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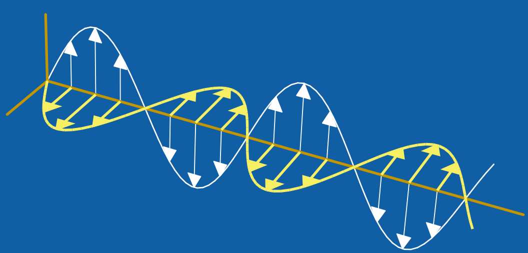

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Abstract

We consider the problem of obtaining information about an inaccessible half-space from electromagnetic measurements made in the accessible half-space. If the measurements are of limited precision, some scatterers will be undetectable because their scattered fields are below the precision of the measuring instrument. How can we make optimal measurements? In other words, what incident fields should we apply that will result in the biggest measurements?

There are many ways to formulate this question, depending on the measuring instruments. In this paper we consider a formulation involving wave-splitting in the accessible half-space: what downgoing wave will result in an upgoing wave of greatest energy? This formulation is most natural for far-field problems.

A closely related question arises in the case when we have a guess about the configuration of the inaccessible half-space. What measurements should we make to determine whether our guess is accurate? In this case we compare the scattered field to the field computed from the guessed configuration. Again we look for the incident field that results in the greatest energy difference.

We show that the optimal incident field can be found by an iterative process involving time reversal "mirrors". For band-limited incident fields and compactly supported scatterers, this iterative process converges to a sum of time-harmonic fields.

1 Introduction

This work concerns the ultimate limits of electromagnetic imaging systems. What measurements should we make to detect the weakest possible scatterers? What measurements should we make to determine that an unknown scatterer is different from a guessed one?

To answer these questions, we must first formulate them more precisely. What sort of measurements do we allow? What do we mean by "detectable"? How do we know what measurements are "best"?

2 Problem formulation

2.1 The equations

The behavior of electromagnetic fields is modelled by the Maxwell equations, which we write in the time domain as

$$\begin{cases} \nabla \times \boldsymbol{\mathcal{E}} = -\dfrac{\partial \boldsymbol{\mathcal{B}}}{\partial t} \\ \nabla \times \boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{J}} + \dfrac{\partial \boldsymbol{\mathcal{D}}}{\partial t} \end{cases}$$

Here $\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{B}}$, and $\boldsymbol{\mathcal{D}}$ are the electric field, magnetic field, magnetic flux density, and electric flux density, respectively, and $\boldsymbol{\mathcal{J}}$ is the current density. We write $\boldsymbol{r} = (x, y, z)$ for a point in $\mathbb{R}^3$, and $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$ for unit vectors in the coordinate directions.

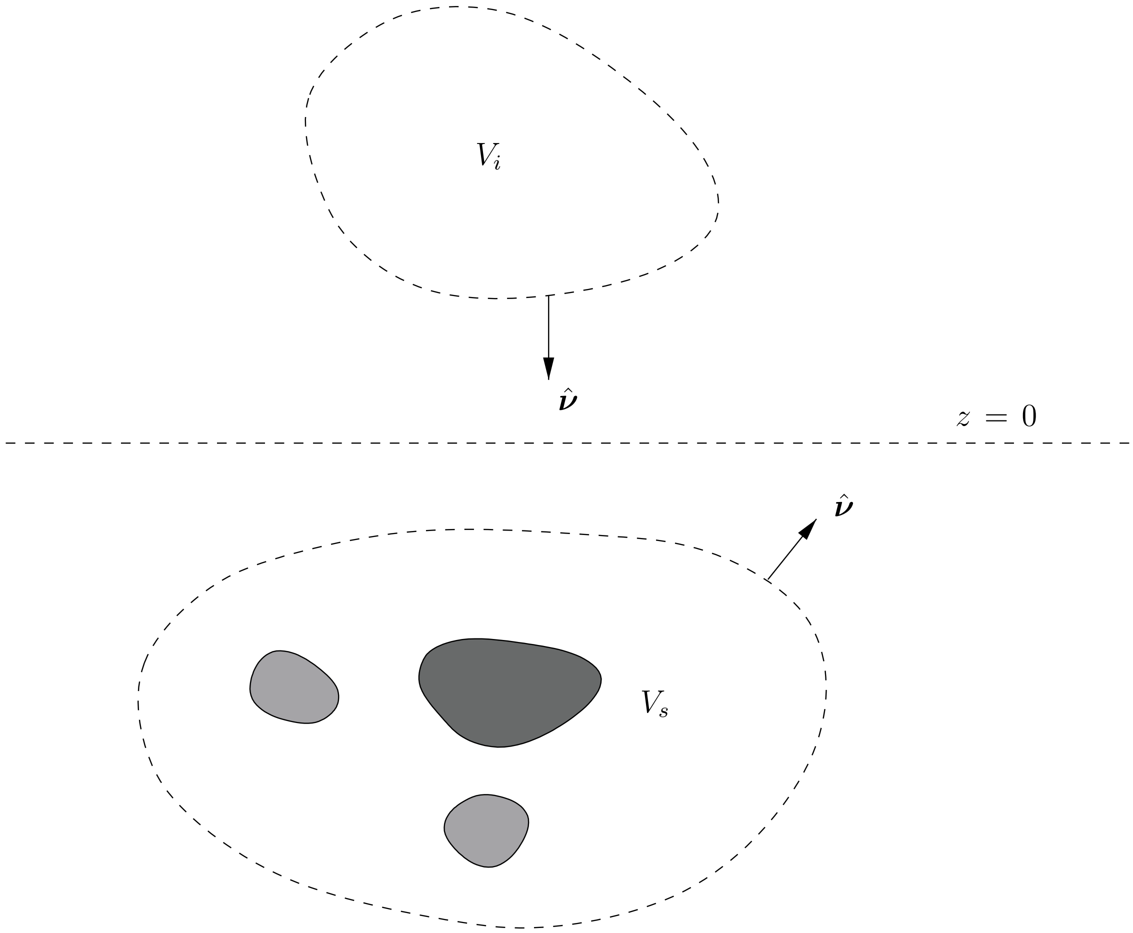

Figure 1: The geometry of the problem. The region V_i contains the sources, and the region V_s contains the scatterers. The lines are dashed to indicate that the surfaces are fictitious.

We assume that the fields are generated and measured in the upper half-space, $z > 0$, which consists of free space away from the sources of the field. We assume that the sources are separated from the boundary $z = 0$, so that there is a layer of free space between the sources and the boundary, see Figure 1. In this region, we have $\boldsymbol{\mathcal{J}} = \mathbf{0}$, and the fields $\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{D}}, \boldsymbol{\mathcal{B}}$, and $\boldsymbol{\mathcal{H}}$ are related to each other by the constitutive relations of free space:

$$\begin{cases} \boldsymbol{\mathcal{D}}(\boldsymbol{r}, t) = \epsilon_0 \boldsymbol{\mathcal{E}}(\boldsymbol{r}, t) \\ \boldsymbol{\mathcal{B}}(\boldsymbol{r}, t) = \mu_0 \boldsymbol{\mathcal{H}}(\boldsymbol{r}, t) \end{cases}$$

Here ϵ_0 and μ_0 are the vacuum permittivity and permeability, respectively.

Thus in the empty layer of the upper half-space, we can write the Maxwell equations as

$$\begin{cases} \nabla \times \boldsymbol{\mathcal{E}} = -\mu_0 \dfrac{\partial \boldsymbol{\mathcal{H}}}{\partial t} \\ \nabla \times \boldsymbol{\mathcal{H}} = \epsilon_0 \dfrac{\partial \boldsymbol{\mathcal{E}}}{\partial t} \end{cases} \tag{2.1}$$

We will make use of the Fourier transform relating the time and frequency domains:

$$\boldsymbol{\mathcal{E}}(\boldsymbol{r},t) = \int \boldsymbol{E}(\boldsymbol{r},k)e^{-ic_0kt}c_0 dk, \tag{2.2}$$

where $c_0 = 1/\sqrt{\epsilon_0\mu_0}$ is the vacuum speed of light, and $k = \omega/c_0$ is the wave number in vacuum. In the empty layer of the upper half-space, the frequency-domain Maxwell equations can be written

$$\begin{cases} \nabla \times \boldsymbol{E} = ik(\eta_0\boldsymbol{H}) \\ \nabla \times (\eta_0\boldsymbol{H}) = -ik\boldsymbol{E} \end{cases} \tag{2.3}$$

where $\eta_0 = \sqrt{\mu_0/\epsilon_0}$ is the vacuum impedance. The advantage of scaling the Maxwell equations in this manner is that the electric field $\boldsymbol{E}$ and the quantity $\eta_0\boldsymbol{H}$ now have the same dimensions (volts/meter).

2.2 Upgoing and downgoing waves

In the empty layer of the upper half-space, where the medium is laterally homogeneous in the variables x and y, it is natural to decompose the electromagnetic field in a spectrum of plane waves. We do this by means of a Fourier transformation of the electric and magnetic fields with respect to the lateral variables x and y. The spatial Fourier transform of a time-harmonic field $\boldsymbol{E}(\boldsymbol{r},k)$ is denoted by

$$\mathbf{E}(z,\boldsymbol{e}_{xy},k) = \iint_{-\infty}^{\infty} \boldsymbol{E}(\boldsymbol{r},k)e^{-ik\boldsymbol{e}_{xy}\cdot\boldsymbol{r}_{xy}}\, dxdy$$

where $\boldsymbol{r}_{xy} = (x,y)$ and

$$\boldsymbol{k}_{xy} = \hat{\boldsymbol{x}}k_x + \hat{\boldsymbol{y}}k_y = k\boldsymbol{e}_{xy} = k(e_x,e_y)$$

is the transverse (tangential) wave vector. We write $e_{xy} = \|\boldsymbol{e}_{xy}\|$. The inverse Fourier transform is defined by

$$\boldsymbol{E}(\boldsymbol{r},k) = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} \mathbf{E}(z,\boldsymbol{e}_{xy},k)e^{ik\boldsymbol{e}_{xy}\cdot\boldsymbol{r}_{xy}}\, k^2 de_x de_y. \tag{2.4}$$

We show in Appendix A that in the upper half-space, the transformed electric field is a sum

$$\mathbf{E}_{xy}(\boldsymbol{e}_{xy},z,k) = \mathbf{E}_{xy}{}^{\uparrow}(\boldsymbol{e}_{xy},k)e^{ike_z z} + \mathbf{E}_{xy}{}^{\downarrow}(\boldsymbol{e}_{xy},k)e^{-ike_z z} \tag{2.5}$$

where

$$e_z(k) = \begin{cases} \sqrt{1-e_{xy}^2} & \text{for } e_{xy} < 1 \\ i\mathrm{sgn}(k)\sqrt{e_{xy}^2-1} & \text{for } e_{xy} > 1. \end{cases} \tag{2.6}$$

We define the vectors $\boldsymbol{e}^{\pm} = \boldsymbol{e}_{xy} \pm \hat{\boldsymbol{z}} e_z$, which satisfy $\boldsymbol{e}^{\pm} \cdot \boldsymbol{e}^{\pm} = 1$.

Equation (2.5) shows how to split the electric field into upgoing and downgoing components. We denote the upgoing three-component transform-domain electric field by $\mathbf{E}^{\uparrow} e^{ike_z z}$ and the downgoing one by $\mathbf{E}^{\downarrow} e^{-ike_z z}$. Then we can transform back to the time-space domain to write a downgoing wave as

$$\begin{aligned}\boldsymbol{\mathcal{E}}^{\downarrow}(\boldsymbol{r}, t) =& \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \iint_{\mathbb{R}^2} \mathbf{E}^{\downarrow}(\boldsymbol{e}_{xy}, k) e^{-ike_z z} e^{ik\boldsymbol{e}_{xy} \cdot \boldsymbol{r}_{xy}} e^{-ic_0 kt}\, k^2 de_x de_y c_0 dk \\ =& \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \iint_{\mathbb{R}^2} \mathbf{E}^{\downarrow}(\boldsymbol{e}_{xy}, k) e^{ik\boldsymbol{e}^{-} \cdot \boldsymbol{r}} e^{-ic_0 kt}\, k^2 de_x de_y c_0 dk,\end{aligned} \tag{2.7}$$

and an upgoing wave as

$$\begin{aligned}\boldsymbol{\mathcal{E}}^{\uparrow}(\boldsymbol{r}, t) =& \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \iint_{\mathbb{R}^2} \mathbf{E}^{\uparrow}(\boldsymbol{e}_{xy}, k) e^{ike_z z} e^{ik\boldsymbol{e}_{xy} \cdot \boldsymbol{r}_{xy}} e^{-ic_0 kt}\, k^2 de_x de_y c_0 dk \\ =& \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \iint_{\mathbb{R}^2} \mathbf{E}^{\uparrow}(\boldsymbol{e}_{xy}, k) e^{ik\boldsymbol{e}^{+} \cdot \boldsymbol{r}} e^{-ic_0 kt}\, k^2 de_x de_y c_0 dk.\end{aligned} \tag{2.8}$$

These equations give us an explicit decomposition of upgoing and downgoing waves in terms of the plane waves $e^{ik\boldsymbol{e}^{-} \cdot \boldsymbol{r}} e^{-ic_0 kt}$ and $e^{ik\boldsymbol{e}^{+} \cdot \boldsymbol{r}} e^{-ic_0 kt}$. These plane waves are either homogeneous, obliquely propagating waves or inhomogeneous (evanescent) waves, depending on whether the transverse wave number ke_{xy}, is less than or greater than the temportal wave number k. For the propagating waves, the vector $\boldsymbol{e}^{\pm}$ gives the direction of propagation.

