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Convex Duality Approach to Robust Stabilization of Uncertain Plants

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Preface

In this thesis we study the problem of designing the controllers that are robust with respect to the parametric uncertainty. The thesis is based upon the four papers.

In Part I "The Rank-One Problem" we consider the class of systems with restriction that the structure of uncertainty is limited to a vector, i.e. Δ is in $\mathbb{C}^m$. This part consists of three articles. In the first article "Canonical Parametrization of the Dual Problem in Robust Optimization: Non-Rational Case" (European Control Conference, Kos, Greece, 2007) we extend the class of the allowed systems. The main result is the canonical parametrization of all destabilizing uncertainties. In the second article "Regularization of the Limiting Optimal Controller in Robust Stabilization" (17th IFAC World Congress, July 6-11, 2008, Seoul, Korea) we present a method of obtaining the suboptimal controller of lower order that provides the stability margin as close to the optimal one as we wish. In the third article "Robust Control via Linear Programming" (to be published) we propose a numerical algorithm for the optimal robust control synthesis. In the special case, when the uncertainty parameter is real-valued, we show that the initial problem can be considered as finite-dimensional in the space of variables (semi-infinite convex programming). These articles are a joint work with Andrey Ghulchak.

Part II "Convex Duality: Matrix Case" is based on the single article - "Canonical Parametrization of the Dual Problem in Robust Optimization: Matrix Case" (to be published). It generalizes the results to the systems with matrix uncertainties.

Acknowledgement

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Contents

Chapter 1

Introduction

"Robust control refers to the control of unknown plants with unknown dynamics subject to unknown disturbances" [17].

There are two sorts of problems in robust control [22]:

- *Analysis*: Given a controller decide if the controlled signals satisfy the demanded properties for all disturbance, noises and model uncertainties.

- *Synthesis*: Design a controller so that the controlled signals satisfy the demanded properties for all disturbance, noises and model uncertainties.

In the thesis we shall be concentrating on the problem of designing the controllers that are robust with respect to the parametric uncertainty.

Most control designs are based on the use of a design model. Quality of a model depends on how close it is to the real one.

A good model should be simple enough to facilitate design yet complex enough to capture the important properties of the true plant.

Usually we will use models that are linear, time-invariant and finite dimensional. Then they are described mathematically as a set of linear ordinary differential equations. But no physical system could possibly correspond to such a model. Physical systems are essentially infinite dimensional, non-linear and time-varying. One way to cover the gap between the design model and the reality is to insert the uncertainties that reflect both our knowledge of the physical mechanism of the plant and our ability to solve the control problems with such representation of uncertainties.

In recent years much research effort has been put into robust control of the uncertain models. However, the design problems appeared to be very hard to solve. For design of controllers that are robust with respect to the parametric uncertainties the different methods have been proposed. Among them are the $\mathbf{H}^\infty$ optimization, the μ synthesis and the Rank-one problem.

1.1 $\mathbf{H}^\infty$ optimization

We consider the system described by the block diagram in Figure 1.1, where the plant G and the controller K are assumed to be real rational and proper. In this figure w, u, z and y are the vector valued signals: w is the input, u is the control signal, z is the output to be controlled and y is the measured output.

Figure 1.1: The standard block diagram
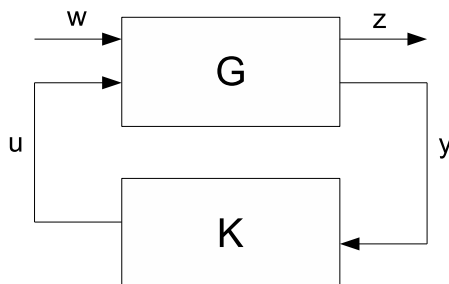
Definition 1. *A feedback system is said to be well-posed if all closed-loop transfer matrices are well-defined and proper.*

We introduce a partition G as

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}.$$

Then Figure (1.1) stands for the algebraic equations

$$z = G_{11}w + G_{12}u$$
$$y = G_{21}w + G_{22}u$$
$$u = Ky.$$

We introduce two additional inputs v_1 and v_2 as in Figure 1.2. The equation relating three inputs w, v_1, v_2 and three signals u, z, y is given by

$$\begin{pmatrix} \mathbb{I} & -G_{12} & \mathbb{O} \\ \mathbb{O} & \mathbb{I} & -K \\ \mathbb{O} & -G_{22} & \mathbb{I} \end{pmatrix} \begin{pmatrix} z \\ u \\ y \end{pmatrix} = \begin{pmatrix} G_{11} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{I} & \mathbb{O} \\ G_{21} & \mathbb{O} & \mathbb{I} \end{pmatrix} \begin{pmatrix} w \\ v_1 \\ v_2 \end{pmatrix}.$$

The proper rational matrix $\begin{pmatrix} \mathbb{I} & -G_{12} & \mathbb{O} \\ \mathbb{O} & \mathbb{I} & -K \\ \mathbb{O} & -G_{22} & \mathbb{I} \end{pmatrix}$ has a proper real-rational inverse for every proper real-rational K. Then the nine transfer matrices from

Figure 1.2: Diagram for stability definition
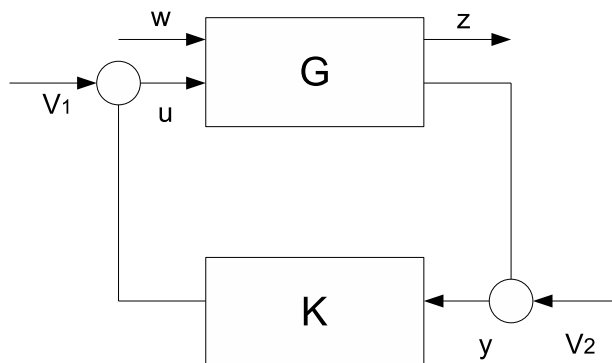
w, v_1, v_2 to u, z, y are proper. If they are stable, then we say that K *stabilizes* G. This is the usual notion of internal stability.

The *standard* $\mathbf{H}^\infty$ *problem* is the following:

to find a real-rational proper K in order to minimize the $\mathbf{H}^\infty$ norm of the transfer matrix from w to z under the constraint that K stabilizes G. The transfer matrix from w to z is a linear-fractional transformation of K:

$$z = [G_{11} + G_{12}K(\mathbb{I} - G_{22}K)^{-1}G_{21}]w.$$

Denote $T_{zw} = [G_{11} + G_{12}K(\mathbb{I} - G_{22}K)^{-1}G_{21}]$.

One example of the standard problem is *a robust stabilization problem.* We consider a system shown in Figure 1.3.

Assume P be strictly proper nominal plant and let R be a scalar valued function in $\mathbf{RH}^\infty$, where $\mathbf{RH}^\infty$ is the set of all functions from $\mathbf{H}^\infty$ that are rational with reals coefficient. Define a family $\mathbf{P}$ of neighboring plants which consists of all strictly proper real-rational matrices $P + \Delta$ having the same numbers of poles in the right-half plane ($\operatorname{Re} s \geq 0$) as P, where the

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Figure 1.3: Feedback system with perturbed plant
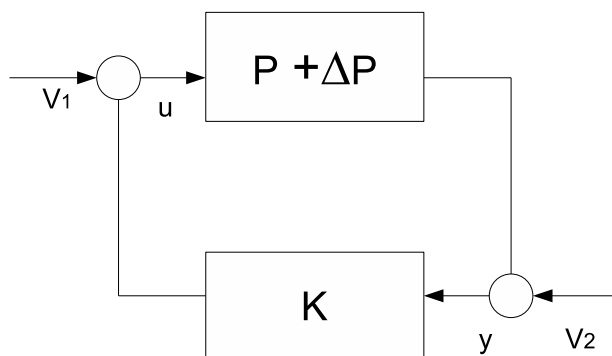
perturbation Δ satisfies the bound

$$\|\Delta(i\omega)\| < |R(i\omega)| \text{ for all } 0 \le \omega \le \infty.$$

For a real-rational proper K the robust stability definition is that K stabilizes all plants in $\mathbf{P}$. How large can $|R|$ be so that internal stability is maintained? The robust stability is guaranteed by a small gain condition.

Consider the interconnected system shown in Figure 1.4 with $M(s)$ a stable transfer matrix.

Theorem 1. *(**Small Gain Theorem**) Suppose $M \in \mathbf{RH}^\infty$ and let $\gamma > 0$. Then interconnected system is well-posed and internally stable for all $\Delta(s) \in \mathbf{RH}^\infty$ with*

1. *$\|\Delta\|_\infty \le 1/\gamma$ if and only if $\|M(s)\|_\infty < \gamma$.*

2. *$\|\Delta\|_\infty < 1/\gamma$ if and only if $\|M(s)\|_\infty \le \gamma$.*

Figure 1.4: M-Δ loop for stability analysis
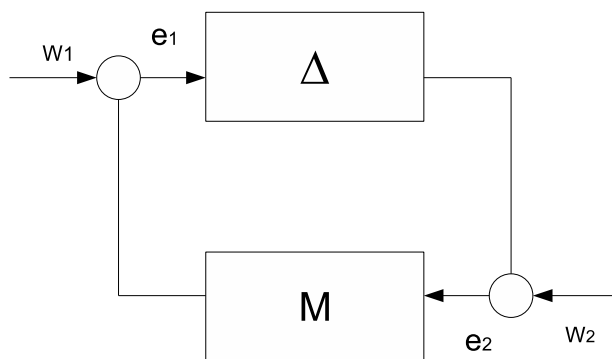
According to a small gain condition a real-rational proper K stabilizes all plants in $\mathbf{P}$ if and only if K stabilizes the nominal plant P and

$$\|RK(\mathbb{I} - PK)^{-1}\|_\infty \leq 1.$$

The $\mathbf{H}^\infty$ control problem and the robust stability problem are equivalent when Δ is a single-block unstructured uncertainty through the application of the small gain theorem.

Our purpose is to find a stabilizing controller $u = Ky$ that minimizes the $\mathbf{H}^\infty$-norm of the closed-loop transfer function T_{zw}. The Youla parametrization of all admissible close-loop transfer functions (see [8])

$$T_{zw} = T_1 + T_2 Q T_3,$$

where $T_1, T_2 \in \mathbf{RH}^\infty$ are defined by the plant P and Q is a new function from $\mathbf{RH}^\infty$. With this parametrization the problem can be stated as follows:

$$\min_{Q \in \mathbf{RH}^\infty} \|T_1 + T_2 Q T_3\|_\infty.$$

For real-rational plants P we can solve the problem by using the well-developed theory (see [8]). However, the problem becomes particularly complicated in the case of non-rational plants. In [10] the weighted sensitivity $\mathbf{H}^\infty$-minimization problem has been investigated and it has been shown that the problem is nontrivial even for the simplest delay systems. Furthermore, the $\mathbf{H}^\infty$ optimization is limited to the class of *unstructured* uncertainty. The attempt to apply the technics to the case with structured uncertainties results in more conservative systems. Thus it is highly desirable to treat the uncertainties as they are and where they are. The structured singular value μ is defined for that purpose.

1.2 μ Synthesis

Consider the block diagonal matrix Δ with two types of blocks: repeated scalar and full blocks. Let S and F represent the number of repeated scalar blocks and the number of full blocks, respectively. We define $\mathbf{\Delta}$ as

$$\mathbf{\Delta} = \{diag[\delta_1\mathbb{I}_{r_1}, ..., \delta_s\mathbb{I}_{r_s}, \Delta_1, ..., \Delta_F] : \ \delta_i \in \mathbb{C}, \ \Delta_j \in \mathbb{C}^{m_j \times m_j}\}.$$

Definition 2. *For matrix $M \in \mathbb{C}^{n \times n}$ we define $\mu_{\mathbf{\Delta}}(M)$ as*

$$\mu_{\mathbf{\Delta}}(M) := \frac{1}{min\{\overline{\sigma}(\Delta) : \Delta \in \mathbf{\Delta}, det(\mathbb{I} - M\Delta) = 0\}},$$

unless no $\Delta \in \mathbf{\Delta}$ makes $\mathbb{I} - M\Delta$ singular, in which case $\mu_{\mathbf{\Delta}}(M) := 0$.

We call μ the structured singular value. The value μ is very useful as a robustness analysis tool in the frequency domain. Suppose $G(s)$ is a stable real rational transfer function. Consider the perturbed system shown in Figure 1.5.

Let $\mathcal{M}(\mathbf{\Delta})$ denote the set of all block diagonal and stable real rational transfer functions that have block structures such as $\mathbf{\Delta}$.

$$\mathcal{M}(\mathbf{\Delta}) := \{\Delta \in \mathbf{RH}^\infty : \Delta(s_0) \in \mathbf{\Delta} \ \text{for all} \ s_0 \in \overline{\mathbb{C}_+}\}.$$

Theorem 2. *Let $\beta > 0$. The loop shown in Figure 1.5 is well-posed and internally stable for all $\Delta \in \mathcal{M}(\mathbf{\Delta})$ with $\|\Delta\|_\infty < \frac{1}{\beta}$ if and only if*

$$\sup_{\omega \in \mathbb{R}} \mu_{\mathbf{\Delta}}(G(i\omega)) \leq \beta.$$

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Figure 1.5: G-Δ loop for stability analysis
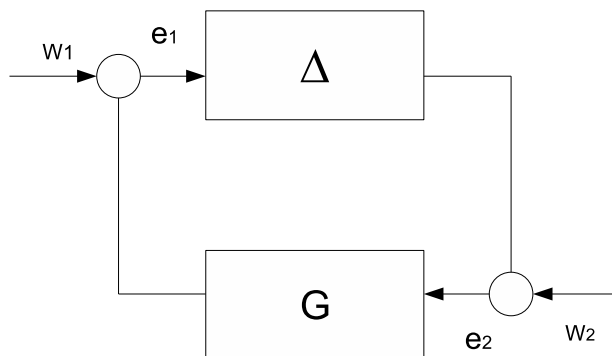
It means that the maximal value of μ determines the size of perturbations such that the perturbed system is still stable.

Consider again the standard block diagram in Figure 1.6. Let M be a transfer function from w to z,

$$M = \mathfrak{F}_l(G, K) = G_{11} + G_{12}K(\mathbb{I} - G_{22}K)^{-1}G_{21},$$

where $\mathfrak{F}_l$ is the standard notation for the lower linear fractional transformation. We will solve the following synthesis problem:

$$\min_K \sup_{\omega \in \mathbb{R}} \mu_{\mathbf{\Delta}}(\mathfrak{F}_l(i\omega)),$$

which is subject to the internal stability of the nominal plant. It appears to be a quite hard task. The upper bound of the structured singular value μ may be obtained by scaling and applying $\|\cdot\|_\infty$. Define class $\mathcal{D}$ as follows.

$$\mathcal{D} = \{D \in \mathbf{H}^\infty : D^{-1} \in \mathbf{H}^\infty, \ D(s)\Delta(s) = \Delta(s)D(s).\}$$

Then we can try to solve the problem

$$\min_K \inf_{D \in \mathcal{D}} \|D\mathfrak{F}_l(G, K)D^{-1}\|_\infty$$

Figure 1.6: The standard block diagram
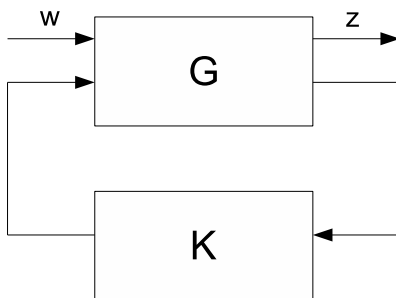
iteratively with respect to K and D. This is so-called *D-K iteration*. For a fixed scaling transfer matrix D we have that $\min_K \|D\mathfrak{F}_l(G,K)D^{-1}\|_\infty$ is a standard $\mathbf{H}^\infty$ optimization problem. For a given stabilizing controller K $\inf_{D\in\mathcal{D}} \|D\mathfrak{F}_l(G,K)D^{-1}\|_\infty$ is a standard convex optimization problem and it can be solved pointwise:

$$\sup_\omega \inf_{D_\omega} \overline{\sigma}[D_\omega \mathfrak{F}_l(G,K)(i\omega)D_\omega^{-1}].$$

The D-K iteration proceeds by performing this two parameter minimizations sequentially: first by minimizing over K with D fixed, then by minimizing pointwise over D with K fixed, then again over K, and again over D, etc.

Though μ synthesis allows for a much more flexible uncertainty structure than $\mathbf{H}^\infty$ optimization does, the design procedures become very involved. Furthermore the joint optimization of D and K is not convex.

The flexible uncertainty structure is also allowed in a rank-one uncertainty model with restriction that the structure of uncertainty is limited to a vector. For a rank-one problem we can apply the convex robust controller parametrization.

1.3 Rank-one uncertainty model.

We consider the loop shown in Figure 1.7, where $\delta \in \mathbb{R}_m$ is an uncertain parameter vector.

Figure 1.7: Rank-one uncertainty model
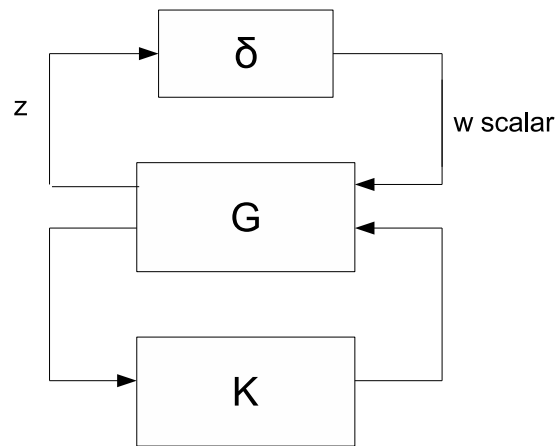
According to the Youla parametrization of internal stabilizing controllers we assume that the transfer function from w to z is given on the form

$$z = (T_1 + T_2 Q)w,$$

where $T_1 \in \mathbf{RH}^\infty_{m \times 1}$ and $T_2 \in \mathbf{RH}^\infty_{m \times n}$ are fixed and parameter Q is any transfer matrix in $\mathbf{RH}^\infty_{n \times 1}$. The system becomes robustly stable if and only if

$$[1 - \delta^\top (T_1 + T_2 Q)]^{-1} \in \mathbf{RH}^\infty \text{ for all } |\delta| \leq \nu,$$

here $|\cdot|$ is some norm on $\mathbb{R}^m$.

The main problem is stated as follows

Given $T_1 \in \mathbf{RH}_{m \times 1}^\infty$, $T_2 \in \mathbf{RH}_{m \times n}^\infty$, find a convex parametrization of all $Q \in \mathbf{RH}_{n \times 1}^\infty$ such that $[1 - \delta^\top (T_1 + T_2 Q)]^{-1} \in \mathbf{RH}^\infty$ for all $\delta \in \mathbb{R}_m$ with $|\delta| \le \nu$.

The problem was solved in [5].

Theorem 3. *Suppose $T_1 \in \mathbf{RH}_{m \times 1}^\infty$, $T_2 \in \mathbf{RH}_{m \times n}^\infty$. Then the following two conditions on the rational matrix Q are equivalent:*

1. *$Q \in \mathbf{RH}_{n \times 1}^\infty$ and for all $\delta \in \mathbb{R}_m$ with $|\delta| \le \nu$*

$$[1 + \delta^\top (T_1 + T_2 Q)]^{-1} \in \mathbf{RH}^\infty.$$

2. *There exist $\alpha \in \mathbf{RH}^\infty$ and $\beta \in \mathbf{RH}^\infty$ such that $Q = \beta / \alpha$ and $\forall \omega \in \mathbb{R} \cup \{\infty\}$,*

$$|Re[T_1 \alpha + T_2 \beta](i\omega)|^d < \nu^{-1} Re\,\alpha(i\omega).$$

Here $|\cdot|^d$ is the dual norm defined by

$$|x|^d = \max\{x^\top y : |y| \le 1\}.$$

The convex parametrization of all controllers that stabilize the system for all possible combinations of parameters makes it possible to use convex optimization to find a robustly stabilizing controller.

Denote

$$F := (\ 1 \quad 0 \quad \ldots \quad 0\ ) \in \mathbb{R}_{n+1},$$
$$G := (\ T_1 \quad T_2\ ) \in \mathbf{RH}^\infty,$$
$$h := \left(\begin{array}{c} \alpha \\ \beta \end{array} \right) \in \mathbf{RH}^\infty.$$

Let $\Delta \ni 0$ be the uncertainty set. We assume that Δ is a compact convex set in $\mathbb{C}_m$. We apply the conformal bilinear transformation of the right half complex plane onto the unit disk $z = (1-s)/(1+s)$. For our purpose it is more convenient to deal with the compact unit disc than with the unbounded right half plane. Due to Theorem 3 the problem of finding a controller is equivalent to the following condition in terms of a function $h \in \mathbf{RH}^\infty$:

$$Re\,(F + \delta^\top G(z))h(z) > 0, \ \forall z \in \mathbb{T}, \ \forall \delta \in \nu \Delta. \tag{1.1}$$

The problem of finding $h \in \mathbf{RH}^\infty$ such that the condition (1.1) holds is considered as a *primal problem*. We would like to solve it for ν as large as possible, that is, for

$$\nu_{opt} = \sup\{\nu | \exists h \in \mathbf{RH}^\infty : \; Re(F + \delta^\top G)h > 0 \; \forall \delta \in \nu\Delta\}. \qquad (1.2)$$

We can construct a finite-dimensional approximation by solving the problem on a finite-dimensional subspace of $\mathbf{RH}^\infty$ and on a finite grid of points $z \in \mathbb{T}$. The solution gives us the lower bound on the optimal uncertainty norm ν_{opt}. To obtain the upper bound on ν_{opt}, the dual problem was introduced in [3] as follows:
.

Theorem 4. *Let $F \in \mathbf{A}_{1\times n}$, $G \in \mathbf{A}_{m\times n}$ and denote $\Phi_\delta = F + \delta^\top G$. Then the optimal value ν_{opt} takes the following form*

$$\nu_{opt} = \min\{\nu_{opt|c}, \nu_{opt|s}\}$$

with the regular part

$$\nu_{opt|c} = \inf\{\nu | \exists w \in \mathbf{L}^1(\mathbb{R}_+)\backslash 0, \delta \in \mathbf{L}_\infty(\nu\Delta) : \Phi_\delta w \in \mathbf{H}_0^1\} \qquad (1.3)$$

and the singular part

$$\nu_{opt|s} \;=\; \inf\{\nu | \exists z \in \mathbb{T}, \; \delta \in \nu\Delta : \Phi_\delta(z) = 0\}, \qquad (1.4)$$

here the disk algebra $\mathbf{A}(Y)$ is by definition $\mathbf{A} = \mathbf{H}^\infty \cap \mathcal{C}$, $\mathcal{C}$ is the set of continuous functions.

As before we can construct a finite-dimensional approximation to the dual problem and obtain the upper bound on ν_{opt}. In [4] it was shown that there is no duality gap between the primal and the dual problems, and the primal-dual method can be used to obtain a suboptimal solution with any predefined level of optimality.

Some theoretical results obtained in [3] show that the convex duality concepts provide us with a convenient method to calculate the optimal stability margin and to design the optimal controller. Unfortunately the class of considered systems was restricted to the rational case. The problem was still of rank one.

In the thesis we continue to study the rank-one problem in Part I. In Chapter 2 we extend the class of allowed systems to the non-rational. The main result is the canonical parametrization of all destabilizing uncertainties in the dual problem. The corresponding result in the rational case was previously stated in terms of unstable zero-pole cancelations. For non-rational systems the situation with common zeros is more complicated. The nominal factors can contain a singular component and cannot be treated by unstable cancelations. We have shown that in the general case the common zeros of the plant factors are naturally replaced by a scalar function with the positive winding number. To illustrate the duality principle, the result is applied to a system with delay. The dual problem can be interpreted as the shortest distance from the nominal plant to all non-stabilizable plants in some metric that has a strong connection to the ν-gap metric.

By dual parametrization obtained in Chapter 2 we can easily calculate the optimal uncertainty bound and the optimal controller. Since the optimal controller is not robustly stabilizing in the strong sense,as it is only a limit of suboptimal robustly stabilizing controllers, we have to regularize the limiting controller. In Chapter 3 we present a method of obtaining the suboptimal controller of lower order that provides the stability margin as close to the optimal one as we wish. The method is illustrated with some scalar examples.

In Chapter 4 we propose the numerical algorithm for the optimal robust control synthesis. The algorithm proposed is a sequence of the standard linear programming problems of growing dimensions which approximate the initial problem. In the special case, when the uncertainty parameter is real-valued, it is shown that the initial problem can be considered as finite-dimensional in the space of variables (semi-infinite convex programming).

In Part II we generalize the results from Chapter 2 in Part I to the system with matrix uncertainties. We obtain a canonical factorization of a plant with unstructured uncertainty in terms of an unitary matrix function with finite winding number and an outer matrix function. We introduce a metric in the space of factorization and discuss connection with $\nu-$gap metric.

Part I

The Rank-One Problem

We start by introducing some basic definitions needed in the forthcoming chapters.

Basic definitions. By $\mathbb{R}$ (or $\mathbb{C}$) we denote the field of real (or complex) numbers. The unit circle and the open unit disc in $\mathbb{C}$ are denoted by $\mathbb{T}$ respectively $\mathbb{D}$

$$\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}, \; \mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}.$$

Let $Y \subset \mathbb{C}^n$ be any measurable set and $1 \leq p \leq \infty$. Denote by $L^p(Y)$ the standard Lebesgue space of functions $f : \mathbb{T} \to Y$ equipped with the norm

$$\|f\|_p = \left\{ \begin{array}{ll} (\int_{\mathbb{T}} |f(z)|^p dm(z))^{\frac{1}{p}}, & 1 \leq p < +\infty, \\ \operatorname{ess\,sup}_{z \in \mathbb{T}} |f(z)|, & p = +\infty \end{array} \right.$$

where $|\cdot|$ denotes the usual 2-norm in $\mathbb{C}^n$

$$|f| = \sqrt{|f_1|^2 + |f_2|^2 + \ldots + |f_n|^2}.$$

The Hardy class $\mathbf{H}^p(Y)$ consists of functions analytic in $\mathbb{D}$ and such that

$$\|f\|_p = \sup_{0 < r < 1} \left(\int_{\mathbb{T}} |f(rz)|^p dm(z) \right)^{\frac{1}{p}} < \infty.$$

The Hardy class $\mathbf{H}^\infty$ is the space of bounded analytic functions in $\mathbb{D}$ with norm

$$\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|.$$

A function θ analytic in $\mathbb{D}$ is called an *inner function* if $\theta \in \mathbf{H}^\infty$ and

$$|\theta(z)| = 1 \text{ for almost all } z \in \mathbb{T}.$$

A function h analytic in $\mathbb{D}$ is called an *outer function* if there exists a real function $g \in \mathbf{L}^1$ and a complex number c of modulus 1 such that

$$h(\lambda) = c \exp \left(\int_{\mathbb{T}} \frac{z + \lambda}{z - \lambda} g(z) dm(z) \right), \; \lambda \in \mathbb{D}$$

If $f \in \mathbf{H}^p$, then f admits the representation $f = \theta h$, where θ is an inner function and h is an outer function in $\mathbf{H}^p$.

We define the *Blaschke product* as follows. For $\lambda \in \mathbb{D}$ we put

$$b_\lambda(z) = \frac{|\lambda|}{\lambda} \frac{\lambda - z}{1 - \overline{\lambda}z}, \quad \lambda \neq 0, \quad and \quad b_0(z) = z.$$

Let $\{\lambda_i\}_{i \geq 0}$ be a sequence in $\mathbb{D}$ satisfying the *Blaschke condition*

$$\sum_{i \geq 0}(1 - |\lambda_i|) < \infty.$$

Let c be a complex number of modulus 1. Then the product

$$B(z) = c \prod_{i \geq 0} b_{\lambda_i}(z)$$

converges for all $z \in \mathbb{D}$ and is not identically equal to 0. Function B is called a *Blaschke product*.

