an open waveguide is that it should support modes that

1. travel along the guide only, *i.e.*, not in transverse directions, and
2. are confined mainly to the core in order to carry finite energy.

By definition, pulses are propagated along the z-axis only, and owing to the absence of optical response in permittivity and permeability operators of the constituents [11], wave-fronts travel with the vacuum speed c_0. The aim is to look for up-going or down-going modes of propagation with the z-dependencies

$$\exp\left(\mp z c_0^{-1}\partial_t \mathcal{N}_z\right), \tag{3.1}$$

respectively, where the real temporal integral operator

$$\mathcal{N}_z = 1 + N_z(t) * \qquad \text{(for all } \boldsymbol{r}_\perp),$$

is referred to as the longitudinal refractive index. The integral kernel $N_z(t)$ is supposed to inherit causality and regularity from the susceptibility kernels; in particular

$$N_z(+0) = 0. \tag{3.2}$$

The propagator (3.1) can, therefore, be factored as

$$\delta\left(t \mp z c_0^{-1}\right) * \exp\left(\mp z c_0^{-1} N_z'(t) *\right) = \delta\left(t \mp z c_0^{-1}\right) * \left(1 + P^{\mp}(z,t) *\right),$$

where the kernels $P^{\mp}(z,t)$, for fixed z, satisfy the temporal Volterra integral equations of the second kind [8]

$$t P^{\mp}(z,t) = \mp t z c_0^{-1} N_z'(t) \mp \left(t z c_0^{-1} N_z' * P^{\mp}\right)(z,t) \tag{3.3}$$

in terms of the kernel $N_z'(t)$. In particular, $P^{\mp}(0,t) = 0$, and, by differentiation of both members of (3.3), $P^{\mp}(z,+0) = \mp z c_0^{-1} N_z'(+0)$. The short-hand notation

$$\partial_z = \mp c_0^{-1}\partial_t \mathcal{N}_z = \mp \mathcal{K}_z \tag{3.4}$$

is used frequently below, depending on whether the mode is up-going or down-going.

For modes with the z-dependencies (3.4), one can write

$$\boldsymbol{Q}^{\pm}(\boldsymbol{r}_\perp, z, t) = \exp\left(\mp z \mathcal{K}_z\right) \boldsymbol{q}^{\pm}(\boldsymbol{r}_\perp, t), \tag{3.5}$$

and the transverse equation (2.8) becomes

$$(\mp\mathcal{K}_z\mathbf{I}_{\perp\perp} - i\mathcal{K}\,\boldsymbol{e}_z \times \mathbf{I}_{\perp\perp}) \cdot \boldsymbol{q}_\perp^\pm = \nabla_\perp q_z^\pm,$$

where $\mathbf{I}_{\perp\perp}$ is the identity operator in the transverse plane. Multiplying both members of this equality by $(\mp\mathcal{K}_z\mathbf{I}_{\perp\perp} + i\mathcal{K}\,\boldsymbol{e}_z \times \mathbf{I}_{\perp\perp})$ gives

$$\left(\mathcal{K}_z^2 - \mathcal{K}^2\right) \boldsymbol{q}_\perp^\pm = (\mp\mathcal{K}_z\mathbf{I}_{\perp\perp} + i\mathcal{K}\,\boldsymbol{e}_z \times \mathbf{I}_{\perp\perp}) \cdot \nabla_\perp q_z^\pm,$$

and assuming that the operator

$$\left(\mathcal{K}_z^2 - \mathcal{K}^2\right)^{-1} = (-\nabla_\perp \cdot \nabla_\perp)^{-1}$$

exists gives

$$\boldsymbol{q}_\perp^\pm = \left(\mathcal{K}_z^2 - \mathcal{K}^2\right)^{-1} (\mp\mathcal{K}_z\mathbf{I}_{\perp\perp} + i\mathcal{K}\,\boldsymbol{e}_z \times \mathbf{I}_{\perp\perp}) \cdot \nabla_\perp q_z^\pm.$$

By (3.5) and the wave equation (2.7), one has

$$-\Delta_\perp q_z^\pm(\boldsymbol{r}_\perp, t) = \left(\mathcal{K}_z^2 - \mathcal{K}^2\right) q_z^\pm(\boldsymbol{r}_\perp, t) \tag{3.6}$$

in the core and in the cladding.

At the boundary,

$$\boldsymbol{\tau} \cdot \boldsymbol{q}_\perp^\pm = -\left(\mathcal{K}_z^2 - \mathcal{K}^2\right)^{-1} \left(\pm\mathcal{K}_z\partial_\tau q_z^\pm + i\mathcal{K}\,\partial_n q_z^\pm\right),$$

where the tangential and normal derivatives are given by

$$\partial_\tau q_z^\pm = \boldsymbol{\tau} \cdot \nabla_\perp q_z^\pm = -\boldsymbol{e}_z \cdot \boldsymbol{n} \times \nabla_\perp q_z^\pm, \qquad \partial_n q_z^\pm = \boldsymbol{n} \cdot \nabla_\perp q_z^\pm.$$

Recall that the quantities

$$\begin{cases} \mathrm{Re}\,(q_z^\pm), \\ \mathrm{Re}\,(i\mathcal{Y} q_z^\pm), \\ \mathrm{Re}\,(\boldsymbol{\tau} \cdot \boldsymbol{q}_\perp^\pm), \\ \mathrm{Re}\,(i\mathcal{Y}\boldsymbol{\tau} \cdot \boldsymbol{q}_\perp^\pm) \end{cases} \tag{3.7}$$

are continuous at the boundary $\partial\Omega$.

3.1 Condition for existence of pulse modes

For a specific mode, (3.4) applies, and [13]

$$\begin{aligned} \mathcal{K}^2 - \partial_z^2 =& \mathcal{K}^2 - \mathcal{K}_z^2 = c_0^{-2}\partial_t^2 \left(2 + (N + N_z)(t)*\right) (N - N_z)(t) * \\ =& c_0^{-2} \left(2 + (N + N_z)(t)*\right) \left((N - N_z)'(+0) + (N - N_z)''(t)*\right), \end{aligned}$$

where the initial conditions (3.2) and (2.4) have been used. The positive square-root of this operator is well defined if only $N'(+0) \neq N_z'(+0)$. Thus,

$$N_1'(+0) \neq N_z'(+0) \neq N_2'(+0). \tag{3.8}$$

From the proceeding sections follows that pulse modes with finite energy cannot propagate unless

$$N_1'(+0) \leq N_z'(+0) \leq N_2'(+0), \tag{3.9}$$

a result that was obtained for the slab waveguide [13]. Combining (3.8) and (3.9) gives

$$N_1'(+0) < N_z'(+0) < N_2'(+0), \tag{3.10}$$

and, in particular,

$$N_1'(+0) < N_2'(+0), \tag{3.11}$$

that is,

$$(\chi_1^e)'(+0) + (\chi_1^m)'(+0) < (\chi_2^e)'(+0) + (\chi_2^m)'(+0),$$

which is the condition for existence of propagating finite energy pulse modes in the slab. A discussion of the condition (3.11) and the corresponding condition for propagating time-harmonic modes of angular frequency ω (in the lossless case) [2]

$$n_2(\omega) < n_1(\omega),$$

where the index of refraction $n_i(\omega)$ is the Fourier transform of the distribution $\delta(t) + N_i(t)$, where $\delta(t)$ is the Dirac delta function ($i = 1, 2$), can be found in [13].