From this decomposition into plane waves, we see that from the Maxwell equations, the plane-wave vector amplitudes $\mathbf{E}^{\uparrow}$ and $\mathbf{H}^{\uparrow}$ satisfy

$$\begin{aligned}\boldsymbol{e}^{+} \times \mathbf{E}^{\uparrow}(\boldsymbol{e}_{xy}, k) =& \eta_0 \mathbf{H}^{\uparrow}(\boldsymbol{e}_{xy}, k) \\ \boldsymbol{e}^{+} \times \eta_0 \mathbf{H}^{\uparrow}(\boldsymbol{e}_{xy}, k) =& - \mathbf{E}^{\uparrow}(\boldsymbol{e}_{xy}, k) \\ \boldsymbol{e}^{+} \cdot \mathbf{E}^{\uparrow}(\boldsymbol{e}_{xy}, k) =& 0 \\ \boldsymbol{e}^{+} \cdot \mathbf{H}^{\uparrow}(\boldsymbol{e}_{xy}, k) =& 0\end{aligned} \tag{2.9}$$

Similar equations hold for the downgoing waves $\mathbf{E}^{\downarrow}$ and $\mathbf{H}^{\downarrow}$.

2.3 Power flow

Our criterion for the best incident field is the field that gives rise to the upgoing wave with the most total energy.

We measure the energy flux through the plane $z = 0$ by means of the Poynting vector, which, for time dependent fields, is

$$\boldsymbol{\mathcal{S}}(\boldsymbol{r}, t) = \boldsymbol{\mathcal{E}}(\boldsymbol{r}, t) \times \boldsymbol{\mathcal{H}}(\boldsymbol{r}, t).$$

The total energy flow into the lower half-space $z < 0$ is

$$W(\boldsymbol{\mathcal{E}}) = \int_{-\infty}^{\infty} \iint\limits_{z=0} -\hat{\boldsymbol{z}} \cdot \left(\boldsymbol{\mathcal{E}}(\boldsymbol{r}_{xy}, 0, t) \times \boldsymbol{\mathcal{H}}(\boldsymbol{r}_{xy}, 0, t)\right) \, dxdy \, dt \tag{2.10}$$

When we apply Parseval's identity to (2.10), we get

$$W(\boldsymbol{\mathcal{E}}) = W(\boldsymbol{E}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \iint\limits_{\mathbb{R}^2} -\hat{\boldsymbol{z}} \cdot \left(\mathbf{E}(\boldsymbol{e}_{xy}, 0, k) \times \mathbf{H}^*(\boldsymbol{e}_{xy}, 0, k)\right) \, k^2 de_x de_y \, c_0 dk \tag{2.11}$$

In this paper we assume that the sources (in the upper half-space) of the downgoing field are far from the scatterers in the lower half-space, so that upgoing and downgoing evanescent waves are not both present at the plane $z = 0$. We show in Appendix B that under these conditions, $W(\mathbf{E}^{\downarrow} + \mathbf{E}^{\uparrow}) = W(\mathbf{E}^{\downarrow}) + W(\mathbf{E}^{\uparrow})$.

For waves that are upgoing, we have expressions (2.9), which together with the vector identity $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b})$, shows that the integrand of (2.11) can be written

$$\begin{aligned}
\hat{\boldsymbol{z}} \cdot \left(\mathbf{E}^{\uparrow}(\boldsymbol{e}_{xy}, k) \times \mathbf{H}^{\uparrow *}(\boldsymbol{e}_{xy}, k)\right) &= \frac{1}{\eta_0} \hat{\boldsymbol{z}} \cdot \left(\mathbf{E}^{\uparrow}(\boldsymbol{e}_{xy}, k) \times (\boldsymbol{e}^{+} \times \mathbf{E}^{\uparrow *}(\boldsymbol{e}_{xy}, k))\right) \\
&= \frac{1}{\eta_0} \hat{\boldsymbol{z}} \cdot \left(\boldsymbol{e}^{+}(\mathbf{E}^{\uparrow} \cdot \mathbf{E}^{\uparrow}) - \mathbf{E}^{\uparrow}(\boldsymbol{e}^{+} \cdot \mathbf{E}^{\uparrow *})\right) \\
&= \frac{1}{\eta_0} (\hat{\boldsymbol{z}} \cdot \boldsymbol{e}^{+}) \left(\mathbf{E}^{\uparrow}(\boldsymbol{e}_{xy}, k) \cdot \mathbf{E}^{\uparrow *}(\boldsymbol{e}_{xy}, k)\right)
\end{aligned}$$

so that the total energy flux of an upgoing wave is

$$W(\mathbf{E}^{\uparrow}) = \frac{-1}{\eta_0 (2\pi)^3} \int_{-\infty}^{\infty} \iint\limits_{\mathbb{R}^2} \left|\mathbf{E}^{\uparrow}(\boldsymbol{e}_{xy}, k)\right|^2 e_z \, k^2 de_x de_y \, c_0 dk. \tag{2.12}$$

We show in Appendix B that the evanescent waves do not contribute to expression (2.12). Thus the domain of integration in (2.12) can be restricted to the set $\{e_{xy} < 1\}$. The corresponding expression for downgoing waves is

$$W(\mathbf{E}^{\downarrow}) = \frac{1}{\eta_0 (2\pi)^3} \int_{-\infty}^{\infty} \iint\limits_{\{e_{xy} < 1\}} \left|\mathbf{E}^{\downarrow}(\boldsymbol{e}_{xy}, k)\right|^2 e_z \, k^2 de_x de_y \, c_0 dk. \tag{2.13}$$

In either case, clearly $|W|$ is a norm on the corresponding space.

On each space (upgoing waves and downgoing waves), we introduce an inner product that corresponds to the flux W:

$$(\mathbf{U}, \mathbf{V})_W = \frac{1}{\eta_0 (2\pi)^3} \int_{-\infty}^{\infty} \iint\limits_{\{e_{xy} < 1\}} \left(\mathbf{U}(\boldsymbol{e}_{xy}, k) \cdot \mathbf{V}^*(\boldsymbol{e}_{xy}, k)\right) e_z \, k^2 de_x de_y \, c_0 dk. \tag{2.14}$$

2.4 Criterion for detectability and distinguishability

2.4.1 Detectability

The downgoing wave that is best for detecting a scatterer is obtained from the optimization problem

$$\sup_{\boldsymbol{\mathcal{E}}^{\downarrow}} \frac{W(\boldsymbol{\mathcal{E}}^{\uparrow})}{W(\boldsymbol{\mathcal{E}}^{\downarrow})} = \sup_{\mathbf{E}^{\downarrow}} \frac{W(\mathbf{E}^{\uparrow})}{W(\mathbf{E}^{\downarrow})} = \sup_{\mathbf{E}^{\downarrow}} \frac{(\mathbf{E}^{\uparrow}, \mathbf{E}^{\uparrow})_W}{(\mathbf{E}^{\downarrow}, \mathbf{E}^{\downarrow})_W} \tag{2.15}$$

For a scatterer to be detectable, (2.15) should be greater than the measurement precision of our instruments (expressed in units so that the denominator of (2.15) is one).

The upgoing and downgoing fields of (2.15) are related by the reflection operator; thus carrying out the optimization of (2.15) requires an analysis of this operator.

2.4.2 The reflection operator

The reflection operator $\boldsymbol{\mathcal{R}}$ for the electric field is the linear operator that maps the downgoing electric field to the upgoing one at $z = 0$:

$$\boldsymbol{\mathcal{E}}^{\uparrow} = \boldsymbol{\mathcal{R}} \cdot \boldsymbol{\mathcal{E}}^{\downarrow} \tag{2.16}$$

The reflection operator is a 3×3 dyadic-valued operator; the dot in (2.16), as thoughout this paper, denotes dyadic contraction. When the dyadics are expressed in a basis, this contraction is simply multiplication of the corresponding matrices.

In the transform domain, we write the reflection operator as $\mathbf{R}$; it satisfies

$$\mathbf{E}^{\uparrow}(\boldsymbol{e}_{xy}, z = 0, k) = \int \mathbf{R}(\boldsymbol{e}_{xy}, \boldsymbol{e}'_{xy}, k) \cdot \mathbf{E}^{\downarrow}(\boldsymbol{e}'_{xy}, z = 0, k) d\boldsymbol{e}'_{xy} \tag{2.17}$$

The kernel $\mathbf{R}(\boldsymbol{e}_{xy}, \boldsymbol{e}'_{xy}, k)$ is defined in terms of the time-domain kernel $\boldsymbol{\mathcal{R}}(\boldsymbol{r}, \boldsymbol{r}', t-\tau)$ by

$$\delta(k-k')\mathbf{R}(\boldsymbol{e}_{xy}, \boldsymbol{e}'_{xy}, k) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\iint_{\mathbb{R}^2}\iint_{\mathbb{R}^2} e^{-ik\boldsymbol{e}_{xy}\cdot\boldsymbol{r}_{xy}} e^{ic_0kt} \boldsymbol{\mathcal{R}}(\boldsymbol{r}, \boldsymbol{r}', t-\tau) e^{-ic_0k'\tau} e^{ik'\boldsymbol{e}'_{xy}\cdot\boldsymbol{r}'_{xy}} d^2\boldsymbol{r}_{xy} d^2\boldsymbol{r}'_{xy} dt d\tau \tag{2.18}$$

2.4.3 Detectability

In terms of the reflection operator, the criterion (2.15) for choosing the downgoing wave that is best for detecting the presence of a scatterer is obtained from

$$\sup_{\boldsymbol{\mathcal{E}}^{\downarrow}} \frac{W(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{E}}^{\downarrow})}{W(\boldsymbol{\mathcal{E}}^{\downarrow})} = \sup_{\mathbf{E}^{\downarrow}} \frac{W(\mathbf{R}\mathbf{E}^{\downarrow})}{W(\mathbf{E}^{\downarrow})} = \sup_{\mathbf{E}^{\downarrow}} \frac{(\mathbf{R}\mathbf{E}^{\downarrow}, \mathbf{R}\mathbf{E}^{\downarrow})_W}{(\mathbf{E}^{\downarrow}, \mathbf{E}^{\downarrow})_W} = \sup_{\mathbf{E}^{\downarrow}} \frac{(\mathbf{E}^{\downarrow}, \mathbf{R}^{\#}\mathbf{R}\mathbf{E}^{\downarrow})_W}{(\mathbf{E}^{\downarrow}, \mathbf{E}^{\downarrow})_W} \tag{2.19}$$

where $\mathbf{R}^{\#}$ denotes the adjoint of $\mathbf{R}$ with respect to the W inner product.

2.4.4 Criterion for distinguishability

Suppose we have a guess as to the scatterer. For our guessed scatterer, we denote the reflection operator by $\boldsymbol{\mathcal{R}}_0$. Then the unknown scatterer can be distinguished from our guess if

$$\begin{aligned}\sup_{\boldsymbol{\mathcal{E}}^{\downarrow}} \frac{W((\boldsymbol{\mathcal{R}} - \boldsymbol{\mathcal{R}}_0)\boldsymbol{\mathcal{E}}^{\downarrow})}{W(\boldsymbol{\mathcal{E}}^{\downarrow})} &= \sup_{\mathbf{E}^{\downarrow}} \frac{((\mathbf{R} - \mathbf{R}_0)\mathbf{E}^{\downarrow}, (\mathbf{R} - \mathbf{R}_0)\mathbf{E}^{\downarrow})_W}{(\mathbf{E}^{\downarrow}, \mathbf{E}^{\downarrow})_W} \\ &= \sup_{\mathbf{E}^{\downarrow}} \frac{(\mathbf{E}^{\downarrow}, (\mathbf{R} - \mathbf{R}_0)^{\#}(\mathbf{R} - \mathbf{R}_0)\mathbf{E}^{\downarrow})_W}{(\mathbf{E}^{\downarrow}, \mathbf{E}^{\downarrow})_W}\end{aligned} \tag{2.20}$$

corresponds to a value greater than the measurement precision of our instruments. Here the # denotes the adjoint in the energy flux inner product.