Let $\mathbf{H}_0^p(Y)$ denote

$$\mathbf{H}_0^p(Y) = z\mathbf{H}^p(Y) = \{f \in \mathbf{H}^p(Y) | f(0) = 0\}.$$

The disk algebra $\mathbf{A}(Y)$ is by definition the subspace of $\mathbf{H}^\infty$ that consists of analytic functions in $\mathbb{D} \subset Y$ that can be extended continuously to the closed unit disk.

The set $\mathbf{RH}^\infty$ is the set of all functions from $\mathbf{H}^\infty$ that are rational with reals coefficient.

The space $\mathcal{C}$ is the space of continuous functions on $\mathbb{T}$.

Define the space $\mathbf{H}^\infty + \mathcal{C}$ as the set of functions $\varphi \in \mathbf{L}^\infty$ such that φ admits a representation $\varphi = f + g$, where $f \in \mathbf{H}^\infty$ and $g \in \mathcal{C}$.

The class of quasicontinuous functions is defined as

$$\mathcal{QC} = (\mathbf{H}^\infty + \mathcal{C}) \cap (\overline{\mathbf{H}^\infty + \mathcal{C}}).$$

The brief notations $\mathbf{A}$, $\mathbf{H}^\infty$ etc. will be used if $Y = \mathbb{C}^n$ and the dimension of the space is clear from context.

The Toeplitz operator T_ϕ and the Hankel operator H_ϕ with the symbol $\phi \in \mathbf{L}^\infty$ are defined by

$$\mathrm{T}_\phi f = \mathbb{P}_+ \phi f, \qquad \mathrm{H}_\phi f = \mathbb{P}_- \phi f, \qquad f \in \mathbf{H}^2,$$

where $\mathbb{P}_+$ is the ortogonal projection from $\mathbf{L}^2(Y)$ onto $\mathbf{H}^2(Y)$ and $\mathbb{P}_- = I - \mathbb{P}_+$.

$\|\mathrm{H}_\phi\|_e = \mathrm{dist}\,(\phi, \mathbf{H}^\infty + \mathcal{C})$ is the essential norm of Hankel operator.

Two matrices $F, G \in \mathbf{A}$ are *left-coprime* if they have equal number of rows and there exist $X, Y \in \mathbf{A}$ such that

$$(\ F \quad G \ ) \begin{pmatrix} X \\ Y \end{pmatrix} = FX + GY = \mathbb{I},$$

i.e. the matrix function $(\ F \quad G \ )$ is right invertible in $\mathbf{A}$.

Let G be a matrix function. We say that the factorization $G = M^{-1}N$ is a *left coprime factorization* if N and M are left-coprime matrices.

The factorization $G = M^{-1}N$ is called a *normalized* left coprime factorization if $NN^* + MM^* = \mathbb{I}$.

The matrix function V is called *co-outer* if $V^\top$ is outer.

The superscript $^\top$ stands for transposition and † stands for pseudoinverse. The bar denotes the complex conjugate and * denotes conjugate transpose. The prefix $\mathcal{B}$ denotes the unit ball in the corresponding space, and $\mathcal{S}$ is the unit sphere.

Chapter 2

Canonical Parametrization of the Dual Problem in Robust Optimization: Non-Rational Case

2.1 Introduction

The duality principle in robust stabilization is a new paradigm that uses convex duality concepts and brings a new insight into robust optimization. Given a nominal plant P and an uncertainty set $\Delta \ni 0$, the general robust controller design problem is to find a controller K that robustly stabilizes the whole family of perturbed plants

$$P_\delta = \frac{N + \delta^\top G_1}{M + \delta^\top G_2}, \ \delta \in \nu\Delta$$

for ν as large as possible.

Denote by $F = (\ N \quad M\ ) \in \mathbf{A}^{1\times n}$ the left coprime factorization of the plant P, by $G = (\ G_1 \quad G_2\ ) \in \mathbf{A}^{m\times n}$ the weight matrix. The set Δ is a convex compact set in $\mathbb{C}^m$. According to a convex parametrization of all robustly stabilizing controllers constructed in [5] we define the *primal problem* as the problem of finding a function $h \in \mathbf{H}^\infty$ such that the condition

$$Re\,(F + \delta^\top G(z))h(z) > 0, \ \forall z \in \mathbb{T}, \ \forall \delta \in \nu\Delta, \tag{2.1}$$

is satisfied. We would like to solve it for ν as large as possible, that is, for

$$\nu_{opt} = \sup\{\nu | \exists h \in \mathbf{RH}^\infty : Re(F + \delta^\top G)h > 0 \ \forall \delta \in \nu\Delta\}.$$

In [3] it was shown that we can allow the large set $\mathbf{H}^\infty$ and

$$\nu_{opt} = \sup\{\nu | \exists h \in \mathbf{H}^\infty : Re(F + \delta^\top G)h > 0 \ \forall \delta \in \nu\Delta\}. \tag{2.2}$$

The *dual problem* for systems with uncertainties of rank one was introduced in [3] as follows.

Theorem 5 (Ghulchak, Rantzer). *Let $F \in \mathbf{A}^{1\times n}$, $G \in \mathbf{A}^{m\times n}$ and denote $\Phi_\delta = F + \delta^\top G$. Then the optimal value ν_{opt} takes the following form*

$$\nu_{opt} = \min\{\nu_{opt|c}, \nu_{opt|s}\}$$

with the regular part

$$\nu_{opt|c} = \inf\{\nu | \exists w \in \mathbf{L}^1(\mathbb{R}_+)\backslash 0, \delta \in \mathbf{L}_\infty(\nu\Delta)\colon \Phi_\delta w \in \mathbf{H}_0^1\} \tag{2.3}$$

and the singular part

$$\nu_{opt|s} \ = \ \inf\{\nu | \exists z \in \mathbb{T}, \ \delta \in \nu\Delta : \Phi_\delta(z) = 0\}. \tag{2.4}$$

In [1] it was shown that the equations in the dual problem can be solved explicitly for the rational systems. For this class of systems it was proposed the dual parametrization of plant factors with destabilizing uncertainties.

In this chapter, we continue to study the duality principle in robust optimization problems of rank one. The result is extended to the class of non-rational systems with continuous nominal factors, for which the dual factorization of all plant factors with destabilizing uncertainties is obtained. We introduce the class $\mathcal{U}$ of all unimodular functions with the well-defined and finite winding numbers

$$\mathcal{U} = \{u \in \mathbf{L}^\infty : |u| \equiv 1, \ \mathrm{wno}\, u > 0\}.$$

The main theorem reads as follows.

Theorem 6. *Let $\Delta = \mathcal{B}\mathbb{C}^m$ and $\nu_{opt|c} < \nu_{opt|s}$. Then the plant numerator and denominator with the worst uncertainty $\Phi_\delta = F + \delta^\top G$ can be factorized in the following way*

$$\Phi_\delta = u\psi,$$

where $u \in \mathcal{U}$ and $\psi \in \mathbf{H}^\infty$ is outer.

The proof of Theorem 6 will be given in section 2.2.

As an example, a system with delay is considered in section 2.3. The example is rather illustrative because the plant factors are essentially non-rational, as they contain a singular component and cannot be simply treated by unstable cancelations.

Finally, in section 2.4 the dual problem is interpreted as a minimization of all destabilizing uncertainties in a certain metric that has interesting connections to the well-known ν-gap metric.

2.2   The Main Result

As we discussed in Theorem 5, the calculation of the optimal ν consists of two parts: singular (2.4) and regular (2.3). The singular part is quite simple: to find the smallest uncertainty vector that brings a common zero to the plant factors on the unit disc. The regular case involves three functional variables and is relatively complicated. In this section, we will discuss the regular case and obtain the canonical factorization of all factors for the perturbed non-stabilizable plant that can be used later to find the solution of our problem.

Let us assume that $\nu_{opt|c} < \nu_{opt|s}$, otherwise the problem can be solved by the singular part.

The main goal of this section is to prove Theorem 6. This will be done in section 2.2.2, but first, in section 2.2.1, we will go through the necessary definitions and preliminary results.

2.2.1 Preliminaries

Recall that $\Phi_\delta = F + \delta^\top G$. We know that F and G are from the disk algebra $\mathbf{A}$. We need supplementary information about δ.

Theorem 7. *Let $\Delta = \mathcal{B}\mathbb{C}^m$ and $\nu_{opt|c} < \nu_{opt|s}$. Then the solution to the regular case (2.3) of the dual problem satisfies $\delta_{opt} \in \overline{\mathbf{H}^\infty}$.*

In order to prove the theorem we need some preliminary results.

We begin with some remarks about solutions of our primal problem and the $\mathbf{H}^\infty$ optimization. The following proposition was proved in [3].

Proposition 1 (Ghulchak). *Let $F \in \mathbf{A}^{1 \times n}$, $G \in \mathbf{A}^{m \times n}$ and $\Delta = \mathcal{B}\mathbb{C}^m$. Then the following statements are equivalent:*

1. *$\exists h \in \mathbf{H}^\infty(\mathbb{C}^{n \times 1})$ such that $\forall z \in \mathbb{T}$ and $\forall \delta \in \nu\Delta$*

$$Re(F(z) + \delta^\top(z)G(z))h(z) > 0.$$

2. *$\exists g \in \mathbf{H}^\infty(\mathbb{C}^{n \times 1})$ such that $Fg = 1$ and*

$$\|Gg\|_\infty < \nu^{-1}.$$

The first problem is our primal problem. From the proposition follows that g is a solution to the second problem if and only if $h = g$ is a solution to the primal problem with $Fh = 1$. Finding optimal ν in (2.2) is reduced now to the following problem

$$\nu_{opt}^{-1} = \inf_{g \in \mathbf{H}^\infty} \{\|Gg\|_\infty | Fg = 1\}.$$

All the solutions to the equation $Fg = 1$, $F \in \mathbf{A}$, can be parameterized as

$$g = g_{part} + Mq,$$

where g_{part} is the particular solution to the equation, M is the basis of the null-space for F and $q \in \mathbf{H}^\infty$. This parametrization is known in the control theory as the Youla parametrization (see [21]).

Then

$$\|Gg\|_\infty = \|Gg_{part} + GMq\|_\infty = \|T_1 - T_2 q\|_\infty,$$
$$\nu_{opt}^{-1} = \inf_{q \in \mathbf{H}^\infty} \|T_1 - T_2 q\|_\infty, \tag{2.5}$$

where $T_1 = Gg_{part}$ and $T_2 = -GM$. This problem is the well-known $\mathbf{H}^\infty$ optimization problem. It means that in the particular case when $\Delta = \mathcal{B}\mathbb{C}^m$ then our primal problem is equivalent to the $\mathbf{H}^\infty$ optimization problem. It turns out that if $F \in \mathbf{A}$ and $G \in \mathbf{A}$ then we can find solution g to $Fg = 1$ such that T_1 and T_2 in the $\mathbf{H}^\infty$ optimization problem are from $\mathbf{A}$ as well.

Lemma 1. *We have $T_1, T_2 \in \mathbf{A}$.*

Proof. Let g_{part} be a particular solution to $Fg = 1$, $F \in \mathbf{A}$. It is known [20] that we can find such a particular solution g_{part} to the equation $Fg = 1$ that $g_{part} \in \mathbf{A}$. Let us show now that we can similarly put $M \in \mathbf{A}$.

The disk algebra $\mathbf{A}$ is the Hermite ring [16]. It means that if $F \in \mathbf{A}$ has a right inverse then F can be complemented to a unimodular matrix $U \in \mathbf{A}$ containing F as a submatrix, i.e. $U = \begin{pmatrix} F \\ B \end{pmatrix}$.

Let us partition the matrix $U^{-1} = \begin{pmatrix} g_{part} & M \end{pmatrix}$ and check that these g_{part} and $M \in \mathbf{A}$ satisfy the necessary equations. Indeed

$$UU^{-1} = \begin{pmatrix} F \\ B \end{pmatrix} \begin{pmatrix} g_{part} & M \end{pmatrix} = \begin{pmatrix} Fg_{part} & FM \\ Bg_{part} & BM \end{pmatrix} = I$$

and $Fg_{part} = 1$, $Bg_{part} = \mathbb{O}^{(n-1) \times 1}$, $FM = \mathbb{O}^{1 \times (n-1)}$, $BM = \mathrm{I}^{(n-1) \times (n-1)}$. Thus we have found $M \in \mathbf{A}_{n \times (n-1)}$ such that $FM = 0$.

Now we will show that $\operatorname{Ker} F = M\mathbf{H}^\infty$.

It is clear that $M\mathbf{H}^\infty \subset \operatorname{Ker} F$. ($\forall q \in \mathbf{H}^\infty$ we have $FMq = 0 \Rightarrow Mq \in \operatorname{Ker} F$.)

Now let $\varphi \in \mathbf{H}^\infty$ be a function from the null-space $\varphi \in \operatorname{Ker} F$. Then

$$U\varphi = \begin{pmatrix} F \\ B \end{pmatrix} \varphi = \begin{pmatrix} 0 \\ B\varphi \end{pmatrix} = \begin{pmatrix} 0 \\ B \end{pmatrix} \varphi$$

and

$$\varphi = U^{-1} \begin{pmatrix} 0 \\ B \end{pmatrix} \varphi = \begin{pmatrix} g_{part} & M \end{pmatrix} \begin{pmatrix} 0 \\ B \end{pmatrix} \varphi = MB\varphi = Mq.$$

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We get $\forall \varphi \in \mathrm{Ker}\ F\ \exists q \in \mathbf{H}^\infty$ such that $\varphi = Mq$. It means that $\mathrm{Ker}\ F \subset M\mathbf{H}^\infty$.

We have proved that $\mathrm{Ker}\ F = M\mathbf{H}^\infty$. It means that M is a basis of the kernel of F and $M \in \mathbf{A}$. $\qquad\square$

We have shown that M is from the disc algebra $\mathbf{A}$ and $g_{part} \in \mathbf{A}$. Then ν_{opt}^{-1} is defined as in (2.5)

$$\nu_{opt}^{-1} = \inf_{q \in \mathbf{H}^\infty} \|T_1 - T_2 q\|_\infty,$$

but now both $T_1 = Gg_{part}$ and $T_2 = -GM$ are from the disk algebra $\mathbf{A}$.

Let us now discuss how we can describe the distance between T_1 and $T_2\mathbf{H}^\infty$ with the help of measures.

In [7] it was shown that

$$\mathrm{dist}\,(T_1, T_2\mathbf{H}^\infty) = \sup\{|\int T_1^\top\, dk| \colon k \in (T_2\mathbf{H}^\infty)^\perp, \|k\| \le 1\}.$$

Recall that $k \in (T_2\mathbf{H}^\infty)^\perp$ means that $|\int (T_2 g)^\top\, dk| = 0$ for all $g \in \mathbf{H}^\infty$. We will show that $\mathrm{dist}\,(T_1, T_2\mathbf{H}^\infty) = \mathrm{dist}\,(T_1, T_2\mathbf{A})$.

Lemma 2. *We have*

$$\inf_{h \in \mathbf{H}^\infty} \|T_1 - T_2 h\|_\infty = \inf_{h \in \mathbf{A}} \|T_1 - T_2 h\|_\infty.$$

Proof. Since $\mathbf{A}$ is a subset of $\mathbf{H}^\infty$ then we have

$$\inf_{h \in \mathbf{H}^\infty} \|T_1 - T_2 h\|_\infty \le \inf_{h \in \mathbf{A}} \|T_1 - T_2 h\|_\infty.$$

In order to prove the opposite inequality, we take $h \in \mathbf{H}^\infty$ as the optimal function, which exists (see [7]), and by $(T_1 - T_2 h)_r$ we denote $T_1(rz) - T_2(rz)h(rz)$, where $r < 1$ and $z \in \mathbb{T}$. According to the maximum modulus theorem (see [20]) we have

$$\|T_1 - T_2 h\|_\infty \ge \|(T_1 - T_2 h)_r\|_\infty = \|T_{1r} - T_{2r} h_r\|_\infty.$$

Now we estimate $\|T_1 - T_2 h_r\|_\infty$.

$$\|T_1 - T_2 h_r\|_\infty = \|T_1 + T_{1r} - T_{1r} - T_2 h_r + T_{2r} h_r - T_{2r} h_r\|_\infty \le$$
$$\|T_{1r} - T_{2r} h_r\|_\infty + \|T_1 - T_{1r}\|_\infty + \|T_2 - T_{2r}\|_\infty \|h_r\|_\infty.$$

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Since T_1 and T_2 are continuous functions, then we have $\|T_1 - T_{1r}\|_\infty \to 0$ and $\|T_2 - T_{2r}\|_\infty \to 0$ as $r \to 1$. By the maximum modulus theorem we have $\|h_r\|_\infty \le \|h\|_\infty$ and we get that

$$\|T_1 - T_2 h_r\|_\infty \le \|T_{1r} - T_{2r} h_r\|_\infty + \varepsilon$$

and

$$\|T_1 - T_2 h\|_\infty \ge \|T_{1r} - T_{2r} h_r\|_\infty \ge \|T_1 - T_2 h_r\|_\infty - \varepsilon.$$

Since $h_r \in \mathbf{A}$ then we have shown that

$$\operatorname{dist}(T_1, T_2 \mathbf{H}^\infty) = \operatorname{dist}(T_1, T_2 \mathbf{A}).$$

$\square$

Thus we have shown that $\operatorname{dist}(T_1, T_2 \mathbf{H}^\infty) = \sup |\int_{\mathbb{T}} T_1^\top d\mu|$, where $d\mu \in (T_2 \mathbf{A})^\perp$ and $\|d\mu\| \le 1$.

Since $\mathbb{T}$ is compact then there exists the optimal measure μ_{opt} such that

$$\operatorname{dist}(T_1, T_2 \mathbf{H}^\infty) = |\int_{\mathbb{T}} T_1^\top d\mu_{opt}|.$$

Since $|\int (T_2 g)^\top d\mu| = |\int g^\top T_2^\top d\mu| = 0$ for all $g \in \mathbf{H}^\infty$, then we get

$$T_2^\top d\mu = f\, dm, \text{ where } f \in \mathbf{H}_0^1.$$

If we decompose the measure $d\mu_{opt}$ into the singular and the absolutely continues parts we get

$$T_2^\top w\, dm + T_2^\top d\mu_s = f\, dm.$$

Here we have two cases:

1. There exists a point $z_0 \in \mathbb{T}$ such that T_2 loses its column-rank at this point. Then $d\mu_s = x \delta_{z_0} dm$, where δ_{z_0} is Dirac's function, and the vector x is such that $T_2^\top(z_0) x = \mathbb{O}$. We call this case the singular case.

2. There is no such a point as above. Then we can take $d\mu_s = 0$. This is the regular case.

Finally

$$\left| \int_{\mathbb{T}} T_1^\top d\mu_{opt} \right| = \begin{cases} |T_1(z_0)|, & d\mu_{opt} = x\delta_{z_0}dm \\ |\int_{\mathbb{T}} T_1^\top w\, dm|, & T_2^\top w \in \mathbf{H}_0^1, \|w\|_1 = 1. \end{cases}$$

Again as in [2] we can divide our problem into two parts: singular and regular.

Consider now the regular case, i.e. $\det T_2^* T_2(z) \neq 0$, $\forall z \in \mathbb{T}$. It is easy to see that the regular case in the $\mathbf{H}^\infty$ optimization problem is equivalent to the regular case (2.3) of our dual problem. Indeed, in the singular case we know that there exists $z \in \mathbb{T}$ such that $\Phi_\delta(z) = 0$, where $\Phi_\delta(z) = F + \delta^\top G$. We consider the unimodular matrix $U = \begin{pmatrix} g_p & M \end{pmatrix}$, where g_p is the particular solution to the equation $Fg = 1$, and M is the kernel of F. Then

$$\Phi_\delta(g_p\ M) = (F + \delta^\top G)(g_p\ M) = (1 + \delta^\top T_1\ \ \delta^\top T_2)$$

and

$$\Phi_\delta(z) = 0 \Leftrightarrow \begin{cases} 1 + \delta^\top T_1(z) = 0 \\ \delta^\top T_2(z) = 0. \end{cases}$$

We have the non-trivial solution of the second equation in the system $(|\delta_{opt}| = \nu)$. This is equivalent to the condition that T_2 loses the rank, i.e. $\det T_2^* T_2(z) = 0$. Furthermore recall that

$$\nu_{opt}^{-1} = \inf_{q \in \mathbf{H}^\infty} \|T_1 - T_2 q\|_\infty.$$

If T_2 loses the rank in z_0, then for all $q \in \mathbf{H}^\infty$ it holds that

$$\|T_1 - T_2 q\|_\infty = \sup_{z \in \mathbb{T}} |T_1(z) - T_2(z)q(z)|_\infty \geq |T_1(z_0)|$$

and

$$\nu_{opt}^{-1} = \inf_{q \in \mathbf{H}^\infty} \|T_1 - T_2 q\|_\infty = |T_1(z_0)|.$$

This means that the condition that T_2 loses the rank is equivalent to the singular case of the dual problem. Furthermore if the solution of the singular part of the dual problem is not trivial ($\nu_{opt|s} < \infty$) then $\nu_{opt} = \nu_{opt|s}$. Since we assume that $\nu_{opt|c} < \nu_{opt|s}$, i.e. the solution is obtained in the regular case, then we can assume that T_2 does not lose the rank. This corresponds to the regular case of the $\mathbf{H}^\infty$ optimization problem.

Now we will discuss some basic properties of the Toeplitz operators [18].

Lemma 3 (Peller). *Let $\varphi \in \mathbf{H}^\infty$. Then the Toeplitz operator T_φ is invertible if and only if φ is invertible in $\mathbf{H}^\infty$*

Theorem 8 (Peller). *Let $\varphi, \psi \in \mathbf{L}^\infty$. Then*

$$\mathrm{T}_{\varphi\psi} = \mathrm{T}_\varphi \mathrm{T}_\psi$$

if and only if $\varphi \in \mathbf{H}^\infty$ or $\overline{\psi} \in \mathbf{H}^\infty$.

Theorem 9 (Peller). *Suppose that Φ is a matrix function in $\mathbf{L}^\infty(\mathbb{M}_{m,n})$ such that H_Φ has a maximizing vector. If $\min(m,n) = 1$, then Φ has a unique best approximation by bounded analytic matrix functions.*

We also need the notion of the Fredholm operator.

Definition 3 (Fredholm operator). *An operator T on a Hilbert space H is called Fredholm if T is invertible modulo compact operators. The index of a Fredholm operator T is defined by*

$$ind\, T = \dim Ker\, T - \dim Ker\, T^*.$$

An operator T is Fredholm if it has closed range and $\dim \mathrm{Ker}\, T < \infty$ and $\dim \mathrm{Ker}\, T^* < \infty$.

Theorem 10 (Peller). *Let n be a positive integer and let U be a unitary-valued matrix function in $\mathbf{L}^\infty(\mathbb{M}_n)$. The operator T_U is Fredholm if and only if $\|\mathrm{H}_U\|_e < 1$ and $\|\mathrm{H}_{U^*}\|_e < 1$.*

In [18] the following lemma was proved:

Lemma 4 (Peller). *Let L be an invariant subspace of multiplication by z on $\mathbf{H}^2$. Then L has finite codimension if and only if $L = B\mathbf{H}^2$ for a Blaschke product B of finite degree.*

In [19] a four block operator Γ_Φ was defined as operator

$$\Gamma_\Phi : \mathbf{H}^2 \oplus \mathbf{L}^2 \to \mathbf{H}^2_- \oplus \mathbf{L}^2$$

such that

$$\Gamma_\Phi \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \mathbb{P}^- \Phi \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},$$

where $\mathbb{P}^-$ is the orthogonal projection from $\mathbf{L}^2 \oplus \mathbf{L}^2$ onto $\mathbf{H}_-^2 \oplus \mathbf{L}^2$, Φ is a block matrix function.

Let Φ be a block matrix function of the form

$$\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}, \tag{2.6}$$

where Φ has size $m \times n$, Φ_{11} has size $m_1 \times n_1$, and Φ_{22} has size $m_2 \times n_2$. The four block problem is to minimize

$$\left\| \begin{pmatrix} \Phi_{11} - Q & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} \right\|_\infty \tag{2.7}$$

over bounded analytic functions Q of size $m_1 \times n_1$. A function Q is called an optimal solution of the four block problem if it minimizes the norm (2.7).

We have the following lemma:

Lemma 5 (Peller, Treil). *Let Φ be a block matrix function of the form (2.6) such that $\|\Gamma_\Phi\|_e < \|\Gamma_\Phi\|$. Let V be a unitary-valued matrix in the form*

$$V = \begin{pmatrix} v_1 & \overline{V}_c & \star \\ v_2 & \mathbb{O} & \star \end{pmatrix},$$

where V_c is an inner and co-outer matrix. Let W be a unitary-valued matrix in the form

$$W^\top = \begin{pmatrix} w_1 & \overline{W}_c & \star \\ w_2 & \mathbb{O} & \star \end{pmatrix},$$

where W_c is an inner and co-outer matrix. Then there exists a unimodular function u_0 such that any optimal solution Q_o of the four block problem satisfies

$$\begin{pmatrix} \Phi_{11} - Q_o & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} = W^* \begin{pmatrix} t_0 u_0 & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \Phi_{11}^{(1)} & \Phi_{12}^{(1)} \\ \mathbb{O} & \Phi_{21}^{(1)} & \Phi_{22}^{(1)} \end{pmatrix} V^*, \tag{2.8}$$

where $\Phi_{11}^{(1)}$ is a matrix function of size $m_1-1 \times n_1-1$. The unimodular function u_0 admits a representation $u_0 = \overline{z}\overline{b}h/h$, where h is an outer function in $\mathbf{H}^2$ and b is a finite Blaschke product. Moreover, the Toeplitz operator T_{u_0} is Fredholm.

Now we are ready to proof Theorem 7.

Proof of Theorem 7. We start with the primal problem

$$\sup_{h \in \mathcal{B}\mathbf{H}^\infty} \inf_{|\delta| \leq \nu} Re(F + \delta^\top G)h > 0.$$

Let h_{opt} be the solution to the problem. Such h_{opt} exists according to Proposition 1. If $Gh = 0$ then the function simplifies to $Re(Fh)$. Thus we assume $Gh \neq 0$. Then the optimal δ has the following expression

$$\delta = -\nu \frac{(Gh_{opt})^*}{|Gh_{opt}|}.$$

Our purpose is to show that $|Gh_{opt}| \equiv const$.

Recall that if h_{opt} is the solution to the primal problem then $g_{opt} = h_{opt}$ is the solution to the $\mathbf{H}^\infty$ optimization problem.

$$\|Gg_{opt}\|_\infty = \inf_{q \in \mathbf{H}^\infty} \|T_1 - T_2 q\|_\infty.$$

As function $T_2 \in \mathbf{H}^\infty$, then T_2 can be factorized into inner and outer factors, i.e. $T_2 = T_{2i}T_{2o}$.