In view of the above inequalities, it is appropriate to introduce dimensionless numbers defined by

$$\lambda_i = 2R^2 c_0^{-2} N_i'(+0) \qquad (i = 1, 2, z), \tag{3.12}$$

where R is a characteristic length parameter of the problem (later specified to be the radius of a circular cylindric geometry).

3.2 Open circular waveguides

The idea of the open waveguide is that the propagating pulse should be confined to the core mainly. Therefore, an appropriate way of formulating the problem (3.6) is

$$\begin{aligned} -\Delta_\perp q_z^\pm(\boldsymbol{r}_\perp, t) &= \left(\sqrt{\mathcal{K}_z^2 - \mathcal{K}_1^2}\right)^2 q_z^\pm(\boldsymbol{r}_\perp, t) \qquad (\boldsymbol{r}_\perp \in \Omega_1)\,, \\ \Delta_\perp q_z^\pm(\boldsymbol{r}_\perp, t) &= \left(\sqrt{\mathcal{K}_2^2 - \mathcal{K}_z^2}\right)^2 q_z^\pm(\boldsymbol{r}_\perp, t) \qquad (\boldsymbol{r}_\perp \in \Omega_2) \end{aligned}$$

in the core and in the cladding, respectively. A brief theory of functions of temporal integral operators is given in section A.

One of the relevant square-root operators is of the form

$$R\sqrt{\mathcal{K}_2^2 - \mathcal{K}_z^2} = \sqrt{\lambda_2 - \lambda_z} + U_2(t)*,$$

where the λ's are given by (3.12) and

$$\begin{aligned} &2\sqrt{\lambda_2-\lambda_z}U_2(t)+\left(U_2*U_2\right)(t)=\left(\lambda_2-\lambda_z\right)\left(N_2(t)+N_z(t)\right)/2 \\ &+\left[2(N_2''(t)-N_z''(t))+\left((N_2''-N_z'')*\left(N_2+N_z\right)\right)(t)\right]\left(Rc_0^{-1}\right)^2. \end{aligned} \tag{3.13}$$

Since $N_z(t)=(H*N_z')(t)$ combined with

$$N_z'(t)=N_z'(+0)H(t)+(H*N_z'')(t),$$

gives

$$N_z(t)=N_z'(+0)tH(t)+((tH)*N_z'')(t),$$

equation (3.13) is a temporal Volterra integral equation of the second kind in the unknown kernels $U_2(t)$ and $N_z''(t)$ for a fixed value of $N_z'(+0)$. Observe that $N_2''(t)$ and for that matter $N_1''(t)$ are known quantities since differentiation of both members of equation (2.4) gives the temporal Volterra integral equation of the second kind

$$2N''(t)+N'(+0)N(t)+(N*N'')(t)=(\chi^e)''(t)+(\chi^m)''(t)+((\chi^e)'*(\chi^m)')(t),$$

where $N(t)$ in the left member and the function in the right member are known, and $2N'(+0)=(\chi^e)'(+0)+(\chi^m)'(+0)$ is known as well.

The other square-root operator of interest is of the form

$$R\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}=\sqrt{\lambda_z-\lambda_1}+U_1(t)*,$$

where

$$\begin{aligned} &2\sqrt{\lambda_z-\lambda_1}U_1(t)+\left(U_1*U_1\right)(t)=\left(\lambda_z-\lambda_1\right)\left(N_z(t)+N_1(t)\right)/2 \\ &+\left[2(N_z''(t)-N_1''(t))+\left((N_z''-N_1'')*\left(N_z+N_1\right)\right)(t)\right]\left(Rc_0^{-1}\right)^2. \end{aligned} \tag{3.14}$$

This is a temporal Volterra integral equation of the second kind in the kernels $U_1(t)$ and $N_z''(t)$ for a fixed value of $N_z'(+0)$.

The boundary conditions for the longitudinal fields are

$$\begin{cases} \operatorname{Re}\left\{q_z^{\pm}(\boldsymbol{r}_\perp,t)\right\}_1=\operatorname{Re}\left\{q_z^{\pm}(\boldsymbol{r}_\perp,t)\right\}_2 & (\boldsymbol{r}_\perp\in\partial\Omega)\,, \\ \operatorname{Re}\left\{i\mathcal{Y}q_z^{\pm}(\boldsymbol{r}_\perp,t)\right\}_1=\operatorname{Re}\left\{i\mathcal{Y}q_z^{\pm}(\boldsymbol{r}_\perp,t)\right\}_2 & (\boldsymbol{r}_\perp\in\partial\Omega)\,, \end{cases}$$

whereas the boundary conditions for the transverse fields are

$$\begin{cases} \operatorname{Re}\Big\{(\sqrt{\mathcal{K}_z^2-\mathcal{K}^2})^{-2}\big(\pm\mathcal{K}_z\partial_\tau q_z^{\pm}(\boldsymbol{r}_\perp,t)+i\mathcal{K}\partial_n q_z^{\pm}(\boldsymbol{r}_\perp,t)\big)\Big\}_1 & \\ =\operatorname{Re}\Big\{-(\sqrt{\mathcal{K}^2-\mathcal{K}_z^2})^{-2}\big(\pm\mathcal{K}_z\partial_\tau q_z^{\pm}(\boldsymbol{r}_\perp,t)+i\mathcal{K}\partial_n q_z^{\pm}(\boldsymbol{r}_\perp,t)\big)\Big\}_2 & (\boldsymbol{r}_\perp\in\partial\Omega)\,, \\ \operatorname{Re}\Big\{i\mathcal{Y}(\sqrt{\mathcal{K}_z^2-\mathcal{K}^2})^{-2}\big(\pm\mathcal{K}_z\partial_\tau q_z^{\pm}(\boldsymbol{r}_\perp,t)+i\mathcal{K}\partial_n q_z^{\pm}(\boldsymbol{r}_\perp,t)\big)\Big\}_1 & \\ =\operatorname{Re}\Big\{-i\mathcal{Y}(\sqrt{\mathcal{K}^2-\mathcal{K}_z^2})^{-2}\big(\pm\mathcal{K}_z\partial_\tau q_z^{\pm}(\boldsymbol{r}_\perp,t)+i\mathcal{K}\partial_n q_z^{\pm}(\boldsymbol{r}_\perp,t)\big)\Big\}_2 & (\boldsymbol{r}_\perp\in\partial\Omega)\,. \end{cases}$$

For a circular geometry with radius R, the appropriate Ansatz is

$$q_z^{\pm} = \begin{cases} J_\nu\left(r\sqrt{\mathcal{K}_z^2 - \mathcal{K}_1^2}\right)\left(J_\nu\left(R\sqrt{\mathcal{K}_z^2 - \mathcal{K}_1^2}\right)\right)^{-1}\left(e^{-i\nu\phi}f_1^{\pm} + e^{i\nu\phi}g_1^{\pm}\right) & (r < R), \\ K_\nu\left(r\sqrt{\mathcal{K}_2^2 - \mathcal{K}_z^2}\right)\left(K_\nu\left(R\sqrt{\mathcal{K}_2^2 - \mathcal{K}_z^2}\right)\right)^{-1}\left(e^{-i\nu\phi}f_2^{\pm} + e^{i\nu\phi}g_2^{\pm}\right) & (r > R), \end{cases}$$

where ν is an arbitrary non-negative integer, J_ν is a Bessel function of the first kind, K_ν is a modified Bessel function of the second kind, and $f_1^{\pm}(t)$, $g_1^{\pm}(t)$, $f_2^{\pm}(t)$, and $g_2^{\pm}(t)$ primarily are arbitrary complex-valued functions. According to the results obtained in section A, the modified Bessel functions operators are given by