2.5 The adjoint of the reflection operator

We compute the adjoint of $\mathbf{R}$ in the flux inner product of (2.14) by transposing $\mathbf{R}$ and interchanging the order of integration:

$$\begin{aligned}(\mathbf{U}, \mathbf{R}\mathbf{V})_W =& \frac{1}{2\pi^3\eta_0} \int_{-\infty}^{\infty} \iint_{e_{xy}<1} \mathbf{U}^{\downarrow}(\boldsymbol{e}_{xy}, k) \cdot \\ & \iint_{e'_{xy}<1} \mathbf{R}^*(\boldsymbol{e}, \boldsymbol{e}', k) \cdot \mathbf{V}^{*\downarrow}(\boldsymbol{e}'_{xy}, k) k^2 \, de'_x de'_y \; e_z k^2 \, de_x de_y \, c_0 dk \\ =& \frac{1}{2\pi^3\eta_0} \int_{-\infty}^{\infty} \iint_{e'_{xy}<1} \iint_{e_{xy}<1} \frac{e_z}{e'_z} \mathbf{R}^{*T}(\boldsymbol{e}_{xy}, \boldsymbol{e}'_{xy}, k) \cdot \mathbf{U}^{\downarrow}(\boldsymbol{e}_{xy}, k) k^2 \, de_x de_y \cdot \\ & \mathbf{V}^{*\downarrow}(\boldsymbol{e}'_{xy}, k) e'_z k^2 \, de'_x de'_y \; c_0 dk\end{aligned}$$

Here the superscript T denotes the matrix transpose. Thus we see that the kernel of the W-adjoint $\mathbf{R}^{\#}$ of $\mathbf{R}$ is

$$\mathbf{R}^{\#}(\boldsymbol{e}'_{xy}, \boldsymbol{e}_{xy}, k) = \frac{e_z}{e'_z} \mathbf{R}^{*T}(\boldsymbol{e}_{xy}, \boldsymbol{e}'_{xy}, k)$$

We show in Appendix D that the reflection operator can be written in terms of the scattering operator $\mathbf{S}$ as

$$\mathbf{R}(\boldsymbol{e}_{xy}, \boldsymbol{e}'_{xy}, k) = \frac{1}{e_z} \mathbf{S}(\boldsymbol{e}^+, \boldsymbol{e}'^-, k),$$

At this point, our only assumption about the scattering medium is that it is linear, so that $\boldsymbol{\mathcal{R}}$ is a linear operator. We now assume that the scattering medium is *reciprocal* (See (E.3)); in this case the scattering operator $\mathbf{S}$ satisfies the *reciprocity relation*

$$\mathbf{S}(\boldsymbol{e}, \boldsymbol{e}', k) = \mathbf{S}^T(-\boldsymbol{e}', -\boldsymbol{e}, k). \tag{2.21}$$

In this case, we see from (2.18) that the kernel $\mathsf{R}^{*T}(-\boldsymbol{e}'_{xy}, -\boldsymbol{e}_{xy}, k)$ corresponds to the time-domain kernel $\boldsymbol{\mathcal{R}}(\boldsymbol{r}, \boldsymbol{r}', \tau - t)$. This latter is the kernel of the operator $\boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{T}}$, where $\boldsymbol{\mathcal{T}}$ denotes the operator of time-reversal: $\boldsymbol{\mathcal{T}}f(t) = f(-t)$.

3 Algorithm for producing the optimal field

Expressions (2.19) and (2.20) give rise to algorithms for producing the optimal field. In the case of (2.19), if $\mathsf{R}^{\#}\mathsf{R}$ were a finite-dimensional matrix $\mathbf{A}$, then we could use the power method [4] to construct the eigenfunction E_∞ of $\mathbf{A}$ corresponding to the largest eigenvalue. In particular, the power method constructs E_∞ as the limit of the sequence $\mathbf{A}^n\mathsf{E}_0/c_n$, where c_n is a certain normalizing factor.

It was shown in [3] that the same method applies to the case of an operator such as $\mathsf{R}^{\#}\mathsf{R}$, an analytic compact-operator-valued function of k. In the case when the adjoint can be interpreted in terms of time reversal so that $\boldsymbol{\mathcal{R}}^{\#}\boldsymbol{\mathcal{R}} = \boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{R}}$, where $\boldsymbol{\mathcal{T}}$ denotes the operator of time-reversal, we have the following algorithm for producing the optimal field.

1. Set $j = 0$. Apply any downgoing field $\boldsymbol{\mathcal{E}}_j^{\downarrow}(\boldsymbol{r}, t)$.
2. Measure the resulting scattered field $\boldsymbol{\mathcal{E}}_j^{\uparrow} = \boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{E}}_j^{\downarrow}$.
3. Time-reverse the measured field, and apply this as a downgoing field:
$$\boldsymbol{\mathcal{E}}_{j+1}^{\downarrow}(\boldsymbol{r}, t) = \boldsymbol{\mathcal{E}}_j^{\uparrow}(\boldsymbol{r}, -t).$$
4. Add one to j. If this new j is even, so that $j = 2n$, divide by c_n (defined below).
5. Go to step 2.

Two iterations, that is, a scattering experiment followed by a time-reversal and another scattering experiment, constitute one application of the operator $\boldsymbol{\mathcal{R}}^{\#}\boldsymbol{\mathcal{R}} = \boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{R}}$ and is thus one step in the power method algorithm for constructing the optimal field.

The normalization c_n is obtained by first choosing an arbitrary test function $\boldsymbol{\psi}$ in a certain space X_B. This space X_B consists of smooth functions of space and time whose temporal Fourier transforms are (uniformly) supported in the frequency band B. The frequency band B should be chosen to lie within the frequency band of the experimental equipment. The normalization factor c_n is then chosen as

$$c_n = (\boldsymbol{\mathcal{E}}_j^{\downarrow}, \boldsymbol{\psi})_W. \tag{3.1}$$

To determine what the above algorithm eventually converges to, we can apply the theorem of [3]. It depends on the fact that the frequency-domain operator $\boldsymbol{R}^{\#}\boldsymbol{R}$ is compact at every k and therefore has a spectral decomposition of the form $\boldsymbol{R}^{\#}\boldsymbol{R} = \sum_l \lambda_l(k)\boldsymbol{P}_l(k)$, where the λ_l are the eigenvalues of the self-adjoint operator $\boldsymbol{R}^{\#}\boldsymbol{R}$ and the $\boldsymbol{P}_l$ are projections onto the corresponding eigenspaces.

The algorithm converges to fixed-frequency waves whose frequency is determined by the values of k at which the eigenvalues λ_l attain a maximum. If the same maximum M is attained at several k, then the limiting field is a sum of fixed-frequency waves. Exactly which terms contribute to the sum depends on the detailed behavior of the λ_l in the neighborhood of the maximum. In particular, if a maximum of λ_l occurs at k_j, and in the neighborhood of k_j, λ_l has a Taylor expansion of the form $M - b_j(k-k_j)^{p_j}$, then we call p_j the *order* of the maximum at k_j.

Theorem 3.1. *Let c_n be given by* (3.1). *Then in the space X_B, $(\boldsymbol{\mathcal{R}}^{\#}\boldsymbol{\mathcal{R}})^n\boldsymbol{\mathcal{E}}_0/c_n$ converges to*

$$\frac{1}{2\pi\sum_{l,i}\beta_l(\boldsymbol{P}_l\boldsymbol{E}_0,\boldsymbol{\psi})(k_i)}\sum_{l,j}\beta_j\boldsymbol{P}_j\boldsymbol{E}_0e^{ik_jc_0t}$$

where the sums are over those indices j and l for which the eigenvalue $\lambda_l(k_j)$ attains its maximum M and has maximal order. The β_j are certain constants depending on b_j, p_j, and k_j (see [3]).

This theorem applies to the case of band-limited signals. It says that in the generic case, the power method converges to a fixed-frequency wave, whose frequency is determined by the frequency k_∞ at which the largest eigenvalue λ_0 attains its maximum in the relevant frequency band. In other words, the algorithm "tunes" automatically to the best available frequency. The spatial shape of the wave is determined by the spectral projector $\boldsymbol{P}_0$ at that frequency. In other words, in the generic case, the spatial shape is given by the eigenfunction corresponding to the eigenvalue $\lambda_0(k_\infty)$. "The generic case" here is the case in which the initial field $\boldsymbol{E}_0$ has a nonzero projection $\boldsymbol{P}_0\boldsymbol{E}_0$ at k_∞, and the case in which λ_0 has only one maximum in the relevant frequency band. If λ_0 happens to have several identical maxima, for example at k_0 and k_1, then the algorithm converges to a field that is a sum of the fields described above. The relative weighting of the different contributions is determined by the projections $\boldsymbol{P}_0(k_0)\boldsymbol{E}_0(k_1)$ and $\boldsymbol{P}_0(k_1)\boldsymbol{E}_0(k_1)$, which also give the spatial shape of the wave.

We note that the only assumptions about the scattering medium are that it is linear and reciprocal. In particular, the theorem applies even in cases when the scatterers are dispersive, dissipative, or bi-anisotropic.

4 An experimental prediction

The algorithm and theorem of the previous section constitute an experimental prediction. In particular, this prediction concerns electromagnetic time-reversal experiments analogous to those done by Fink *et al.* [9, 10] for acoustics. The theorem predicts that an interative time-reversal process will "tune" itself to a fixed-frequency wave whose frequency gives the maximum scattering.

In particular, the iterative time-reversal process should "tune" to the resonant frequency of a Lorentz medium.

5 Acknowledgments

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Appendix A Wave-splitting calculations

The analysis presented in this appendix is a review of the results presented in [11, 12].

In the upper half-space, the Fourier components of the electric and magnetic fields can be decomposed in their transverse and vertical components as

$$\begin{cases} \mathbf{E}(z, \boldsymbol{e}_{xy}, k) = \mathbf{E}_{xy}(z, \boldsymbol{e}_{xy}, k) + \hat{\boldsymbol{z}}\ \mathsf{E}_z(z, \boldsymbol{e}_{xy}, k) \\ \mathbf{H}(z, \boldsymbol{e}_{xy}, k) = \mathbf{H}_{xy}(z, \boldsymbol{e}_{xy}, k) + \hat{\boldsymbol{z}}\ \mathsf{H}_z(z, \boldsymbol{e}_{xy}, k) \end{cases}$$

and similarly for the flux densities.

In the transformed variables, we rewrite the Maxwell equations by making the substitution $\nabla \longrightarrow i\boldsymbol{k}_{xy} + \hat{\boldsymbol{z}}\partial_z = ik\boldsymbol{e}_{xy} + \hat{\boldsymbol{z}}\partial_z$ in (2.3). We obtain

$$\frac{d}{dz}\begin{pmatrix} \hat{z} \times \mathbf{E} \\ \hat{z} \times \eta_0 \mathbf{H} \end{pmatrix} = -ik \begin{pmatrix} \boldsymbol{e}_{xy} \times \mathbf{E} \\ \boldsymbol{e}_{xy} \times \eta_0 \mathbf{H} \end{pmatrix} + ik \begin{pmatrix} \eta_0 \mathbf{H} \\ -\mathbf{E} \end{pmatrix} \tag{A.1}$$

This is a system of six linear, coupled ordinary differential equations in the variable z.