Since

$$\begin{pmatrix} T_{2i}^* \\ I - T_{2i}T_{2i}^* \end{pmatrix}^* \begin{pmatrix} T_{2i}^* \\ I - T_{2i}T_{2i}^* \end{pmatrix} = \begin{pmatrix} T_{2i} & I - T_{2i}T_{2i}^* \end{pmatrix} \begin{pmatrix} T_{2i}^* \\ I - T_{2i}T_{2i}^* \end{pmatrix} =$$

$$T_{2i}T_{2i}^* + \mathbb{I} - T_{2i}T_{2i}^* - T_{2i}T_{2i}^* + T_{2i}T_{2i}^*T_{2i}T_{2i}^* = \mathbb{I} - T_{2i}T_{2i}^* + T_{2i}T_{2i}^* = \mathbb{I},$$

then we have

$$\inf_{q \in \mathbf{H}^\infty} \|T_1 - T_2 q\|_\infty = \inf_{q \in \mathbf{H}^\infty} \left\| \begin{pmatrix} T_{2i}^* \\ I - T_{2i}T_{2i}^* \end{pmatrix} (T_1 - T_2 q) \right\|_\infty =$$

$$\inf_{q \in \mathbf{H}^\infty} \left\| \begin{pmatrix} T_{2i}^* T_1 - T_{2o} q \\ (I - T_{2i}T_{2i}^*) T_1 \end{pmatrix} \right\|_\infty.$$

We will show that any inner function in the disk algebra $\mathbf{A}$ is the finite Blaschke product. Indeed, T_{2o} is invertible in $\mathbf{H}^\infty$, then by Lemma 3 and Theorem 8 the Toeplitz operator $\mathrm{T}_{T_{2o}}$ is invertible and $\mathrm{T}_{T_{2i}} = \mathrm{T}_{T_2} \mathrm{T}_{T_{2o}}^{-1}$. Since $T_{2i} \in \mathbf{A} = \mathbf{H}^\infty \cap \mathcal{C}$, then we have that the essential norm of the Hankel operators $\mathrm{H}_{T_{2i}}$ and $\mathrm{H}_{\overline{T_{2i}}}$ are zeros. The function T_{2i} is unimodular. According

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to Theorem 10 the Toeplitz operator $T_{T_{2i}}$ is Fredholm. Then $T_{2i}\mathbf{H}^2$ has finite codimension ($\operatorname{codim} X = \dim(X^\perp)$). Indeed, take φ such that $\varphi \in \mathbf{H}^2 \ominus T_{2i}\mathbf{H}^2$. It means that $\varphi \perp T_{2i}f$ for all $f \in \mathbf{H}^2$ and $\int f^* T_{2i}^* \varphi \; dm = 0$ and $T_{2i}^*\varphi \perp \mathbf{H}^2$. Hence, $\mathbb{P}_+ T_{2i}^*\varphi = 0$ and $\varphi \in \operatorname{Ker} T_{T_{2i}}^*$. Since $T_{T_{2i}}$ is Fredholm, then dimension of $\operatorname{Ker} T_{T_{2i}}^*$ is finite and codimension of $T_{2i}\mathbf{H}^2$ is finite.

By Lemma 4 function T_{2i} is a Blaschke product of finite degree.

Let $\Phi = \begin{pmatrix} T_{2i}^* T_1 \\ (I - T_{2i}T_{2i}^*)T_1 \end{pmatrix}$.

Since $T_1, T_2 \in \mathbf{A}$, it follows that $\Phi \in \mathbf{H}^\infty + \mathcal{C}$. Recall that $\|\Gamma_\Phi\|_e = \operatorname{dist}(\Phi, \mathbf{H}^\infty + \mathcal{C})$ is the essential norm of Hankel operator. As $\Phi \in \mathbf{H}^\infty + \mathcal{C}$ then we have that $\|\Gamma_\Phi\|_e < \|\Gamma_\Phi\|$.

Recall that we consider the regular case of the dual problem. It is equivalent to the condition that $\det T_2^* T_2(z) \neq 0 \; \forall z \in \mathbb{T}$. Then T_{2o} has a right inverse in $\mathbf{H}^\infty$ and

$$\inf_{q\in\mathbf{H}^\infty} \left\| \begin{pmatrix} T_{2i}^* T_1 - T_{2o}q \\ (I - T_{2i}T_{2i}^*)T_1 \end{pmatrix} \right\|_\infty = \inf_{q_1\in\mathbf{H}^\infty} \left\| \begin{pmatrix} T_{2i}^* T_1 - q_1 \\ (I - T_{2i}T_{2i}^*)T_1 \end{pmatrix} \right\|_\infty.$$

According to Lemma 5 and Theorem 9 we have that there exists the unique optimal solution to this approximation problem and the solution can be parameterized as follows

$$\begin{pmatrix} T_{2i}^* T_1 - q_{1opt} \\ (I - T_{2i}T_{2i}^*)T_1 \end{pmatrix} = \begin{pmatrix} t_0 u_0 & 0 \end{pmatrix} \begin{pmatrix} v & \overline{V_c} \end{pmatrix}^*,$$

where $u_0 = \overline{z}\overline{b}\overline{h}/h$, h is an outer function, b is a finite Blaschke product, V is an unitary matrix $V = \begin{pmatrix} v & \overline{V_c} \end{pmatrix}$, with an inner vector function v and a co-outer matrix function V_c.

For q_{1opt} we have $T_{2i}^* T_1 - q_{1opt} = t_0 u_0 v^*$.

Recall that $|Gg_{opt}| = |T_1 - T_2 q_{opt}|$. Thus

$$|Gg_{opt}| = |t_0 u_0 v^*| \equiv const,$$

where g_{opt} is the solution to $\mathbf{H}^\infty$ optimization. Then $|Gh_{opt}| \equiv const$, where h_{opt} is the solution to the primal problem.

Since

$$\delta = -\nu \frac{(Gh_{opt})^*}{|Gh_{opt}|}, \; |Gh_{opt}| \equiv const$$

then $\delta \in \overline{\mathbf{H}^\infty}$.

The proof of Theorem 7 is finished. $\qquad\square$

We need some more definitions and theoretical results from [18]. First we define the winding number for a continuous function.

Definition 4 (wno for a continuous function). *Let φ be a function on $\mathcal{C}(\mathbb{T})$ that does not vanish on $\mathbb{T}$. The winding number $wno\,\varphi$ with respect to the origin is defined in the following way. Consider a continuous branch of the argument $\arg_\varphi$ of the function $t \to \varphi(e^{it})$, $t \in [0, 2\pi]$,*

$$\arg_\varphi \in \mathcal{C}([0, 2\pi]), \quad \frac{\varphi(e^{it})}{|\varphi(e^{it})|} = \exp(i \arg_\varphi), \ t \in [0, 2\pi].$$

Then

$$wno\,\varphi = \frac{1}{2\pi}(\arg_\varphi(2\pi) - \arg_\varphi(0)).$$

The winding number for the functions invertible in $\mathbf{H}^\infty + \mathcal{C}$ is defined as follows.

Definition 5 (wno for a $\mathbf{H}^\infty + \mathcal{C}$ function). *Let φ be an invertible function in $\mathbf{H}^\infty + \mathcal{C}$. Given $r < 1$, let $\varphi_r(z) := \varphi(rz)$, $z \in \mathbb{T}$. Let r_0 be a number in $(0, 1)$ such that*

$$\inf\{|\varphi(z)| : \ |z| > r_0\} > 0.$$

Then $wno\,\varphi$ is by definition $wno\,\varphi_r$, where r is an arbitrary number in $(r_0, 1)$.

Now we give the definition of a badly approximable function and a thematic function.

Definition 6 (badly approximable function). *Consider the matrix function $\Phi \in \mathbf{L}^\infty$ such that H_Φ has a maximazing vector. Φ is called a badly approximable matrix function if*

$$\|\Phi\|_\infty = dist_{\mathbf{L}^\infty}(\Phi, \mathbf{H}^\infty).$$

Definition 7 (thematic function). *Let n be a positive integer. Suppose that Υ is an $n \times 1$ inner and co-outer matrix function and Θ is an $n \times (n-1)$ inner and co-outer matrix function. If the matrix function*

$$\Omega = \begin{pmatrix} \Upsilon & \overline{\Theta} \end{pmatrix}$$

is unitary-valued, then it is called the thematic matrix function.

Theorem 11 (Peller). *Suppose that $\Phi \in \mathbf{L}^{\infty}(\mathbb{M}_{m,n})$ and $\mathrm{H}_{\Phi} \neq \mathbb{O}$. The following statements are equivalent*

- *Φ is badly approximable and H_{Φ} has a maximazing vector;*

- *Φ admits a factorization of the form*

$$\Phi = W^{*} \begin{pmatrix} tu & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix} V^{*}, \qquad (2.9)$$

where $t > 0$, V and $W^{\top}$ are thematic matrix functions, $\|\Psi\|_{\infty} \leq t$, and u is a scalar unimodular function of the form $u = \frac{\bar{z}\bar{\theta}h}{h}$ for an inner function θ and an outer function $h \in \mathbf{H}^{2}$.

Recall that $\mathcal{QC}$ is the class of quasicontinuous functions defined as

$$\mathcal{QC} = (\mathbf{H}^{\infty} + \mathcal{C}) \cap (\overline{\mathbf{H}^{\infty} + \mathcal{C}}).$$

Theorem 12 (Peller). *Let $\Phi \in \mathbf{H}^{\infty} + \mathcal{C}(\mathbb{M}_{m,n})$ and $\mathrm{H}_{\Phi} \neq \mathbb{O}$. If F is a best approximation of Φ by bounded analytic matrix functions and*

$$\Phi - F = W^{*} \begin{pmatrix} tu & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix} V^{*},$$

then $u \in \mathcal{QC}$.

2.2.2 The main theorem

We are now ready to prove the main result. Recall that in the regular case, i.e. $\nu_{opt|c} < \nu_{opt|s}$, we need to find the functions $w_{opt} \in \mathbf{L}^{1}(\mathbb{R}_{+}) \backslash 0$ and $\delta_{opt} \in \mathbf{L}_{\infty}(\nu\Delta)$ such that $\Phi_{\delta} w \in \mathbf{H}_{0}^{1}$. The last condition means that there exists a vector function $p \in \mathbf{H}^{1}$ such that

$$\Phi_{\delta}(z)w(z) = zp(z).$$

Note that in the regular case we have $|\Phi_{\delta}| \geq \varepsilon$, which implies, in particular, that $\log(w) \in \mathbf{L}^{1}$. A real positive function with this property can be factorized as $w = f^{*}f$, where f is a scalar outer function. Then Φ_{δ} can be written as

$$\Phi_{\delta} = z\frac{p}{f^{*}f} = z\frac{f}{f^{*}}\frac{p}{f^{2}}.$$

Furthermore, the function p can be factorized into inner and outer factors $p = p_i p_0$, where p_i is scalar, and the function Φ_δ may be written as

$$\Phi_\delta = z p_i \frac{f}{f^*} \frac{p_o}{f^2} = u\psi, \tag{2.10}$$

where $u = z p_i \frac{f}{f^*}$ is a scalar function and $\psi \in \mathbf{H}^\infty$ is an outer (row-)function. In the rational case, the rational function u collects all common zeros of the plant factors and gives a factorization of the dual problem via unstable zero-pole cancelations. In our case, the situation with the common zeros is more complicated.

We show that in the case when the uncertainty set is the ball $\Delta = \mathcal{B}\mathbb{C}^m$, the winding number for the function u is well-defined. It plays the same role as the unstable cancelations in the rational case.

We introduce the class $\mathcal{U}$ of all unimodular functions with the well-defined and finite winding numbers

$$\mathcal{U} = \{u \in \mathbf{L}^\infty : |u| \equiv 1, \text{ wno } u > 0\}.$$

Theorem 13 (Main result). *Let $\Delta = \mathcal{B}\mathbb{C}^m$ and $\nu_{opt|c} < \nu_{opt|s}$. Then the plant numerator and denominator with the worst uncertainty $\Phi_\delta = F + \delta^\top G$ can be factorized in the following way*

$$\Phi_\delta = u\psi, \tag{2.11}$$

where $u \in \mathcal{U}$ and $\psi \in \mathbf{H}^\infty$ is outer.

Proof. Recall that we can parameterize Φ_δ such that

$$\Phi_\delta = u\psi,$$

where $u = z p_i \frac{f}{f^*}$, f is a scalar outer function, p_i is a scalar inner function and $\psi \in \mathbf{H}^\infty$ is an outer vector function. We have to show that $u \in \mathcal{U}$.

Let ϕ be a scalar outer function such that $|\psi| = |\phi|$. We can write $\overline{\Phi_\delta}$ in the following way:

$$\frac{\overline{\Phi_\delta}}{\phi} = \frac{\overline{u\phi}}{\phi} \frac{\overline{\psi}}{\overline{\phi}}.$$

Now we show that $\frac{\overline{\Phi_\delta}}{\phi}$ is a badly approximable function.

Introduce the following notation:

$$u_1 := \frac{\overline{u\phi}}{\phi} \quad \text{and} \quad V^* := \begin{pmatrix} \frac{\overline{\psi}}{\phi} \\ \Theta^\top \end{pmatrix},$$

where Θ is $n \times (n-1)$ inner and co-outer matrix function. Let $t_0 = 1$. Then we can factorize $\frac{\overline{\Phi_\delta}}{\phi}$ in the following way:

$$\frac{\overline{\Phi_\delta}}{\phi} = \begin{pmatrix} t_0 u_1 & 0 \end{pmatrix} V^*.$$

We will show that $V = \begin{pmatrix} \frac{\psi}{\phi}^\top & \overline{\Theta} \end{pmatrix}$ is a thematic function.

We have that $(\frac{\psi}{\phi})^\top$ is an inner and co-outer column vector. The scalar outer function ϕ is invertible in $\mathbf{H}^\infty$. Then $(\frac{\psi}{\phi})^\top \in \mathbf{H}^\infty$. Since

$$(\frac{\psi}{\phi}^\top)^* \frac{\psi}{\phi}^\top = \frac{|\psi|^2}{|\phi|^2} = 1,$$

we get that $(\frac{\psi}{\phi})^\top$ is an inner function. Furthermore $((\frac{\psi}{\phi})^\top)^\top = \frac{\psi}{\phi}$, ψ is outer and $\frac{1}{\phi} \in \mathbf{H}^\infty$, ϕ is outer and $|\phi| > 0$. Thus we have that $(\frac{\psi}{\phi})^\top$ is a co-outer function.

In [18] it was shown that any inner and co-outer $n \times 1$ matrix function can be completed to a thematic matrix function. According to this result there is Θ such that Θ is $n \times (n-1)$ inner and co-outer matrix function and $V = \begin{pmatrix} \frac{\psi}{\varphi}^\top & \overline{\Theta} \end{pmatrix}$ is unitary-valued. It means exactly that V is a thematic function.

We get that

$$\frac{\overline{\Phi_\delta}}{\phi} = \begin{pmatrix} t_0 u_1 & 0 \end{pmatrix} V^*,$$

where $t_0 = 1$, $u_1 = \frac{\overline{z p_i f \phi}}{f \phi}$ and V is a thematic function. We obtain the same factorization as in (2.9) with W identically equal to 1 and with submatrix Ψ of size 0. Then $\frac{\overline{\Phi_\delta}}{\phi}$ is badly approximable.

Finally we know that $F \in \mathbf{A}$ and $G \in \mathbf{A}$. From Lemma 7 we get that $\delta \in \overline{\mathbf{H}^\infty}$. Therefore

$$\overline{\Phi_\delta} = \overline{F} + \overline{\delta^\top G} \in \mathbf{H}^\infty + \mathcal{C}.$$

As $\overline{\frac{\Phi_\delta}{\phi}}$ is badly approximable, then using Theorem 12 we get that $u_1 \in \mathcal{QC}$.

Recall that $u_1 = \overline{\frac{zp_if\phi}{f\phi}}$. Since the outer function ϕ is invertible in $\mathbf{H}^\infty$, then the Toeplitz operator $T_{\frac{\overline\phi}{\phi}}$ can be written as

$$T_{\frac{\overline\phi}{\phi}} = T_{\overline\phi} T_{\frac{1}{\phi}} = T_\phi^* T_{\frac{1}{\phi}}.$$

Since both T_ϕ^* and $T_{\frac{1}{\phi}}$ are invertible, then $T_{\frac{\overline\phi}{\phi}}$ is invertible too and $T_{\frac{\overline\phi}{\phi}}$ is Fredholm. From [18] we know that the Toeplitz operator T_φ is invertible if and only if it is Fredholm and $\operatorname{ind} T_\varphi = 0$. Thus

$$\operatorname{ind} T_{\frac{\overline\phi}{\phi}} = 0.$$

Since $u_1 \in \mathcal{QC}$, we have T_{u_1} is Fredholm and

$$\operatorname{wno} u_1 = -\operatorname{ind} T_{u_1} > -\infty.$$

As $u_1 \in \mathbf{L}^\infty$, then from [18] we have that either $\operatorname{Ker} T_{u_1} = \mathbb{O}$ or $\operatorname{Ker} T_{u_1}^* = \mathbb{O}$. Since $u_1 = \overline{\frac{zp_if\phi}{f\phi}}$, it is clear that $f\phi \in \operatorname{Ker} T_{u_1}$. Therefore $\operatorname{Ker} T_{u_1}^* = \mathbb{O}$ and $\operatorname{ind} T_{u_1} > 0$ which means that $\operatorname{wno} u_1 < 0$.

Thus we have

$$-\infty < \operatorname{wno} \overline{z}\,\frac{\overline{f}}{f}\overline{p_i}\frac{\overline\phi}{\phi} = \operatorname{wno} \overline{z}\,\frac{\overline{f}}{f}\overline{p_i} + \operatorname{wno} \frac{\overline\phi}{\phi} = \operatorname{wno} \overline{u} < 0.$$

Then we get the factorization of Φ_δ,

$$\Phi_\delta = u\psi,$$

where u is from the class $\mathcal{U}$,

$$\mathcal{U} = \{u \in \mathbf{L}^\infty : |u| \equiv 1,\ \operatorname{wno} u > 0\}$$

and $\psi \in \mathbf{H}^\infty$ is outer. $\qquad\qquad\square$

According to this theorem the plant with the worst uncertainty becomes

$$\frac{N(z) + \delta(z)^\top G_1(z)}{M(z) + \delta(z)^\top G_2(z)} = \frac{u(z)\psi_1(z)}{u(z)\psi_2(z)}$$

and by Cauchy's argument principle the condition wno $u > 0$ means precisely that, as in the rational case, *the numerator and denominator of the plant with the worst uncertainty have more common unstable zeros than all unstable poles*, but now the number of zeros may be infinite or even it may be limiting zeros coming from a singular function. The last kind of zeros is the most difficult to treat. Nevertheless, using this parametrization we can obtain the expression for δ_{opt} and calculate ν_{opt} by taking infimum over all such u and ψ. We will show how this works for a system with delay.

2.3 Example: A System with Delay.

In this section, we will show how the problem to optimize the stability margin and to find a controller, that robustly stabilizes the whole family of perturbed plants can be solved for the system with delay using the dual parametrization. Several simple examples for the rational case were solved in [2]. Now we will extend the class of systems that can be solved using the duality principle.

In [10] the authors have considered the $\mathbf{H}^\infty$-sensitivity minimization problem for a linear time-invariant delay system. The corresponding robust stabilization problem for the uncertain plant

$$P_\Delta = \frac{e^{-hs}}{1 + \frac{\Delta}{as+1}} \tag{2.12}$$

can be obtained by the small gain theorem. It turned out that the sensitivity minimization problem even for the simplest delay system is nontrivial. We will consider a similar problem and solve it using the factorization obtained.

Consider the uncertain plant

$$P_\delta = \frac{\frac{e^{-\tau s}}{s+a_0}}{1 + \frac{\delta}{s+a_0}}. \tag{2.13}$$

The minor modification of the numerator is necessary to formally satisfy the continuity assumption, however, it occurs to be irrelevant in this case, and the solution to both problems will be (almost) the same.

Comparing to the rational transfer functions we have here the singular function $e^{-\tau s}$ with a limiting zero at ∞ (essential singularity). It makes the problem much more difficult. It is no longer possible to talk about numbers

of zeros and poles. One has to use the factorization (2.11) and to work with the winding number instead.

It is easy to see that the singular part has no solution. Indeed, we have to find $z \in \mathbb{T}$ such that $\Phi_\delta(z) = 0$. The numerator $\frac{e^{-\tau s}}{s+a_0}$ is zero only at infinity, but we cannot get zero at infinity in the denominator since δ is bounded. It means that it is enough to consider the regular case only.

We will find functions u and ψ such that $u \in \mathcal{U}$, $\psi \in \mathbf{H}^\infty$ is outer and $\Phi_\delta = u\psi$. In our example, it means the following

$$\begin{cases} \frac{e^{-\tau s}}{s+a_0} = u\psi_1 \\ 1 + \frac{\delta}{s+a_0} = u\psi_2. \end{cases} \tag{2.14}$$

Let us make several simple observations:

1) The factor $(s+a_0)$ can be absorbed into ψ_1 with the equivalent system being

$$\begin{cases} e^{-\tau s} = u\psi_1 \\ 1 + \frac{\delta}{s+a_0} = u\psi_2, \end{cases} \tag{2.15}$$

where the condition $\psi_1 \in \mathbf{H}^\infty$ is still valid. The last system is precisely the same as we would get from (2.12), so the dual solution for both plants (2.12) and (2.13) after this little change is the same.

2) The condition $|u| \equiv 1$ does not affect the winding number, so we can first find u with $|u| > 0$ and the required winding number, and then normalize it later.

3) ψ_1 can be taken as an entire function. As due to $\psi_1 \in \mathbf{H}^\infty$, ψ_1 has no unstable poles. Moreover, if ψ_1 has some stable poles, they can be canceled with stable zeros of u. This operation does not affect the winding number either.

Now we get

$$\delta = (s + a_0)(u\psi_2 - 1) = (s + a_0)(\frac{e^{-\tau s}}{\psi_1}\psi_2 - 1) \in \mathbf{L}^\infty,$$

$$u = \frac{e^{-\tau s}}{\psi_1}, \quad \text{wno } u > 0.$$

Thus it follows the two conditions on ψ_1 and ψ_2

$$\begin{cases} \frac{e^{-\tau s}}{\psi_1}\psi_2 \to 1 \text{ as } \omega \to \infty, \\ 0 < \text{wno } \frac{e^{-\tau s}}{\psi_1} < \infty. \end{cases}$$

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As $\omega \to \infty$, that $e^{-\tau s}$ turns around zero infinitely many times. Using that wno $\frac{e^{-\tau s}}{\psi_1} < \infty$, we search for the entire function ψ_1 of exponential type $-\tau$, i.e. in the following form

$$\psi_1 = e^{-\tau s} f + c,$$

where f is a rational function. We get

$$\delta = (s+a_0)\frac{e^{-\tau s}(\psi_2 - f) - c}{e^{-\tau s} f + c} = \frac{s + a_0}{-s + a_0} \frac{e^{-\tau s}(\psi_2 - f) - c}{\frac{e^{-\tau s} f}{-s+a_0} + \frac{c}{-s+a_0}}$$

$$= \frac{s + a_0}{-s + a_0} e^{\tau s} \frac{e^{-\tau s}(\psi_2 - f) - c}{\frac{f}{-s+a_0} + \frac{c e^{\tau s}}{-s+a_0}}. \tag{2.16}$$

It is known (see [8]) that in the regular case the solution is all-pass, i.e. $|\delta(\imath\omega)| = \nu$ for a constant ν. Hence, from the representation (2.16) we get the following system

$$\begin{cases} \psi_2 - f = \nu \frac{\bar{c}}{s+a_0} \\ c = -\nu \frac{\bar{f}}{s+a_0}. \end{cases}$$

As

$$\bar{c} = -\nu \frac{f}{-s + a_0} \qquad \Rightarrow \qquad \psi_2 - f = -\nu^2 \frac{f}{a_0^2 - s^2}$$

we get

$$\psi_2 = \frac{(a_0^2 - s^2 - \nu^2)f}{a_0^2 - s^2}.$$

To satisfy the condition $\psi_2 \in \mathbf{H}^\infty$ we have to cancel the unstable pole. It means that f has the factor $a_0 - s$, so f can be factorized as $f = (a_0 - s)f_1$, where f_1 is a rational function. Thus

$$\psi_2 = \frac{(a_0^2 - s^2 - \nu^2)f_1}{a_0 + s}.$$

The function

$$\psi_1 = e^{-\tau s}(a_0 - s)f_1 + c = e^{-\tau s}(a_0 - s)f_1 - \nu \overline{f_1}$$

is entire. It means that if f_1 has a pole, it has to be canceled with a zero of $e^{-\tau s}(a_0 - s)f_1 - \nu \overline{f_1}$.

From the expression for ψ_2

$$\psi_2 = \frac{(a_0^2 - s^2 - \nu^2)f_1}{a_0 + s},$$

it is clear that f_1 can have only two unstable poles on the boundary, namely zeros of $a_0^2 - s^2 - \nu^2$, and no more.

Suppose f_1 has some stable poles s_k. Then they are zeros of

$$e^{-\tau s}(a_0 - s)f_1 - \nu\overline{f_1}$$

and

$$e^{-\tau s}(a_0 - s)f_1 = \nu\overline{f_1}, \text{ when } s = s_k.$$

If f_1 has stable poles then $\overline{f_1}$ has unstable poles. It is impossible because $e^{-\tau s}(a_0 - s)f_1$ has no unstable poles. So f_1 can only have poles that are zeros of $a_0^2 - s^2 - \nu^2$.

Take $\varphi = (a_0^2 - s^2 - \nu^2)f_1$. Then φ does not have any poles, so φ is just a polynomial. The expression for ψ_2 is $\psi_2 = \frac{\varphi}{s+a_0}$ and ψ_2 has to be bounded, so $\deg \varphi \leq 1$.

Now substitution ψ_1 and ψ_2 into the expression for u yields

$$u = \frac{e^{-\tau s}(a_0^2 - s^2 - \nu^2)}{e^{\tau s}(-s + a_0)\varphi - \nu\overline{\varphi}}.$$

We will find such u that $\text{wno}\, u > 0 \Rightarrow$

$$\text{wno}\, \frac{e^{-\tau s}\varphi(a_0^2 - s^2 - \nu^2)}{e^{-\tau s}(-s + a_0)\frac{\varphi}{\overline{\varphi}} - \nu} > 0.$$

Both $e^{-\tau s}\varphi$ and $e^{-\tau s}(-s + a_0)\frac{\varphi}{\overline{\varphi}}$ have infinitely many encirclements of the origin when s follows the standard contour, indented into the right half plane, but the function in the denominator is moved by the constant ν such that the overall denominator $e^{-\tau s}(-s + a_0)\frac{\varphi}{\overline{\varphi}} - \nu$ has less encirclement than $e^{-\tau s}\varphi$. The length of this movement ν should be minimized according to the dual problem (4.7). It is clear that for this purpose the denominator of u will have exactly one encirclement less than the numerator, i.e. $\text{wno}\, u = 1$. Hence, ν should be chosen as the shortest distance from the Nyquist curve to the origin along the real axis, that is, as

$$\nu = \nu_{opt} = |e^{-\tau\imath\omega_0}(-\imath\omega_0 + a_0)\frac{\varphi}{\overline{\varphi}}| = \sqrt{\omega_0^2 + a_0^2},$$

where ω_0 is the first positive frequency such that

$$e^{-\tau\imath\omega_0}(-\imath\omega_0 + a_0)\frac{\varphi}{\overline{\varphi}} \in \mathbb{R}.$$

The last condition implies

$$-\tau\omega_0 - \arctan\frac{\omega_0}{a_0} + 2\arg\varphi = \pi k. \tag{2.17}$$

Recall that $\deg\varphi \le 1$. Moreover, φ is a stable polynomial because an unstable zero would increase the number of encirclements of the origin, so we would have to increase ν also in order to move $e^{-\tau s}(-s + a_0)$ further. As $\omega_0 > 0$, then $\arg\varphi > 0$ and the minimization of ν and, hence, ω_0 in (2.17) implies that $\arg\varphi = 0$, i.e. φ is a constant. Moreover we get that $\omega_0 > 0$ satisfies

$$\tan\tau\omega_0 \;=\; -\frac{\omega_0}{a_0}, \text{ where } \frac{\pi}{2} \;<\; \tau\omega_0 \;<\; \pi.$$

The optimal stability margin is $\nu_{opt} = \sqrt{a_0^2 + \omega_0^2}$.