$$K_\nu\left(r\sqrt{\mathcal{K}_2^2 - \mathcal{K}_z^2}\right) = K_\nu\left(r/R\sqrt{\lambda_2 - \lambda_z}\right) + V_2(r,t)*,$$

where the kernels $V_2(r,t)$ for fixed r satisfy the temporal Volterra integral equations of the second kind

$$\begin{aligned} &(r/R)^2\left(\sqrt{\lambda_2 - \lambda_z} + U_2(t)*\right)^2 (tW_2(r,t)) \\ &+ r/R\left(\sqrt{\lambda_2 - \lambda_z} + U_2(t)*\right)(tV_2(r,t)) \\ &- \left((r/R)^2\left(\sqrt{\lambda_2 - \lambda_z} + U_2(t)*\right)^2 + \nu^2\right) \\ &\left(K_\nu\left(r/R\sqrt{\lambda_2 - \lambda_z}\right) + V_2(r,t)*\right)\left(trR^{-1}U_2(t)\right) = 0, \end{aligned} \tag{3.15}$$

where

$$tV_2(r,t) = \left(K'_\nu\left(r/R\sqrt{\lambda_2 - \lambda_z}\right) + W_2(r,t)*\right)\left(trR^{-1}U_2(t)\right) \tag{3.16}$$

in terms of the kernel $U_2(t)$. Similarly, the Bessel functions operators are given by

$$J_\nu\left(r\sqrt{\mathcal{K}_z^2 - \mathcal{K}_1^2}\right) = J_\nu\left(r/R\sqrt{\lambda_z - \lambda_1}\right) + V_1(r,t)*,$$

where the kernels $V_1(r,t)$ for fixed r satisfy the temporal Volterra integral equations of the second kind

$$\begin{aligned} &(r/R)^2\left(\sqrt{\lambda_z - \lambda_1} + U_1(t)*\right)^2 (tW_1(r,t)) \\ &+ r/R\left(\sqrt{\lambda_z - \lambda_1} + U_1(t)*\right)(tV_1(r,t)) \\ &+ \left((r/R)^2\left(\sqrt{\lambda_z - \lambda_1} + U_1(t)*\right)^2 - \nu^2\right) \\ &\left(J_\nu\left(r/R\sqrt{\lambda_z - \lambda_1}\right) + V_1(r,t)*\right)\left(trR^{-1}U_1(t)\right) = 0, \end{aligned} \tag{3.17}$$

where

$$tV_1(r,t) = \left(J_\nu'\left(r/R\sqrt{\lambda_z-\lambda_1}\right) + W_1(r,t)*\right)\left(trR^{-1}U_1(t)\right) \tag{3.18}$$

in terms of the kernel $U_1(t)$.

Observe, that, for $r = R$, equations (3.13), (3.15), (3.16),(3.14), (3.17), and (3.18) constitute six coupled Volterra integral equations of the second kind in the seven kernels $N_z''(t)$, $U_1(t)$, $U_2(t)$, $V_1(R,t)$, $V_2(R,t)$, $W_1(R,t)$, and $W_2(R,t)$. The dispersion relation presented below provides us with a seventh Volterra integral equation of the second kind in the seven kernels above. Solving these equations simultaneously determines the modes of propagation.

The boundary conditions for the longitudinal field components imply that

$$\begin{cases} f_1^\pm + \left(g_1^\pm\right)^* = f_2^\pm + \left(g_2^\pm\right)^*, \\ \mathcal{Y}_1\left(f_1^\pm - \left(g_1^\pm\right)^*\right) = \mathcal{Y}_2\left(f_2^\pm - \left(g_2^\pm\right)^*\right). \end{cases} \tag{3.19}$$

Differentiating the longitudinal field components gives

$$\partial_\tau q_z^\pm = \begin{cases} i\frac{\nu}{r}J_\nu\left(r\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\right)\left(J_\nu\left(R\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\right)\right)^{-1}\left(e^{-i\nu\phi}f_1^\pm - e^{i\nu\phi}g_1^\pm\right) & (r<R), \\ i\frac{\nu}{r}K_\nu\left(r\sqrt{\mathcal{K}_2^2-\mathcal{K}_z^2}\right)\left(K_\nu\left(R\sqrt{\mathcal{K}_2^2-\mathcal{K}_z^2}\right)\right)^{-1}\left(e^{-i\nu\phi}f_2^\pm - e^{i\nu\phi}g_2^\pm\right) & (r>R) \end{cases}$$

and

$$\partial_n q_z^\pm = \begin{cases} \partial_r J_\nu\left(r\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\right)\left(J_\nu\left(R\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\right)\right)^{-1}\left(e^{-i\nu\phi}f_1^\pm + e^{i\nu\phi}g_1^\pm\right) & (r<R), \\ \partial_r K_\nu\left(r\sqrt{\mathcal{K}_2^2-\mathcal{K}_z^2}\right)\left(K_\nu\left(R\sqrt{\mathcal{K}_2^2-\mathcal{K}_z^2}\right)\right)^{-1}\left(e^{-i\nu\phi}f_2^\pm + e^{i\nu\phi}g_2^\pm\right) & (r>R). \end{cases}$$

Using the boundary conditions for the transverse fields gives

$$\begin{aligned} &\pm\left(\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\right)^{-2}\frac{\nu}{R}\mathcal{K}_z\left(f_1^\pm + \left(g_1^\pm\right)^*\right) \\ &+\left(\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\right)^{-2}\mathcal{K}_1\partial_r J_\nu\left(R\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\right)\left(J_\nu\left(R\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\right)\right)^{-1}\left(f_1^\pm - \left(g_1^\pm\right)^*\right) \\ =&\mp\left(\sqrt{\mathcal{K}_2^2-\mathcal{K}_z^2}\right)^{-2}\frac{\nu}{R}\mathcal{K}_z\left(f_2^\pm + \left(g_2^\pm\right)^*\right) \\ &-\left(\sqrt{\mathcal{K}_2^2-\mathcal{K}_z^2}\right)^{-2}\mathcal{K}_2\partial_r K_\nu\left(R\sqrt{\mathcal{K}_2^2-\mathcal{K}_z^2}\right)\left(K_\nu\left(R\sqrt{\mathcal{K}_2^2-\mathcal{K}_z^2}\right)\right)^{-1}\left(f_2^\pm - \left(g_2^\pm\right)^*\right) \end{aligned}$$

and

$$
\begin{aligned}
&\pm \mathcal{Y}_1 \left(\sqrt{\mathcal{K}_z^2 - \mathcal{K}_1^2}\right)^{-2} \frac{\nu}{R}\mathcal{K}_z \left(f_1^\pm - \left(g_1^\pm\right)^*\right) \\
&+ \mathcal{Y}_1 \left(\sqrt{\mathcal{K}_z^2 - \mathcal{K}_1^2}\right)^{-2} \mathcal{K}_1 \partial_r J_\nu \left(R\sqrt{\mathcal{K}_z^2 - \mathcal{K}_1^2}\right) \left(J_\nu \left(R\sqrt{\mathcal{K}_z^2 - \mathcal{K}_1^2}\right)\right)^{-1} \left(f_1^\pm + \left(g_1^\pm\right)^*\right) \\
= &\mp \mathcal{Y}_2 \left(\sqrt{\mathcal{K}_2^2 - \mathcal{K}_z^2}\right)^{-2} \frac{\nu}{R}\mathcal{K}_z \left(f_2^\pm - \left(g_2^\pm\right)^*\right) \\
&- \mathcal{Y}_2 \left(\sqrt{\mathcal{K}_2^2 - \mathcal{K}_z^2}\right)^{-2} \mathcal{K}_2 \partial_r K_\nu \left(R\sqrt{\mathcal{K}_2^2 - \mathcal{K}_z^2}\right) \left(K_\nu \left(R\sqrt{\mathcal{K}_2^2 - \mathcal{K}_z^2}\right)\right)^{-1} \left(f_2^\pm + \left(g_2^\pm\right)^*\right).
\end{aligned}
$$