We separate the transverse components of (A.1) from the z components, and use the fact that $\hat{z} \times \mathbf{E} = \mathfrak{J}\mathbf{E}_{xy}$, where $\mathfrak{J} = \hat{\boldsymbol{z}} \times \mathbf{I}_2$ is the dyadic of rotation by $\pi/2$ in the x-y plane, namely

$$\mathfrak{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

In addition, we use the fact that $\boldsymbol{e}_{xy} \times \boldsymbol{a} = -a_z \mathfrak{J} \cdot \boldsymbol{e}_{xy} + (\boldsymbol{a}_{xy} \cdot \mathfrak{J} \cdot \boldsymbol{e}_{xy})\hat{z}$.

$$\frac{d}{dz}\begin{pmatrix} \mathfrak{J} \cdot \mathbf{E}_{xy}(z) \\ \mathfrak{J} \cdot \eta_0 \mathbf{H}_{xy}(z) \end{pmatrix} = ik \begin{pmatrix} \mathfrak{J} \cdot \boldsymbol{e}_{xy}\ \mathsf{E}_z(z) \\ \mathfrak{J} \cdot \boldsymbol{e}_{xy} \eta_0\ \mathsf{H}_z(z) \end{pmatrix} + ik \begin{pmatrix} \eta_0 \mathbf{H}_{xy}(z) \\ -\mathbf{E}_{xy}(z) \end{pmatrix} \tag{A.2}$$

and

$$\begin{pmatrix} \mathsf{E}_z(z) \\ \eta_0\ \mathsf{H}_z(z) \end{pmatrix} = \begin{pmatrix} -\eta_0 \mathbf{H}_{xy}(z) \cdot \mathfrak{J} \cdot \boldsymbol{e}_{xy} \\ \mathbf{E}_{xy}(z) \cdot \mathfrak{J} \cdot \boldsymbol{e}_{xy} \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}_{xy} \cdot \mathfrak{J} \cdot \eta_0 \mathbf{H}_{xy}(z) \\ -\boldsymbol{e}_{xy} \cdot \mathfrak{J} \cdot \mathbf{E}_{xy}(z) \end{pmatrix} \tag{A.3}$$

We use (A.3) in (A.2) to obtain a system of equations involving only the transverse components of the electric and magnetic fields:

$$\frac{d}{dz}\begin{pmatrix} \mathfrak{J} \cdot \mathbf{E}_{xy}(z) \\ \mathfrak{J} \cdot \eta_0 \mathbf{H}_{xy}(z) \end{pmatrix} = ik \begin{pmatrix} \mathbf{0} & \mathbf{I}_2 + \mathfrak{J} \cdot \boldsymbol{e}_{xy}\boldsymbol{e}_{xy} \cdot \mathfrak{J} \\ -(\mathbf{I}_2 + \mathfrak{J} \cdot \boldsymbol{e}_{xy}\boldsymbol{e}_{xy} \cdot \mathfrak{J}) & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{E}_{xy}(z) \\ \eta_0 \mathbf{H}_{xy}(z), \end{pmatrix} \tag{A.4}$$

Here we use the dyadic notation $\mathfrak{J} \cdot \boldsymbol{e}_{xy}\boldsymbol{e}_{xy} \cdot \mathfrak{J} \cdot \boldsymbol{v} = \mathfrak{J} \cdot \boldsymbol{e}_{xy}(\boldsymbol{e}_{xy} \cdot \mathfrak{J} \cdot \boldsymbol{v})$; $\mathbf{I}_2$ denotes the 2×2 identity operator. We multiply both equations of (A.4) by $-\mathfrak{J}$ and use the relation $\mathfrak{J} \cdot \mathfrak{J} = -\mathbf{I}_2$ to obtain

$$\frac{d}{dz}\begin{pmatrix} \mathsf{E}_{xy}(z) \\ \eta_0 \mathsf{H}_{xy}(z) \end{pmatrix} = ik\mathbf{M} \cdot \begin{pmatrix} \mathsf{E}_{xy}(z) \\ \eta_0 \mathsf{H}_{xy}(z), \end{pmatrix} \tag{A.5}$$

where the matrix $\mathbf{M}$ is

$$\mathbf{M} = \begin{pmatrix} \mathbf{0} & (-\mathbf{I}_2 + \boldsymbol{e}_{xy}\boldsymbol{e}_{xy}) \cdot \mathfrak{J} \\ (\mathbf{I}_2 - \boldsymbol{e}_{xy}\boldsymbol{e}_{xy}) \cdot \mathfrak{J} & \mathbf{0} \end{pmatrix}. \tag{A.6}$$

The eigenvalues of $\mathbf{M}$ are $\pm\sqrt{1 - e_{xy}^2}$, and the general solution of (A.5) is therefore given by (2.5).

To find the magnetic field corresponding to (2.5), we use (2.5) in the first component of (A.5):

$$\begin{aligned} &(-\mathbf{I}_2 + \boldsymbol{e}_{xy}\boldsymbol{e}_{xy}) \cdot \eta_0 \mathfrak{J} \cdot \mathsf{H}_{xy}(\boldsymbol{e}_{xy}, z, k) \\ &\qquad = e_z \left(\mathsf{E}_{xy}{}^{\uparrow}(\boldsymbol{e}_{xy}, k) e^{ike_z z} - \mathsf{E}_{xy}{}^{\downarrow}(\boldsymbol{e}_{xy}, k) e^{-ike_z z} \right). \end{aligned} \tag{A.7}$$

In (A.7) we then use the identity

$$\begin{aligned} (\mathbf{I}_2 - \boldsymbol{e}_{xy}\boldsymbol{e}_{xy}) \cdot \left(\mathbf{I}_2 + \frac{\boldsymbol{e}_{xy}\boldsymbol{e}_{xy}}{e_z^2} \right) &= \mathbf{I}_2 + \frac{\boldsymbol{e}_{xy}\boldsymbol{e}_{xy}}{e_z^2} - \boldsymbol{e}_{xy}\boldsymbol{e}_{xy}\frac{e_{xy}^2}{e_z^2} - \boldsymbol{e}_{xy}\boldsymbol{e}_{xy} \\ &= \mathbf{I}_2 + \frac{\boldsymbol{e}_{xy}\boldsymbol{e}_{xy}}{e_z^2}\left(1 - e_{xy}^2 - e_z^2\right) = \mathbf{I}_2. \end{aligned} \tag{A.8}$$

We thus obtain for the magnetic field the expression

$$\eta_0 \mathsf{H}_{xy}(\boldsymbol{e}_{xy}, z, k) = \mathfrak{J} \cdot \mathfrak{P} e_z \cdot \left(\mathsf{E}_{xy}{}^{\uparrow}(\boldsymbol{e}_{xy}, k) e^{ike_z z} - \mathsf{E}_{xy}{}^{\downarrow}(\boldsymbol{e}_{xy}, k) e^{-ike_z z} \right) \tag{A.9}$$

where the admittance dyadic is

$$\mathfrak{P} = \mathbf{I}_2 + \frac{\boldsymbol{e}_{xy}\boldsymbol{e}_{xy}}{e_z^2}$$

Finally, the z components of the electric and magnetic fields can be computed from (A.3) and (A.9).

Appendix B Properties of the energy flux W

B.1 Flux of waves that are both upgoing and downgoing

We show in this section that $W(\boldsymbol{E}^{\downarrow} + \boldsymbol{E}^{\uparrow}) = W(\boldsymbol{E}^{\downarrow}) + W(\boldsymbol{E}^{\uparrow})$ if either $E^{\downarrow}$ or $E^{\uparrow}$ has no evanescent components.

The integrand of (2.10) can be written

$$\hat{\boldsymbol{z}} \cdot \boldsymbol{\mathcal{S}}(\boldsymbol{r}, t) = -\boldsymbol{\mathcal{E}}(\boldsymbol{r}, t) \cdot (\hat{\boldsymbol{z}} \times \boldsymbol{\mathcal{H}}(\boldsymbol{r}, t)) = -\boldsymbol{\mathcal{E}}_{xy}(\boldsymbol{r}, t) \cdot (\mathfrak{J} \cdot \boldsymbol{\mathcal{H}}_{xy}(\boldsymbol{r}, t)),$$

which implies that (2.11) can be written

$$W(\boldsymbol{\mathcal{E}}) = -\int_{-\infty}^{\infty} \iint\limits_{z=0} \boldsymbol{\mathcal{E}}_{xy}(\boldsymbol{r}_{xy}, 0, t) \cdot (\mathfrak{J} \cdot \boldsymbol{\mathcal{H}}_{xy}(\boldsymbol{r}_{xy}, 0, t))\; dxdy\, dt \tag{B.1}$$

When we apply Parseval's identity to (2.10), we get

$$W(\boldsymbol{E}) = -\frac{1}{2\pi^3} \int_{-\infty}^{\infty} \iint\limits_{\mathbb{R}^2} \mathbf{E}_{xy}^*(\boldsymbol{e}_{xy}, 0, k) \cdot (\mathfrak{J} \cdot \mathbf{H}_{xy}(\boldsymbol{e}_{xy}, 0, k))\; k^2 de_x de_y\, c_0 dk, \tag{B.2}$$

Using $\mathbf{E}_{xy} = \mathbf{E}_{xy}^{\uparrow} + \mathbf{E}_{xy}^{\downarrow}$ and (A.9) in (B.2) gives us

$$\begin{aligned} W(\mathbf{E}_{xy}^{\uparrow} + \mathbf{E}_{xy}^{\downarrow}) = \frac{1}{(2\pi)^3 \eta_0} \int_{-\infty}^{\infty} \iint\limits_{\mathbb{R}^2} & \big(\mathbf{E}_{xy}{}^{\uparrow}(\boldsymbol{e}_{xy}, k) + \mathbf{E}_{xy}{}^{\downarrow}(\boldsymbol{e}_{xy}, k)\big)^* \\ & \cdot \mathfrak{P} \cdot \big(\mathbf{E}_{xy}{}^{\uparrow}(\boldsymbol{e}_{xy}, k) - \mathbf{E}_{xy}{}^{\downarrow}(\boldsymbol{e}_{xy}, k)\big)\, e_z k^2 de_x de_y\, c_0 dk, \end{aligned}$$

which can be written

$$\begin{aligned} W(\mathbf{E}_{xy}^{\uparrow} + \mathbf{E}_{xy}^{\downarrow}) &= W(\mathbf{E}_{xy}^{\uparrow}) + W(\mathbf{E}_{xy}^{\downarrow}) + \\ & \frac{1}{(2\pi)^3 \eta_0} \int_{-\infty}^{\infty} \iint\limits_{\mathbb{R}^2} \Big[\mathbf{E}_{xy}^{\downarrow *} \cdot (\boldsymbol{e}_{xy}, k) \mathfrak{P} \cdot \mathbf{E}_{xy}^{\uparrow}(\boldsymbol{e}_{xy}, k) \\ & - \mathbf{E}_{xy}^{\uparrow *}(\boldsymbol{e}_{xy}, k) \cdot \mathfrak{P} \cdot \mathbf{E}_{xy}^{\downarrow}(\boldsymbol{e}_{xy}, k)\Big] e_z k^2 de_x de_y\, c_0 dk. \end{aligned} \tag{B.3}$$

The k integral of (B.3) we split into two parts, corresponding to $k > 0$ and $k < 0$. In the $k < 0$ integral we make the change of variables $k \to -k$ and use the symmetry relations

$$\mathbf{E}_{xy}^{\uparrow\downarrow}(\boldsymbol{e}_{xy}, -k) = \mathbf{E}_{xy}^{\uparrow\downarrow *}(\boldsymbol{e}_{xy}, k), \qquad e_z(-k) = e_z^*(k), \tag{B.4}$$

which follow from the fact that the time-domain fields $\boldsymbol{\mathcal{E}}^{\uparrow\downarrow}$ are real. These operations show that the integral of (B.3) can be written

$$\begin{aligned} I = & \frac{1}{(2\pi)^3 \eta_0} 2\,\mathrm{Re} \int_{0}^{\infty} \iint\limits_{\mathbb{R}^2} \Big[\mathbf{E}_{xy}^{\downarrow *}(\boldsymbol{e}_{xy}, k) \cdot \mathfrak{P} \cdot \mathbf{E}_{xy}^{\uparrow}(\boldsymbol{e}_{xy}, k) \\ & - \mathbf{E}_{xy}^{\uparrow *}(\boldsymbol{e}_{xy}, k) \cdot \mathfrak{P} \cdot \mathbf{E}_{xy}^{\downarrow}(\boldsymbol{e}_{xy}, k)\Big] e_z k^2 de_x de_y\, c_0 dk \\ = & \frac{1}{(2\pi)^3 \eta_0} 2\,\mathrm{Re} \int_{0}^{\infty} \iint\limits_{\mathbb{R}^2} 2i\,\mathrm{Im}\Big[\mathbf{E}_{xy}^{\downarrow *}(\boldsymbol{e}_{xy}, k) \cdot \mathfrak{P} \cdot \mathbf{E}_{xy}^{\uparrow}(\boldsymbol{e}_{xy}, k)\Big] e_z\; k^2 de_x de_y\, c_0 dk \end{aligned} \tag{B.5}$$

since $\mathfrak{P} = \mathbf{I}_2 + \boldsymbol{e}_{xy}\boldsymbol{e}_{xy}/e_z^2$ is a real, symmetric dyadic. We see that if the imaginary part of the factor in square brackets is zero when e_z is purely imaginary, that is to say when $|e_{xy}| > 1$, then (B.5) is zero. In particular, the factor in square brackets is zero if either $\mathbf{E}_{xy}^{\downarrow}$ or $\mathbf{E}_{xy}^{\uparrow}$ is zero for $|e_{xy}| > 1$.

B.2 Energy Flux of Evanescent Waves

We show in this section that evanescent waves do not contribute to the energy flux of upgoing or downgoing waves.

In (2.12) or (2.13) we split the k integral into pieces corresponding to $k < 0$ and $k > 0$. In the $k < 0$ integral, we make the substitution $k \to -k$ and use the symmetry relations (B.4). The factor $\left|\mathsf{E}^{\uparrow\downarrow}(\boldsymbol{e}_{xy}, k)\right|^2$ remains unchanged, but for evanescent waves, e_z changes sign. Thus the terms corresponding to evanescent waves cancel each other.

Appendix C Connection to sources and scatterers

In this section we show how source currents and scatterers give rise to downgoing and upgoing fields. For this we need the electric field Green's dyadic.

C.1 The electric field Green's dyadic.

The electric field Green's dyadic satisfies

$$\nabla \times \nabla \times \mathbf{G} - k^2\mathbf{G} = \mathbf{I}\delta, \tag{C.1}$$

where $\mathbf{I}$ is the 3×3 identity dyadic and $\mathbf{G}$ is a 3×3 dyadic-valued function. The dyadic $\mathbf{G}$ also satisfies outgoing radiation condtions.