Moreover, a necessary factorization of the dual problem is

$$\begin{pmatrix} \frac{e^{-\tau s}}{s+a_0} \\ 1 + \frac{\delta}{s+a_0} \end{pmatrix} = \frac{e^{-\tau s}(s^2 + \omega_0^2)}{e^{-\tau s}(s - a_0) - \sqrt{a_0^2 + \omega_0^2}} \begin{pmatrix} \frac{e^{-\tau s}(s-a_0) - \sqrt{a_0^2 + \omega_0^2}}{(s^2 + \omega_0^2)(s+a_0)} \\ \frac{1}{s+a_0} \end{pmatrix} =$$

$$\frac{\frac{e^{-\tau s}(s^2 + \omega_0^2)}{s+a_0}}{e^{-\tau s}(s - a_0) - \sqrt{a_0^2 + \omega_0^2}} \begin{pmatrix} \frac{e^{-\tau s}(s-a_0) - \sqrt{a_0^2 + \omega_0^2}}{(s^2 + \omega_0^2)} \\ 1 \end{pmatrix}.$$

In order to obtain the canonical parametrization, the function u should be normalized to unimodular.

The above example shows that, even in such a "bad" situation, when we have singular zeros in the plant, the function $u \in \mathbf{L}^\infty$ is still "nice", in fact, continuous and wno $u = 1$.
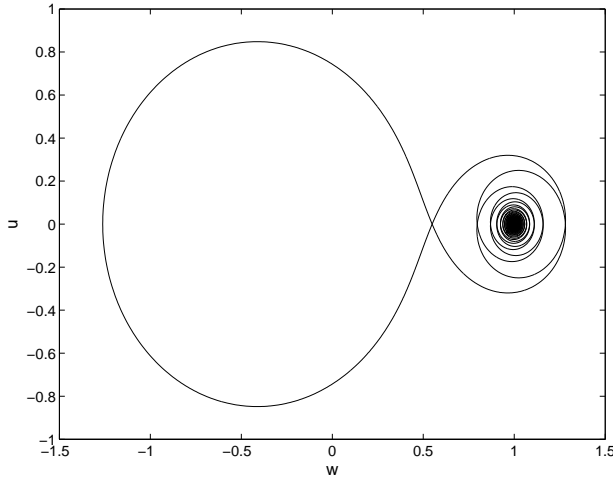
The optimal controller can be also obtained by the dual factorization as

$$K = \frac{s^2 + \omega_0^2}{e^{-\tau s}(s - a_0) - \sqrt{a_0^2 + \omega_0^2}}.$$

Recall that the optimization problem in [10] corresponds to robust stabilizability of the plant (2.12). The result of [10] is identical to our example modulo the small gain theorem. Indeed, in [10]

$$\|\Delta\| = \sqrt{1 + \frac{a^2 y_{ah}^2}{h^2}},$$

where y_{ah} is a unique root of the equation

$$\tan y + \frac{ay}{h} = 0$$

lying between $\pi/2$ and π.

For $\tau = h$, $\omega_0 \tau = y$, $a_0 = 1/a$, $\delta = \Delta/a$ the plant (2.12) give us the same system as (2.15) and we get the same result for the stability margin as in our case.

2.4   The classes of equivalences and the distance between the classes

Consider the dual parametrization

$$F + G\delta = \psi u, \ u \in \mathcal{U}, \psi \in \mathbf{H}^\infty.$$

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In this section, we will use the transposed plant factors compared to the previous sections and denote them by the same F and G.

Recall that for the optimal δ we have wno $u > 0$, i.e. on the boundary of the stability region, we have a non-stabilizable plant, the plant that has an unstable pole/zero cancelation. This gives an idea how to define the classes of equivalence in the factor space. Then we can introduce a distance between them.

Let $\mathcal{U}_0$ be the class of all unimodular functions with well-defined winding numbers given by

$$\mathcal{U}_0 = \{u \in \mathbf{L}^\infty : |u| \equiv 1, \ \text{wno}\, u = 0\}.$$

Suppose that we have two plants P_1 and P_2. Let $F_1 = \begin{bmatrix} N_1 \\ M_1 \end{bmatrix}$ be a coprime factorization of P_1 and $F_2 = \begin{bmatrix} N_2 \\ M_2 \end{bmatrix}$ be a coprime factorization of P_2. We will call two factorizations *equivalent*, $F_1 \sim F_2$, if we can find Q such that $Q \in \mathcal{U}_0$ and $F_2 = F_1 Q$.

We define the distance between equivalent classes as follows:

$$\delta(F_1, F_2) = \inf\{\|\delta\|_\infty : F_1 + G\delta \in \text{class}\,(F_2)\}.$$

Lemma 6. *δ is a metric on the factor space of the plant factorizations.*

Proof. Function $\delta(\cdot, \cdot)$ is a metric on the space of factorizations if, for all factorizations F_1, F_2 and F_3, we have

1. $\delta(F_1, F_2) \geq 0$, with $\delta(F_1, F_2) = 0$ if and only if $F_1 \sim F_2$;

2. $\delta(F_1, F_2) = \delta(F_2, F_1)$;

3. $\delta(F_1, F_2) \leq \delta(F_1, F_3) + \delta(F_3, F_2)$.

1. It is clear that $\delta(F_1, F_2) = \inf \|\delta\|_\infty \geq 0$. Next, if $F_1 \sim F_2$ then $F_1 \in \text{class}\,(F_2)$ and $\delta(F_1, F_2) = 0$.

If now $\delta(F_1, F_2) = 0$ then we can find $Q \in \mathcal{U}_0$ satisfying $F_1 + G\delta = F_2 Q$, where δ satisfies $\inf \|\delta\|_\infty = 0$. It means that $F_1 \in \text{class}\,(F_2)$ and $F_1 \sim F_2$.

To prove 2 we take δ such that $F_1 + G\delta = F_2 Q$. Then by multiplying the both sides with Q^* and we get $(F_1 + G\delta)Q^* = F_2 Q Q^* = F_2$ and

$$F_2 - G\delta Q^* = F_1 Q^* \ \Rightarrow \ F_2 - G\delta Q^* \in \text{class}\,(F_1),$$

since wno $Q^* = -$wno $Q = 0$. Finally we get

$$\delta(F_1, F_2) = \inf \|\delta\|_\infty = \inf \|\delta Q^*\|_\infty = \delta(F_2, F_1).$$

To prove 3 we take δ_{13} such that we can find Q_1 which yields $F_1 + G\delta_{13} = F_3 Q_1$. Take δ_{23} such that we can find Q_2 and $F_3 + G\delta_{23} = F_2 Q_2$. Then by straightforward calculations we get

$$F_1 + G\delta_{13} = F_2 Q_2 Q_1 - G\delta_{23} Q_1$$

$$F_1 + G(\delta_{13} + \delta_{23} Q_1) = F_2 Q_2 Q_1,$$

where wno $Q_2 Q_1 = $ wno $Q_2 + $ wno $Q_1 = 0$. It means that $F_1 \in$ class (F_2).

Then $\delta(F_1, F_2) \leq \|\delta_{13} + \delta_{23} Q\|_\infty \leq \|\delta_{13}\|_\infty + \|\delta_{23} Q\|_\infty = \|\delta_{13}\|_\infty + \|\delta_{23}\|_\infty$. By calculating infimum over δ_{13} and δ_{23} we get $\delta(F_1, F_2) \leq \delta(F_1, F_3) + \delta(F_3, F_2)$. $\qquad\square$

Now we can get an expression for our metric $\delta(F_1, F_2)$. By the definition of $\delta(F_1, F_2)$ we are looking for such a δ that $F_1 + G\delta \in$ class (F_2). It means that there exists $Q \in \mathcal{U}_0$ such that $F_1 + G\delta = F_2 Q$. We have to take an infimum over all such Q in order to find the optimal δ, i.e. the distance between equivalent classes. Then we get

$$\delta(F_1, F_2) = \inf_{Q \in \mathcal{U}_0} \|G^\dagger F_1 - G^\dagger F_2 Q\|_\infty,$$

where $G^\dagger$ is a pseudoinverse of G.

In [13] the ν-gap metric between two plants P_1 and P_2 was introduced as

$$\delta_\nu(P_1, P_2) = \inf_{\substack{Q, Q^{-1} \in \mathbf{L}_\infty \\ \text{wno} \det(Q) = 0}} \|G_1 - G_2 Q\|_\infty,$$

where G_1 and G_2 are the *normalized* coprime factorizations of P_1 respectively P_2.

In particular, if $G = I$ and F_1 is the normalized coprime factorization of the nominal plant P_1, it is straightforward to see that $\delta \geq \delta_\nu$. Furthermore, the distance from P_1 to the set of all non-stabilizable plants is the same ν_{opt} in both metrics. In all other cases, the ν-gap metric does not generally provide the optimal stability margin, whereas the δ-metric does. In this sense, the δ-metric is more appropriate to robust stabilization of parametrically uncertain plants.

2.5　Conclusion

In this chapter we studied the duality principle in the robust optimization problem of rank one. We extended the class of systems that can be considered. We discussed how the dual estimate of the stability margin can be computed using the canonical factorization. We illustrated the technique on the system with a pure delay.

We introduced a metric in the factor space of factorizations of the plant as a distance between the equivalent classes. It has been shown that this metric has close connection with the ν-gap metric.

Chapter 3

Regularization of the Limiting Optimal Controller in Robust Stabilization

3.1 Introduction

The convex duality principle has brought a new insight into robust stabilization. The dual problem for systems with uncertainties of rank one has been introduced in [3]. Several examples have been solved in [2] to illustrate the power and simplicity of the principle. A method based on unstable cancelations has been presented to calculate optimal stability margin and to design optimal controller. The great advantage of this method is the low order of the optimal controller. However the optimal controller is a limit of suboptimal robustly stabilizing controllers, which is never robustly stabilizing itself. Different *ad hoc* ideas have been used in [2] to obtain the suboptimal controller, for example by analyzing Bode/Nyquist plot, but it would be of great interest to derive a method that would work in general. In this chapter we give the method to design the suboptimal controller that provides the stability margin arbitrary close to the optimal one and still is of low order.

This chapter is organized as follows. In section 3.2 we describe the primal/dual problem and present some preliminary results. In section 3.3 we propose the algorithm to find the suboptimal controller. In order to illustrate how this algorithm works we solve two examples in sections 3.4 and 3.5.

3.2 Preliminaries

Let P be a nominal plant and let $\Delta \ni 0$ be an uncertainty set. Recall that the general robust controller design problem is to find a controller K that robustly stabilizes the whole family of perturbed plants

$$P_\delta = \frac{N + \delta^\top G_1}{M + \delta^\top G_2}, \ \delta \in \nu\Delta$$

for ν as large as possible. The problem of finding a controller is equivalent to the following condition in terms of a function $h \in \mathbf{H}^\infty$:

$$\mathrm{Re} \ (F + \delta^\top G(z))h(z) > 0, \ \forall z \in \mathbb{T}, \ \forall \delta \in \nu\Delta. \tag{3.1}$$

Here $F = \begin{pmatrix} N & M \end{pmatrix} \in \mathbf{A}^{1 \times n}$ is the coprime factorization of the plant P, the weight matrix $G = \begin{pmatrix} G_1 & G_2 \end{pmatrix} \in \mathbf{A}^{m \times n}$ and the set Δ is a convex compact set in $\mathbb{C}^m$. The problem of finding $h \in \mathbf{H}^\infty$ such that condition (3.1) holds

is considered as a primal problem. We would like to solve it for ν as large as possible, i.e., for

$$\nu_{opt} = \sup\{\nu | \exists h \in \mathbf{H}^\infty : \mathrm{Re}\,(F + \delta^\top G)h > 0\ \forall \delta \in \nu\Delta\}.$$

In [3] the dual problem was introduced in the following theorem.

Theorem 14 (Ghulchak, Rantzer). *Let $F \in \mathbf{A}^{1\times n}$, $G \in \mathbf{A}^{m\times n}$ and denote $\Phi_\delta = F + \delta^\top G$. Then the optimal value ν_{opt} takes the following form*

$$\nu_{opt} = \min\{\nu_{opt|c}, \nu_{opt|s}\},$$

with the regular part

$$\nu_{opt|c} = \inf\{\nu | \exists w \in \mathbf{L}^1(\mathbb{R}_+)\backslash 0, \delta \in \mathbf{L}_\infty(\nu\Delta) : \Phi_\delta w \in \mathbf{H}_0^1\} \qquad (3.2)$$

and the singular part

$$\nu_{opt|s} \;=\; \inf\{\nu | \exists z \in \mathbb{T},\ \delta \in \nu\Delta : \Phi_\delta(z) = 0\}. \qquad (3.3)$$

Suppose we solved the primal/dual problem, i.e. we found the maximal stability margin ν_{opt}. We know that we can determine the optimal controller using the alignment principle for convex optimization. According to the alignment principle (see [12]) we get

$$(N + \delta_{opt}^\top G_1)h_1 - (M + \delta_{opt}^\top G_2)h_2 \equiv 0,$$

and the optimal controller

$$h_{opt} = \frac{h_1}{h_2} = \frac{M + \delta_{opt}^\top G_2}{N + \delta_{opt}^\top G_1}. \qquad (3.4)$$

It means that the optimal controller is equal to the inverted plant with the worst uncertainty.

In Chapter 2 it was proposed the dual parametrization of the plant factors with destabilizing uncertainties. According to this parametrization *the numerator and denominator of the plant with the worst uncertainty have more common unstable zeros than all unstable poles.* Recall that the plant with the worst uncertainty becomes

$$\frac{N(z) + \delta(z)^\top G_1(z)}{M(z) + \delta(z)^\top G_2(z)} = \frac{u(z)\psi_1(z)}{u(z)\psi_2(z)}. \qquad (3.5)$$

The occurrence of common unstable zeros leads to a zero/pole cancelation when the optimal controller will be designed according to (3.4). All common unstable zeros will be canceled in the ratio (3.4) and the optimal controller will have a low order. But the optimal controller will not be robustly stabilizing.

Now our task will be to find a suboptimal controller h_ε such that

$$\text{Re } (F + \delta^\top G(z))h_\varepsilon(z) > 0$$

and h_ε provides robust stability for $\nu_{opt} - \varepsilon$. Preserving the low order property of the optimal controller we will construct a suboptimal controller that will be robustly stabilizing.

3.3   The main result

We begin by increasing the stability area.

Consider $F, G \in \mathbf{A}$ on the unit ball $\mathcal{B}$. Suppose that F and $G \in \mathbf{A}$ on the bigger area $\mathcal{B}_\varepsilon$. By changing the variable z to $w = (1 - \varepsilon)z$ we get new F_ε and G_ε. We will solve the primal/dual problem for $F_\varepsilon + \delta^\top G_\varepsilon$ and obtain the maximal stability margin ν_ε and optimal controller h_ε. As we use the duality principle and we use the dual parametrization of the plant factors with destabilizing uncertainties, then we get the suboptimal controller of low order. To show that the suboptimal controller provides the stability margin close to the optimal one it is enough to show that $\nu_\varepsilon \to \nu_{opt}$ if $\varepsilon \to 0$. We have the following theorem.

Theorem 15. $\nu_\varepsilon \to \nu_{opt}$ *if* $\varepsilon \to 0$. *Moreover* h_ε *stabilizes the system* $F + \delta^\top G$.

Proof. Denote $\mathbf{H}_\infty^+ = \mathbf{H}_\infty(\mathcal{B}_\varepsilon)$ and let $\mathbb{T}^+$ denote the boundary of $\mathcal{B}_\varepsilon$. Using the change of the variable as before: $w = (1 - \varepsilon)z$, and the assumption that F and G are analytic functions in bigger area $\mathcal{B}_\varepsilon$, then the solution to the primal problem with F_ε and G_ε for $h \in \mathbf{H}_\infty$ and $z \in \mathbb{T}$ is the same as the solution to the primal problem with F and G for $h \in \mathbf{H}_\infty^+$ and $z \in \mathbb{T}^+$, i.e.

$$\sup_{h \in \mathbf{H}_\infty^+} \inf_{\delta \in \Delta_\nu} \inf_{z \in \mathbb{T}^+} \text{Re} \, (F + \delta^\top G)h = \sup_{h \in \mathbf{H}_\infty} \inf_{\delta \in \Delta_\nu} \inf_{z \in \mathbb{T}} \text{Re} \, (F_\varepsilon + \delta^\top G_\varepsilon)h.$$

According to the mean value theorem for harmonic functions we get

$$\inf_{z \in \mathbb{T}} \text{Re} \, (F + \delta^\top G)h \geq \inf_{z \in \mathbb{T}^+} \text{Re} \, (F + \delta^\top G)h.$$

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As $\mathbf{H}_\infty^+ \subset \mathbf{H}_\infty$, we get

$$\sup_{h\in\mathbf{H}_\infty} \inf_{\delta\in\Delta_\nu} \inf_{z\in\mathbb{T}} \operatorname{Re}(F + \delta^\top G)h \geq \sup_{h\in\mathbf{H}_\infty^+} \inf_{\delta\in\Delta_\nu} \inf_{z\in\mathbb{T}^+} \operatorname{Re}(F + \delta^\top G)h$$

and

$$\sup_{h\in\mathbf{H}_\infty} \inf_{\delta\in\Delta_\nu} \inf_{z\in\mathbb{T}} \operatorname{Re}(F + \delta^\top G)h \geq \sup_{h\in\mathbf{H}_\infty} \inf_{\delta\in\Delta_\nu} \inf_{z\in\mathbb{T}} \operatorname{Re}(F_\varepsilon + \delta^\top G_\varepsilon)h. \qquad (3.6)$$

We call the problem

$$\sup_{h\in\mathbf{H}_\infty} \inf_{\delta\in\Delta_\nu} \inf_{z\in\mathbb{T}} \operatorname{Re}(F + \delta^\top G)h > 0 \qquad (3.7)$$

the first problem and the problem

$$\sup_{h\in\mathbf{H}_\infty} \inf_{\delta\in\Delta_\nu} \inf_{z\in\mathbb{T}} \operatorname{Re}(F_\varepsilon + \delta^\top G_\varepsilon)h > 0 \qquad (3.8)$$

the second problem.

In order to show the first part of the theorem we choose ν such that the second problem (3.8) has a solution. By ν_ε we denote the optimal stability margin in the second problem, i.e.

$$\operatorname{Re}(F_\varepsilon + \delta_{opt}^\top G_\varepsilon)h_{opt} = 0,$$

where $|\delta_{opt}| = \nu_\varepsilon$. It's clear that $\nu_\varepsilon > \nu$. But if the second problem has a solution for ν, then according to the inequality (3.6), the first problem (3.7) has a solution too and $\nu_{opt} > \nu$. We get that

$$\nu_\varepsilon > \nu \Rightarrow \nu_{opt} > \nu$$

and it means that $\nu_{opt} \geq \nu_\varepsilon$.

Now we will show that there is ε_ν such that $\nu_\varepsilon \geq \nu_{opt} - \varepsilon_\nu$.

$$\operatorname{Re}(F_\varepsilon + \delta^\top G_\varepsilon)h = \operatorname{Re}(F_\varepsilon - F + F + \delta^\top(G_\varepsilon - G + G))h =$$
$$\operatorname{Re}(F + \delta^\top G)h + \operatorname{Re}(F_\varepsilon - F + \delta^\top(G_\varepsilon - G))h.$$

Using the properties of sup and inf we get that

$$\inf_{z\in\mathbb{T}} \operatorname{Re}(F_\varepsilon + \delta^\top G_\varepsilon)h \geq \inf_{z\in\mathbb{T}} \operatorname{Re}(F + \delta^\top G)h - \sup_{z\in\mathbb{T}} |\operatorname{Re}(F_\varepsilon - F + \delta^\top(G_\varepsilon - G))h| \geq$$
$$\inf_{z\in\mathbb{T}} \operatorname{Re}(F + \delta^\top G)h - \sup_{z\in\mathbb{T}} |\operatorname{Re}(F_\varepsilon - F)h| - \sup_{z\in\mathbb{T}} |\operatorname{Re}(\delta^\top(G_\varepsilon - G))h| \geq$$
$$\inf_{z\in\mathbb{T}} \operatorname{Re}(F + \delta^\top G)h - \|F_\varepsilon - F\|_\infty \|h\|_\infty - |\delta| \|G_\varepsilon - G\|_\infty \|h\|_\infty$$

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and

$$\sup_{h\in\mathbf{H}_\infty}\inf_{\delta\in\Delta_\nu}\inf_{z\in\mathbb{T}}\operatorname{Re}(F_\varepsilon+\delta^\top G_\varepsilon)h \geq$$

$$\sup_{h\in\mathbf{H}_\infty}(\inf_{\delta\in\Delta_\nu}\inf_{z\in\mathbb{T}}\operatorname{Re}(F+\delta^\top G)h-\|F_\varepsilon-F\|_\infty\|h\|_\infty-\sup_{\delta\in\Delta_\nu}|\delta|\|G_\varepsilon-G\|_\infty\|h\|_\infty).$$

Since F and G are continuous then $\|F_\varepsilon-F\|_\infty\to 0$ and $\|G_\varepsilon-G\|_\infty\to 0$ as $\varepsilon\to 0$. Then $\sup_{\delta\in\Delta_\nu}|\delta|$ is bounded. It means that

$$\sup_{h\in\mathbf{H}_\infty}\inf_{\delta\in\Delta_\nu}\inf_{z\in\mathbb{T}}\operatorname{Re}(F_\varepsilon+\delta^\top G_\varepsilon)h \geq \sup_{h\in\mathbf{H}_\infty}\inf_{\delta\in\Delta_\nu}\inf_{z\in\mathbb{T}}\operatorname{Re}(F+\delta^\top G)h-\varepsilon_1.$$

We take ν such that the first problem (3.7) has a solution. Then there exists ε_2 such that

$$\sup_{h\in\mathbf{H}_\infty}\inf_{\delta\in\Delta_\nu}\inf_{z\in\mathbb{T}}\operatorname{Re}(F+\delta^\top G)h-\varepsilon_2 > 0,$$

i.e. the second problem (3.8) has a solution too if ε is sufficiently close to 0. It means that $\exists\varepsilon$ such that $\nu_\varepsilon > \nu$.

We take now ν close to ν_{opt}, i.e. $\nu = \nu_{opt}-\varepsilon_\nu$ and repeat the above argumentation. We get

$$\forall\varepsilon_\nu > 0\ \exists\varepsilon : \nu_\varepsilon > \nu = \nu_{opt}-\varepsilon_\nu.$$

We have shown that there exists ε such that $\nu_{opt}-\varepsilon_\nu < \nu_\varepsilon \leq \nu_{opt}$. It means that if $\varepsilon_\nu \to 0$, then $\nu_\varepsilon \to \nu_{opt}$. It is clear that ε has to be sufficiently small.

Now we will show that h_ε stabilizes the system $F+\delta^\top G$.

Recall the inequality (3.6),

$$\sup_{h\in\mathbf{H}_\infty}\inf_{\delta\in\Delta_\nu}\inf_{z\in\mathbb{T}}\operatorname{Re}(F+\delta^\top G)h \geq \sup_{h\in\mathbf{H}_\infty}\inf_{\delta\in\Delta_\nu}\inf_{z\in\mathbb{T}}\operatorname{Re}(F_\varepsilon+\delta^\top G_\varepsilon)h.$$

It is clear that if h stabilizes the second system for a fix δ, then the same h will stabilize the first system for the same δ. It means that if we have found the optimal controller $h_{opt,\varepsilon}$ that stabilizes the second system for all δ such that $|\delta| \leq \nu_\varepsilon$, then the same controller will stabilize the first system for the same δ. We have

$$\operatorname{Re}(F_\varepsilon(w)+\delta^\top G_\varepsilon(w))h_{opt,\varepsilon}(w) = \operatorname{Re}(F(\frac{w}{1-\varepsilon})+\delta^\top G(\frac{w}{1-\varepsilon}))h_{opt,\varepsilon}(w).$$

Now we change the variable back $z = \frac{w}{1-\varepsilon}$ and we get

$$\operatorname{Re}(F(\frac{w}{1-\varepsilon})+\delta^\top G(\frac{w}{1-\varepsilon}))h_{opt,\varepsilon}(w) = \operatorname{Re}(F(z)+\delta^\top G(z))h_\varepsilon(z),$$

where $h_\varepsilon = h_{opt,\varepsilon}((1 - \varepsilon)z)$ is the suboptimal controller that stabilizes the system $F + \delta^\top G$ with the stability margin $\nu_\varepsilon > \nu_{opt} - \varepsilon_\nu$, i.e. arbitrary close to the optimal one. $\qquad\square$

According to the method we have described above we can now propose an algorithm to find the suboptimal controller. Usually the system we consider is defined in the right half plane and we will begin with the conformal bilinear transformation of the right half complex plane onto the unit disk.

1. Transform the right half plane onto unit disk by changing the variable $z = \frac{1-s}{1+s}$.

2. Increase the stability area to B_ε by changing the variable $w = (1 - \varepsilon)z$.

3. Transform the stability area B_ε onto the right half plane by changing the variable $s_{new} = \frac{1-w}{1+w}$.

4. Solve the primal/dual problem for $F_\varepsilon + \delta^\top G_\varepsilon$ and obtain the maximal stability margin ν_ε and the optimal controller h.

5. Change back to the original stability area by changing the variable $s = \frac{((2-\varepsilon)s_{new}+\varepsilon)}{(2-\varepsilon+\varepsilon s_{new})}$ and obtain the suboptimal controller h_ε.

To illustrate how this algorithm works we will solve two examples: the first one is the robust stabilizability of the plant $P_\delta = \frac{s-\delta}{(s-1)^2}$ and the second one is the gain margin optimization for the plant $P_\delta = \delta \frac{s-1}{(s+1)(s-2)}$.

3.4 Design of suboptimal controller to $P_\delta = \frac{s-\delta}{(s-1)^2}$.

The robust stabilizability of $P_\delta = \frac{s-\delta}{(s-1)^2}$ on the uncertainty set $|\delta| \leq \nu$ was solved in [2]. The stability margin ν_{opt} was calculated and the optimal controller that achieves the optimal level of stability was designed. However the controller was not proper. Now we will show how we can design a robustly stabilizing low order suboptimal controller that provides the stability margin as close to the optimal one as we wish. We will consider the case of complex δ.

According to the proposed algorithm we start with three changes of variables and obtain the following system:

$$G_{\delta,\varepsilon} = \frac{\frac{(2-\varepsilon)s-\varepsilon}{2-\varepsilon-\varepsilon s} - \delta}{\frac{4(s-1)^2}{(2-\varepsilon-\varepsilon s)^2}}$$

$$= \frac{((2-\varepsilon)s-\varepsilon)(2-\varepsilon-\varepsilon s) - \delta(2-\varepsilon-\varepsilon s)^2}{4(s-1)^2}.$$

We will solve the problem to calculate the maximal stability margin ν_{opt} and to design the optimal controller for $G_{\delta,\varepsilon}$. We use the duality principle. Recall that according to this principle the numerator and the denominator of the plant with the worst uncertainty have more common unstable zeros than all unstable poles.

The denominator of $G_{\delta,\varepsilon}$ has double zero at $s = 1$. It means that the number of possible common unstable zeros cannot exceed two, and the numerator and the denominator can have at most one unstable pole which must be a pole of δ. Apart from this unstable pole, δ can have a double pole at $s = \frac{2-\varepsilon}{\varepsilon}$. Summarizing the duality principle we get $\nu_{opt} = \inf \|\delta\|_\infty$ over all δ that have three unstable poles (two of them are known) and we provide the plant with two unstable cancelations.

We can choose

$$\delta(s) = \frac{\gamma s^3 + bs^2 + cs + d}{(2-\varepsilon-\varepsilon s)^2(s-a)}, \; a > 0. \tag{3.9}$$

We have to choose δ proper, otherwise $\|\delta\|_\infty$ will be unbounded. By a we denote the unstable pole that the plant $G_{\varepsilon,\delta}$ can have. Recall that it has to be the only one unstable pole. We put δ in the plants equation and we get the numerator

$$\frac{s^3(-\varepsilon(2-\varepsilon)-\gamma) + s^2(a\varepsilon(2-\varepsilon)+(2-\varepsilon)^2+\varepsilon^2-b)}{(2-\varepsilon-\varepsilon s)^2(s-a)}+$$

$$\frac{s(-a(2-\varepsilon)^2-a\varepsilon^2-\varepsilon(2-\varepsilon)-c)+(a\varepsilon(2-\varepsilon)-d)}{(2-\varepsilon-\varepsilon s)^2(s-a)}=$$

$$\frac{(s-1)^2(\beta s + k)}{(2-\varepsilon-\varepsilon s)^2(s-a)},$$

where the last equality comes from the necessity to have double zero at 1. Then

$$G_{\delta,\varepsilon} = \frac{(\beta s + k)}{4(s-a)}.$$

The suboptimal controller will have the order one.