The functions $\left(f_2^\pm \pm \left(g_2^\pm\right)^*\right)$ can be eliminated using (3.19), resulting in two equations in the unknowns $\left(f_1^\pm \pm \left(g_1^\pm\right)^*\right)$. Consequently, one obtains the system of integral equations

$$
\begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{pmatrix} \begin{pmatrix} f_1^\pm + \left(g_1^\pm\right)^* \\ f_1^\pm - \left(g_1^\pm\right)^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
$$

where

$$
\begin{cases}
\mathcal{A}_{11}^\pm = \mathcal{A}_{22}^\pm = \pm \dfrac{\nu}{R}\mathcal{K}_z \left(\left(\sqrt{\mathcal{K}_z^2 - \mathcal{K}_1^2}\right)^{-2} + \left(\sqrt{\mathcal{K}_2^2 - \mathcal{K}_z^2}\right)^{-2}\right), \\
\mathcal{A}_{12}^\pm = \mathcal{K}_1 \left(\sqrt{\mathcal{K}_z^2 - \mathcal{K}_1^2}\right)^{-2} \partial_r J_\nu \left(R\sqrt{\mathcal{K}_z^2 - \mathcal{K}_1^2}\right) \left(J_\nu \left(R\sqrt{\mathcal{K}_z^2 - \mathcal{K}_1^2}\right)\right)^{-1} \\
\qquad + \mathcal{K}_2 \left(\sqrt{\mathcal{K}_2^2 - \mathcal{K}_z^2}\right)^{-2} \partial_r K_\nu \left(R\sqrt{\mathcal{K}_2^2 - \mathcal{K}_z^2}\right) \left(K_\nu \left(R\sqrt{\mathcal{K}_2^2 - \mathcal{K}_z^2}\right)\right)^{-1} \mathcal{Y}_2^{-1}\mathcal{Y}_1, \\
\mathcal{A}_{21}^\pm = \mathcal{K}_1 \left(\sqrt{\mathcal{K}_z^2 - \mathcal{K}_1^2}\right)^{-2} \partial_r J_\nu \left(R\sqrt{\mathcal{K}_z^2 - \mathcal{K}_1^2}\right) \left(J_\nu \left(R\sqrt{\mathcal{K}_z^2 - \mathcal{K}_1^2}\right)\right)^{-1} \\
\qquad + \mathcal{K}_2 \left(\sqrt{\mathcal{K}_2^2 - \mathcal{K}_z^2}\right)^{-2} \partial_r K_\nu \left(R\sqrt{\mathcal{K}_2^2 - \mathcal{K}_z^2}\right) \left(K_\nu \left(R\sqrt{\mathcal{K}_2^2 - \mathcal{K}_z^2}\right)\right)^{-1} \mathcal{Y}_1^{-1}\mathcal{Y}_2,
\end{cases}
$$

Letting $f_1^\pm + \left(g_1^\pm\right)^*$ be arbitrary, $f_1^\pm - \left(g_1^\pm\right)^*$ are determined by

$$
f_1^\pm - \left(g_1^\pm\right)^* = -\left(\mathcal{A}_{11}^\pm\right)^{-1} \mathcal{A}_{12}^\pm \left(f_1^\pm + \left(g_1^\pm\right)^*\right).
$$

The conditions for non-trivial solutions are

$$
\mathcal{A}_{11}^\pm \mathcal{A}_{22}^\pm - \mathcal{A}_{12}^\pm \mathcal{A}_{21}^\pm = 0.
$$

Two different kinds are distinguishable, namely

$$
\begin{aligned}
&\mathcal{K}_1\left(\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\right)^{-1} J_\nu'\left(R\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\right)\left(J_\nu\left(R\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\right)\right)^{-1}\\
&=-\mathcal{K}_2\left(\sqrt{\mathcal{K}_2^2-\mathcal{K}_z^2}\right)^{-1} K_\nu'\left(R\sqrt{\mathcal{K}_2^2-\mathcal{K}_z^2}\right)\left(K_\nu\left(R\sqrt{\mathcal{K}_2^2-\mathcal{K}_z^2}\right)\right)^{-1}\mathcal{M}_+\\
&\quad\pm\sqrt{\left(\mathcal{K}_2\left(\sqrt{\mathcal{K}_2^2-\mathcal{K}_z^2}\right)^{-1} K_\nu'\left(R\sqrt{\mathcal{K}_2^2-\mathcal{K}_z^2}\right)\left(K_\nu\left(R\sqrt{\mathcal{K}_2^2-\mathcal{K}_z^2}\right)\right)^{-1}\mathcal{M}_-\right)^2+\mathcal{A}_{11}^2},
\end{aligned}
\tag{3.20}
$$

where

$$
\mathcal{M}_\pm=\frac{\mathcal{Y}_1^{-1}\mathcal{Y}_2\pm\mathcal{Y}_2^{-1}\mathcal{Y}_1}{2}.
$$

This is the dispersion equation for the open waveguide.

In particular, the initial values that determine the modes are given by

$$
\begin{aligned}
&\frac{J_\nu'\left(\sqrt{\lambda_z-\lambda_1}\right)}{\sqrt{\lambda_z-\lambda_1}J_\nu\left(\sqrt{\lambda_z-\lambda_1}\right)}\mp\frac{\nu}{\left(\sqrt{\lambda_z-\lambda_1}\right)^2}\\
&=-\frac{K_\nu'\left(\sqrt{\lambda_2-\lambda_z}\right)}{\sqrt{\lambda_2-\lambda_z}K_\nu\left(\sqrt{\lambda_2-\lambda_z}\right)}\pm\frac{\nu}{\left(\sqrt{\lambda_2-\lambda_z}\right)^2}\qquad(\lambda_1<\lambda_z<\lambda_2),
\end{aligned}
$$

that is[1],

$$
\mp\frac{J_{\nu\pm1}\left(\sqrt{\lambda_z-\lambda_1}\right)}{\sqrt{\lambda_z-\lambda_1}J_\nu\left(\sqrt{\lambda_z-\lambda_1}\right)}=\frac{K_{\nu\pm1}\left(\sqrt{\lambda_2-\lambda_z}\right)}{\sqrt{\lambda_2-\lambda_z}K_\nu\left(\sqrt{\lambda_2-\lambda_z}\right)}\qquad(\lambda_1<\lambda_z<\lambda_2),\tag{3.21}
$$

where the upper equation is valid when $\nu=0$ and the inequalities $\lambda_1<\lambda_z<\lambda_2$ are assumed to hold in both cases, implying that the right and left members are both real[2]. This is indeed the case; for if $\lambda_z>\lambda_2$, then[3]

$$
\begin{aligned}
&\frac{J_\nu'\left(\sqrt{\lambda_z-\lambda_1}\right)}{\sqrt{\lambda_z-\lambda_1}J_\nu\left(\sqrt{\lambda_z-\lambda_1}\right)}\mp\frac{\nu}{\left(\sqrt{\lambda_z-\lambda_1}\right)^2}\\
&=-\frac{\left(H_\nu^{(1)}\right)'\left(-\sqrt{\lambda_z-\lambda_2}\right)}{\sqrt{\lambda_z-\lambda_2}H_\nu^{(1)}\left(-\sqrt{\lambda_z-\lambda_2}\right)}\mp\frac{\nu}{\left(\sqrt{\lambda_z-\lambda_2}\right)^2}\qquad(\lambda_z>\lambda_2),
\end{aligned}
$$

[1] Use the formulae $J_\nu'(z)=\mp J_{\nu\pm1}(z)\pm\nu J_\nu(z)/z$ and $K_\nu'(z)=-K_{\nu\pm1}(z)\pm\nu K_\nu(z)/z$.