This Green's dyadic is given by

$$\mathbf{G}(k, |\boldsymbol{r} - \boldsymbol{r}'|) = \left(\mathbf{I} + k^{-2}\nabla\nabla\right) g(k, |\boldsymbol{r} - \boldsymbol{r}'|) \tag{C.2}$$

where the dyadic $\nabla\nabla$ is a dyadic-valued operator whose i, jth element is $\partial_{x_i}\partial_{x_j}$ (in the notation $(x, y, z) = (x_1, x_2, x_3)$). Here g is the scalar Green's function

$$g(k, |\boldsymbol{r} - \boldsymbol{r}'|) = \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r} - \boldsymbol{r}'|}. \tag{C.3}$$

Note that each column of the matrix $\nabla\nabla g_0$ is the gradient of something. We can easily check that $\mathbf{G}$ satisfies (C.1):

$$\begin{aligned}\nabla \times \nabla \times \mathbf{G} - k^2\mathbf{G} &= \left(\nabla \times \nabla \times (\mathbf{I}g) + k^{-2}\nabla \times \nabla \times (\nabla\nabla g)\right) - k^2\left(g\mathbf{I} + k^{-2}\nabla\nabla g\right)\\ &= \nabla\nabla g - \nabla^2 g\mathbf{I} - k^2 g\mathbf{I} - \nabla\nabla g\\ &= \delta\mathbf{I}\end{aligned} \tag{C.4}$$

where the term $\nabla \times (\nabla \times \nabla\nabla g)$ vanishes because the expression in parentheses is the curl of a gradient, and where we have used the identity $\nabla \times \nabla \times \boldsymbol{a} = \nabla(\nabla \cdot \boldsymbol{a}) - \nabla^2\boldsymbol{a}$ together with the identity $\nabla(\nabla \cdot \mathbf{I}g) = \nabla\nabla g$ to simplify the term $\nabla \times \nabla(\mathbf{I}g)$.

The scalar Green's function g can be written in terms of its Fourier transform [1, 2] in the lateral variables x and y:

$$g(k, |\boldsymbol{r} - \boldsymbol{r}'|) = \frac{1}{(2\pi)^2}\iint \frac{-1}{2ike_z} e^{ik\boldsymbol{e}^{\pm}\cdot(\boldsymbol{r}-\boldsymbol{r}')} k^2 de_x de_y \tag{C.5}$$

where the plus sign in the exponent is used for $z > z'$ and the minus sign for $z < z'$.

From (C.5) and (C.2), we can obtain a similar expression for the electric field Green's function (C.2) [1, 7]:

$$\mathbf{G}(k, |\boldsymbol{r} - \boldsymbol{r}'|) = \frac{1}{(2\pi)^2} \iint \frac{-1}{2ike_z} (\mathbf{I}_3 - \boldsymbol{e}^{\pm}\boldsymbol{e}^{\pm}) e^{ik\boldsymbol{e}^{\pm}\cdot(\boldsymbol{r}-\boldsymbol{r}')} k^2 de_x de_y. \tag{C.6}$$

The dyad appearing in the integrand of (C.6), namely

$$\mathbf{P}_{\perp}(\boldsymbol{e}^{\pm}) = \mathbf{I}_3 - \boldsymbol{e}^{\pm}\boldsymbol{e}^{\pm}, \tag{C.7}$$

is the operator that projects a vector onto its components orthogonal to $\boldsymbol{e}^{\pm}$.

C.2 The electric field in terms of its sources

We use two different approaches to representing a field in terms of its sources: a volume integral formulation and a surface integal formulation. The volume integral formulation is perhaps more familiar, and fits easily with the Lippmann-Schwinger integral equation. The surface integral formulation, however, is more general. It makes no assumptions about the medium contained within the surface.

C.2.1 Volume representation

The field that arises from a source current density $\boldsymbol{J}$ in free space is determined from the frequency-domain version of the Maxwell equations, which in this case are

$$\begin{aligned} \nabla \times \boldsymbol{E} &= ik(\eta_0 \boldsymbol{H}) \\ \nabla \times (\eta_0 \boldsymbol{H}) &= \eta_0 \boldsymbol{J} - ik\boldsymbol{E} \end{aligned} \tag{C.8}$$

Taking the curl of the first equation of (C.8) and using the second equation results in

$$\nabla \times \nabla \times \boldsymbol{E} = ik\eta_0 \boldsymbol{J} + k^2 \boldsymbol{E}. \tag{C.9}$$

From (C.1) and (C.9) we see that we can write the electric field as

$$\boldsymbol{E}(\boldsymbol{r}) = ik\eta_0 \iiint \mathbf{G}(k, |\boldsymbol{r} - \boldsymbol{r}'|) \boldsymbol{J}(\boldsymbol{r}') d^3\boldsymbol{r}'. \tag{C.10}$$

Using (C.6), with the notation (C.7), in (C.10), we obtain depending on whether z is below (minus sign) or above (plus sign) the support of $\boldsymbol{J}$.

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{r}) &= ik\eta_0 \iiint \frac{1}{(2\pi)^2} \iint \frac{-1}{2ike_z} \mathbf{P}_{\perp}(\boldsymbol{e}^{\pm}) e^{ik\boldsymbol{e}^{\pm}\cdot(\boldsymbol{r}-\boldsymbol{r}')} k^2 de_x de_y \cdot \boldsymbol{J}(\boldsymbol{r}') d^3\boldsymbol{r}' \\ &= \frac{1}{(2\pi)^2} \iint \left[\frac{-\eta_0}{2e_z} \mathbf{P}_{\perp}(\boldsymbol{e}^{\pm}) \cdot \widetilde{\boldsymbol{J}}(k\boldsymbol{e}^{\pm}) \right] e^{ik\boldsymbol{e}^{\pm}\cdot\boldsymbol{r}} k^2 de_x de_y, \end{aligned} \tag{C.11}$$

where $\widetilde{\boldsymbol{J}}$ denotes the three-dimensional Fourier transform of $\boldsymbol{J}$. Comparing (C.11) with (2.8) and (2.7), we see that the quantity in square brackets in (C.11) is $\mathsf{E}^{\uparrow}$ or $\mathsf{E}^{\downarrow}$, depending on whether we have used the plus sign or the minus sign in the argument of $\widetilde{\boldsymbol{J}}$, which in turn depends on whether z is below or above the support of $\boldsymbol{J}$.

Example: sources on a plane. If, for example, the sources are on a plane $z' = h$, so that $\boldsymbol{J}(\boldsymbol{r}') = \boldsymbol{j}(\boldsymbol{r}'_{xy})\delta(z-h)$, and $z < h$, then

$$\begin{aligned}\mathbf{E}^{\downarrow}(\boldsymbol{e}_{xy}, k) &= \frac{-\eta_0}{2e_z}\mathbf{P}_{\perp}(\boldsymbol{e}^-) \cdot \widetilde{\boldsymbol{J}}(k\boldsymbol{e}^-) \\ &= \frac{-\eta_0}{2e_z}\mathbf{P}_{\perp}(\boldsymbol{e}^-) \cdot \int \boldsymbol{J}(\boldsymbol{r}')e^{-ik\boldsymbol{e}^-\cdot\boldsymbol{r}'}d^3\boldsymbol{r}' \\ &= \frac{-\eta_0}{2e_z}\mathbf{P}_{\perp}(\boldsymbol{e}^-) \cdot \int \boldsymbol{j}(\boldsymbol{r}'_{xy})e^{-ik\boldsymbol{e}_{xy}\cdot\boldsymbol{r}'_{xy}}d^2\boldsymbol{r}'_{xy}e^{ike_z h}\end{aligned} \tag{C.12}$$

We note that for evanescent waves, the exponential at the end of the last line of (C.12) is $\exp(-|k|h\sqrt{e_{xy}^2 - 1})$, which decays exponentially with large h.

Similarly, an upgoing field due to sources on the plane $z' = -h$ corresponds to

$$\mathbf{E}^{\uparrow}(\boldsymbol{e}_{xy}, k) = \frac{-\eta_0}{2e_z}\mathbf{P}_{\perp}(\boldsymbol{e}^+) \cdot \int \boldsymbol{j}(\boldsymbol{r}'_{xy})e^{-ik\boldsymbol{e}_{xy}\cdot\boldsymbol{r}'_{xy}}d^2\boldsymbol{r}'_{xy}e^{-ike_z(-h)} \tag{C.13}$$

whose evanescent components again decay with large h.

The exponential decay of the evanescent components of (C.12) and (C.13) show that for sources far from the scatterers, the interaction of upgoing and downgoing evanescent waves is negligible.

Equations (C.12) and (C.13) can be inverted for $\boldsymbol{j}$ to obtain sources that give rise to the corresponding downgoing or upgoing waves. We note that both sides of (C.12) are orthogonal to $\boldsymbol{e}^-$; similarly both sides of (C.13) are orthogonal to $\boldsymbol{e}^+$.

Example: scattering from dielectrics. Scattering from a dielectric can be interpreted in terms of sources as follows.

We assume that the medium in the lower half-space has a magnetic permeability that is the same as that of free space, and the electric conductivity σ and permittivity ϵ differ from their vacuum values (0 and ϵ_0, respectively) only in a bounded region. In this case, the frequency-domain version of the Maxwell equations are

$$\nabla \times \boldsymbol{E} = ik(\eta_0\boldsymbol{H}) \tag{C.14}$$

$$\nabla \times (\eta_0\boldsymbol{H}) = (\eta_0\sigma - ik\epsilon_r)\boldsymbol{E} \tag{C.15}$$

where $\epsilon_r = \epsilon/\epsilon_0$.

Since the magnetic permeability is a constant μ_0, taking the curl of (C.14) and using (C.15) results in

$$\nabla \times \nabla \times \boldsymbol{E} = i\omega\mu_0(\sigma - i\omega\epsilon)\boldsymbol{E} = (\omega^2\mu_0\epsilon + i\omega\mu_0\sigma)\boldsymbol{E} \tag{C.16}$$

We rewrite (C.16) as

$$\nabla \times \nabla \times \boldsymbol{E} - k^2\boldsymbol{E} = V\boldsymbol{E} \tag{C.17}$$

where $V = \omega^2\mu_0(\epsilon - \epsilon_0) + i\omega\mu_0\sigma$ and as before $k^2 = \omega^2\mu_0\epsilon_0$.

We write $\boldsymbol{E} = \boldsymbol{E}^i + \boldsymbol{E}^s$ where the incident field $\boldsymbol{E}^i$ satisfies the unperturbed equation

$$\nabla \times \nabla \times \boldsymbol{E}^i - k^2 \boldsymbol{E}^i = \mathbf{0}.$$

The scattered field $\boldsymbol{E}^s$ then satisfies

$$\nabla \times \nabla \times \boldsymbol{E}^s - k^2 \boldsymbol{E}^s = V \boldsymbol{E},$$

which with the help of (C.1) can be written

$$\boldsymbol{E}^s(\boldsymbol{r}, k) = \int \mathbf{G}(k, |\boldsymbol{r} - \boldsymbol{r}'|) V(\boldsymbol{r}') \boldsymbol{E}(\boldsymbol{r}', k) d^3\boldsymbol{r}'. \tag{C.18}$$

We note that (C.18) is of the form (C.10), where $ik\eta_0 \boldsymbol{J} = V\boldsymbol{E}$ corresponds to the effective currents in the scatterer that are induced by the incident field. The Fourier transform of $ik\eta_0 \boldsymbol{J}$ is

$$ik\eta_0 \widetilde{\boldsymbol{J}}(k\boldsymbol{e}^+) = \int e^{-ik\boldsymbol{e}^+ \cdot \boldsymbol{r}'} V(\boldsymbol{r}') \boldsymbol{E}(\boldsymbol{r}', k) d^3\boldsymbol{r}' \tag{C.19}$$

From (C.11) we therefore see that the upgoing wave $\mathsf{E}^\uparrow = \mathsf{E}^s$ can be written

$$\mathsf{E}^\uparrow(\boldsymbol{e}_{xy}, z, k) = \frac{-1}{2ike_z} \mathbf{P}_\perp(\boldsymbol{e}^+) \cdot \int e^{-ik\boldsymbol{e}^+ \cdot \boldsymbol{r}'} V(\boldsymbol{r}') \boldsymbol{E}(\boldsymbol{r}', k) d^3\boldsymbol{r}' \; e^{ike_z z}.$$

C.2.2 Surface representation

Let V_i be a region of sources to the incident field, and V_s be a region containing all the scatterers; see Figure 1. We denote the boundaries of V_i and V_s by S_i and S_s, respectively. From Green's theorem and radiation conditions, the field between V_i and V_s is [6, 8, 13]

$$\begin{aligned}
&\frac{i}{k} \nabla \times \left\{ \nabla \times \iint_{S_i} g(k, |\boldsymbol{r} - \boldsymbol{r}'|) \left(\hat{\boldsymbol{\nu}}(\boldsymbol{r}') \times \eta_0 \boldsymbol{H}(\boldsymbol{r}') \right) dS' \right\} \\
&+ \nabla \times \iint_{S_i} g(k, |\boldsymbol{r} - \boldsymbol{r}'|) \left(\hat{\boldsymbol{\nu}}(\boldsymbol{r}') \times \boldsymbol{E}(\boldsymbol{r}') \right) dS' \\
&+ \frac{i}{k} \nabla \times \left\{ \nabla \times \iint_{S_s} g(k, |\boldsymbol{r} - \boldsymbol{r}'|) \left(\hat{\boldsymbol{\nu}}(\boldsymbol{r}') \times \eta_0 \boldsymbol{H}(\boldsymbol{r}') \right) dS' \right\} \\
&+ \nabla \times \iint_{S_s} g(k, |\boldsymbol{r} - \boldsymbol{r}'|) \left(\hat{\boldsymbol{\nu}}(\boldsymbol{r}') \times \boldsymbol{E}(\boldsymbol{r}') \right) dS' \\
&= \begin{cases} \boldsymbol{E}(\boldsymbol{r}), & \boldsymbol{r} \text{ outside } S_i \text{ and } S_s \\ \mathbf{0}, & \boldsymbol{r} \text{ inside } S_i \text{ or } S_s \end{cases}
\end{aligned} \tag{C.20}$$

where $\hat{\boldsymbol{\nu}}$ denotes the outward unit normal vector and g is given by (C.3).