In order to design the controller we have to find 8 variables. We have 4 equations due to the unstable cancelation condition (3.5). Another four equations we get using the property of δ. We know (see [8]), that in the complex case, δ_{opt} will be an all-pass function with $|\delta(j\omega)| = \nu_{opt}$. Now we are able to find δ_{opt}.

Since calculations are quite "heavy", we will skip them here and we only present the results for different ε.

For $\varepsilon = 0.1$ we get $a = 2.606$ and $\nu_{opt} = 0.361$. The suboptimal controller is

$$h_\varepsilon = \frac{(s - 2.959)}{0.005s + 1.124}.$$

Now take $\varepsilon = 0.01$. For this ε we found $a = 2.431$ and $\nu_{opt} = 0.409$. The suboptimal controller is

$$h_\varepsilon = \frac{(s - 2.728)}{0.048s + 5.033}.$$

The last value we will take is $\varepsilon = 0.001$. Then we get $a = 2.416$ and $\nu_{opt} = 0.4137$. The suboptimal controller will be

$$h_\varepsilon = \frac{(s - 2.707)}{0.023s + 4.569}.$$

Note that all suboptimal controllers are robustly stabilizing and have the order one.

In [2] it was calculated the maximal stability margin and found the optimal controller that achieves this level of stability. The unstable pole in the numerator is $a = 1 + \sqrt{2}$, the stability margin is $\nu_{opt} = \sqrt{2} - 1 \approx 0.414$ and the optimal controller is $K_{opt} = s - a$. If we compare our results with the optimal one we see that in case $\varepsilon = 0.1$ the suboptimal controller provides robust stability for $|\delta| \leq 0.361 \approx \nu_{opt} - 0.053$, if $\varepsilon = 0.01$ then the suboptimal controller achieves robust stability for $|\delta| \leq 0.409 \approx \nu_{opt} - 0.005$ and, finally if $\varepsilon = 0.001$ then the suboptimal controller provides robust stability for $|\delta| \leq 0.4137 \approx \nu_{opt} - 0.0003$. Thus we can find the first order suboptimal controller that provides the stability margin arbitrary close to the optimal one.

3.5 The gain margin optimization.

In this section we consider the gain margin problem for the plant

$$G_\delta(s) = \delta G(s) = \delta \frac{s-1}{(s+1)(s-2)}. \tag{3.10}$$

In [2] it was calculated the best possible gain k ($k_{opt} = 4$) such that the uncertain plant (3.10) is robustly stabilizable on the uncertainty set $1 \leq \delta \leq k$ and it was found the second order optimal controller that provides the gain margin. It was also proposed the suboptimal controller which provides robust stability of all gains $\delta \in [1, 3.9]$. Now we will show how we can design the suboptimal controller for all suboptimal levels $k \leq k_{subopt}$ with k_{subopt} arbitrary close to k_{opt}.

As before we start with the three changes of variables and obtain the following system:

$$G_{\delta,\varepsilon} = \delta \frac{(s-1)(2-\varepsilon-\varepsilon s)}{(1-\varepsilon)(s+1)((2+\varepsilon)s - 4 + \varepsilon)}. \tag{3.11}$$

The denominator of (3.11) has one unstable zero at $s = \frac{4-\varepsilon}{2+\varepsilon}$. It means that the numerator can not have any unstable poles at all and $\delta(s-1)(2-\varepsilon-\varepsilon s)$ is be an analytical function with zero at $s = \frac{4-\varepsilon}{2+\varepsilon}$. It implies that function δ must contain the unstable factors $s-1$ and $2-\varepsilon-\varepsilon s$ in the denominator and the factor $(2+\varepsilon)s - (4-\varepsilon)$ in the numerator. Moreover δ must be real on the imaginary axis and proper. We have only one possibility for δ :

$$\delta(s) = \gamma \frac{((2+\varepsilon)^2 s^2 - (4-\varepsilon)^2)(b^2 - s^2)}{(s^2-1)((2-\varepsilon)^2 - \varepsilon^2 s^2)}$$

for some constant γ. Then

$$G_{\delta,\varepsilon} = \delta G = \gamma \frac{((2+\varepsilon)s + (4-\varepsilon))(b^2 - s^2)}{(s+1)^2(1-\varepsilon)((2-\varepsilon) + \varepsilon s)}.$$

The suboptimal controller is of the third order.

On the imaginary axis the values of the function should lie in the interval $[1, k]$,

$$\delta(j\omega) = \gamma \frac{((2+\varepsilon)^2 \omega^2 + (4-\varepsilon)^2)(b^2 + \omega^2)}{(\omega^2+1)((2-\varepsilon)^2 - \varepsilon^2 \omega^2)} \in [1, k].$$

We solve the problem for different small $\varepsilon > 0$.

We take first $\varepsilon = 0.1$. The optimal k_ε is 2.7375 ($\approx k_{opt} - 1.26$) which corresponds to $\gamma = 0.00621$. The suboptimal controller is

$$h_\varepsilon = \frac{1.2685(0.105s + 1)(s + 1)^2}{(0.1496s + 1)(s + 1.74)(1 - 0.045s)}.$$

For $\varepsilon = 0.01$ we get that $k_\varepsilon = 3.8272$ ($\approx k_{opt} - 0.17$) which corresponds to $\gamma = 0.000095$. The suboptimal controller is

$$h_\varepsilon = \frac{1.0297(0.01005s + 1)(s + 1)^2}{(0.015s + 1)(s + 1.970)(1 - 0.005s)}.$$

Finally for $\varepsilon = 0.001$ we will come very close to k_{opt} which is 4. Now we calculate $k_\varepsilon = 3.9821$
($\approx k_{opt} - 0.018$) which corresponds to $\gamma = 0.99452 \cdot 10^{-6}$. The suboptimal controller is

$$h_\varepsilon = \frac{1.003(0.0010s + 1)(s + 1)^2}{(0.0015s + 1)(s + 1.997)(1 - 0.0005s)}.$$

Note that we have designed the low-order suboptimal controller. We can obtain k_ε very close to the optimal k and still have the controller of low order.

3.6 Conclusion

In this chapter we have proposed the algorithm to construct the suboptimal controller that robustly stabilizes the system with rank one uncertainty. We have shown that the suboptimal controller provides the stability margin arbitrary close to the optimal one, whereas the low order of the controller is preserved.

Chapter 4

Robust Control via Linear Programming

4.1 Introduction

Large number of analysis and synthesis problems in robust control were stated in terms of convex optimization. In particular, in [5] it was shown that the robust stabilization problem under the parametric uncertainties has convex formulation if the characteristic polynomial depends linearly on the uncertainty parameters.

Consider the uncertainty as an artificial feedback loop

$$
G_\delta = \left\{ \begin{array}{c} \left(\begin{array}{c} y \\ z \end{array} \right) = G \left(\begin{array}{c} w \\ u \end{array} \right) \\ w = \nu \delta^\top z, \end{array} \right.
$$

where G is the nominal plant, w is the scalar input and δ is the uncertain vector. The objective is to robustly stabilize the plant for all δ satisfying the norm bound $|\delta|_{pr} \leq 1$, where $|\cdot|_{pr}$ stands for a vector norm in $\mathbb{R}^m$ (*pr* for *primary*).

All closed-loop transfer functions from w to z are of the form $T_{zw} = T_1 + T_2 Q$, where Q is stable and T_1, T_2 are determined by G. The condition for robust stability becomes

$$
[1 + \nu \delta^\top (T_1 + T_2 Q)]^{-1} \in \mathbf{RH}^\infty, \forall \delta : |\delta|_{pr} \leq 1.
$$

A convex parametrization of all robustly stabilizing controllers was constructed in [5]. We slightly modify the result and state the theorem as follows.

Theorem 16. *Suppose $T_1 \in \mathbf{RH}^\infty_{m \times 1}$, $T_2 \in \mathbf{RH}^\infty_{m \times n}$. Then the following two conditions on the rational matrix Q are equivalent:*

1. $Q \in \mathbf{RH}^\infty_{n \times 1}$ and for all $\delta \in \mathbb{R}^m$ with $|\delta|_{pr} \leq 1$

$$
[1 + \nu \delta^\top (T_1 + T_2 Q)]^{-1} \in \mathbf{RH}^\infty.
$$

2. There exist $\alpha \in \mathbf{RH}^\infty$ and $\beta \in \mathbf{RH}^\infty$ such that $Q = \beta / \alpha$ and $\forall \omega \in \mathbb{R} \cup \{\infty\}$

$$
|Re[T_1 \alpha + T_2 \beta](j\omega)|_{du} < \nu^{-1} Re\, \alpha(j\omega).
$$

Here the dual norm $|\cdot|_{du}$ defined as $|x|_{du} = \sup\{\delta^\top x : |\delta|_{pr} \leq 1.\}$ (du for dual)

In [3] it was shown that we can allow the large set $\mathbf{H}^\infty$, i.e. $\alpha, \beta \in \mathbf{H}^\infty$. Due to Theorem 16 and the result from [3] the problem of finding a controller is equivalent to the following condition in terms of a function $h \in \mathbf{H}^\infty$,

$$Re\,(F + \nu\delta^\top G(z))h(z) > 0, \ \forall z \in \mathbb{T}, \ \forall \delta \in \Delta. \tag{4.1}$$

In [4] it has been developed the algorithm that solves the problem (4.1) for the maximal possible ν. The algorithm is the combination of two finite-dimensional approximations of the primal and dual infinite-dimensional problems. In this chapter we improve the algorithm. We propose how to choose the next step in the optimization of the uncertainty bound. We discuss the case when the uncertainty is real-valued and we obtain the finite dimensional solution of the problem (4.1).

The chapter is organized as follows. The dual form of the problem is presented in section 4.2. In section 4.3 we describe the primal convex programming algorithm in case when the uncertainty norm bound ν is given. In section 4.4 we discuss the special case, when the uncertainty vector is real-valued. The numerical example is considered in section 4.5.

4.2   Preliminaries

The problem of finding $h \in \mathbf{H}^\infty$ such that condition (4.1) holds is considered as a *primal problem*. We would like to solve the problem for ν as large as possible.

Denote

$$\gamma_{opt}(\nu) = \sup_{h \in \mathcal{B}\mathbf{H}^\infty} ess \inf_{z \in \mathbb{T}} \inf_{\delta \in \Delta} \mathrm{Re}\,(F(z) + \nu\delta^\top G(z))h(z).$$

We choose the unit ball in $\mathbf{H}^\infty$ as an optimization set since the function

$$\mathrm{Re}\,(F(z) + \nu\delta^\top G(z))h(z)$$

depends linearly on h and then any bounded set containing the origin as an interior point can be chosen. However, the unit ball has an easier interpretation from the classical results and is more appropriate for our task.

Note that

$$\inf_{\delta \in \Delta} \mathrm{Re}\,(F(z) + \nu\delta^\top G(z))h(z) = \mathrm{Re}\,F(z)h(z) - \nu \sup_{\delta \in -\Delta} \mathrm{Re}\,\delta^\top G(z)h(z).$$

In [1] it was shown that for a convex compact Δ
$\sup_{\delta \in \Delta} \operatorname{Re} \delta^\top G(z) h(z) \geq 0$ for all z. It means that instead of the condition $|h| \leq 1$ we can demand that $|Fh| \leq 1$.

As $\operatorname{Re} Fh > 0$, then F has a right inverse. The disk algebra $\mathbf{A}$ is the Hermite ring [16]. It means that if F has a right inverse then F can be complemented to a unimodular matrix $U \in \mathbf{A}$ containing F as a submatrix
$$U = \begin{pmatrix} F \\ B \end{pmatrix}.$$

We denote
$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} F \\ B \end{pmatrix} h.$$

Then $h = U^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ and $Gh = GU^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. We partition the matrix

$GU^{-1} = \begin{pmatrix} G_1 & G_2 \end{pmatrix}$. Then $Gh = \begin{pmatrix} G_1 & G_2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = G_1 \alpha + G_2 \beta$ and

$$\gamma_{opt}(\nu) = \sup_{\alpha \in \mathcal{B}\mathbf{H}^\infty, \beta \in \mathbf{H}^\infty} ess \inf_{z \in \mathbb{T}} \inf_{\delta \in \Delta} \operatorname{Re}\left(\alpha(z) + \nu \delta^\top (G_1(z)\alpha(z) + G_2(z)\beta(z))\right).$$

Now we present the main result on duality.

Theorem 17. *We have*

$$\gamma_{opt}(\nu) = \inf_{\substack{w \in \mathcal{S}\mathbf{L}^1(\mathbb{R}_+) \\ \delta \in \mathbf{L}^\infty(\Delta) \\ \delta^\top G_2 w \in \mathbf{H}_0^1}} \inf_{p \in \mathbf{H}_0^1} \|(1 + \nu \delta^\top G_1)w - p\|_1.$$

Proof. In [3] it was shown that

$$ess \inf_{z \in \mathbb{T}} \inf_{\delta \in \Delta} \varphi_\delta(z) = \inf_{w \in \mathcal{S}\mathbf{L}^1(\mathbb{R}_+)} \inf_{\delta \in \mathbf{L}^\infty(\Delta)} \int_{\mathbb{T}} \varphi_\delta(z) w \, dm,$$

where $\varphi_\delta(z) = \operatorname{Re}\left(\alpha(z) + \nu \delta^\top (G_1(z)\alpha(z) + G_2(z)\beta(z))\right)$. Thus

$$\sup_{\alpha \in \mathcal{B}\mathbf{H}^\infty, \beta \in \mathbf{H}^\infty} ess \inf_{z \in \mathbb{T}} \inf_{\delta \in \Delta} \operatorname{Re}\left(\alpha(z) + \nu \delta^\top (G_1(z)\alpha(z) + G_2(z)\beta(z))\right) =$$

$$\sup_{\alpha \in \mathcal{B}\mathbf{H}^\infty, \beta \in \mathbf{H}^\infty} \inf_{w \in \mathcal{S}\mathbf{L}^1(\mathbb{R}_+)} \inf_{\delta \in \Delta} \operatorname{Re} \int_{\mathbb{T}} (\alpha + \nu \delta^\top (G_1\alpha + G_2\beta)) w \, dm.$$

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As Δ is compact, then $\inf_{\delta\in\Delta}$ is achieved on Δ and we can assume that δ is a bounded function, i.e. $\delta \in \mathbf{L}^\infty(\Delta)$. Thus

$$\inf_{\delta\in\Delta} \int_{\mathbb{T}} (\alpha(z) + \nu\delta^\top(G_1(z)\alpha(z) + G_2(z)\beta(z)))w(z)dm =$$

$$\inf_{\delta\in\mathbf{L}^\infty(\Delta)} \int_{\mathbb{T}} (\alpha(z) + \nu\delta^\top(z)(G_1(z)\alpha(z) + G_2(z)\beta(z)))w(z)dm.$$

Now we introduce two measures μ and ξ defined as follows:

$$\mu(E) = \int_E wdm$$

$$\xi(E) = \Delta\mu(E) = \int_E \delta^\top wdm = \int_E \delta^\top d\mu.$$

Then

$$\sup_{\alpha\in\mathcal{B}\mathbf{H}^\infty,\beta\in\mathbf{H}^\infty} \inf_{w\in S\mathbf{L}^1(\mathbb{R}_+)} \inf_{\delta\in\mathbf{L}^\infty(\Delta)} \mathrm{Re} \int_{\mathbb{T}} (\alpha + \nu\delta^\top(G_1\alpha + G_2\beta))wdm =$$

$$\sup_{\alpha\in\mathcal{B}\mathbf{H}^\infty,\beta\in\mathbf{H}^\infty} \inf_{\begin{pmatrix}\mu\\\xi\end{pmatrix}\in M} \mathrm{Re} \int_{\mathbb{T}} (\alpha d\mu + \nu d\xi(G_1\alpha + G_2\beta)),$$

where M is the space of measures.

The set $(\mathcal{B}\mathbf{H}^\infty, \mathbf{H}^\infty)$ is convex.

The set $\begin{pmatrix}\mu\\\xi\end{pmatrix}$ is convex and weakly compact since Δ is a convex compact set and the set

$$\mu(A) = \int_A w(z)dm(z)$$

is weakly compact.

The function $\mathrm{Re} \int_{\mathbb{T}}(\alpha d\mu + \nu d\xi(G_1\alpha + G_2\beta))$ is concave on $h = \begin{pmatrix}\alpha\\\beta\end{pmatrix}$, convex and continuous on $\begin{pmatrix}\mu\\\xi\end{pmatrix}$.

By the Ky Fan's min-max theorem (see [9]), the order of sup and inf can

be interchanged.

$$\sup_{\alpha \in \mathcal{B}\mathbf{H}^\infty, \beta \in \mathbf{H}^\infty} \inf_{\left(\begin{array}{c} \mu \\ \xi \end{array}\right) \in M} \mathrm{Re} \int_{\mathbb{T}} (\alpha d\mu + \nu d\xi (G_1\alpha + G_2\beta)) =$$

$$\inf_{\left(\begin{array}{c} \mu \\ \xi \end{array}\right) \in M} \sup_{\alpha \in \mathcal{B}\mathbf{H}^\infty, \beta \in \mathbf{H}^\infty} \mathrm{Re} \int_{\mathbb{T}} (\alpha d\mu + \nu d\xi (G_1\alpha + G_2\beta)).$$

We denote $df_1(\mu, \xi, \nu) = d\mu + \nu d\xi G_1 = (1 + \nu \delta^\top G_1) w dm$ and $df_2(\xi) = d\xi G_2 = \delta^\top G_2 w dm$. Then

$$\inf_{\left(\begin{array}{c} \mu \\ \xi \end{array}\right) \in M} \sup_{\alpha \in \mathcal{B}\mathbf{H}^\infty, \beta \in \mathbf{H}^\infty} \mathrm{Re} \int_{\mathbb{T}} (\alpha d\mu + \nu d\xi (G_1\alpha + G_2\beta)) =$$

$$\inf_{\left(\begin{array}{c} \mu \\ \xi \end{array}\right) \in M} (\sup_{\alpha \in \mathcal{B}\mathbf{H}^\infty} \mathrm{Re} \int_{\mathbb{T}} df_1(\mu, \xi, \nu)\alpha + \nu \sup_{\beta \in \mathbf{H}^\infty} \mathrm{Re} \int_{\mathbb{T}} df_2(\xi)\beta).$$

Note that

$$\sup_{\alpha \in \mathcal{B}\mathbf{H}^\infty} \mathrm{Re} \int_{\mathbb{T}} df_1(\mu, \xi, \nu)\alpha = |\int_{\mathbb{T}} df_1(\mu, \xi, \nu)\alpha| \text{ and}$$

$$\sup_{\beta \in \mathbf{H}^\infty} \mathrm{Re} \int_{\mathbb{T}} df_2(\xi)\beta = |\int_{\mathbb{T}} df_2(\xi)\beta|.$$

Indeed,

$$|\int_{\mathbb{T}} fh \ dm| = \int_{\mathbb{T}} fhu \ dm = \mathrm{Re} \int_{\mathbb{T}} fhu \ dm,$$

where $u = \exp(-i \arg(\int_{\mathbb{T}} fh \ dm))$. If $h \in \mathbf{H}^\infty$ then $hu \in \mathbf{H}^\infty$ and we have

$$\sup_{h \in \mathbf{H}^\infty} |\int_{\mathbb{T}} fh \ dm| = \sup_{h \in \mathbf{H}^\infty} \mathrm{Re} \int_{\mathbb{T}} fh \ dm.$$

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Thus

$$\inf_{\left(\begin{array}{c}\mu\\\xi\end{array}\right)\in M}(\sup_{\alpha\in\mathcal{B}\mathbf{H}^\infty}\mathrm{Re}\int_\mathbb{T}df_1(\mu,\xi,\nu)\alpha+\nu\sup_{\beta\in\mathbf{H}^\infty}\mathrm{Re}\int_\mathbb{T}df_2(\xi)\beta)=$$

$$\inf_{\left(\begin{array}{c}\mu\\\xi\end{array}\right)\in M}(\sup_{\alpha\in\mathcal{B}\mathbf{H}^\infty}|\int_\mathbb{T}df_1(\mu,\xi,\nu)\alpha|+\nu\sup_{\beta\in\mathbf{H}^\infty}|\int_\mathbb{T}df_2(\xi)\beta|).$$

Recall that $df_2 = \delta^\top G_2 w dm$. Note that $\delta^\top G_2 w \in \mathbf{H}_0^1$, as otherwise, by choosing appropriate β we can get infinity in the second term. Therefore $|\int_\mathbb{T} df_2(\xi)\beta| = 0$. We denote by M_1 the space of measures defined as follows:

$$M_1 = \left\{\left(\begin{array}{c}\mu\\\xi\end{array}\right): \begin{array}{l}\mu(E) = \int_E w dm,\ w\in S\mathbf{L}^1(\mathbb{R}_+)\\\xi(E) = \int_E \delta^\top w dm,\ \delta\in\mathbf{L}^\infty(\Delta),\ \delta^\top G_2 w\in\mathbf{H}_0^1\end{array}\right\}.$$

Then we get

$$\inf_{\left(\begin{array}{c}\mu\\\xi\end{array}\right)\in M}(\sup_{\alpha\in\mathcal{B}\mathbf{H}^\infty}|\int_\mathbb{T}df_1(\mu,\xi,\nu)\alpha|+\nu\sup_{\beta\in\mathbf{H}^\infty}|\int_\mathbb{T}df_2(\xi)\beta|)=$$

$$\inf_{\left(\begin{array}{c}\mu\\\xi\end{array}\right)\in M_1}\sup_{\alpha\in\mathcal{B}\mathbf{H}^\infty}|\int_\mathbb{T}df_1(\mu,\xi,\nu)\alpha|=\inf_{\left(\begin{array}{c}\mu\\\xi\end{array}\right)\in M_1}\inf_{p\in\mathbf{H}_0^1}\|f_1-p\|_1=$$

$$\inf_{\substack{w\in S\mathbf{L}^1(\mathbb{R}_+)\\\delta\in\mathbf{L}^\infty(\Delta_\nu)\\\delta^\top G_2 w\in\mathbf{H}_0^1}}\inf_{p\in\mathbf{H}_0^1}\|(1+\nu\delta^\top G_1)w-p\|_1=\gamma_{opt}(\nu).$$

$\square$

Using Theorem 17 we state the dual problem as follows.

Dual problem. Given $F = (\ 1\ \ 0\ )$, $G = (\ G_1\ \ G_2\ )$ and a convex compact set Δ, find a sequence of functions $\{(w_i,\ \delta_i,\ p_i)\}$ such that $w_i\in S\mathbf{L}^1(\mathbb{R}_+)$, $\delta_i\in\mathbf{L}^\infty(\Delta)$, $\delta_i^\top G_2 w_i\in\mathbf{H}_0^1$, $p_i\in\mathbf{H}_0^1$, and

$$\|(1+\nu\delta_i^\top G_1)w_i-p_i\|\to 0,\ \text{as } i\to\infty.$$

In the same way as in [3] the dual problem can be split into two parts: one regular and one singular.

Theorem 18. *Let $F \in \mathbf{A}^{1 \times n}$, $G \in \mathbf{A}^{m \times n}$ and denote $\Phi_\delta = F + \nu \delta^\top G$. Then the optimal value ν_{opt} takes the following form*

$$\nu_{opt} = \min\{\nu_{opt|c}, \nu_{opt|s}\}$$

with the regular part

$$\nu_{opt|c} = \inf\{\nu | \exists w \in \mathcal{S}\mathbf{L}^1(\mathbb{R}_+)\backslash 0, \delta \in \mathbf{L}^\infty(\Delta) \colon \Phi_\delta w \in \mathbf{H}_0^1\} \qquad (4.2)$$

and the singular part

$$\nu_{opt|s} = \inf\{\nu | \exists z \in \mathbb{T}, \ \delta \in \Delta : \Phi_\delta(z) = 0\}. \qquad (4.3)$$

4.3 Linear programming algorithm for a given uncertainty bound

Note that the primal problem is infinite dimensional. In [4] it was proposed a numerical method to solve the problem by successive finite-dimensional approximation. In this session we will slightly modify this algorithm. We show how we should choose the next step in optimization of the uncertainty bound in order to guarantee the existence of solution.

Theorem 19. *The function $\gamma_{opt}(\nu)$ satisfies the following inequality:*

$$0 \le \gamma_{opt}(\nu) \le 1 - \frac{\nu}{\nu_{opt}}.$$

Proof. It is obvious that $\gamma(\nu) \ge 0$.

Now we show that $\gamma_{opt}(\nu) \le 1 - \frac{\nu}{\nu_{opt}}$.

From the proof of Theorem 17 we have that

$$\gamma_{opt}(\nu) = \inf_{\substack{\delta \in \mathbf{L}^\infty(\Delta) \\ w \in S\mathbf{L}^1(\mathbb{R}_+)}} \sup_{\alpha \in \mathcal{B}\mathbf{H}^\infty} \mathrm{Re} \int_\mathbb{T} (w + \nu \delta^\top w G_1) \alpha \, dm.$$

For particular w_{opt} and $(\delta^\top w)_{opt}$ we have

$$\gamma_{opt}(\nu) \le \sup_{\alpha \in \mathcal{B}\mathbf{H}^\infty} \mathrm{Re} \int_\mathbb{T} (w_{opt} + \nu(\delta^\top w)_{opt} G_1) \alpha \, dm =$$

$$\sup_{\alpha \in \mathcal{B}\mathbf{H}^\infty} \mathrm{Re} \, \Big(\int_\mathbb{T} w_{opt} \alpha \, dm + \nu \int_\mathbb{T} (\delta^\top w)_{opt} G_1 \alpha \, dm.$$

For the optimal stability margin ν_{opt}, the optimal w and $\delta^\top w$ we know that $w_{opt} + \nu_{opt}(\delta^\top w)_{opt}G_1 \in \mathbf{H}_0^1$. It means that there exists $p \in \mathbf{H}^1$ such that

$$w_{opt} + \nu_{opt}(\delta^\top w)_{opt}G_1 = zp.$$

We multiply the equation by α and get

$$(w_{opt} + \nu_{opt}(\delta^\top w)_{opt}G_1)\alpha = zp\alpha.$$

The function $zp\alpha$ is analytic. Then

$$\int_{\mathbb{T}} (w_{opt} + \nu_{opt}(\delta^\top w)_{opt}G_1)\alpha = 0,$$

$$\int_{\mathbb{T}} w_{opt}\alpha + \nu_{opt}\int_{\mathbb{T}}(\delta^\top w)_{opt}G_1\alpha = 0,$$

$$\int_{\mathbb{T}}(\delta^\top w)_{opt}G_1\alpha = -\frac{1}{\nu_{opt}}\int_{\mathbb{T}} w_{opt}\alpha,$$

and

$$\gamma_{opt}(\nu) \leq \sup_{\alpha \in \mathcal{B}\mathbf{H}^\infty} \mathrm{Re}\,\Big(\int_{\mathbb{T}} w_{opt}\alpha dm - \frac{\nu}{\nu_{opt}}\int_{\mathbb{T}} w_{opt}\alpha dm\Big)$$

$$=(1 - \frac{\nu}{\nu_{opt}})\sup_{\alpha \in \mathcal{B}\mathbf{H}^\infty} \mathrm{Re}\int_{\mathbb{T}} w_{opt}\alpha dm$$

$$=(1 - \frac{\nu}{\nu_{opt}})\sup_{\alpha \in \mathcal{B}\mathbf{H}^\infty} |\int_{\mathbb{T}} w_{opt}\alpha dm|$$

$$\leq(1 - \frac{\nu}{\nu_{opt}})\sup_{\alpha \in \mathcal{B}\mathbf{H}^\infty} \int_{\mathbb{T}} w_{opt}|\alpha|dm$$

$$\leq(1 - \frac{\nu}{\nu_{opt}})\int_{\mathbb{T}} w_{opt}dm = 1 - \frac{\nu}{\nu_{opt}}.$$

$\square$

Let h_{opt} be the optimal solution to the primal problem and recall that $\Phi_\delta(z) = F(z) + \nu\delta^\top G(z)$.