[2] Recall that the Bessel functions and the modified Bessel functions of the first and second kinds are real for real arguments, and that, by definition, $I_\nu(z)=i^{-n}J_\nu(iz)$.

[3] Recall that, by definition, $K_\nu(z)=(\pi/2)\cdot i^{n+1}H_\nu^{(1)}(iz)$.

in which the right member is imaginary[4], whereas the left member is real, and if $\lambda_z < \lambda_1$, then, by definition,

$$-\frac{I_{\nu\pm1}\left(\sqrt{\lambda_1-\lambda_z}\right)}{\sqrt{\lambda_1-\lambda_z}I_\nu\left(\sqrt{\lambda_1-\lambda_z}\right)} = \frac{K_{\nu\pm1}\left(\sqrt{\lambda_2-\lambda_z}\right)}{\sqrt{\lambda_2-\lambda_z}K_\nu\left(\sqrt{\lambda_2-\lambda_z}\right)} \qquad (\lambda_1, \lambda_2 > \lambda_z),$$

in which the left member is negative, whereas the right member is positive[5]. In either case, no solutions exist, and, consequently, equation (3.21) holds. Observe that inequality (3.10) follows.

Before discussing the solutions of equation (3.21), it is appropriate to recall some elementary properties of the Bessel functions of the first kind that will be used tacitly in the next paragraph. Denoting the positive zeros of J_ν in increasing order by $\xi_{\nu k}$, $k = 1, 2, 3, \cdots$, one has, as a consequence of Rolle's theorem and some Bessel function recursion formulae, for any k, that

$$0 < \xi_{\nu k} < \xi_{(\nu+1)k} < \xi_{\nu k+1} \qquad (\nu = 0, 1, 2, 3, \cdots).$$

Moreover, $J'_\nu(\xi_{\nu k}) \neq 0$ for all k, so that J_ν assumes both positive and negative values in a neighborhood of any of its zeros.

Equation (3.21) has at most a finite number of solutions. The right member is a positive, continuous function that is finite at all points λ_z except at the endpoint $\lambda_z = \lambda_2$ for all the upper cases and for the lower case when $\nu = 1$. For the lower cases $\nu > 1$, one has

$$\lim_{\lambda_z \to \lambda_2} \frac{K_{\nu-1}\left(\sqrt{\lambda_2-\lambda_z}\right)}{\sqrt{\lambda_2-\lambda_z}K_\nu\left(\sqrt{\lambda_2-\lambda_z}\right)} = \frac{1}{2(\nu-1)}.$$

The left member is a piecewise continuous function that is infinite at the endpoint $\lambda_z = \lambda_1$ for all the lower cases with the limit value $+\infty$, and at the zeros $\lambda_z = \lambda_1 + \xi_{\nu k}^2$ of $J_\nu\left(\sqrt{\lambda_z-\lambda_1}\right)$ with the limit values $-\infty$ and $+\infty$ depending on whether one approaches the zero from the left or from the right. For the upper cases, one has

$$\lim_{\lambda_z \to \lambda_1} -\frac{J_{\nu+1}\left(\sqrt{\lambda_z-\lambda_1}\right)}{\sqrt{\lambda_z-\lambda_1}J_\nu\left(\sqrt{\lambda_z-\lambda_1}\right)} = -\frac{1}{2(\nu+1)}.$$

The zeros of the left member are those of $J_{\nu\pm1}\left(\sqrt{\lambda_z-\lambda_1}\right)$ for the upper and lower cases, respectively, that is, $\lambda_z = \lambda_1 + \xi_{(\nu\pm1)k}^2$. Using these facts, one can draw the following conclusions:

Upper case. No solutions exist if $\lambda_2 \leq \lambda_1 + \xi_{\nu 1}^2$ and at least k solutions exist if $\lambda_2 \geq \lambda_1 + \xi_{(\nu+1)k}^2$. The different situations are illustrated in Figure 1 and Figure 2.

---

[4]In order that the right member be real, there must be a real number a, such that $\left(H_\nu^{(1)}\right)'(x) = a\left(H_\nu^{(1)}\right)(x)$ for $x = \sqrt{\lambda_z-\lambda_2}$, and, consequently, $J'_\nu(x) = aJ_\nu(x)$ and $Y'_\nu(x) = aY_\nu(x)$. This implies that the Wronskian $W(J_\nu, Y_\nu)(x) \equiv J_\nu Y'_\nu(x) - Y_\nu(x)J'_\nu(x) = 0$, contradicting the well-known result that $W(J_\nu, Y_\nu)(x) = -1/x$.

[5]Recall that the modified Bessel functions I_ν and K_ν are positive for (real) positive arguments.

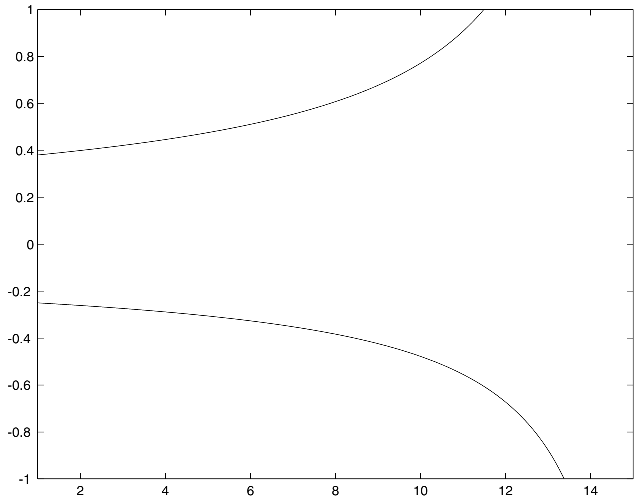

Figure 1: Right and left members of equation (3.21) for the upper case when $\nu = 1$, $\lambda_1 = 1$, and $\lambda_2 = 15$. There are no solutions.

Lower case. When $\nu = 1$, at least one solution exists. When $\nu > 1$, then at least k solutions exist if $\lambda_2 \geq \lambda_1 + \xi^2_{(\nu-1)k}$. The situation for $\nu = 1$ is depicted in Figure 3 and Figure 4.

Concluding, solving equation (3.21) gives the set of possible initial derivatives $N_z'(+0)$ of the longitudinal refractive kernel that determines the variety of pulse modes. Once $N_z'(+0)$ has been obtained, equations (3.20), (3.13), (3.15), (3.16), (3.14), (3.17), and (3.18) constitute a system of seven coupled Volterra integral equations of the second kind in the seven kernels $N_z''(t)$, $U_1(t)$, $U_2(t)$, $V_1(R,t)$, $V_2(R,t)$, $W_1(R,t)$, and $W_2(R,t)$. These seven equations are to be solved simultaneously, and the solutions determine the pulse modes. The problem is thus similar to the corresponding one for the slab waveguide [13].