We assume that the source region lies above the scattering region. Thus in the empty layer between them, we have

$$\begin{aligned}\boldsymbol{E}^{\downarrow}(\boldsymbol{r}) = &\frac{i}{k}\nabla\times\left\{\nabla\times\iint\limits_{S_i} g(k,|\boldsymbol{r}-\boldsymbol{r}'|)\,(\hat{\boldsymbol{\nu}}(\boldsymbol{r}')\times\eta_0\boldsymbol{H}(\boldsymbol{r}'))\;dS'\right\}\\ &+\nabla\times\iint\limits_{S_i} g(k,|\boldsymbol{r}-\boldsymbol{r}'|)\,(\hat{\boldsymbol{\nu}}(\boldsymbol{r}')\times\boldsymbol{E}(\boldsymbol{r}'))\;dS'\end{aligned}\tag{C.21}$$

and

$$\begin{aligned}\boldsymbol{E}^{\uparrow}(\boldsymbol{r}) = &\frac{i}{k}\nabla\times\left\{\nabla\times\iint\limits_{S_s} g(k,|\boldsymbol{r}-\boldsymbol{r}'|)\,(\hat{\boldsymbol{\nu}}(\boldsymbol{r}')\times\eta_0\boldsymbol{H}(\boldsymbol{r}'))\;dS'\right\}\\ &+\nabla\times\iint\limits_{S_s} g(k,|\boldsymbol{r}-\boldsymbol{r}'|)\,(\hat{\boldsymbol{\nu}}(\boldsymbol{r}')\times\boldsymbol{E}(\boldsymbol{r}'))\;dS'\end{aligned}\tag{C.22}$$

The spatial Fourier transforms of $\boldsymbol{E}^{\downarrow}$ and $\boldsymbol{E}^{\uparrow}$ are then obtained by using (C.5) with a minus sign in (C.21), and (C.5) with a plus sign in (C.22), respectively. Comparing the results with (2.8) and (2.7), we see that [7]

$$\begin{aligned}\mathsf{E}^{\uparrow}(k,\boldsymbol{e}_{xy}) = &-ik\boldsymbol{e}^{+}\times\left\{\boldsymbol{e}^{+}\times\iint\limits_{S_s}\frac{-1}{2ike_z}e^{-ik\boldsymbol{e}^{+}\cdot\boldsymbol{r}'}\,(\hat{\boldsymbol{\nu}}(\boldsymbol{r}')\times\eta_0\boldsymbol{H}(\boldsymbol{r}'))\;dS'\right\}\\ &+ik\boldsymbol{e}^{+}\times\iint\limits_{S_s}\frac{-1}{2ike_z}e^{-ik\boldsymbol{e}^{+}\cdot\boldsymbol{r}'}\,(\hat{\boldsymbol{\nu}}(\boldsymbol{r}')\times\boldsymbol{E}(\boldsymbol{r}'))\;dS'\end{aligned}\tag{C.23}$$

and

$$\begin{aligned}\mathsf{E}^{\downarrow}(k,\boldsymbol{e}_{xy}) = &-ik\boldsymbol{e}^{-}\times\left\{\boldsymbol{e}^{-}\times\iint\limits_{S_i}\frac{-1}{2ike_z}e^{-ik\boldsymbol{e}^{-}\cdot\boldsymbol{r}'}\,(\hat{\boldsymbol{\nu}}(\boldsymbol{r}')\times\eta_0\boldsymbol{H}(\boldsymbol{r}'))\;dS'\right\}\\ &+ik\boldsymbol{e}^{-}\times\iint\limits_{S_i}\frac{-1}{2ike_z}e^{-ik\boldsymbol{e}^{-}\cdot\boldsymbol{r}'}\,(\hat{\boldsymbol{\nu}}(\boldsymbol{r}')\times\boldsymbol{E}(\boldsymbol{r}'))\;dS'\end{aligned}\tag{C.24}$$

Certain of the operations in (C.23) and (C.24) have a physical interpretation. In particular, calculations analagous to those of (C.4), which led to (C.6), show that we can replace $\boldsymbol{e}^{\pm}\times\boldsymbol{e}^{\pm}\times$ in (C.23) and (C.24) by $-\mathbf{P}_{\perp}(\boldsymbol{e}^{\pm})\cdot$. We also note that the operator $\boldsymbol{e}^{\pm}\times$ can be considered as a projection onto the plane perpendicular to $\boldsymbol{e}^{\pm}$ (i.e., the operator $\mathbf{P}_{\perp}(\boldsymbol{e}^{\pm})$) followed by a rotation of $\pi/2$.

Appendix D The far-field amplitude, the scattering operator, and the reflection operator

In this appendix we consider the general problem of scattering of a fixed-frequency plane wave by a bounded object.

We write the electric field as

$$\boldsymbol{E} = \boldsymbol{E}^i + \boldsymbol{E}^s, \tag{D.1}$$

where the incident field $\boldsymbol{E}^i$ is given by

$$\boldsymbol{E}^i(\boldsymbol{r}) = \boldsymbol{E}^0 e^{ik\hat{\boldsymbol{e}}\cdot\boldsymbol{r}}$$

D.1 The far-field amplitude

In the far field, the scattered field $\boldsymbol{E}^s$ has the large-r asymptotic behavior

$$\boldsymbol{E}^s(\boldsymbol{r}) = \boldsymbol{F}(\hat{\boldsymbol{r}}, \boldsymbol{E}^0, \hat{\boldsymbol{e}}, k)\frac{e^{ikr}}{kr} + O((kr)^{-2})$$

Formulas for $\boldsymbol{F}$ can be found from (C.10) or (C.22). For this, we make use of the far-field expansion of g, which is [8]

$$g(k, |\boldsymbol{r} - \boldsymbol{r}'|) = \frac{e^{ikr}}{4\pi r} e^{-ik\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'} \left(1 + O(kd^2/r)\right)(1 + O(d/r)),$$

where $d = \max|\boldsymbol{r}'|$. To obtain a far-field expansion for the Green's dyadic, we need also the derivatives

$$\frac{1}{k^2}\nabla g(k, |\boldsymbol{r} - \boldsymbol{r}'|) = i\hat{\boldsymbol{r}}\frac{e^{ikr}}{4\pi kr} e^{-ik\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'} \left(1 + O(kd^2/r)\right)(1 + O(d/r))\left(1 + O((kr)^{-1})\right)$$

and

$$\frac{1}{k^2}\nabla\nabla g(k, |\boldsymbol{r} - \boldsymbol{r}'|) = -\hat{\boldsymbol{r}}\hat{\boldsymbol{r}}\frac{e^{ikr}}{4\pi r} e^{-ik\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'} \left(1 + O(kd^2/r)\right)(1 + O(d/r))\left(1 + O((kr)^{-1})\right).$$

These calculations give rise to the expansion

$$\mathbf{G}(k, |\boldsymbol{r} - \boldsymbol{r}'|) = \frac{e^{ikr}}{4\pi r}\mathbf{P}_\perp(\hat{\boldsymbol{r}}) e^{-ik\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'} \left(1 + O(kd^2/r)\right)(1 + O(d/r))\left(1 + O((kr)^{-1}\right). \tag{D.2}$$

If $\boldsymbol{E}^s = \boldsymbol{E}^\uparrow$ is given by (C.10), then from (C.2) and (D.2), we see that [5]

$$\begin{aligned} \boldsymbol{F}(\hat{\boldsymbol{r}}, \boldsymbol{E}^0, \hat{\boldsymbol{e}}, k) &= \frac{ik^2\eta_0}{4\pi}\boldsymbol{P}_\perp(\hat{\boldsymbol{r}}) \cdot \iiint e^{-ik\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'} \boldsymbol{J}(\boldsymbol{r}', \boldsymbol{E}^0, \hat{\boldsymbol{e}}) d^3\boldsymbol{r}' \\ &= \frac{ik^2\eta_0}{4\pi}\boldsymbol{P}_\perp(\hat{\boldsymbol{r}}) \cdot \widetilde{\boldsymbol{J}}(k\hat{\boldsymbol{r}}, \boldsymbol{E}^0, \hat{\boldsymbol{e}}) \end{aligned} \tag{D.3}$$

If $\boldsymbol{E}^s = \boldsymbol{E}^\uparrow$ is given by the surface integral (C.22), then [8]

$$\boldsymbol{F}(\hat{\boldsymbol{r}}, \boldsymbol{E}^0, \hat{\boldsymbol{e}}, k) = \frac{ik^2}{4\pi}\hat{\boldsymbol{r}} \times \iint_{S_s} \left[\hat{\boldsymbol{\nu}}(\boldsymbol{r}') \times \boldsymbol{E}(\boldsymbol{r}') - \hat{\boldsymbol{r}} \times (\hat{\boldsymbol{\nu}}(\boldsymbol{r}') \times \eta_0 \boldsymbol{H}(\boldsymbol{r}'))\right] e^{-ik\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'} dS' \tag{D.4}$$

D.2 The scattering operator

Because the Maxwell equations are linear, the far-field amplitude must depend linearly on the incident field and in particular on the incident polarization $\boldsymbol{E}^0$. We can thus write the far-field amplitude as

$$\boldsymbol{F}(\hat{\boldsymbol{r}}, \boldsymbol{E}^0, \hat{\boldsymbol{e}}, k) = \mathbf{S}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{e}}, k) \cdot \boldsymbol{E}^0; \tag{D.5}$$

$\mathbf{S}$ is called the *scattering dyadic* [8].

D.3 The reflection operator

To obtain an explicit representation for the reflection operator, we choose $\mathsf{E}^{\downarrow}$ in (2.17) to be a delta function, and then work out explicitly what the scattered field $\mathsf{E}^{\uparrow}$ is. In particular, we take $\mathsf{E}^{\downarrow}(\boldsymbol{e}_{xy}, z, k) = \boldsymbol{E}^0 \delta(k(\boldsymbol{e}_{xy} - \boldsymbol{e}'_{xy})) \exp(-ike_z z)$, which corresponds to the frequency-domain field $\boldsymbol{E}^{\downarrow}(\boldsymbol{r}, k) = \boldsymbol{E}^0 \exp(ik\boldsymbol{e}' \cdot \boldsymbol{r})$, where $\boldsymbol{e}' = (e'_{xy}, -e'_z)$. With this notation, the corresponding upgoing field, which we denote by $\mathsf{E}^{\uparrow}(\boldsymbol{e}_{xy}, z, k, \boldsymbol{E}^0, \boldsymbol{e}')$, is related to the reflection operator by

$$\mathsf{E}^{\uparrow}(\boldsymbol{e}_{xy}, z = 0, k, \boldsymbol{E}^0, \boldsymbol{e}') = \mathsf{R}(\boldsymbol{e}_{xy}, \boldsymbol{e}'_{xy}, k) \cdot \boldsymbol{E}^0$$

We have calculated the upgoing field and the corresponding far-field amplitude in two different ways, with a volume representation (D.3) and with a surface representation (D.4).