Theorem 20. *In the regular case the function* $\inf_{\delta \in \Delta} Re\,\Phi_\delta(z)h_{opt}(z)$ *is constant.*

Proof. Note first that the function $\inf_{\delta \in \Delta} \operatorname{Re} \Phi_\delta(z) h_{opt}(z) > 0$. According to the definition of the optimal γ we have

$$
\begin{aligned}
\gamma_{opt} &= \inf_{z \in \mathbb{T}} \inf_{\delta \in \Delta} \operatorname{Re} \Phi_\delta(z) h_{opt}(z) \\
&= \inf_{w \in S\mathbf{L}^1(\mathbb{R}_+)} \int_{\mathbb{T}} \inf_{\delta \in \Delta} \operatorname{Re} \Phi_\delta(z) h_{opt}(z) w(z) dm \\
&\leq \inf_{w \in S\mathbf{L}^1(\mathbb{R}_+)} \inf_{\delta \in \Delta} \operatorname{Re} \int_{\mathbb{T}} \Phi_\delta w(z) h_{opt}(z) dm \\
&= \inf_{w \in S\mathbf{L}^1(\mathbb{R}_+)} \inf_{\delta \in \Delta} \operatorname{Re} \int_{\mathbb{T}} (\Phi_\delta w(z) - p_{opt}(z)) h_{opt}(z) dm \\
&\leq \inf_{\delta \in \Delta} \int_{\mathbb{T}} |\Phi_\delta w_{opt}(z) - p_{opt}(z)| dm = \gamma_{opt},
\end{aligned}
$$

and all inequalities become equalities. Therefore

$$
\int_{\mathbb{T}} \inf_{z \in \mathbb{T}} \inf_{\delta \in \Delta} \operatorname{Re} \Phi_\delta(z) h_{opt}(z) w_{opt}(z) dm = \int_{\mathbb{T}} \inf_{\delta \in \Delta} \operatorname{Re} \Phi_\delta(z) h_{opt}(z) w_{opt}(z) dm,
$$

$$
\int_{\mathbb{T}} (\inf_{\delta \in \Delta} \operatorname{Re} \Phi_\delta(z) h_{opt}(z) - \inf_{z \in \mathbb{T}} \inf_{\delta \in \Delta} \operatorname{Re} \Phi_\delta(z) h_{opt}(z)) w_{opt}(z) dm = 0 \Rightarrow
$$

$$
(\inf_{\delta \in \Delta} \operatorname{Re} \Phi_\delta(z) h_{opt}(z) - \inf_{z \in \mathbb{T}} \inf_{\delta \in \Delta} \operatorname{Re} \Phi_\delta(z) h_{opt}(z)) w_{opt}(z) = 0.
$$

As we consider the regular case and $w_{opt} \neq 0$, then we have that

$$
\inf_{\delta \in \Delta} \operatorname{Re} \Phi_\delta(z) h_{opt}(z) - \inf_{z \in \mathbb{T}} \inf_{\delta \in \Delta} \operatorname{Re} \Phi_\delta(z) h_{opt}(z) = 0
$$

and the function $\inf_{\delta \in \Delta} \operatorname{Re} \Phi_\delta(z) h_{opt}(z)$ is constant. $\qquad \square$

Now we are able to present the algorithm for searching the solution of the primal problem. Recall that we have formulated the robust controller design problem for a given uncertainty bound ν as follows.

Primal problem: Given $\nu > 0$ find a function $h \in \mathcal{B}\mathbf{H}^\infty$ such that $\forall z \in \mathbb{T}$

$$
\operatorname{Re}(Fh(z) + \nu\delta(G(z)h(z))) > 0, \tag{4.4}
$$

where

$$
\begin{aligned}
F &= (\ 1 \quad 0 \quad \ldots \quad 0\ ) \in \mathbb{R}^{n+1} \\
G &= (\ G_1 \quad G_2\ ) \in \mathbf{RH}^\infty \\
h &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.
\end{aligned}
$$

We want to find the optimal uncertainty bound and the optimal regulator. For this purpose we try to find the optimal h for every given uncertainty bound ν, i.e. we are looking for a function $h_0 \in \mathcal{B}\mathbf{H}^\infty$ such that the condition (4.4) holds and

$$\gamma_\nu(z) = \sup_{h \in \mathcal{B}\mathbf{H}^\infty} \mathrm{Re}\,(Fh(z) + \nu\delta(G(z)h(z))) = \mathrm{Re}\,(Fh_0(z) + \nu\delta(G(z)h_0(z))). \tag{4.5}$$

According to Theorem 20, the function γ_ν should be constant.

If we have the solution, then the bound ν is a lower bound for the optimal ν.

As in [4] we construct a finite-dimensional approximation by solving the problem on a finite-dimensional subspace of $\mathbf{RH}^\infty$ and on finite grid of points $z \in \mathbb{T}$.

We consider a basis $\{\varphi_i\}_{i=0}^\infty$, such that $\varphi_i = z^i$. Then the span of functions, $\{\varphi_i\}_{i=0}^{N-1}$

$$H_N = \{h | h(z) = \sum_{i=0}^{N-1} h_i\varphi_i(z), h_i \in \mathbb{R}^{n+1}\},$$

forms an $N(n+1)$-th dimensional subspace of $\mathbf{RH}^\infty$. The optimal bound for this subspace is

$$\nu_N = \sup\{\nu | \exists h \in H_N : \gamma_\nu(z) > 0, \ \forall z \in \mathbb{T}\}.$$

We consider a finite grid of points $Z_K = \{z_k\}_{k=1}^K$ of the upper half of the unit circle. The condition (4.5) for a function $h \in H_N$ over the grid Z_K takes the form

$$\gamma(\{h_i\}_{i=0}^{N-1}, z_k) > 0, \ \forall z_k \in Z_k. \tag{4.6}$$

We propose the following algorithm:

1. Take $N = 1$, $K \geq 2$ and $\{0, \pi\} \subset Z_k$.

2. For given N, K, find a solution h to (4.6). If the problem is infeasible increase N by 1 and repeat.

3. Check the condition (4.5) for all z in the upper half of $\mathbb{T}$. If it does not hold, increase K by adding some of "bad" points to the set Z_k and go to Step 2.

4. Check that $\max \gamma_\nu - \min \gamma_\nu \leq$ tolerance. If it is not satisfied, increase N by 1 and go to Step 2, otherwise STOP.

In Step 3 we use the idea presented in [4]. We check that the function γ_ν does not vary very fast by looking at its derivative. Then as "bad" points to add to set Z_k we choose the local negative minima of γ_ν.

If the given ν is a lower bound for ν_{opt} then the algorithm finds a feasible solution. Suppose that we have solved the problem for some given ν. We choose next value of ν according to the result in Theorem 19. As

$$\gamma_\nu \leq 1 - \frac{\nu}{\nu_{opt}}$$

hence we choose the next ν as follows:

$$\nu_{new} = \frac{\nu}{1 - \gamma_\nu}, \ \nu_{new} \leq \nu_{opt}.$$

Therefore there is a feasible solution of the primal problem for ν_{new}.

4.4 Optimization of the uncertainty bound via dual problem for a real-valued uncertainty.

As we have mentioned above, a feasible solution of the primal problem gives a lower bound ν to the optimal value ν_{opt}. However, the algorithm is unable to indicate if there is no solution for a given ν since at each step we solve a finite-dimensional approximation. In order to obtain the upper bound to the optimal value ν_{opt} we use the duality result. Recall that the duality result stands as follows.

Theorem 21. *Let $F \in \mathbf{A}^{1 \times n}$, $G \in \mathbf{A}^{m \times n}$ and denote $\Phi_\delta = F + \nu \delta^\top G$. Then the optimal value ν_{opt} takes the following form*

$$\nu_{opt} = \min\{\nu_{opt|c}, \nu_{opt|s}\},$$

with the regular part

$$\nu_{opt|c} = \inf\{\nu | \exists w \in \mathcal{S}\mathbf{L}^1(\mathbb{R}_+)\backslash 0, \delta \in \mathbf{L}_\infty(\Delta) \colon \Phi_\delta w \in \mathbf{H}_0^1\} \qquad (4.7)$$

and the singular part

$$\nu_{opt|s} \ = \ \inf\{\nu | \exists z \in \mathbb{T}, \ \delta \in \Delta : \Phi_\delta(z) = 0\}. \qquad (4.8)$$

Calculation of an upper bound for $\nu_{opt|s}$ can be organized as a finite-dimensional convex programming at each z. But for $\nu_{opt|c}$ it becomes infinite-dimensional. In [4] it was proposed the dual algorithm to estimate $\nu_{opt|c}$. In this section we discuss the solution of the dual problem for the real-valued uncertainty. It turns out that in this case the dual problem becomes a finite-dimensional in the space of variables (semi-infinite convex programming).

Let us start with a slight modification of the considered optimization set. Recall that

$$\gamma_{opt}(\nu) = \sup_{\alpha \in \mathcal{B}\mathbf{H}^\infty, \beta \in \mathbf{H}^\infty} ess \inf_{z \in \mathbb{T}} \inf_{\delta \in \Delta} \operatorname{Re}\left(\alpha(z) + \nu \delta^\top (G_1(z)\alpha(z) + G_2(z)\beta(z))\right).$$

We optimize over the unit ball in $\mathbf{H}^\infty$. Now we will consider a set which is more appropriate for our task. We denote by $\mathbf{H}_1^\infty$ the subclass of $\mathbf{H}^\infty$ which contains the analytical functions h such that $h(0) = 1$, i.e.

$$\mathbf{H}_1^\infty = \{h : \; h \in \mathbf{H}^\infty, \; h(0) = 1\}.$$

Now we use the class $\mathbf{H}_1^\infty$ as an optimization set. It is clear that the optimal value ν_{opt} is the same if we optimize either over $\mathcal{B}\mathbf{H}^\infty$ or over $\mathbf{H}_1^\infty$.

Denote by

$$\gamma_{opt}^1(\nu) := \inf_{\substack{w \in \mathbf{L}^1(\mathbb{R}_+) \\ \delta \in \mathcal{S}\mathbf{L}^\infty(\Delta_\nu) \\ \delta^\top G_2 w \in \mathbf{H}_0^1}} \sup_{\alpha \in \mathbf{H}_1^\infty} \operatorname{Re} \int_{\mathbb{T}} (1 + \nu \delta^\top G_1)\alpha w \, dm.$$

Note that $\gamma_{opt}^1(\nu_{opt}) = \gamma_{opt}(\nu_{opt})$. Recall that $\gamma_{opt}(\nu) \geq 0$ for all ν. However $\gamma_{opt}^1(\nu)$ can assume positive values as well as negative. This property of $\gamma_{opt}^1(\nu)$ is more convenient for our purpose to find the optimal value ν_{opt}.

As $\alpha \in \mathbf{H}_1^\infty$, we can decompose α in the power series, i.e.

$$\alpha = 1 + \sum_{k=1}^\infty \alpha_k z^k$$

and we get

$$\gamma^1_{opt}(\nu) = \inf_{\substack{w \in \mathcal{SL}^1(\mathbb{R}_+) \\ \delta \in \mathbf{L}^\infty(\Delta_\nu) \\ \delta^\top G_2 w \in \mathbf{H}^1_0}} \sup_{\alpha \in \mathbf{H}^\infty_1} \mathrm{Re}\,(\int_{\mathbb{T}}(1 + \nu\delta^\top G_1)w dm +$$

$$\sum_{k=1}^\infty \int_{\mathbb{T}}(1 + \nu\delta^\top G_1)w\alpha_k z^k dm)$$

It is clear that $(1 + \nu\delta^\top G_1)w \in \mathbf{H}^1$, as otherwise, by choosing an appropriate α, we can get infinity. Therefore, we find $p(z) \in \mathbf{H}^1$ such that $(1 + \nu\delta^\top G_1)w = p$ and we get

$$\gamma^1_{opt}(\nu) = \inf_{p \in \mathbf{H}^1} \mathrm{Re}\, p(0).$$

According to the discussion above we have the following conditions on the functions δ and w

$$\begin{aligned} (1 + \nu\delta^\top G_1)w &= p_1 \\ \delta^\top G_2 w &= zp_2, \end{aligned} \tag{4.9}$$

where $p_1, p_2 \in \mathbf{H}^1$.

Now we consider the case when δ is a real-valued vector. We show that in this case the dual problem becomes finite dimensional in space of variables (semi-definite convex programming).

As $G_2 \in \mathbf{A}$, hence we can factorize G_2 as $G_2 = G_{2_i}G_{2_o}$, where G_{2_i} is an inner factor and G_{2_o} is an outer factor. Then

$$\delta^\top w G_{2_i}G_{2_o} = zp_2.$$

As G_{2_o} is right-invertible in $\mathbf{H}^\infty$, then we get

$$\begin{aligned} \delta^\top w G_{2_i} &= zp_2 G_{2_o}^{-1}, \\ \delta^\top w G_{2_i} &= zp_2' := zp_2, \\ \delta^\top w G_{2_i}G_{2_i}^* &= zp_2 G_{2_i}^*. \end{aligned}$$

As function G_2 is from the disk algebra $\mathbf{A}$, then the inner function G_{2_i} is a Blashke-Potapov product. It means that G_{2_i} can be written in the following way

$$G_{2_i} = U \prod_{j=1}^{k} \begin{pmatrix} \mathbb{I}_{r_j} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \frac{\lambda_j - z}{1 - \overline{\lambda_j} z} \mathbb{I}_{q_j} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{I}_{s_j}, \end{pmatrix},$$

where $r_j + q_j + s_j = n$ and U is an unitary function. Then

$$zp_2 G_{2_i}^* = zp_2 \prod_{j=1}^{k} \begin{pmatrix} \mathbb{I}_{r_j} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & -\frac{\overline{\lambda_j} - \overline{z}}{1 - \lambda_j \overline{z}} \mathbb{I}_{q_j} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{I}_{s_j}, \end{pmatrix} U^*$$

$$= zp_2 \prod_{j=1}^{k} \begin{pmatrix} \mathbb{I}_{r_j} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & -\frac{z^{-1}(1 - \overline{\lambda_j} z)^2}{|z - \lambda_j|^2} \mathbb{I}_{q_j} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{I}_{s_j}, \end{pmatrix} U^*.$$

Let $p_2^j = \frac{\hat{p}_2^j}{(1 - \overline{\lambda_j} z)^2}$, where $\hat{p}_2^j$ is a polynomial. Then we have

$$\delta^\top w G_{2_i} G_{2_i}^* = zp_2 \prod_{j=1}^{k} \begin{pmatrix} \mathbb{I}_{r_j} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & -\frac{z^{-1}(1 - \overline{\lambda_j} z)^2}{|z - \lambda_j|^2} \mathbb{I}_{q_j} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{I}_{s_j}, \end{pmatrix} U^*$$

$$= \frac{zz^{-k}}{\prod_{j=1}^{k} |z - \lambda_j|^2} \begin{pmatrix} q_1 & \cdots & q_n \end{pmatrix} U^*,$$

where

$$q_i = z^{k_i} \prod_{j=1}^{k} (|z - \lambda_j|^2)^{s_i} \hat{p}_2^i.$$

As δ and w are real-valued, then every $\hat{p}_2^j$ has to be such that the following equality is satisfied

$$\frac{z^{1-k}}{\prod_{j=1}^{k} |z - \lambda_j|^2} (z^{k_1} \prod_{j=1}^{k} (|z - \lambda_j|^2)^{s_1} \hat{p}_2^1 \ldots z^{k_n} \prod_{j=1}^{k} (|z - \lambda_j|^2)^{s_n} \hat{p}_2^n) U^* =$$

$$\frac{z^{k-1}}{\prod_{j=1}^{k} |z - \lambda_j|^2} (z^{-k_1} \prod_{j=1}^{k} (|z - \lambda_j|^2)^{s_1} \overline{\hat{p}_2^1} \ldots z^{-k_n} \prod_{j=1}^{k} (|z - \lambda_j|^2)^{s_n} \overline{\hat{p}_2^n}) U^\top.$$

$$(4.10)$$

It means that every $\hat{p}_2^j$ is a polynomial of degree $\leq 2(k-1)$. Now put the expression of the $\delta^\top w$ into the first equation in (4.9).

$$
\begin{aligned}
w &= p_1 - \nu \delta^\top w G_1 \\
&= p_1 - \nu \frac{z^{1-k}}{\prod_{j=1}^k |z - \lambda_j|^2} \begin{pmatrix} q_1 & \cdots & q_n \end{pmatrix} U^* (G_{2_i} G_{2_i}^*)^{-1} G_1 \\
&= \frac{z^{-k} \hat{p}_1 \prod (z - \lambda_j) - \nu z^{1-k} \begin{pmatrix} q_1 & \cdots & q_n \end{pmatrix} U^* (G_{2_i} G_{2_i}^*)^{-1} G_1}{\prod_{j=1}^k |z - \lambda_j|^2},
\end{aligned}
$$

where $p_1 = \frac{\hat{p}_1}{\prod_{j=1}^k (1 - \overline{\lambda_j} z)}$ and $\hat{p}_1$ is a polynomial. As w is real-valued, then $\hat{p}_1$ has to be such that the following equality is satisfied

$$
\begin{aligned}
z^{-k} \hat{p}_1 \prod (z - \lambda_j) - \nu z^{1-k} \begin{pmatrix} q_1 & \cdots & q_n \end{pmatrix} U^* G_1 = \\
z^m \overline{\hat{p}_1} \prod (\overline{z - \lambda_j}) - \nu z^{1-k} \begin{pmatrix} q_1 & \cdots & q_n \end{pmatrix} U^\top \overline{G_1}.
\end{aligned}
\tag{4.11}
$$

Therefore $\hat{p}_1$ is a polynomial of degree $\leq \max(k, \max(\deg G_{1_j}) + k - 1)$. As $\int_\mathbb{T} w \, dm = 1$, then we have one more condition on p_1 and p_2:

$$
\int_\mathbb{T} p_1 - \nu \delta w G_1 dm = \int_\mathbb{T} p_1 - \nu z p_2 U^{-1} (G_{2_i} G_{2_i}^*)^{-1} T_1 dm = 1
\tag{4.12}
$$

According to the discussion above we formulate the main result as follows.

Main result: Let the uncertainties vector δ be real-valued. Given the uncertainty bound ν, solve the following problem

$$
\min \operatorname{Re} p_1(0).
$$

Here $p_1 = \frac{\hat{p}_1}{\prod_{j=1}^k (1 - \overline{\lambda_j} z)}$ is such that $\hat{p}_1$ is a polynomial,
$\deg \hat{p}_1 \leq \max(k, \max(\deg G_{1_j}) + k - 1)$, and satisfies equation (4.11),
k is a number of zeros λ_j of function $\det B_2$, where B_2 is such that $G_{2_i} = U B_2$
$\hat{p}_2$ is a polynomial, $\deg \hat{p}_2 \leq 2(k-1)$, satisfying the equation (4.10),
the functions $p_1 = \frac{\hat{p}_1}{\prod_{j=1}^k (1 - \overline{\lambda_j} z)}$ and
$p_2 = \frac{z^{-k}}{\prod_{j=1}^k |z - \lambda_j|^2} \left(z^{k_1} \prod_{j=1}^k (|z - \lambda_j|^2)^{s_1} \hat{p}_2^1 \ldots z^{k_n} \prod_{j=1}^k (|z - \lambda_j|^2)^{s_n} \hat{p}_2^n \right)$ satisfy the equation (4.12).

We will show how we solve that problem using a numerical example.

4.5 Numerical example: Robust stabilization, real-valued uncertainty.

In this section we solve the robust stabilization problem for the system with real-valued uncertainty by the primal-dual method. The problem is formulated as follows.

Given uncertain plant P

$$\begin{pmatrix} y \\ z \end{pmatrix} = P \begin{pmatrix} w \\ u \end{pmatrix},$$

$$w = \nu \delta^\top z,$$

where $\delta \in \Delta$ for some convex compact $\Delta \ni 0$, the problem is to find a controller $u = Ky$ that robustly stabilizes the plant for ν as a large as possible. In case when w is a scalar input, the Youla parametrization leads to the following equivalent problem: find a function $Q \in \mathbf{RH}^\infty$ that maximizes ν such that

$$1 + \nu \delta^\top (T_1(z) + T_2(z)Q(z)) \neq 0 \, \forall z \in \mathbb{T}, \, \forall \delta \in \Delta.$$

By Theorem 16 the problem can be reduced to the form (4.1). In our case we take $\Delta = [-1, 1]$, i.e. the uncertainty vector δ is real-valued.

Let $T_1 = z^2 + 1.5z + 0.7$ and $T_2 = z^2 + \frac{1}{2}$. As T_1/T_2 is not in $\mathbf{H}^\infty$, then the solution to the optimization problem is not trivial.
First we will solve the primal problem. We are searching for a function h such that
$$\gamma_\nu(z) = \mathrm{Re}\,(Fh(z) + \nu \delta(G(z)h(z))) > 0,$$
where $F = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and $G = \begin{pmatrix} T_1 & T_2 \end{pmatrix}$.

We demand that function γ_ν is almost constant. We start at $\nu = 0.3$, find the solution, and increase the value of ν by Theorem 19. In Figure (4.1) we plot the function γ for different values of ν. We stop at $\nu = 2.55$. We could continue to approximate ν_{opt} but the degree of the function h would increase and it would be hard for the linear solver to find an approximation. We have $\gamma = 2.093e - 005$ and the optimal controller

$$Q_{opt} = \frac{\beta_{opt}}{\alpha_{opt}} = \frac{k(z^2 + 1.566z + 1.322)(z^2 - 1.195z + 2.648)(z^2 + 1.125z + 2.644)}{(z^2 + 0.1202z + 1.026)(z^2 + 1.183z + 2.595)(z^2 - 1.183z + 2.6)},$$
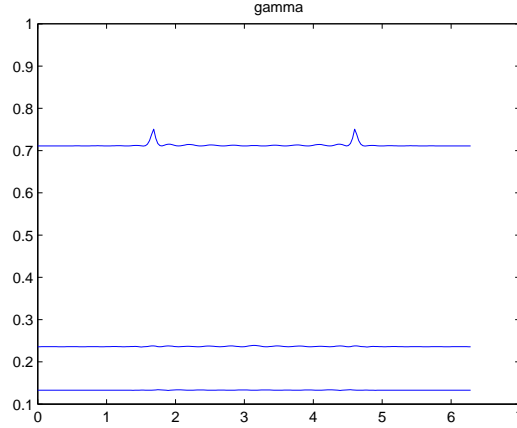
where $k = -1.0052$.

Figure 4.1: The function γ for different ν

Denote $R = \prod_{j=1}^{m}(z - \lambda_j)$, where $\{\lambda_j\}$ is the set of all zeros of T_2. In our case $R = z^2 + \frac{1}{2}$. Using the main result (see section 4.4) we have

$$\delta w R = z p_2,$$
$$\delta w R R^* = z p_2 R^*,$$
$$\varphi(z) := \delta w |R|^2 = z p_2 R^* = z p_2 z^{-2}(1 + \frac{1}{2}z^2) = z^{-1}\hat{p}_2,$$

where $p_2 = \frac{\hat{p}_2}{1+\frac{1}{2}z^2}$.

The function $\varphi(z) = z^{-1}\hat{p}_2$ has to be real, i.e. $z^{-1}\hat{p}_2 = z\overline{\hat{p}_2}$. It means that $\varphi(z)$ is a quasi-polynomial of degree 1 with symmetrical coefficients, i.e.

$$\varphi(z) = c_0 z^{-1} + c_1 + c_0 z$$

and

$$\delta w = \frac{c_0 z^{-1} + c_1 + c_0 z}{|R|^2}.$$

Now

$$w = p_1 - \nu \delta w T_1 = \frac{p_1 |R|^2 - \nu \varphi T_1}{|R|^2} = \frac{\hat{p}_1 z^{-2} R - \nu \varphi T_1}{|R|^2} = \frac{\psi}{|R|^2},$$

where $p_1 = \frac{\hat{p}_1}{1+\frac{1}{2}z^2}$. The function $\psi = \hat{p}_1 z^{-2} R - \nu \varphi T_1$ is real. Therefore $\psi(z)$ is a quasi-polynomial of degree 2 with symmetrical coefficients, i.e.

$$\psi(z) = d_0 z^{-2} + d_1 z^{-1} + d_2 + d_1 z + d_0 z^2.$$

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Now our task is to minimize $\operatorname{Re} p_1(0)$. Using the calculations above we obtain that

$$p_1(z) = \frac{\hat{p}_1(z)}{1 + \frac{1}{2}z^2} = \frac{(\psi + \nu\varphi T_1)z^2}{R(1 + \frac{1}{2}z^2)}$$

and

$$p_1(0) = 2d_0.$$

The functions φ and ψ have to satisfy the following conditions

$$1) \quad |\varphi| \leq \psi$$

$$2) \quad \int_{\mathbb{T}} \frac{\psi}{|R|^2} dm = 1$$

$$3) \quad \psi(\frac{-i}{\sqrt{2}}) + \nu\varphi T_1((\frac{-i}{\sqrt{2}})) = 0$$

$$4) \quad \psi(\frac{i}{\sqrt{2}}) + \nu\varphi T_1((\frac{i}{\sqrt{2}})) = 0 \qquad (4.13)$$

The first two conditions in (4.13) are due to the conditions on δ and w, i.e. $|\delta w| \leq w$ and $\int_{\mathbb{T}} w \, dm = 1$. The last two conditions came from the fact that p_1 has to be an analytic function, i.e. all unstable poles need to be canceled.

As the problem is still infinite-dimensional on z, then we must consider a finite grid of points on $\mathbb{T}$. We get the finite-dimensional linear program

$$\min_X fX \text{ subject to}$$

$$A_{12}X \leq 0$$

$$A_{22}X = 1$$

$$A_{32}X = 0,$$

where vector $X = (\begin{array}{ccccc} c_0 & c_1 & d_0 & d_1 & d_2 \end{array})$ absorbs the coefficients of the functions φ and ψ and $f = (\begin{array}{ccccc} 0 & 0 & 1 & 0 & 0 \end{array})$.

We run the algorithm in the linear programming form for different values of ν. We stop the optimization at $\nu = 2.5812$ when $\gamma^1(\nu) = 6.907e - 006$. We calculate the optimal uncertainty

$$\delta_{opt} = \frac{46341.8z(z^2 + 0.1181z + 1)}{(z + 2702)(z + 34.27)(z + 0.02918)(z + 0.0003701)}.$$

Using the principle formulated in [2]: *the optimal controller is equal to the inverted plant with the optimal uncertainty strategy*, we get the following optimal controller

$$Q_{opt} = \frac{z^2 + 1.618z + 1.354}{z^2 + 0.1181z + 1}.$$

Note that the controller is of lower order.