It remains to solve the problem presented in this article and the problem in [13] in full detail numerically for realistic susceptibility kernels. This is left as an open problem.

Appendix A Functions of causal convolution operators

In this section some functions of causal convolution operators are defined and convolution equations for these functions derived.

If c is complex constant and if $C(t)$ is a complex function that vanishes for $t < 0$ and is bounded and continuous for $t < 0$, then

$$\mathcal{C} = c + C(t)* = (c\delta(t) + C(t))\,*$$

is said to be a causal convolution operator with the kernel $c\delta(t) + C(t)$. The operator $\mathcal{C}$ is said to of the first kind if $c = 0$; otherwise, it is of the second kind.

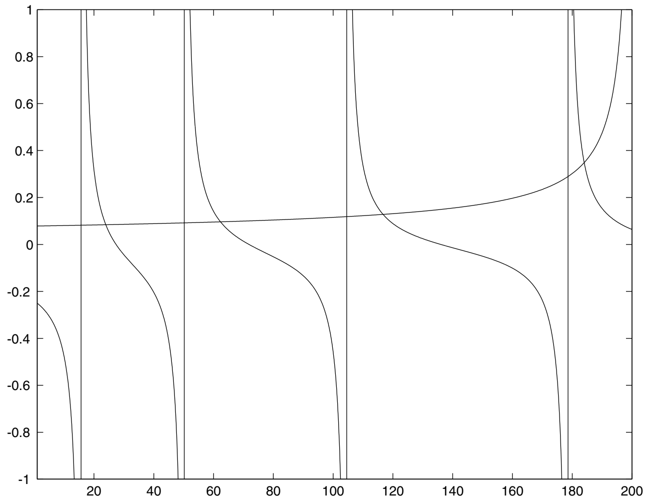

Figure 2: Right and left members of equation (3.21) for the upper case when $\nu = 1$, $\lambda_1 = 1$, and $\lambda_2 = 200$. There are four solutions.

Recall the simple facts that convolution is commutative and that causal convolution is associative. Among the causal convolution operators, there exists a causal convolution operator of the second kind, referred to as the identity operator and denoted by $\mathcal{I} = \delta(t)*$, with the property that $\mathcal{I}\mathcal{C} = \mathcal{C}$ for each causal convolution operator $\mathcal{C}$ of the first or of the second kind. To each causal convolution operator $\mathcal{C}$ of the second kind, there is a causal convolution operator of the second kind, referred to as the inverse of $\mathcal{C}$ and denoted by $\mathcal{C}^{-1}$, with the property that $\mathcal{C}\mathcal{C}^{-1} = \mathcal{I}$. By introducing $F(z) = z^{-1}$, one has $\mathcal{C}F(\mathcal{C}) = \mathcal{I}$ and $F(\mathcal{C}) = F(c) + F_{\mathcal{C}}(t)*$, where

$$cF_{\mathcal{C}}(t) + F(c)C(t) + F_{\mathcal{C}}(t) * C(t) = 0. \tag{A.1}$$

The inverse of $\mathcal{C}$ can also be written explicitly as

$$\mathcal{C}^{-1} = c^{-1}\left(1 + \frac{C(t)}{c}*\right)^{-1} = c^{-1} + c^{-1}\sum_{n=1}^{\infty}(-1)^n\left(\frac{C(t)}{c}*\right)^n,$$

where the series converges uniformly in each bounded interval. The set of kernels of causal convolution operators of the second kind and the convolution operation $*$ constitute an Abelian group, a fact that often, tacitly, will be used below. Unless stated otherwise, $\mathcal{C}$ will denote a given second-kind causal convolution operator.

A.1 Entire functions

Let $F(z)$ be an entire function, let $f(z) = F'(z)$ be the derivative of $F(z)$, and define

$$\begin{cases} F(\mathcal{C}) = \displaystyle\sum_{n=0}^{\infty} F_n\mathcal{C}^n, \\ f(\mathcal{C}) = F'(\mathcal{C}) = \displaystyle\sum_{n=0}^{\infty} nF_n\mathcal{C}^{n-1}, \end{cases}$$

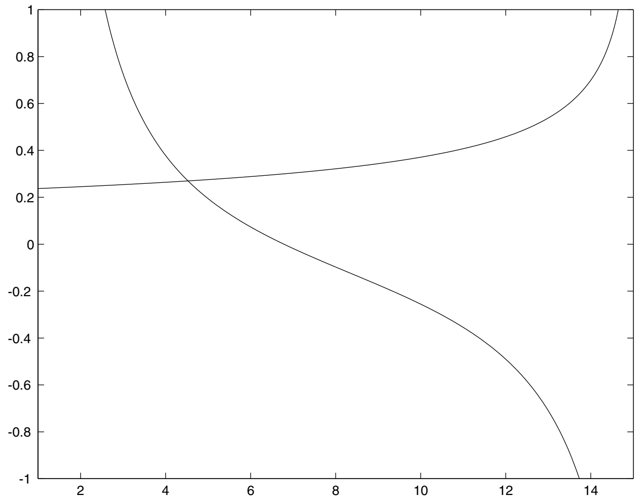

Figure 3: Right and left members of equation (3.21) for the lower case when $\nu = 1$, $\lambda_1 = 1$, and $\lambda_2 = 15$. There is one solution.

where the complex numbers F_n are the coefficients in the Taylor expansion of $F(z)$:

$$F(z) = \sum_{n=0}^{\infty} F_n z^n.$$

Then $F(\mathcal{C})$ and $f(\mathcal{C})$ are causal convolution operators of the form

$$\begin{cases} F(\mathcal{C}) = F(c) + F_{\mathcal{C}}(t)* = (F(c)\delta(t) + F_{\mathcal{C}}(t))\,*, \\ f(\mathcal{C}) = f(c) + f_{\mathcal{C}}(t)* = (f(c)\delta(t) + f_{\mathcal{C}}(t))\,*, \end{cases}$$

where the causal kernels $F_{\mathcal{C}}(t)$ and $f_{\mathcal{C}}(t)$ are related as

$$tF_{\mathcal{C}}(t) = f(c)tC(t) + f_{\mathcal{C}}(t) * (tC(t))\,.$$

For use of the rule

$$t\delta(t) = 0 \tag{A.2}$$

and repeated use of the rule

$$t(U(t) * V(t)) = (tU(t)) * V(t) + U(t) * (tV(t)) \tag{A.3}$$

gives

$$t\left(\sum_{n=0}^{\infty} F_n \mathcal{C}^n \delta(t)\right) = \sum_{n=0}^{\infty} nF_n \mathcal{C}^{n-1}\left(t\left(\mathcal{C}\delta(t)\right)\right),$$

that is,

$$t\left(F(\mathcal{C})\delta(t)\right) = F'(\mathcal{C})\left(t\left(\mathcal{C}\delta(t)\right)\right), \tag{A.4}$$

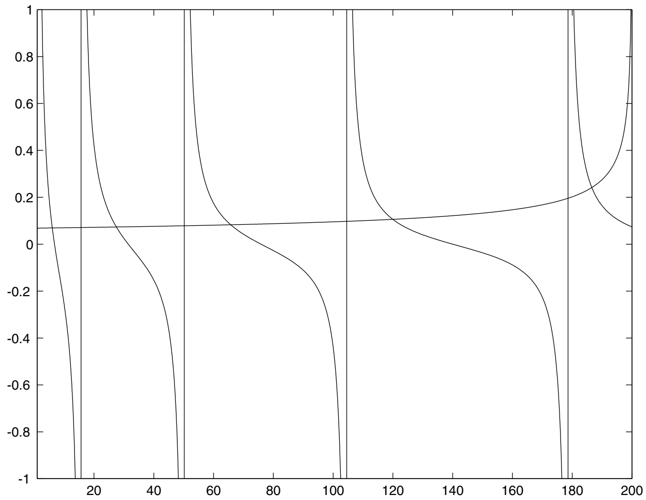

Figure 4: Right and left members of equation (3.21) for the lower case when $\nu = 1$, $\lambda_1 = 1$, and $\lambda_2 = 200$. There are five solutions.

which is equivalent to the wanted formula. Observe that $\mathcal{C}$ may be of the first kind.