D.3.1 Volume representation

From (C.11), we find that R can be written

$$\mathsf{R}(\boldsymbol{e}_{xy}, \boldsymbol{e}'_{xy}, k) \cdot \boldsymbol{E}^0 = \frac{-\eta_0}{2e_z} \mathbf{P}_{\perp}(\boldsymbol{e}^+) \cdot \widetilde{\boldsymbol{J}}(k\boldsymbol{e}^+, \boldsymbol{E}^0, \boldsymbol{e}'^-), \tag{D.6}$$

where $\boldsymbol{J}(\boldsymbol{r}, \boldsymbol{E}^0, \boldsymbol{e}'^-)$ are the effective currents induced in the scattering medium by the incident field $\boldsymbol{E}^{\downarrow}(\boldsymbol{r}, k) = \boldsymbol{E}^0 \exp(ik\boldsymbol{e}'^- \cdot \boldsymbol{r})$. A comparison of (D.6) with (D.3) shows that

$$\mathsf{R}(\boldsymbol{e}_{xy}, \boldsymbol{e}'_{xy}, k) \cdot \boldsymbol{E}^0 = \frac{2\pi i}{k^2 e_z} \boldsymbol{F}(\boldsymbol{e}^+, \boldsymbol{E}^0, \boldsymbol{e}'^-, k).$$

In the case of scattering by a dielectric, where (C.19) holds, (D.6) becomes

$$\mathsf{R}(\boldsymbol{e}_{xy}, \boldsymbol{e}'_{xy}, k) \cdot \boldsymbol{E}^0 = \frac{-1}{2ike_z} \mathbf{P}_{\perp}(\boldsymbol{e}^+) \cdot \int e^{-ik\boldsymbol{e}^+ \cdot \boldsymbol{r}'} V(\boldsymbol{r}') \boldsymbol{E}(\boldsymbol{r}', k, \boldsymbol{E}^0, \boldsymbol{e}'^-) d^3\boldsymbol{r}'. \tag{D.7}$$

D.3.2 Surface representation

A comparison of (D.4) and (C.23) shows also that

$$\mathsf{R}(\boldsymbol{e}_{xy}, \boldsymbol{e}'_{xy}, k) \cdot \boldsymbol{E}^0 = \frac{2\pi i}{k^2 e_z} \boldsymbol{F}(\boldsymbol{e}^+, \boldsymbol{E}^0, \boldsymbol{e}'^-, k).$$

Appendix E Reciprocity

In this section we prove the reciprocity relation (2.21). We denote by $\boldsymbol{E}^a$ and $\boldsymbol{E}^b$ any two fields of the form (D.1) where

$$\begin{cases} \boldsymbol{E}_i^a(\boldsymbol{r}) = \boldsymbol{E}_0^a e^{ik\hat{\mathbf{e}}^a\cdot\boldsymbol{r}} \\ \boldsymbol{E}_i^b(\boldsymbol{r}) = \boldsymbol{E}_0^b e^{ik\hat{\mathbf{e}}^b\cdot\boldsymbol{r}}, \end{cases} \tag{E.1}$$

where we have made the change of notation $\boldsymbol{E}^i \to \boldsymbol{E}_i$.

Lemma E.1. *If S is a smooth surface enclosing a reciprocal medium, the identity*

$$\iint_S \left\{\boldsymbol{E}^a \times \boldsymbol{H}^b - \boldsymbol{E}^b \times \boldsymbol{H}^a\right\} \cdot \hat{\boldsymbol{\nu}}\, dS = 0 \tag{E.2}$$

holds. In other words, if inside S we have the constitutive relations

$$\begin{cases} \boldsymbol{D}(\boldsymbol{r}) = \epsilon_0\Big\{\boldsymbol{\epsilon}(\boldsymbol{r})\cdot\boldsymbol{E}(\boldsymbol{r}) + \eta_0\boldsymbol{\xi}(\boldsymbol{r})\cdot\boldsymbol{H}(\boldsymbol{r})\Big\} \\ \boldsymbol{B}(\boldsymbol{r}) = \dfrac{1}{c_0}\Big\{\boldsymbol{\zeta}(\boldsymbol{r})\cdot\boldsymbol{E}(\boldsymbol{r}) + \eta_0\boldsymbol{\mu}(\boldsymbol{r})\cdot\boldsymbol{H}(\boldsymbol{r})\Big\} \end{cases}$$

where the reciprocity conditions hold

$$\begin{cases} \boldsymbol{\epsilon}(\boldsymbol{r}) = \boldsymbol{\epsilon}^T(\boldsymbol{r}) \\ \boldsymbol{\mu}(\boldsymbol{r}) = \boldsymbol{\mu}^T(\boldsymbol{r}) \\ \boldsymbol{\xi}(\boldsymbol{r}) = -\boldsymbol{\zeta}^T(\boldsymbol{r}) \end{cases} \tag{E.3}$$

then (E.2) *holds.*

Proof: By Gauss' theorem, the left side of (E.2) is

$$I = \iiint_{V_s} \nabla\cdot\left\{\boldsymbol{E}^a \times \boldsymbol{H}^b - \boldsymbol{E}^b \times \boldsymbol{H}^a\right\}\, dv$$

The integrand is

$$\begin{aligned} \nabla\cdot &\left\{\boldsymbol{E}^a(\boldsymbol{r}) \times \boldsymbol{H}^b(\boldsymbol{r}) - \boldsymbol{E}^b(\boldsymbol{r}) \times \boldsymbol{H}^a(\boldsymbol{r})\right\} \\ &= (\nabla\times\boldsymbol{E}^a(\boldsymbol{r}))\cdot\boldsymbol{H}^b(\boldsymbol{r}) - \boldsymbol{E}^a(\boldsymbol{r})\cdot(\nabla\times\boldsymbol{H}^b(\boldsymbol{r})) \\ &- (\nabla\times\boldsymbol{E}^b(\boldsymbol{r}))\cdot\boldsymbol{H}^a(\boldsymbol{r}) + \boldsymbol{E}^b(\boldsymbol{r})\cdot(\nabla\times\boldsymbol{H}^a(\boldsymbol{r})). \end{aligned} \tag{E.4}$$

In (E.4) we use the Maxwell equations and the constitutive relations to obtain

$$
\begin{aligned}
&\nabla\cdot \Big\{\boldsymbol{E}^a(\boldsymbol{r})\times \boldsymbol{H}^b(\boldsymbol{r}) - \boldsymbol{E}^b(\boldsymbol{r})\times \boldsymbol{H}^a(\boldsymbol{r})\Big\}\\
&= i\omega\frac{1}{c_0}\boldsymbol{H}^b(\boldsymbol{r})\cdot\Big\{\boldsymbol{\zeta}(\boldsymbol{r})\cdot\boldsymbol{E}^a(\boldsymbol{r}) + \eta_0\boldsymbol{\mu}(\boldsymbol{r})\cdot\boldsymbol{H}^a(\boldsymbol{r})\Big\}\\
&+ i\omega\epsilon_0\boldsymbol{E}^a(\boldsymbol{r})\cdot\Big\{\boldsymbol{\epsilon}(\boldsymbol{r})\cdot\boldsymbol{E}^b(\boldsymbol{r}) + \eta_0\boldsymbol{\xi}(\boldsymbol{r})\cdot\boldsymbol{H}^b(\boldsymbol{r})\Big\}\\
&- i\omega\frac{1}{c_0}\boldsymbol{H}^a(\boldsymbol{r})\cdot\Big\{\boldsymbol{\zeta}(\boldsymbol{r})\cdot\boldsymbol{E}^b(\boldsymbol{r}) + \eta_0\boldsymbol{\mu}(\boldsymbol{r})\cdot\boldsymbol{H}^b(\boldsymbol{r})\Big\}\\
&- i\omega\epsilon_0\boldsymbol{E}^b\cdot\Big\{\boldsymbol{\epsilon}(\boldsymbol{r})\cdot\boldsymbol{E}^a(\boldsymbol{r}) + \eta_0\boldsymbol{\xi}(\boldsymbol{r})\cdot\boldsymbol{H}^a(\boldsymbol{r})\Big\}\\
&= i\omega\epsilon_0\Big\{\eta_0\boldsymbol{H}^b(\boldsymbol{r})\cdot(\boldsymbol{\zeta}(\boldsymbol{r})+\boldsymbol{\xi}^T(\boldsymbol{r}))\cdot\boldsymbol{E}^a(\boldsymbol{r}) + \eta_0^2\boldsymbol{H}^b(\boldsymbol{r})\cdot(\boldsymbol{\mu}(\boldsymbol{r})-\boldsymbol{\mu}^T(\boldsymbol{r}))\cdot\boldsymbol{H}^a(\boldsymbol{r})\\
&+ \boldsymbol{E}^a(\boldsymbol{r})\cdot(\boldsymbol{\epsilon}(\boldsymbol{r})-\boldsymbol{\epsilon}^T(\boldsymbol{r}))\cdot\boldsymbol{E}^b(\boldsymbol{r}) - \eta_0\boldsymbol{H}^a(\boldsymbol{r})\cdot(\boldsymbol{\zeta}(\boldsymbol{r})+\boldsymbol{\xi}^T(\boldsymbol{r}))\cdot\boldsymbol{E}^b(\boldsymbol{r})\Big\}
\end{aligned}
\tag{E.5}
$$

The conditions (E.3) for the reciprocity of the medium now show that the integrand (E.5) is identically zero. QED.

Theorem E.1. *(Reciprocity) Suppose that for $\boldsymbol{E}_i^a$ and $\boldsymbol{E}_i^b$ of the form* (E.1), *we have*

$$\iint_{S_s}\big\{\boldsymbol{E}^a\times\boldsymbol{H}^b - \boldsymbol{E}^b\times\boldsymbol{H}^a\big\}\cdot\hat{\boldsymbol{\nu}}\,dS = 0, \tag{E.6}$$

where S_s denotes a smooth surface within which all the scatterers are located. Then

$$\mathbf{S}(-\hat{\boldsymbol{e}}^a,\hat{\boldsymbol{e}}^b) = \mathbf{S}^T(-\hat{\boldsymbol{e}}^b,\hat{\boldsymbol{e}}^a). \tag{E.7}$$

This theorem says that the transpose of the scattering dyadic for one set of incident and observation directions is identical to the scattering dyadic if the incident and the observation directions are reversed and interchanged.

Proof: The proof uses the following identity:

$$\iint_{S_R}\big\{\boldsymbol{E}^a\times\boldsymbol{H}^b - \boldsymbol{E}^b\times\boldsymbol{H}^a\big\}\cdot\hat{\boldsymbol{\nu}}\,dS = \iint_{S_s}\big\{\boldsymbol{E}^a\times\boldsymbol{H}^b - \boldsymbol{E}^b\times\boldsymbol{H}^a\big\}\cdot\hat{\boldsymbol{\nu}}\,dS, \tag{E.8}$$

where S_R denotes the sphere of radius R, R being chosen large enough so that S_s lies entirely within S_R. This identity follows from Lemma E.1 in the case in which the surface S is composed of two parts, S_s and S_R, between which we have vacuum (which is a reciprocal medium). By hypothesis, the right side of (E.8) is zero.

On the left side of (E.8), we substitute the following expressions for the electric fields on S_R:

$$
\begin{cases}\boldsymbol{E}_i^a(\boldsymbol{r}) = \boldsymbol{E}_0^a e^{ik\hat{\boldsymbol{e}}^a\cdot\boldsymbol{r}}\\ \boldsymbol{E}_i^b(\boldsymbol{r}) = \boldsymbol{E}_0^b e^{ik\hat{\boldsymbol{e}}^b\cdot\boldsymbol{r}}\end{cases}
\qquad
\begin{cases}\boldsymbol{E}_s^a(\boldsymbol{r}) = \boldsymbol{F}(\hat{\boldsymbol{r}},\hat{\boldsymbol{e}}^a)\dfrac{e^{ikR}}{kR} + o((kR)^{-1})\\ \boldsymbol{E}_s^b(\boldsymbol{r}) = \boldsymbol{F}(\hat{\boldsymbol{r}},\hat{\boldsymbol{e}}^b)\dfrac{e^{ikR}}{kR} + o((kR)^{-1})\end{cases}
$$

and the magnetic counterparts

$$\begin{cases}\eta_0 \boldsymbol{H}_i^a(\boldsymbol{r}) = \hat{\boldsymbol{e}}^a \times \boldsymbol{E}_0^a e^{ik\hat{\mathbf{e}}^a\cdot\boldsymbol{r}} \\ \eta_0 \boldsymbol{H}_i^b(\boldsymbol{r}) = \hat{\boldsymbol{e}}^b \times \boldsymbol{E}_0^b e^{ik\hat{\mathbf{e}}^b\cdot\boldsymbol{r}}\end{cases} \qquad \begin{cases}\eta_0 \boldsymbol{H}_s^a(\boldsymbol{r}) = \hat{\boldsymbol{r}} \times \boldsymbol{F}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{e}}^a)\dfrac{e^{ikR}}{kR} + o((kR)^{-1}) \\ \eta_0 \boldsymbol{H}_s^b(\boldsymbol{r}) = \hat{\boldsymbol{r}} \times \boldsymbol{F}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{e}}^b)\dfrac{e^{ikR}}{kR} + o((kR)^{-1})\end{cases}$$

The integral over S_R can be written as a sum of three terms.

$$\iint\limits_{S_R} \left\{\boldsymbol{E}^a \times \boldsymbol{H}^b - \boldsymbol{E}^b \times \boldsymbol{H}^a\right\} \cdot \hat{\boldsymbol{\nu}} \ , dS = I_1 + I_2 + I_3$$

where

$$\begin{cases} I_1 = \displaystyle\iint\limits_{S_R} \left\{\boldsymbol{E}_i^a \times \boldsymbol{H}_i^b - \boldsymbol{E}_i^b \times \boldsymbol{H}_i^a\right\} \cdot \hat{\boldsymbol{\nu}}\, dS \\ I_2 = \displaystyle\iint\limits_{S_R} \left\{\boldsymbol{E}_s^a \times \boldsymbol{H}_s^b - \boldsymbol{E}_s^b \times \boldsymbol{H}_s^a\right\} \cdot \hat{\boldsymbol{\nu}}\, dS \\ I_3 = \displaystyle\iint\limits_{S_R} \left\{\boldsymbol{E}_i^a \times \boldsymbol{H}_s^b - \boldsymbol{E}_i^b \times \boldsymbol{H}_s^a + \boldsymbol{E}_s^a \times \boldsymbol{H}_i^b - \boldsymbol{E}_s^b \times \boldsymbol{H}_i^a\right\} \cdot \hat{\boldsymbol{\nu}}\, dS \end{cases}$$

The first integral I_1 vanishes by Lemma E.1, where the medium within $S = S_R$ is vacuum.