The plot of the closed-loop pole trajectories is shown in Figure (4.2).
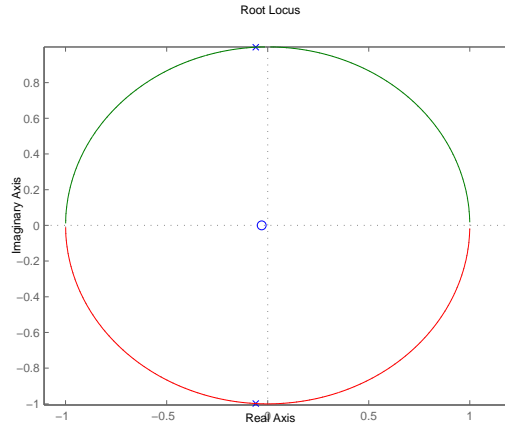
Figure 4.2: The closed-loop pole trajectories

Now we will consider a more complicated T_1, i.e. $T_1 = \frac{z^5 + 3*z^4 + 2*z^3 + 4*z^2 + 5*z + 3}{z^3 - z^2 - 4*z + 12}$ and solve the same robust stabilization problem. Here we stop the primal algorithm at $\nu = 10$ when $\gamma = 4.0828e{-}005$. The order of optimization reached is 19×88 and the optimal controller is of order 19. However, the dual algorithm gives us the optimal bound $\nu = 10.1784$ when $\gamma^1 = 1.7776e - 006$ and the optimal controller has a lower order

$$Q_{opt} = \frac{-(z + 2.601)(z^2 + 1.553z + 1.229)(z^2 - 0.976z + 1.811)}{(z + 2.506)(z^2 + 0.178z + 1)(z^2 - 3.506z + 4.788)}.$$

The plot of the closed-loop pole trajectories is shown in Figure (4.3).

4.6 Conclusions and future works.

In this chapter we presented the Linear Programming algorithm which solves the problem of robustly stabilizing controller design when the uncertainty pa-
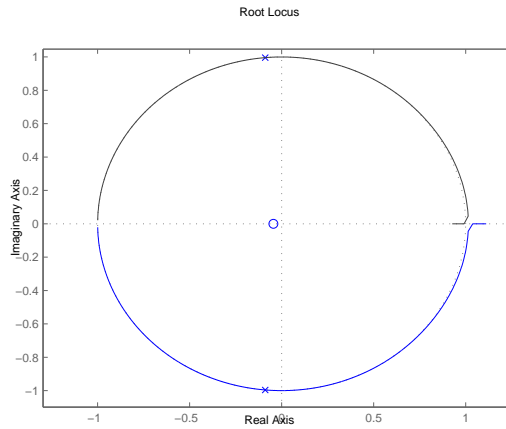
Figure 4.3: The closed-loop pole trajectories

rameter appears linearly in the closed-loop characteristic polynomial (rank-one problem). The initial problem is infinite-dimensional. The algorithm was derived as finite-dimensional approximation of the initial problem. We discussed the special case when an uncertainty vector is real-valued. In this case we showed that estimating of the uncertainty bound ν became the optimization problem on a finite-dimensional parameter. It should be interesting to develop the algorithm for a matrix uncertainty.

Part II

Convex Duality: Matrix Case

Chapter 5

Canonical Parametrization of the Dual Problem in Robust Optimization: Matrix Case

5.1 Introduction

Large research efforts have been devoted to robustness questions for linear control systems with parametric uncertainty. It was proposed different methods for design of controllers that are robust with respect to parametric uncertainty (for example "D-K iteration"). $\mathbf{H}^\infty$ optimization gives a very efficient tool for synthesis of robust controller, but became complicated in the case of nonrational plants. The problem to find a controller that provides the maximal stability margin to a rational system under rank-one uncertainties was studied in [1]. The name "rank one" problem comes from the fact that δ, an uncertain parameter, is a vector. The uncertainty set is an arbitrary convex compact set in the finite-dimensional vector space. It was formulated the primal/dual problem and shown that the dual problem can be stated as minimization of the "length" of uncertainties that destabilize the plant. The result was stated in terms of unstable zero-pole cancelation: *the numerator and denominator of the plant with the worst uncertainty have more common unstable zeros than all unstable poles.* In Chapter 2 the result was extended to the class of non-rational systems with continuous nominal factors. It was obtained the canonical parametrization of all destabilizing unstructured uncertainties in the dual problem. The duality principle provides an easy method to calculate the optimal stability margin and to design the optimal controllers. However the result was limited to the rank-one systems. In this chapter we will extend the result to the matrix case.

This chapter is organized as follows. In section 5.3 the dual/primal problems for the matrix case are described. Section 5.4 compares the primal problem with $\mathbf{H}^\infty$ optimization. Here we also present the important result that there is no duality gap between the primal and the dual problems in case of unstructured uncertainties. The main result, which is the canonical parametrization of the dual problem, is presented in section 5.5. Finally, in section 5.6, we introduce the $\delta-$metric and show that the $\delta-$metric has close connection to the $\nu-$gap metric.

We begin by introducing the basic definitions needed in this chapter similar to the definitions given in Part I.

5.2   Basic Definitions

By $\mathbb{R}$ (or $\mathbb{C}$) we denote the field of real (or complex) numbers. The unit circle and the open unit disc in $\mathbb{C}$ are denoted by $\mathbb{T}$ respectively $\mathbb{D}$:

$$\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}, \ \mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}.$$

For any measurable $Y \subset \mathbb{C}^{m \times n}$, the notation $L^p(Y)$ stands for the standard Lebesgue space of matrix-valued functions $F : \mathbb{T} \to Y$ equipped with the norm

$$\|F\|_p = \left\{ \begin{array}{l} (\int_{\mathbb{T}} \sum_{i=1}^n \sigma_i^p(F) dm)^{1/p}, \ 1 \le p < +\infty, \\ \text{ess sup}_{z \in \mathbb{T}} \overline{\sigma}(F(z)), p = +\infty, \end{array} \right.$$

where by σ_i we denote the singular values and $\overline{\sigma}$ stands for the maximum singular value.

$\mathbf{H}^p(Y)$, $1 \le p < +\infty$ denotes Hardy space of matrix-valued functions $F(z)$ analytic in $\mathbb{D}$ and such that

$$\|F\|_p = \sup_{0 < r < 1} (\int_{\mathbb{T}} \sum_{i=1}^n \sigma_i^p(F(rz)) dm)^{1/p} < \infty.$$

The Hardy class $\mathbf{H}^\infty$ is the space of bounded matrix-valued functions $F(z)$ which are analytic in $\mathbb{D}$ with norm

$$\|F\|_\infty = \sup_{z \in \mathbb{D}} \overline{\sigma}(F(z)).$$

A matrix-valued function is analytic in $\mathbb{D}$ if every element of the matrix is analytic in $\mathbb{D}$.

$\mathbf{H}_0^p(Y)$ denote

$$\mathbf{H}_0^p(Y) = z\mathbf{H}^p(Y) = \{F \in \mathbf{H}^p(Y) | F(0) = 0\}.$$

The space $\mathcal{C}$ is the space of continuous functions on $\mathbb{T}$.

The space $\mathbf{H}^\infty + \mathcal{C}$ is the set of functions $\Phi \in \mathbf{L}^\infty$ such that Φ admits a representation $\Phi = F + G$, where $F \in \mathbf{H}^\infty$ and $G \in \mathcal{C}$.

The class of *quasicontinuous* functions is defined by

$$\mathcal{QC} = (\mathbf{H}^\infty + \mathcal{C}) \cap (\overline{\mathbf{H}^\infty + \mathcal{C}}).$$

Let f be a function in $\mathbf{L}^1$ on the unit circle and let I be a subarc of $\mathbb{T}$. Put

$$f_I = \frac{1}{m(I)} \int_I f \, dm,$$

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the mean value of f over I. The space BMO of functions of *bounded mean oscillation* consists of functions $f \in \mathbf{L}^1$ such that

$$\sup_I \frac{1}{m(I)} \int_I |f - f_I| \, dm < \infty.$$

The space VMO of functions of *vanishing mean oscillation* consists of functions $f \in BMO$ for which

$$\lim_{m(I) \to 0} \frac{1}{m(I)} \int_I |f - f_I| \, dm = 0.$$

The disk algebra $\mathbf{A}(Y)$ is the subspace of $\mathbf{H}^\infty$ that consists of analytic functions in $\mathbb{D} \subset Y$ that can be extended continuously to the closed unit disk.

The set $\mathbf{RH}^\infty$ is the set of all functions from $\mathbf{H}^\infty$ that are rational with reals coefficient.

The brief notations $\mathbf{A}$, $\mathbf{H}^\infty$ etc. will be used if $Y = \mathbb{C}^{m \times n}$ and the dimension of the space is clear from the context.

The Toeplitz operator T_Φ and the Hankel operator H_Φ with the symbol $\Phi \in \mathbf{L}^\infty$ are defined by

$$\mathrm{T}_\Phi f = \mathbb{P}_+ \Phi f, \qquad \mathrm{H}_\Phi f = \mathbb{P}_- \Phi f, \qquad f \in \mathbf{H}^2,$$

where $\mathbb{P}_+$ is the ortogonal projection from $\mathbf{L}^2(Y)$ onto $\mathbf{H}^2(Y)$ and $\mathbb{P}_- = I - \mathbb{P}_+$.

We denote $\|\mathrm{H}_\Phi\|_e = \mathrm{dist}\,(\Phi, \mathbf{H}^\infty + \mathcal{C})$ the essential norm of Hankel operator.

An operator V from a Hilbert space $\mathcal{H}$ to a Hilbert space $\mathcal{K}$ is called a *partial isometry* if the restriction of V to $(\mathrm{Ker}\, V)^\perp$ maps isometrically $(\mathrm{Ker}\, V)^\perp$ onto Range V.

A function Θ is called *inner* if $\Theta^*(\zeta)\Theta(\zeta) = \mathbb{I}$ for almost all $\zeta \in \mathbb{T}$.

A function $\Phi \in \mathbf{H}^2$ is called *outer* if the functions of the form Φq, where q ranges over the analytic polynomial, are dense in $\mathbf{H}^2$.

Two matrices $F, G \in \mathbf{A}$ are *left-coprime* if they have equal number of rows and there exist $X, Y \in \mathbf{A}$ such that

$$(\; F \quad G \;) \begin{pmatrix} X \\ Y \end{pmatrix} = FX + GY = \mathbb{I},$$

i.e. the matrix function $(\; F \quad G \;)$ is right invertible in $\mathbf{A}$.

Let G be a matrix function. We say that the factorization $G = M^{-1}N$ is a *left coprime factorization* if N and M are left-coprime matrices.

The factorization $G = M^{-1}N$ is called a *normalized* left coprime factorization if $NN^* + MM^* = \mathbb{I}$.

The matrix function V is called *co-outer* if $V^\top$ is outer.

The superscript $^\top$ stands for transposition and † for pseudoinverse. A bar denotes the complex conjugate and * denotes conjugate transpose. The prefix $\mathcal{B}$ denotes the unit ball in the corresponding space, and $\mathcal{S}$ is the unit sphere.

5.3   Primal/dual problem

Given a nominal plant P and an uncertainty set $\Delta \ni 0$, the general robust controller design problem is to find a controller K that robustly stabilizes the whole family of perturbed plants

$$P_\delta = (M + \delta^\top G_2)^{-1}(N + \delta^\top G_1), \ \delta \in \nu\Delta \tag{5.1}$$

for ν as large as possible. Here $F = (\ N \quad M\ ) \in \mathbf{A}^{m \times n}$ is the left coprime factorization of the plant P, the weight matrix $G = (\ G_1 \quad G_2\ ) \in \mathbf{A}^{m \times n}$ and the set Δ is a convex compact set in $\mathbb{C}^{m \times m}$.

In the Rank-one model the problem of finding a controller is equivalent to the condition in terms of a function $h \in \mathbf{H}^\infty$ (see [5], [3]):
Let $F \in \mathbf{A}^{1 \times n}$, $G \in \mathbf{A}^{m \times n}$, and let $\Delta_\nu \subset \mathbb{C}^m$ be a convex compact set. Denote $\Phi_\delta = F + \delta^\top G$. Find a function $h \in \mathbf{H}^\infty(\mathbb{C}^{n \times 1})$ such that

$$Re\,\Phi_\delta(z)h(z) > 0 \ \ \forall z \in \mathbb{T}, \forall \delta \in \Delta_\nu. \tag{5.2}$$

The problem of finding $h \in \mathbf{H}^\infty$ such that the condition (5.2) holds is considered as a *primal problem.* We would like to solve it for ν as large as possible, that is, for

$$\nu_{opt|p} = \sup\{\nu | \exists h \in \mathbf{H}^\infty : \ Re(F + \delta^\top G)h > 0 \ \forall \delta \in \nu\Delta\}. \tag{5.3}$$

In the matrix case we can formulate the primal problem in the similar way but since F, G and h are matrix-valued and $\Delta_\nu \subset \mathbb{C}^{m \times m}$, the positivity of $Re\,\Phi_\delta h \in \mathbb{C}^{m \times m}$ means that $x^* Re\,\Phi_\delta h x > 0$ for all nonzero vectors $x \in \mathbb{C}^m$.

A solution to the primal problem gives the lower bound $\nu_{opt|p}$ to the optimal value ν_{opt} in the problem of robust stabilization (5.1). Note that $\nu_{opt|p}$ will be the upper bound to the structured singular value μ.

As in the scalar case we introduce the dual problem.

Theorem 22. *The following duality inequality is valid:*
$$\sup_{h\in\mathcal{B}\mathbf{H}^\infty} \inf_{\delta\in\Delta_\nu} ess\inf_{z\in\mathbb{T}} \min_{x\in\mathcal{C}^m} x^* Re\,\Phi_\delta(z)h(z)x \leq \inf_{\delta\in\mathbf{L}^\infty(\Delta_\nu)} \inf_{W\in\mathcal{S}\mathbf{L}^1(\mathbb{R}_+)} \inf_{P\in\mathbf{H}^1_0} \|W\Phi_\delta - P\|_1.$$

Proof. According to the trace properties
$$x^* Re\,\Phi_\delta(z)h(z)x = \operatorname{Tr} Re\,\Phi_\delta(z)h(z)xx^*.$$

As in the scalar case we have
$$ess\inf_{z\in\mathbb{T}} \min_{x\in\mathcal{C}^m} \operatorname{Tr} Re\,\Phi_\delta(z)h(z)xx^* = \min_{x\in\mathcal{C}^m} \inf_{w\in\mathcal{S}\mathbf{L}^1(\mathbb{R}_+)} \int_\mathbb{T} \operatorname{Tr} Re\,\Phi_\delta(z)h(z)xx^* w\,dm =$$
$$\inf_{W\in\mathcal{S}\mathbf{L}^1(\mathbb{R}_+)} \int_\mathbb{T} \operatorname{Tr} Re\,\Phi_\delta(z)h(z)W\,dm,$$

where $W = xx^* w$.
We get
$$\sup_{h\in\mathcal{B}\mathbf{H}^\infty} \inf_{\delta\in\Delta_\nu} ess\inf_{z\in\mathbb{T}} \min_{x\in\mathcal{C}^m} x^* Re\,\Phi_\delta(z)h(z)x =$$
$$\sup_{h\in\mathcal{B}\mathbf{H}^\infty} \inf_{W\in\mathcal{S}\mathbf{L}^1(\mathbb{R}_+)} \inf_{\delta\in\Delta_\nu} \int_\mathbb{T} \operatorname{Tr} Re\,\Phi_\delta hW\,dm.$$

Since Δ_ν is a compact set, inf is attained on Δ_ν. Then we can take infimum over all bounded functions from $\mathbf{L}^\infty$ with valuers in Δ_ν and
$$\inf_{\delta\in\Delta_\nu} \int_\mathbb{T} \operatorname{Tr} Re\,\Phi_\delta hW\,dm = \inf_{\delta\in\mathbf{L}^\infty(\Delta_\nu)} \int_\mathbb{T} \operatorname{Tr} Re\,\Phi_\delta hW\,dm.$$

According to the property of the trace we have $\operatorname{Tr} Re\,\Phi_\delta hW = Re\operatorname{Tr}\Phi_\delta hW$

and then

$$\sup_{\substack{h \in \mathcal{B}\mathbf{H}^\infty}} \inf_{\substack{\delta \in \mathbf{L}^\infty(\Delta_\nu), \\ W \in \mathcal{S}\mathbf{L}^1(\mathbb{R}_+)}} \int_{\mathbb{T}} \operatorname{Tr} \operatorname{Re} \Phi_\delta h W \, dm =$$

$$\sup_{\substack{h \in \mathcal{B}\mathbf{H}^\infty}} \inf_{\substack{\delta \in \mathbf{L}^\infty(\Delta_\nu), \\ W \in \mathcal{S}\mathbf{L}^1(\mathbb{R}_+)}} \operatorname{Re} \int_{\mathbb{T}} \operatorname{Tr} W \Phi_\delta h \, dm \leq$$

$$\inf_{\substack{\delta \in \mathbf{L}^\infty(\Delta_\nu), \\ W \in \mathcal{S}\mathbf{L}^1(\mathbb{R}_+)}} \sup_{\substack{h \in \mathcal{B}\mathbf{H}^\infty}} \operatorname{Re} \int_{\mathbb{T}} \operatorname{Tr} W \Phi_\delta h \, dm =$$

$$\inf_{\substack{\delta \in \mathbf{L}^\infty(\Delta_\nu), \\ W \in \mathcal{S}\mathbf{L}^1(\mathbb{R}_+)}} \sup_{\substack{h \in \mathcal{B}\mathbf{H}^\infty}} |\int_{\mathbb{T}} \operatorname{Tr} W \Phi_\delta h \, dm| = \inf_{\substack{\delta \in \mathbf{L}^\infty(\Delta_\nu), \\ W \in \mathcal{S}\mathbf{L}^1(\mathbb{R}_+)}} \inf_{P \in \mathbf{H}_0^1} \|\Phi_\delta^\top W - P\|_1.$$

The last equality follows from the Banach duality relation [7]. $\qquad\square$

Using Theorem 22 we formulate the *dual problem* as follows.

Given $F, G \in \mathbf{A}$ and a convex compact set Δ_ν, find a sequence of functions $\{(W_i, \delta_i, P_i)\}_{i=0}^{+\infty}$ such that $W_i \in \mathcal{S}\mathbf{L}^1(\mathbb{R}_+)$, $\delta_i \in \mathbf{L}^\infty(\Delta_\nu)$, $P_i \in \mathbf{H}_0^1$, and

$$\|(F + \delta_i^\top G)^\top W_i - P_i\|_1 \to 0, \text{ as } i \to +\infty.$$

The dual problem can be naturally decomposed into two parts: one regular and one singular.

Theorem 23. *The optimal dual value ν_{opt} has the representation*

$$\nu_{opt} = \min\{\nu_{opt|s}, \nu_{opt|c}\},$$

where

$$\nu_{opt|s} = \inf\{\nu| \exists z \in \mathbb{T}, \exists \delta \in \Delta_\nu : \Phi_\delta(z) \text{ loses the column rank}\},$$

$$\nu_{opt|c} = \inf\{\nu| \exists W \in \mathcal{S}\mathbf{L}^1(\mathbb{R}_+), \exists \delta \in \mathbf{L}^\infty(\Delta_\nu) : \Phi_\delta^\top W \in \mathbf{H}_0^1\}.$$

Proof. The proof of the theorem is a slight modification of the proof for the scalar case [3]. Indeed, we can decompose W on a regular and singular part such that

$$\|\Phi_\delta^\top W - P\|_1 = \|\Phi_\delta^\top W_c - P\|_1 + \int_E \operatorname{Tr} |\Phi_\delta^\top W_s| \, dm,$$

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where W_c is a regular function in $\mathbf{L}^1(\mathbb{R}_+)$ and W_s is a generalized function which is equal to zero on $\mathbb{T} \setminus E$ and $m(E) = 0$. By $|\cdot|$ we denote $|F| = \sqrt{F^*F}$ for matrix-valued function F. W_{opt} can be chosen in such way that either the regular part W_c or the singular part W_s is zero. Then

$$\inf_{P \in \mathbf{H}_0^1} \inf_W \|\Phi_\delta^\top W - P\|_1 = \min \left\{ \inf_{P \in \mathbf{H}_0^1} \|\Phi_\delta^\top W_c - P\|_1, \int_E \mathrm{Tr}\, |\Phi_\delta^\top| d\mu_s \right\}.$$

For the optimal δ we have either $\inf_{P \in \mathbf{H}_0^1} \|\Phi_{\delta_{opt}}^\top W_c - P\|_1 = 0$ or $\int_E |\Phi_{\delta_{opt}}^\top| d\mu_s = 0$. According to the Banach duality relation [7] there exists the optimal function P_{opt} such that

$$\inf_{P \in \mathbf{H}_0^1} \|\Phi_{\delta_{opt}}^\top W_c - P\|_1 = \|\Phi_{\delta_{opt}}^\top W_c - P_{opt}\|_1.$$

Then if $\inf_{P \in \mathbf{H}_0^1} \|\Phi_{\delta_{opt}}^\top W_c - P\|_1 = 0$ we have

$$\nu_{opt|c} = \inf\{\nu| \; \exists W \in \mathbf{L}^1(\mathbb{R}_+), \exists \delta \in \mathbf{L}^\infty(\Delta_\nu) : \Phi_\delta^\top W \in \mathbf{H}_0^1\}.$$

Consider the case $\int_E \mathrm{Tr}\, |\Phi_{\delta_{opt}}^\top| d\mu_s = 0$. If $\Phi_{\delta_{opt}}$ does not lose the rank then $\mathrm{Tr}\, |\Phi_{\delta_{opt}}^\top W_s| > \epsilon$ and $\int_E \mathrm{Tr}\, |\Phi_{\delta_{opt}}^\top| d\mu_s \neq 0$. In order to fulfill the condition, $\Phi_{\delta_{opt}}$ has to lose the column rank and we can choose $d\mu_s$ as $d\mu_s = xx^*\delta_{z_0}dm$, where δ_{z_0} is the Dirac's function and the vector x is such that $\Phi_{\delta_{opt}}^\top(z_0)x = \mathbb{O}$. In this case we get

$$\nu_{opt|s} = \inf\{\nu| \; \exists z \in \mathbb{T}, \exists \delta \in \Delta_\nu : \Phi_\delta(z) \, loses \, the \, column \, rank\}$$

and

$$\nu_{opt} = min\{\nu_{opt|s}, \nu_{opt|c}\}.$$

$\square$

5.4 Primal problem versus $\mathbf{H}^\infty$ optimization

In this section we consider the case of unstructured uncertainty when the set of uncertainties is the ball, $\Delta = \mathcal{B}\mathbb{C}^{m \times m}$. Under this construction our primal problem is equivalent to the $\mathbf{H}^\infty$ optimization problem. Furthermore we prove that there is no duality gap between the primal and the dual problems.

Theorem 24. *Let $F \in \mathbf{A}^{m \times n}$, $G \in \mathbf{A}^{m \times n}$ and $\Delta = \mathcal{B}\mathbb{C}^{m \times m}$. Then the following statements are equivalent:*

1) $\exists h \in \mathbf{H}^{\infty}(\mathcal{C}^{n \times m})$ such that for $\forall z \in \mathbb{T}$ and for $\forall \delta \in \Delta_{\nu}$,

$$Re(F(z) + \delta^{\top} G(z))h(z) > 0.$$

2) $\exists g \in \mathbf{H}^{\infty}(\mathcal{C}^{n \times m})$ such that $Fg = I$ and $\|Gg\|_{\infty} < \nu^{-1}$.

Proof. We start by showing that 1) $\Rightarrow$ 2).

Let h be a solution of the first problem. Then $Re(F(z) + \delta^{\top} G(z))h(z) > 0$. We will show that $((F + \delta^{\top} G)h)^{-1} \in \mathbf{H}^{\infty}$. It's enough to show that

$$\det(F(s) + \delta^{\top} G(s))h(s) \neq 0 \text{ for } \forall s \in \mathbb{D}.$$

As $(F(s) + \delta^{\top} G(s))h(s)$ is an analytic function, then $Re(F(s) + \delta^{\top} G(s))h(s)$ is a harmonic function and by the mean value theorem [20] $Re(F(s) + \delta^{\top} G(s))h(s) > 0$.
By the Ostrowski-Taussky inequality [6] for the matrices with positive definite real parts, $\det Re(F + \delta^{\top} G)h \leq |\det(F + \delta^{\top} G)h|$. Then $|\det(F + \delta^{\top} G)h| > 0$ and $((F + \delta^{\top} G)h)^{-1} \in \mathbf{H}^{\infty}$.
Taking δ as zero matrix we get that $(Fh)^{-1} \in \mathbf{H}^{\infty}$. We will show that $g = h(Fh)^{-1} \in \mathbf{H}^{\infty}$ is solution of the second problem.

Indeed, $Fg = Fh(Fh)^{-1} = I$. Then

$$(I + \delta^{\top} Gg)^{-1} = (Fh)(Fh + \delta^{\top} Gh)^{-1} \in \mathbf{H}^{\infty}, \ \forall \delta \in \Delta_{\nu}.$$

For each z we can take

$$\delta^{\top} = -\alpha \frac{(G(z)g(z))^{*}}{\|G(z)g(z)\|_{2}}.$$

Here $\| \cdot \|_{2}$ is the norm that is defined for a constant matrix A as follows: $\|A\|_{2} = \sqrt{\lambda_{max}(A^{*}A)}$, where λ_{max} is the maximal eigenvalue of matrix $A^{*}A$. As $\delta \in \Delta_{\nu}$, then we have $\|\delta\|_{2} \leq \nu$ and $\alpha \in [0, \nu]$. As $I + \delta^{\top} Gg$ does not lose the rang, then $1 - \alpha\|G(z)g(z)\|_{2} > 0$, $\|G(z)g(z)\|_{2} < \frac{1}{\alpha}$ for all z and $\|Gg\|_{\infty} < \nu^{-1}$.

Now we show the implication 2) $\Rightarrow$ 1).
Let g be a solution of the second problem. We will show that $h = g$ is a solution of the first problem. We have

$$Re(Fh + \delta^{\top} Gh) = Re(I + \delta^{\top} Gh) = I + Re(\delta^{\top} Gh).$$

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We will show that $I + Re(\delta^\top G h) > 0$. It is enough to show that $\underline{\lambda}(I + Re(\delta^\top G h)) > 0$, where $\underline{\lambda}(I + Re(\delta^\top G h))$ is the minimal eigenvalue of matrix $I + Re(\delta^\top G h)$. We have

$$\underline{\lambda}(I + Re(\delta^\top G h)) = 1 - \overline{\lambda}(Re(\delta^\top G h)) = 1 - \overline{\sigma}(Re(\delta^\top G h)) \geq$$

$$1 - \overline{\sigma}\frac{(\delta^\top G h)}{2} - \overline{\sigma}\frac{(G h)^*(\delta^\top)^*}{2} \geq 1 - \frac{\overline{\sigma}(\delta^\top)\overline{\sigma}(G h)}{2} - \frac{\overline{\sigma}((G h)^*)\overline{\sigma}((\delta^\top)^*)}{2} >$$

$$1 - \frac{\nu\nu^{-1}}{2} - \frac{\nu^{-1}\nu}{2} = 0,$$

since $\|\delta\|_2 \leq \nu$, $\|G g\|_\infty < \nu^{-1}$.

$\square$

Now the problem of finding the optimal primal ν is reduced to the following problem

$$\nu_{opt}^{-1} = \inf_{g \in \mathbf{H}^\infty}\{\|G g\|_\infty | F g = I\}.$$

All the solutions of the equation $F g = I$, $F \in \mathbf{A}$, can be parameterized as $g = g_{part} + M Q$, where g_{part} is a particular solution of the equation, M is a basis of the null-space for F and $Q \in \mathbf{H}^\infty$.

Then

$$\|G g\|_\infty = \|G g_{part} + G M Q\|_\infty = \|T_1 - T_2 Q\|_\infty,$$
$$\nu_{opt}^{-1} = \inf_{Q \in \mathbf{H}^\infty} \|T_1 - T_2 Q\|_\infty,$$

where $T_1 = G g_{part}$ and $T_2 = -G M$. This problem is the well-known $\mathbf{H}^\infty$ optimization problem. It means that in the particular case when $\Delta = \mathcal{B}\mathbb{C}^{m \times m}$ our primal problem is equivalent to the $\mathbf{H}^\infty$ optimization problem. In Chapter 2 was shown that, if $F \in \mathbf{A}$ and $G \in \mathbf{A}$, then we can find a solution g to $F g = I$, such that T_1 and T_2 in the $\mathbf{H}^\infty$ optimization problem are from $\mathbf{A}$ as well.