Equation (A.4) applies to the exponential function, $F(z) = \exp(z) = f(z)$:

$$t\left(F(\mathcal{C})\delta(t)\right) = F(\mathcal{C})\left(t\left(\mathcal{C}\delta(t)\right)\right), \tag{A.5}$$

that is,

$$tF_{\mathcal{C}}(t) = \exp(c)tC(t) + F_{\mathcal{C}}(t) * (tC(t)) .$$

Formula (A.4) is now to to be generalized for an operator $\mathcal{C}$ of the second kind. Set $G(z) = F(z^{-1})$, where $F(z)$ is entire. Since $\mathcal{C}^{-1}$ exists, formula (A.4) gives

$$t\left(F(\mathcal{C}^{-1})\delta(t)\right) = F'(\mathcal{C}^{-1})\left(t\left(\mathcal{C}^{-1}\delta(t)\right)\right),$$

which, using that $F'(z^{-1}) = -G'(z)z^2$ and the rules (A.2) and (A.3), results in

$$t\left(G(\mathcal{C})\delta(t)\right) = -G'(\mathcal{C})\mathcal{C}\left(\mathcal{C}\delta(t)\right) * \left(t\left(\mathcal{C}^{-1}\delta(t)\right)\right) = G'(\mathcal{C})\mathcal{C}\left(\mathcal{C}^{-1}\delta(t)\right) * \left(t\left(\mathcal{C}\delta(t)\right)\right),$$

that is,

$$t\left(G(\mathcal{C})\delta(t)\right) = G'(\mathcal{C})\left(t\left(\mathcal{C}\delta(t)\right)\right).$$

Consequently, formula (A.4) holds for $G(z) = F(z^{-1})$, where $F(z)$ is entire.

A useful example is given by the powers of z, that is, $F(z) = z^n$, where n is an integer. Since this function satisfies the Euler equation $zF'(z) = nF(z)$, (A.4) gives

$$\mathcal{C}\left(t\left(F(\mathcal{C})\delta(t)\right)\right) = nF(\mathcal{C})\left(t\left(\mathcal{C}\delta(t)\right)\right),$$

that is,

$$ctF_{\mathcal{C}}(t) + C(t) * (tF_{\mathcal{C}}(t)) = nF(c)tC(t) + nF_{\mathcal{C}}(t) * (tC(t)) .$$

Observe that this equation trivially holds when $n = 1$, $n = 0$, and $n = -1$, cf. (A.1).

A.2 General results

Formula (A.4) is of great generality. For if $F(z)$ and $G(z)$ are complex functions such that satisfy (A.4) and if α and β are complex numbers, then $(\alpha F+\beta G)(z)$ and $(F\cdot G)(z)$ satisfy (A.4). The first of these assertions is immediate. The second formula,

$$t\left(F(\mathcal{C})G(\mathcal{C})\delta(t)\right)=\left(F(\mathcal{C})G'(\mathcal{C})+G(\mathcal{C})F'(\mathcal{C})\right)\left(t\left(\mathcal{C}\delta(t)\right)\right),$$

follows by sidewise addition of the formulae

$$\begin{cases} G(\mathcal{C})\left(t\left(F(\mathcal{C})\delta(t)\right)\right)=G(\mathcal{C})F'(\mathcal{C})\left(t\left(\mathcal{C}\delta(t)\right)\right), \\ F(\mathcal{C})\left(t\left(G(\mathcal{C})\delta(t)\right)\right)=F(\mathcal{C})G'(\mathcal{C})\left(t\left(\mathcal{C}\delta(t)\right)\right), \end{cases}$$

and using (A.3).

A.3 The logarithm and other transcendental functions

Recall that the principal branch of the logarithm of a number z in the along the negative real axis cut complex plane is the complex number defined by

$$\ln(z)=\int_1^z \frac{dt}{t},$$

where the path of integration from 1 and z must not pass the origin or the negative real axis, but is otherwise arbitrary. Choosing the straight line between the points 1 and z as the path of integration gives an expression that proves useful in defining the principal branch of the logarithm of a causal convolution operator of the second kind, namely

$$\ln(z)=\int_0^1 ds\,(1+s(z-1))^{-1}(z-1).$$

The logarithm of causal convolution operator $\mathcal{C}=c+C(t)*$, where c belongs to the along the negative real axis cut complex plane, is the causal convolution operator

$$\ln(\mathcal{C})=\ln(c+C(t)*)=\int_0^1 ds\left(1+\frac{s}{1+s(c-1)}C(t)*\right)^{-1}\frac{c-1+C(t)*}{1+s(c-1)}.$$

By expanding the inverse operator as

$$\left(1+\frac{s}{1+s(c-1)}C(t)*\right)^{-1}=\sum_{n=0}^{\infty}(-1)^n\left(\frac{s}{1+s(c-1)}\right)^n(C(t)*)^n$$

one gets, after manipulation,

$$\ln(\mathcal{C})=\ln(c)+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{C(t)}{c}*\right)^n a(c,n),$$

where[6]

$$a(c,n) = nc^n \int_0^1 ds \frac{s^{n-1}}{(1+s(c-1))^{n+1}} = 1.$$

This formula shows that

$$\ln(\mathcal{C}) = \ln(c) + \ln(\mathcal{C}/c)$$

as expected and that

$$t\left(\ln(\mathcal{C})\delta(t)\right) = \mathcal{C}^{-1}\left(t\left(\mathcal{C}\delta(t)\right)\right), \tag{A.6}$$

that is, (A.4) holds with $F(z) = \ln(z)$ and $F'(z) = z^{-1}$. This result could also have been obtained by putting

$$\exp(\ln(\mathcal{C})) := \mathcal{C}$$

and substituting this into the equality

$$t\left(\exp(\ln(\mathcal{C}))\delta(t)\right) = \exp(\ln(\mathcal{C}))\left(t\left(\ln(\mathcal{C})\delta(t)\right)\right).$$

As an example of other transcendental functions, consider $f(z) = z^a$ (principal branch), where a is a complex number, that is $f(z) = \exp(a\ln(z))$. Then using (A.5) and (A.6) gives

$$t\left(\mathcal{C}^a\delta(t)\right) = \mathcal{C}^a\left(t\left(a\ln(\mathcal{C})\delta(t)\right)\right) = a\mathcal{C}^a\mathcal{C}^{-1}\left(t\left(\mathcal{C}\delta(t)\right)\right),$$

that is, (A.4) holds with $F(z) = z^a$ and $F'(z) = az^az^{-1}$. For the square root function ($a = 1/2$) and for the square function ($a = 2$), this equation, via (A.3), reduces to the familiar equations $\mathcal{C}^{1/2}\mathcal{C}^{1/2} = \mathcal{C}$ and $\mathcal{C}\mathcal{C} = \mathcal{C}^2$, respectively.