The second integral I_2 is also seen to vanish in the limit $R \to \infty$, since the integrand is proportional to

$$\begin{aligned}&\Big\{\boldsymbol{F}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{e}}^a) \times (\hat{\boldsymbol{r}} \times \boldsymbol{F}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{e}}^b)) - \boldsymbol{F}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{e}}^b) \times (\hat{\boldsymbol{r}} \times \boldsymbol{F}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{e}}^a))\Big\} \cdot \hat{\boldsymbol{r}} + o((kR)^{-2}) \\ &\quad = o((kR)^{-2})\end{aligned}$$

where we have used the “BAC–CAB” identity $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b})$ and $\hat{\boldsymbol{r}} \cdot \boldsymbol{F}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{e}}) = 0$.

The remaining integral has to be evaluated more explicitly. The dominant part of the integral I_3 as $R \to \infty$ is

$$\begin{aligned}\eta_0 I_3 = R^2 \frac{e^{ikR}}{kR} \iint\limits_{\Omega} \Big\{&\boldsymbol{E}_0^a \times (\hat{\boldsymbol{r}} \times \boldsymbol{F}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{e}}^b)) e^{ik\hat{\mathbf{e}}^a\cdot\boldsymbol{r}} - \boldsymbol{E}_0^b \times (\hat{\boldsymbol{r}} \times \boldsymbol{F}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{e}}^a)) e^{ik\hat{\mathbf{e}}^b\cdot\boldsymbol{r}} \\ &+ \boldsymbol{F}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{e}}^a) \times (\hat{\boldsymbol{e}}^b \times \boldsymbol{E}_0^b) e^{ik\hat{\mathbf{e}}^b\cdot\boldsymbol{r}} - \boldsymbol{F}^b(\hat{\boldsymbol{r}}, \hat{\boldsymbol{e}}^b) \times (\hat{\boldsymbol{e}}^a \times \boldsymbol{E}_0^a) e^{ik\hat{\mathbf{e}}^a\cdot\boldsymbol{r}}\Big\} \cdot \hat{\boldsymbol{r}}\, d\Omega \\ &+ o(R^2(kR)^{-1}) \\ = R^2 \frac{e^{ikR}}{kR} \iint\limits_{\Omega} \Big\{&(\boldsymbol{E}_0^a \cdot \boldsymbol{F}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{e}}^b))(1 - \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{e}}^a) e^{ik\hat{\mathbf{e}}^a\cdot\boldsymbol{r}} - (\boldsymbol{E}_0^b \cdot \boldsymbol{F}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{e}}^a))(1 - \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{e}}^b) e^{ik\hat{\mathbf{e}}^b\cdot\boldsymbol{r}} \\ &- (\boldsymbol{E}_0^b \cdot \hat{\boldsymbol{r}})(\boldsymbol{F}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{e}}^a) \cdot \hat{\boldsymbol{e}}^b) e^{ik\hat{\mathbf{e}}^b\cdot\boldsymbol{r}} + (\boldsymbol{E}_0^a \cdot \hat{\boldsymbol{r}})(\boldsymbol{F}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{e}}^b) \cdot \hat{\boldsymbol{e}}^a) e^{ik\hat{\mathbf{e}}^a\cdot\boldsymbol{r}}\Big\}\, d\Omega \\ &+ o(R^2(kR)^{-1}),\end{aligned}$$

where $d\Omega$ denotes the measure on the unit sphere Ω and where we have again used the BAC–CAB identity and $\hat{\boldsymbol{r}} \cdot \boldsymbol{F}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{e}}) = 0$.

The generic integral that occurs here is

$$\begin{aligned}
\iint_{\Omega} f(\hat{\boldsymbol{r}}) e^{ik\hat{\boldsymbol{e}}\cdot\boldsymbol{r}}\, d\Omega &= \int_0^{2\pi}\int_0^{\pi} f(\theta,\phi) e^{ikr\cos\theta} \sin\theta\, d\theta d\phi \\
&= \frac{i}{kr}\int_0^{2\pi}\Big\{ f(\theta,\phi)e^{ikr\cos\theta}\Big|_{\theta=0}^{\theta=\pi} - \int_0^{\pi} \frac{\partial f(\theta,\phi)}{\partial\theta} e^{ikr\cos\theta}\, d\theta\Big\} d\phi \\
&= \frac{2\pi i}{kr}\big(f(\hat{\boldsymbol{r}}=-\hat{\boldsymbol{e}})e^{-ikr} - f(\hat{\boldsymbol{r}}=\hat{\boldsymbol{e}})e^{ikr}\big) + O((kr)^{-2})
\end{aligned}$$

The I_3 integral then is

$$\begin{aligned}
\eta_0 I_3 = \frac{2\pi i}{k^2}\Big\{ & 2\boldsymbol{E}_0^a \cdot \boldsymbol{F}(-\hat{\boldsymbol{e}}^a, \hat{\boldsymbol{e}}^b) - 2\boldsymbol{E}_0^b \cdot \boldsymbol{F}(-\hat{\boldsymbol{e}}^b, \hat{\boldsymbol{e}}^a) \\
&+ (\boldsymbol{E}_0^b\cdot\hat{\boldsymbol{e}}^b)\left[\boldsymbol{F}(-\hat{\boldsymbol{e}}^b, \hat{\boldsymbol{e}}^a)\cdot(-\hat{\boldsymbol{e}}^b)\right] - (\boldsymbol{E}_0^a\cdot\hat{\boldsymbol{e}}^a)\left[\boldsymbol{F}(-\hat{\boldsymbol{e}}^a, \hat{\boldsymbol{e}}^b)\cdot(-\hat{\boldsymbol{e}}^a)\right] \\
&+ (\boldsymbol{E}_0^b\cdot\hat{\boldsymbol{e}}^b)\left[\boldsymbol{F}(\hat{\boldsymbol{e}}^b, \hat{\boldsymbol{e}}^a)\cdot\hat{\boldsymbol{e}}^b\right] e^{2ikR} - (\boldsymbol{E}_0^a\cdot\hat{\boldsymbol{e}}^a)\left[\boldsymbol{F}(\hat{\boldsymbol{e}}^a, \hat{\boldsymbol{e}}^b)\cdot\hat{\boldsymbol{e}}^a\right] e^{2ikR}\Big\} + o(1) \\
= \frac{4\pi i}{k^2}\Big\{ & \boldsymbol{E}_0^a \cdot \boldsymbol{F}(-\hat{\boldsymbol{e}}^a, \hat{\boldsymbol{e}}^b) - \boldsymbol{E}_0^b \cdot \boldsymbol{F}(-\hat{\boldsymbol{e}}^b, \hat{\boldsymbol{e}}^a)\Big\} + o(1),
\end{aligned} \tag{E.9}$$

where all the terms in square brackets vanish because $\hat{\boldsymbol{r}} \cdot \boldsymbol{F}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{e}}) = 0$.

Since both sides of (E.8) are assumed to be zero, the expression in brackets in (E.9) must also be zero. This shows that

$$\boldsymbol{E}_0^a \cdot \mathbf{S}(-\hat{\boldsymbol{e}}^a, \hat{\boldsymbol{e}}^b)\cdot \boldsymbol{E}_0^b = \boldsymbol{E}_0^b \cdot \mathbf{S}(-\hat{\boldsymbol{e}}^b, \hat{\boldsymbol{e}}^a)\cdot \boldsymbol{E}_0^a, \quad \text{for all } \boldsymbol{E}_0^{a,b}$$

which is equivalent to (E.7). QED

This result holds for all scatterers for which the identity (E.6) holds. We showed in Theorem D.1 that (E.6) holds for a reciprocal medium inside S_s. Note also that (E.6) holds also for a smooth perfectly conducting object.

Notice that the integrand in the integral of (E.6) is continuous over the boundary surface S_s. We see this from

$$\begin{aligned}
(\boldsymbol{E}_+^a \times \boldsymbol{H}_+^b)\cdot\hat{\boldsymbol{\nu}} &= (\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_+^a)\cdot \boldsymbol{H}_+^b = (\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_-^a)\cdot \boldsymbol{H}_+^b \\
&= -(\hat{\boldsymbol{\nu}} \times \boldsymbol{H}_+^b)\cdot \boldsymbol{E}_-^a = -(\hat{\boldsymbol{\nu}} \times \boldsymbol{H}_-^b)\cdot \boldsymbol{E}_-^a = (\boldsymbol{E}_-^a \times \boldsymbol{H}_-^b)\cdot\hat{\boldsymbol{\nu}}
\end{aligned}$$

and similarly for the second term.

References

[1] A. Boström, G. Kristensson, and S. Ström. Transformation properties of plane, spherical and cylindrical scalar and vector wave functions. In V. V. Varadan, A. Lakhtakia, and V. K. Varadan, editors, *Field Representations and Introduction to Scattering*, Acoustic, Electromagnetic and Elastic Wave Scattering, chapter 4, pages 165–210. Elsevier Science Publishers, Amsterdam, 1991.

[2] M. Cheney and D. Isaacson. Inverse problems for a perturbed dissipative half-space. *Inverse Problems*, **11**, 865–888, 1995.

[3] M. Cheney, D. Isaacson, and M. Lassas. Optimal acoustic measurements. *SIAM J. Appl. Math.*, 2001.
(To appear. Also available at `http://xxx.lanl.gov/abs/math.AP/0009037` and `http://www.rpi.edu/~chenem/acoust.ps`.).

[4] E. Isaacson and H. B. Keller. *The Analysis of Numerical Methods*. John Wiley & Sons, New York, 1966. (Also published by Dover Publications, 1994).

[5] J. A. Kong. *Electromagnetic Wave Theory*. John Wiley & Sons, New York, 1986.

[6] G. Kristensson. Electromagnetic scattering from buried inhomogeneities—a general three-dimensional formalism. *J. Appl Phys.*, **51**(7), 3486–3500, 1980.

[7] G. Kristensson, M. Åkerberg, and S. Poulsen. Scattering from a frequency selective surface supported by a bianisotropic substrate. Technical Report LUTEDX/(TEAT-7085)/1–28/(2000), Lund Institute of Technology, Department of Applied Electronics, Electromagnetic Theory, P.O. Box 118, S-211 00 Lund, Sweden, 2000.

[8] G. Kristensson. *Spridningsteori med antenntillämpningar*. Studentlitteratur, Lund, 1999. (In Swedish).

[9] C. Prada and M. Fink. Eigenmodes of the time reversal operator: A solution to selective focusing in multiple-target media. *Wave Motion*, **20**, 151–163, 1994.

[10] C. Prada, J.-L. Thomas, and M. Fink. The iterative time reversal process: Analysis of the convergence. *J. Acoust. Soc. Am.*, **97**, 62–71, 1995.

[11] S. Rikte, G. Kristensson, and M. Andersson. Propagation in bianisotropic media—reflection and transmission. *IEE Proc.-H Microwaves, Antennas and Propagation*, 2001. (to appear).

[12] S. Rikte, G. Kristensson, and M. Andersson. Propagation in bianisotropic media—reflection and transmission. Technical Report LUTEDX/(TEAT-7067)/1–31/(1998), Lund Institute of Technology, Department of Electromagnetic Theory, P.O. Box 118, S-211 00 Lund, Sweden, 1998.

[13] S. Ström. Introduction to integral representations and integral equations for time-harmonic acoustic, electromagnetic and elastodynamic wave fields. In V. V. Varadan, A. Lakhtakia, and V. K. Varadan, editors, *Field Representations and Introduction to Scattering*, Acoustic, Electromagnetic and Elastic Wave Scattering, chapter 2, pages 37–141. Elsevier Science Publishers, Amsterdam, 1991.