[6]This is evident when $c = 1$. If $c \neq 1$, then the binomial theorem gives

$$\begin{aligned}
a(c,n) &= \frac{nc^n}{(c-1)^{n-1}} \int_0^1 ds \sum_{k=0}^{n-1} \binom{n-1}{k} (1+s(c-1))^{-2-k}(-1)^k \\
&= \frac{c^n}{(c-1)^n} \sum_{k=0}^{n-1} \frac{n}{k+1} \binom{n-1}{k} \left(c^{-1-k}-1\right)(-1)^{k+1} \\
&= \frac{c^n}{(c-1)^n} \sum_{k=0}^{n-1} \binom{n}{k+1} \left(c^{-1-k}-1\right)(-1)^{k+1} \\
&= \frac{c^n}{(c-1)^n} \sum_{k=1}^{n} \binom{n}{k} \left(c^{-k}-1\right)(-1)^{k} \\
&= \frac{1}{(c-1)^n} \left(\sum_{k=0}^{n} \binom{n}{k} (-1)^k c^{n-k} - c^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k \right) \\
&= \frac{1}{(c-1)^n} \left((c-1)^n - c^n (1-1)^n \right) = 1.
\end{aligned}$$

A.4 Bessel functions

Since the Bessel functions

$$J_n(z) = \left(\frac{z}{2}\right)^n \sum_{j=0}^{\infty} (-1)^j \frac{1}{j!(n+j)!} \left(\frac{z}{2}\right)^{2j}$$

and the modified Bessel functions

$$I_n(z) = \left(\frac{z}{2}\right)^n \sum_{j=0}^{\infty} \frac{1}{j!(n+j)!} \left(\frac{z}{2}\right)^{2j}$$

of the first kind and integer order are entire functions, equation (A.4) can be applied to their first derivatives:

$$t\left(F'(\mathcal{C})\delta(t)\right) = F''(\mathcal{C})\left(t\left(\mathcal{C}\delta(t)\right)\right). \tag{A.7}$$

Combining (A.4) and (A.7) with the respective second-order differential equation gives useful identities. For instance, the Bessel equation

$$z^2 J_n''(z) + z J_n'(z) + \left(z^2 - n^2\right) J_n(z) = 0$$

transforms into — recall the notation $j_n(z) = J_n'(z)$ —

$$\begin{cases} \mathcal{C}^2\left(t\left(j_n(\mathcal{C})\delta(t)\right)\right) + \mathcal{C}\left(t\left(J_n(\mathcal{C})\delta(t)\right)\right) + \left(\mathcal{C}^2 - n^2\right) J_n(\mathcal{C})\left(t\left(\mathcal{C}\delta(t)\right)\right) = 0, \\ t\left(J_n(\mathcal{C})\delta(t)\right) = j_n(\mathcal{C})\left(t\left(\mathcal{C}\delta(t)\right)\right), \end{cases}$$

which, since

$$\begin{cases} J_n(\mathcal{C}) = J_n(c) + J_{n\mathcal{C}}(t)* = \left(J_n(c)\delta(t) + J_{n\mathcal{C}}(t)\right)*, \\ j_n(\mathcal{C}) = j_n(c) + j_{n\mathcal{C}}(t)* = \left(j_n(c)\delta(t) + j_{n\mathcal{C}}(t)\right)* \end{cases}$$

constitutes a system of convolution equations in the kernels $J_{n\mathcal{C}}(t)$ and $j_{n\mathcal{C}}(t)$:

$$\begin{cases} (c + C(t)*)^2\left(t j_{n\mathcal{C}}(t)\right) + (c + C(t)*)\left(t J_{n\mathcal{C}}(t)\right) \\ \quad + \left((c + C(t)*)^2 - n^2\right)\left(J_n(c) + J_{n\mathcal{C}}(t)*\right)\left(tC(t)\right) = 0, \\ t J_{n\mathcal{C}}(t) = \left(j_n(c) + j_{n\mathcal{C}}(t)*\right)\left(tC(t)\right). \end{cases}$$

The ascending series for the modified Bessel functions of the second kind and integer order are

$$\begin{aligned} K_n(z) = {} & \frac{1}{2}\left(\frac{z}{2}\right)^{-n} \sum_{j=0}^{n-1} (-1)^j \frac{(n-j-1)!}{j!} \left(\frac{z}{2}\right)^{2j} \\ & + (-1)^{n-1} \ln\left(\frac{z}{2}\right) I_n(z) \\ & + (-1)^n \frac{1}{2}\left(\frac{z}{2}\right)^n \sum_{j=0}^{\infty} \frac{1}{j!(n+j)!} \left(\psi(j+1) + \psi(n+j+1)\right)\left(\frac{z}{2}\right)^{2j}, \end{aligned}$$

where $I_n(z)$ are the modified Bessel function of the first kind and integer order and

$$\begin{cases} \psi(1) = -\gamma, \\ \psi(n) = -\gamma + \displaystyle\sum_{i=1}^{n-1} \frac{1}{i}, \end{cases}$$

i.e.,

$$K_n(z) = z^{-n} P_n(z) + (-1)^{n+1} \ln(z) I_n(z) + A_n(z),$$

where $P_n(z)$ is a polynomial and $A_n(z)$ is analytic. Since equation (A.4) is valid for the functions $P_n(z)$, $\ln(z)$, $I_n(z)$, and $A_n(z)$, equation (A.4) holds also for $K_n(z)$ according to the results stated in section A.2. Moreover, (A.4) holds for the derivative $k_n(z) = K_n'(z)$. Therefore, by introducing

$$\begin{cases} K_n(\mathcal{C}) = K_n(c) + K_{n\mathcal{C}}(t)* = (K_n(c)\delta(t) + K_{n\mathcal{C}}(t))\,*, \\ k_n(\mathcal{C}) = k_n(c) + k_{n\mathcal{C}}(t)* = (k_n(c)\delta(t) + k_{n\mathcal{C}}(t))\,* \end{cases}$$

and using the modified Bessel equation

$$z^2 K_n''(z) + z K_n'(z) - \left(z^2 + n^2\right) K_n(z) = 0,$$

one obtains a system of convolution equations in the kernels $K_{n\mathcal{C}}(t)$ and $k_{n\mathcal{C}}(t)$, namely

$$\begin{cases} \mathcal{C}^2 \left(t \left(k_n(\mathcal{C})\delta(t)\right)\right) + \mathcal{C} \left(t \left(K_n(\mathcal{C})\delta(t)\right)\right) - \left(\mathcal{C}^2 + n^2\right) K_n(\mathcal{C}) \left(t \left(\mathcal{C}\delta(t)\right)\right) = 0, \\ t \left(K_n(\mathcal{C})\delta(t)\right) = k_n(\mathcal{C}) \left(t \left(\mathcal{C}\delta(t)\right)\right) \end{cases}$$

or

$$\begin{cases} (c + C(t)*)^2 \left(t k_{n\mathcal{C}}(t)\right) + (c + C(t)*) \left(t K_{n\mathcal{C}}(t)\right) \\ \quad - \left((c + C(t)*)^2 + n^2\right) \left(K_n(c) + K_{n\mathcal{C}}(t)*\right) \left(t C(t)\right) = 0, \\ t K_{n\mathcal{C}}(t) = \left(k_n(c) + k_{n\mathcal{C}}(t)*\right) \left(t C(t)\right). \end{cases}$$

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