Using the result from [7] we get that

$$\text{dist}(T_1, T_2 \mathbf{H}^\infty) = \sup\{|\langle T_1, k\rangle| \colon k \in (T_2 \mathbf{H}^\infty)^\perp, \|k\| \leq 1\},$$

where $\langle T_1, k\rangle = \int \text{Tr}\, T_1^\top dk$. Recall that $k \in (T_2 \mathbf{H}^\infty)^\perp$ means that $\langle T_2 f, k\rangle = 0$ for all $f \in \mathbf{H}^\infty$.

In the same way as in the scalar case it can be shown that $\text{dist}(T_1, T_2 \mathbf{H}^\infty) = \text{dist}(T_1, T_2 \mathbf{A})$, i.e.

$$\inf_{h \in \mathbf{H}^\infty} \|T_1 - T_2 h\|_\infty = \inf_{h \in \mathbf{A}} \|T_1 - T_2 h\|_\infty.$$

Hence dist $(T_1, T_2\mathbf{H}^\infty) = \sup \int_\mathbb{T} \operatorname{Tr} T_1^\top d\varphi$, where $\|d\varphi\| \leq 1$ and $d\varphi \in (T_2\mathbf{A})^\perp$.

Here the norm is the induced norm described as follows.

$$\|d\varphi\| = \sup_{F \in \mathcal{C}, \|F\|_\infty \leq 1} \left| \int \operatorname{Tr} F^\top d\varphi \right|.$$

Now we are able to show that there is no duality gap between the primal and the dual problems in case of unstructured uncertainties.

Theorem 25. *Let $F \in \mathbf{A}^{m \times n}$, $G \in \mathbf{A}^{m \times n}$ and $\Delta = \mathcal{BC}^m$. Then there is no duality gap between the primal and the dual problem.*

Proof. Recall that the optimal dual ν can be calculated as

$$\nu_{opt|d} = \inf\{\|\delta\|_\infty : \Phi_\delta^\top d\mu = P_1 dm, P_1 \in \mathbf{H}_0^1, \|d\mu\|_1 = 1\}. \qquad (5.4)$$

The optimal primal ν is equal to the optimal margin ν in the $\mathbf{H}^\infty$ optimization problem and

$$\nu_{opt|p} = \inf\{\frac{1}{|\int_\mathbb{T} \operatorname{Tr} T_1^\top d\varphi|}, \|d\varphi\| \leq 1, T_2^\top d\varphi = P_2 dm, P_2 \in \mathbf{H}_0^1\}. \qquad (5.5)$$

We will show that $\nu_{opt|d} = \nu_{opt|p}$.

We start by showing that $\nu_{opt|p} \leq \nu_{opt|d}$.

Assume that we have found the optimal $d\mu$ in (5.4). Now we construct the measure $d\varphi$ in (5.5) with the demanded property. We multiply $\Phi_\delta = F + \delta^\top G$ by the unimodular matrix $U = \begin{pmatrix} g_p & M \end{pmatrix}$, where g_p is the particular solution to the equation $Fg = I$, and M is the kernel of F. The existence of such matrix U was discussed in Chapter 2, section 2.2.1. Then

$$\Phi_\delta(g_p \ M) = (F + \delta G)(g_p \ M) = (I + \delta^\top T_1 \ \delta^\top T_2)$$

and

$$(g_p \ M)^\top \Phi_\delta^\top d\mu_{opt} = \begin{cases} (I + T_1^\top \delta) d\mu_{opt} = P_1^1 dm \\ T_2^\top \delta d\mu_{opt} = P_1^2 dm, \end{cases}$$

where $\begin{pmatrix} P_1^1 \\ P_1^2 \end{pmatrix} = P_1 \in \mathbf{H}_0^1$

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Then, as $d\varphi$ we can put $d\varphi = -\dfrac{\delta d\mu_{opt}}{\|\delta\|_{\infty}}$. We claim that $d\varphi$ has the demanded properties.

Suppose first that $d\mu_{opt}$ is the regular measure. Then

$$\|d\varphi\| = \sup_{F \in \mathcal{C}, \|F\|_{\infty} \leq 1} \left| \int \mathrm{Tr}\, F^{\top} d\varphi \right| = \sup_{F \in \mathcal{C}, \|F\|_{\infty} \leq 1} \left| \int \mathrm{Tr}\, F^{\top} \frac{\delta d\mu_{opt}}{\|\delta\|_{\infty}} \right|$$

$$\leq \sup_{F \in \mathcal{C}, \|F\|_{\infty} \leq 1} \int \sum_{i=1}^{m} |\lambda_i(F^{\top} \frac{\delta d\mu_{opt}}{\|\delta\|_{\infty}})|,$$

here λ_i denotes the eigenvalue. Since $\sum_{i=1}^{m} |\lambda_i| \leq \sum_{i=1}^{m} \sigma_i$ (see [11]), where σ_i are the singular values, then we have that

$$\sup_{F \in \mathcal{C}, \|F\|_{\infty} \leq 1} \int \sum_{i=1}^{m} |\lambda_i(F^{\top} \frac{\delta d\mu_{opt}}{\|\delta\|_{\infty}})| \leq$$

$$\sup_{F \in \mathcal{C}, \|F\|_{\infty} \leq 1} \int \sum_{i=1}^{m} \sigma_i(F^{\top} \frac{\delta d\mu_{opt}}{\|\delta\|_{\infty}}) \leq \int \sum_{i=1}^{m} \sigma_i(\frac{\delta d\mu_{opt}}{\|\delta\|_{\infty}}) =$$

$$\int \mathrm{Tr}\, \sqrt{\frac{(d\mu_{opt})^* \delta^* \delta d\mu_{opt}}{\|\delta\|_{\infty}^2}} \leq \int \mathrm{Tr}\, \sqrt{(d\mu_{opt})^* d\mu_{opt}} = 1.$$

By $\sqrt{d\mu_{opt}^* d\mu_{opt}}$ we mean, as in [15], the unique nonnegative hermitian matrix H such that $H^2 = M^* M$, where M such that $d\mu_{opt} = M dm$.

If $d\mu_{opt}$ is the singular, i.e. $d\mu_{opt} = d\mu_s = x x^* \delta_{z_0} dm$, where δ_{z_0} is the Dirac's function and the vector x is such that $\Phi_{\delta_{opt}}^{\top}(z_0) x = \mathbb{O}$, then

$$\|d\varphi\| = \sup_{F \in \mathcal{C}, \|F\|_{\infty} \leq 1} \left| \int \mathrm{Tr}\, F^{\top} d\varphi \right| = \sup_{F \in \mathcal{C}, \|F\|_{\infty} \leq 1} \left| \int \mathrm{Tr}\, F^{\top} x x^* \delta_{z_0} dm \right| \leq 1.$$

We get that $\|d\varphi\| \leq 1$.

The second property id obvious,

$$T_2^{\top} d\varphi = \frac{T_2^{\top} \delta d\mu_{opt}}{\|\delta\|_{\infty}} = \frac{P_1^2 dm}{\|\delta\|_{\infty}} = P_2 dm.$$

For this measure $d\varphi$ we have

$$\left| \int \mathrm{Tr}\, T_1^{\top} d\varphi \right| = \left| \int \mathrm{Tr}\, T_1^{\top} \frac{(-\delta) d\mu_{opt}}{\|\delta\|_{\infty}} \right| = \frac{1}{\|\delta\|_{\infty}} \left| \int \mathrm{Tr}\, (-T_1^{\top} \delta) d\mu_{opt} \right| =$$

$$\left| \frac{1}{\|\delta\|_{\infty}} \int \mathrm{Tr}\, d\mu_{opt} \right| = \frac{1}{\|\delta\|_{\infty}}$$

and $\dfrac{1}{\left| \int \operatorname{Tr} T_1^\top d\varphi \right|} = \|\delta\|_\infty$. It means that $\nu_{opt|p} \le \nu_{opt|d}$.

Now we will show that $\nu_{opt|d} \le \nu_{opt|p}$. We decompose our problem into two problems - regular and singular. Suppose that we have found the optimal measure $d\varphi$ such that $d\varphi = \Phi dm$ is regular. Recall that $\|d\varphi\| \le 1$ and $T_2^\top \Phi = P_2$, $P_2 \in \mathbf{H}_0^1$. We will find the measure $(d\mu, dx)$ satisfying the following system:
$$\begin{cases} d\mu + T_1^\top dx = P_1^1 dm, \\ \qquad T_2^\top dx = P_1^2 dm. \end{cases}$$

We choose $d\mu$ such that $d\mu = (\Phi^*\Phi)^{\frac{1}{2}} dm$ and $dx = -\nu_{opt|p} d\varphi$.

Let $Q \in \mathbf{H}^\infty$ be the optimal function, i.e. $\operatorname{dist}(T_1, T_2\mathbf{H}^\infty) = \|T_1 - T_2 Q\|_\infty$. Such Q exists (see [7]). According to the alignment principle [12]

$$\|T_1 - T_2 Q\|_\infty = |\int \operatorname{Tr} T_1^\top \Phi dm| = |\int \operatorname{Tr}(T_1 - T_2 Q)^\top \Phi dm| =$$

$$\|T_1 - T_2 Q\|_\infty |\int \operatorname{Tr} \frac{(T_1 - T_2 Q)^\top}{\|T_1 - T_2 Q\|_\infty} \Phi dm| \le \|T_1 - T_2 Q\|_\infty \int \operatorname{Tr} \sqrt{\Phi^*\Phi} dm =$$

$$\|T_1 - T_2 Q\|_\infty \|\Phi\|_1 \le \|T_1 - T_2 Q\|_\infty.$$

It means that $\|\Phi\|_1 = 1$ and $\dfrac{(T_1 - T_2 Q)}{\|T_1 - T_2 Q\|_\infty} = \overline{\Phi}(\Phi^\top \overline{\Phi})^{\ddagger} = \overline{\Phi}((\Phi^*\Phi)^\top)^{\ddagger}$. We define pseudoinverse $(\Phi^*\Phi)^\dagger$ on the range of $\Phi^*\Phi$ as the inverse and the zero on the kernel of $\Phi^*\Phi$. Now we have to check that $d\mu$ and dx have the demanded properties:

1) $\|d\mu\|_1 = 1$,

2) $d\mu + T_1^\top dx = P_1^1 dm$, $P_1^1 \in \mathbf{H}_0^1$,

3) $T_2^\top dx = P_1^2 dm$, $P_1^2 \in \mathbf{H}_0^1$.

Indeed, 1) $\|d\mu\|_1 = \int \operatorname{Tr} \sqrt{(\Phi^*\Phi)} dm = \|\Phi\|_1 = 1$. The first property is satisfied.

2) As $\dfrac{(T_1 - T_2 Q)}{\|T_1 - T_2 Q\|_\infty} = \overline{\Phi}(\Phi^\top \overline{\Phi})^{\ddagger}$, then we have $\dfrac{(T_1 - T_2 Q)^\top \Phi}{\|T_1 - T_2 Q\|_\infty} = \sqrt{(\Phi^*\Phi)}$ and

$$d\mu + T_1^\top dx = (\sqrt{(\Phi^*\Phi)} - \nu_{opt|p} T_1^\top \Phi) dm = \left(\frac{(T_1 - T_2 Q)^\top \Phi}{\nu_{opt|p}^{-1}} - \nu_{opt|p} T_1^\top \Phi \right) dm =$$

$$- \nu_{opt|p} (T_2 Q)^\top \Phi dm = P_1^1 dm.$$

The second property is satisfied.

3) Now we have

$$T_2^\top dx = -\nu_{opt|p} T_2^\top d\varphi = -\nu_{opt|p} P_2 dm = P_1^2 dm.$$

We have checked that all properties are satisfied.

Define δ as the density $\delta d\mu = dx$. Then $\delta\sqrt{(\Phi^*\Phi)} = -\nu_{opt|p}\Phi$ and $\delta = -\nu_{opt|p}\Phi(\Phi^*\Phi)^{\frac{\dagger}{2}}$.

Now we calculate the infinity norm of δ. We have

$$\delta^*\delta = \nu_{opt|p}^2 (\Phi^*\Phi)^{\frac{\dagger}{2}} \Phi^*\Phi(\Phi^*\Phi)^{\frac{\dagger}{2}} = \nu_{opt|p}^2 \mathbb{I}$$

and $\|\delta\|_\infty = \nu_{opt|p}$. It means that $\nu_{opt|d} \leq \nu_{opt|p}$.

Suppose now that our measure $d\varphi_s$ is singular, i.e. $d\varphi_s = yy^*\delta_{z_0}dm$, where δ_{z_0} is the Dirac function and y is such that $T_2^\top(z_0)y = \mathbb{O}$. Then $d\mu_s = d\varphi_s$ and $dx_s = -\nu_{opt|p}d\varphi_s$. Again we have to check all the properties:

1) $\|d\mu_s\|_1 = 1$,

2) $d\mu_s + T_1^\top dx_s = \mathbb{O}$,

3) $T_2^\top dx_s = \mathbb{O}$.

Note that

$$\|T_1 - T_2Q\|_\infty \left| \int \mathrm{Tr}\, \frac{(T_1 - T_2Q)^\top}{\|T_1 - T_2Q\|_\infty} d\mu_s \right| = \|T_1 - T_2Q\|_\infty \left| \int \mathrm{Tr}\, \frac{T_1^\top}{\|T_1 - T_2Q\|_\infty} d\mu_s \right| \leq \|T_1 - T_2Q\|_\infty \|d\mu_s\|.$$

Therefore $\|d\mu_s\|_1 = 1$ and

$$d\mu_s = \frac{T_1^\top d\mu_s}{\|T_1 - T_2Q\|_\infty} = \nu_{opt|p} T_1^\top d\mu_s.$$

Then

$$d\mu_s + T_1^\top dx_s = \nu_{opt|p}T_1^\top d\mu_s - \nu_{opt|p}T_1^\top d\varphi_s = \nu_{opt|p}T_1^\top d\varphi_s - \nu_{opt|p}T_1^\top d\varphi_s = \mathbb{O}.$$

The third property is obvious:

$$T_2^\top dx_s = -\nu_{opt|p}T_2^\top(z_0)y = \mathbb{O}.$$

Again we define δ as the density $\delta d\mu_s = dx_s$. Then $\delta yy^* \delta_{z_0} dm = -\nu_{opt|p} yy^* \delta_{z_0} dm$ and δ can be choosen as $\delta = -\nu_{opt|p} I$. Then $\|\delta\|_\infty = \nu_{opt|p}$. We have shown that also in the singular case $\nu_{opt|d} \leq \nu_{opt|p}$.

$\square$

Corollary 1. *The singular and the regular cases of the dual problem is equivalent to the singular and the regular cases of the $\mathbf{H}^\infty$ optimization problem.*

Proof. From the proof of Theorem 25 we have the following connection between the optimal measure $d\mu$ in the dual problem and the optimal measure $d\varphi$ in the $\mathbf{H}^\infty$ optimization problem

$$d\varphi = -\frac{\delta d\mu}{\|\delta\|_\infty}.$$

It means that if the optimal measure in the dual problem is regular then the optimal measure in the $\mathbf{H}^\infty$ optimization problem will be the regular as well, and vice versa. The same is valid in the singular case. $\square$

5.5 The main result

The purpose of this section is to obtain the canonical parametrization of the dual problem. We begin with a useful lemma

Lemma 7. *Let $\Delta = \mathcal{B}\mathbb{C}^{m \times m}$ and $\nu_{opt|c} < \nu_{opt|s}$. Then the solution to the regular part of the dual problem satisfies $\delta_{opt} \in \overline{\mathbf{H}^\infty}$.*

Proof. Consider the primal problem

$$\sup_{h \in \mathcal{B}\mathbf{H}^\infty} \inf_{\|\delta\| \leq \nu} Re(F + \delta^\top G)h > 0$$

and let $Gh \neq 0$. Let h_{opt} be the solution of the problem. Then the optimal δ has the following expression

$$\delta^\top = -\nu \frac{(Gh_{opt})^*}{\overline{\sigma}(Gh_{opt})}.$$

We will show that $\overline{\sigma}(Gh_{opt}) \equiv const$.

Recall that if h_{opt} is such that $Fh_{opt} = I$ is the solution of the primal problem then $g_{opt} = h_{opt}$ is the solution of $\mathbf{H}^\infty$ optimization problem. Then

$$\|Gg_{opt}\|_\infty = \inf_{Q \in \mathbf{H}^\infty} \|T_1 - T_2 Q\|_\infty = \inf_{Q \in \mathbf{H}^\infty} \left\| \begin{pmatrix} T_{2i}^* \\ I - T_{2i}T_{2i}^* \end{pmatrix} (T_1 - T_2 Q) \right\|_\infty$$

$$\inf_{Q \in \mathbf{H}^\infty} \left\| \begin{pmatrix} T_{2i}^* T_1 - T_{2o}Q \\ (I - T_{2i}T_{2i}^*)T_1 \end{pmatrix} \right\|_\infty .$$

It was shown in Chapter 2 that inner functions in the disk algebra $\mathbf{A}$ are finite Blashke products. It implies that $T_{2i}^* \in \mathcal{C}$.

Let $\Phi = \begin{pmatrix} T_{2i}^* T_1 \\ (I - T_{2i}T_{2i}^*)T_1 \end{pmatrix}$. In [19] a four block operator Γ_Φ was defined as operator

$$\Gamma_\Phi : \mathbf{H}^2 \oplus \mathbf{L}^2 \rightarrow \mathbf{H}_-^2 \oplus \mathbf{L}^2$$

such that

$$\Gamma_\Phi \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \mathbb{P}^- \Phi \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},$$

where $\mathbb{P}^-$ is the ortogonal projection from $\mathbf{L}^2 \oplus \mathbf{L}^2$ onto $\mathbf{H}_-^2 \oplus \mathbf{L}^2$, Φ is a block matrix function.

The function $\Phi \in \mathbf{H}^\infty + \mathcal{C}$. Therefore the essential norm of operator Γ_Φ is zero and $\|\Gamma_\Phi\|_e < \|\Gamma_\Phi\|$.

Since we consider the regular case of the dual problem and it is equivalent to the condition that $\det T_2^* T_2(z) \neq 0 \; \forall z \in \mathbb{T}$, then T_{2o} has an inverse in $\mathbf{H}^\infty$ and

$$\inf_{Q \in \mathbf{H}^\infty} \left\| \begin{pmatrix} T_{2i}^* T_1 - T_{2o}Q \\ (I - T_{2i}T_{2i}^*)T_1 \end{pmatrix} \right\|_\infty = \inf_{Q_1 \in \mathbf{H}^\infty} \left\| \begin{pmatrix} T_{2i}^* T_1 - Q_1 \\ (I - T_{2i}T_{2i}^*)T_1 \end{pmatrix} \right\|_\infty .$$

In [19] it has been shown that if Q_{1opt} is an optimal solution to this approximation problem then the matrix function $\begin{pmatrix} T_{2i}^* T_1 - Q_{1opt} \\ (I - T_{2i}T_{2i}^*)T_1 \end{pmatrix}$ admits the following representation

$$\begin{pmatrix} T_{2i}^* T_1 - Q_{1opt} \\ (I - T_{2i}T_{2i}^*)T_1 \end{pmatrix} = W^* \begin{pmatrix} t_0 u_0 & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \Phi_{11}^{(1)} - Q_2 & \Phi_{12}^{(1)} \\ \mathbb{O} & \Phi_{21}^{(1)} & \Phi_{22}^{(1)} \end{pmatrix} V^*,$$

where $u_0 = \overline{z}\overline{b}\overline{h}/h$, h is a scalar outer function, b is a finite Blashke product, V, W are unitary matricies $V = \begin{pmatrix} v & \overline{V_c} & \star \end{pmatrix}$, $W^\top = \begin{pmatrix} w_1 & \overline{W_c} & \star \\ w_2 & \mathbb{O} & \star \end{pmatrix}$, with

inner vector functions v, w_1, w_2, co-outer matrix functions V_c, W_c and $Q_2 \in \mathbf{H}^\infty$ such that $\left\| \begin{array}{cc} \Phi_{11}^{(1)} - Q_2 & \Phi_{12}^{(1)} \\ \Phi_{21}^{(1)} & \Phi_{22}^{(1)} \end{array} \right\|_\infty \leq t_0$.

For Q_{1opt} we have

$$\overline{\sigma}(T_{2i}^* T_1 - Q_{1opt}) = \overline{\sigma}\left(\left(\begin{array}{ccc} \overline{w} & W_c & \star \end{array} \right) \left(\begin{array}{ccc} t_0 u_0 & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \Phi_{11}^{(1)} - Q_2 & \Phi_{12}^{(1)} \\ \mathbb{O} & \Phi_{21}^{(1)} & \Phi_{22}^{(1)} \end{array} \right) \left(\begin{array}{c} v^* \\ V_c^\top \\ \star \end{array} \right) \right) \Rightarrow$$

$$\overline{\sigma}(G g_{opt}(z)) = \overline{\sigma}(T_{2i}^* T_1 - Q_{1opt}) = \overline{\sigma}(\overline{w} t_0 u_0 v^*(z)) \equiv const,$$

where $g_{opt}(z)$ is the solution to $\mathbf{H}^\infty$ optimization. Then $\overline{\sigma}(G h_{opt}(z)) \equiv const$, where h_{opt} is the solution to the primal problem.
Since

$$\delta^\top = -\nu \frac{(G h_{opt})^*}{\overline{\sigma}(G h_{opt}(z))},$$

$\overline{\sigma}(G h_{opt}(z)) \equiv const$, then $\delta \in \overline{\mathbf{H}^\infty}$.

$\square$

Introduce the class $\mathcal{U}$ of all unitar matrix functions with well-defined and finite winding numbers,

$$\mathcal{U} = \{ U \in \mathbf{L}^\infty : U^* U = I, \text{ wno} \det U > 0 \}.$$

We prove now the main theorem.

Theorem 26 (Main result). *Let $\Delta = \mathcal{B}\mathbb{C}^{m \times m}$ and $\nu_{opt|c} < \nu_{opt|s}$. Then the plant numerator and the denominator with the worst uncertainty $\Phi_\delta = F + \delta^\top G$ can be factorized in the following way*

$$\Phi_\delta = U \Psi, \tag{5.6}$$

where $U \in \mathcal{U}$ and $\Psi \in \mathbf{H}^\infty$ is outer.

Proof. We will proof the theorem in several steps.

1) We will get expression for Φ_δ. Recall that in the regular case we have $W \Phi_\delta = zP$, where W is a positive semi-definite matrix function with summable entries. Let W have rank r. Note that in the regular case Φ_δ does not lose the rank which implies that $\log Det\, W \in \mathbf{L}^1$, where $Det\, W$ is the determinant of W as a transformation on its range. Since P is an analytic

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function, the range $\mathfrak{R}$ of W is an analytic function. According to Theorem 13 in [14] W can be factorized as $W = (VB)(VB)^*$, where V is a co-outer partial isometry mapping a fixed space of r dimensions onto the range $\mathfrak{R}$ of W, and B is an outer function of full rank r. We get

$$VBB^*V^*\Phi_\delta = zP.$$

We can find $P_1 \in \mathbf{H}^1$ such that $P = VP_1$. Then

$$V^*\Phi_\delta = (BB^*)^{-1}zP_1$$

and

$$\Phi_\delta = V((BB^*)^{-1}zP_1)) + V_\perp \Upsilon = \begin{pmatrix} V & V_\perp \end{pmatrix} \begin{pmatrix} (BB^*)^{-1}zP_1 \\ \Upsilon \end{pmatrix},$$

where $V_\perp$ is the complement of V to the unitary matrix. We choose $V_\perp$ with some special properties, namely $(V_\perp)^* \in \mathbf{H}^\infty$ is co-inner and outer. According to Theorem 14.1.1 [18] we can always find such $V_\perp$ with demanded properties if V is inner and co-outer. The matrix function $\Xi = \begin{pmatrix} V & V_\perp \end{pmatrix}$ is called an $r-$balanced matrix function. In [18] was shown that if Ξ is a balanced matrix function then $\det \Xi$ is the constant function.

2) Let $\tilde{\Phi}_\delta = (BB^*)^{-1}zP_1$. We will show that $\tilde{\Phi}_\delta$ can be factorized in the following way

$$\tilde{\Phi}_\delta = \tilde{U}\tilde{\Psi},$$

where $\tilde{U} \in \mathcal{U}$ and $\tilde{\Psi} \in \mathbf{H}^\infty$ is outer.

Take outer function C such that $B^*B = CC^*$ and get

$$\tilde{\Phi}_\delta = (BB^*)^{-1}zP_1 = B^{-*}zCC^{-1}B^{-1}P_1 = B^{-*}zCP_2,$$

where $P_2 \in \mathbf{H}^\infty$. We can factorize P_2 as $P_2 = P_{2i}P_{2o}$, where P_{2i} is inner (square) function and P_{2o} is outer function. Recall that $\delta \in \overline{\mathbf{H}^\infty}$, and F, G are from the disk-algebra $\mathbf{A}$. Then $\tilde{\Phi}_\delta^* \in \mathbf{H}^\infty + \mathcal{C}$ and

$$\tilde{\Phi}_\delta^* = P_{2o}^*P_{2i}^*C^*\bar{z}B^{-1}.$$

Take outer function T such that $P_{2o}P_{2o}^* = TT^*$ such that T is invertible in $\mathbf{H}^\infty$, and we get

$$\tilde{\Phi}_\delta^* = P_{2o}^*T^{-*}T^*P_{2i}^*C^*\bar{z}B^{-1}.$$

Such T exists according to Theorem 13 in [14].

Since $T^{-1}P_{2o}P_{2o}^*T^{-*} = T^{-1}TT^*T^{-*} = I$, then $P_{2o}^*T^{-*}$ is an unitar function and

$$T^{-1}P_{2o}\tilde{\Phi}_\delta^* = T^*P_{2i}^*C^*\bar{z}B^{-1} = (CP_{2i}T)^*\bar{z}B^{-1}$$

and

$$T^{-1}P_{2o}\tilde{\Phi}_\delta^*B = (CP_{2i}T)^*\bar{z}.$$

Now we will normalize the rows of $CP_{2i}T \in \mathbf{H}^\infty$. In every row j let θ_j be the greatest common inner divisor of the entries, i.e. θ_j is a scalar inner function such that all entries of the row are divisible by θ_j and any other common inner divisor of all entries of the row is also a divisor of θ_j. Let f_j be a scalar outer function such that

$$(CP_{2i}T)_j^*(CP_{2i}T)_j = f_j^*f_j.$$

Then we can factorize $CP_{2i}T$ as follows

$$CP_{2i}T = \begin{pmatrix} \theta_1 f_1 & & 0 \\ & \ddots & \\ 0 & & \theta_r f_r \end{pmatrix} C\tilde{P}_{2i}T.$$

Every row of $C\tilde{P}_{2i}T$ is normalized, i.e. $C\tilde{P}_{2i}T(C\tilde{P}_{2i}T)^* = \begin{pmatrix} 1 & & \star \\ & \ddots & \\ \star & & 1 \end{pmatrix}$. Multiply $T^{-1}P_{2o}\tilde{\Phi}_\delta^*B$ by S^{-1}, where $S = \begin{pmatrix} f_1 & & 0 \\ & \ddots & \\ 0 & & f_r \end{pmatrix}$.

Then

$$T^{-1}P_{2o}\tilde{\Phi}_\delta^*BS^{-1} = (C\tilde{P}_{2i}T)^* \begin{pmatrix} \dfrac{\bar{f}_1\bar{z}\bar{\theta}_1}{f_1} & & 0 \\ & \ddots & \\ 0 & & \dfrac{\bar{f}_r\bar{z}\bar{\theta}_r}{f_1} \end{pmatrix}.$$

Let $u_j = \dfrac{\bar{f}_j\bar{z}\bar{\theta}_j}{f_j}$. Now we will show that u_j is a quasicontinuous function. Since $C\tilde{P}_{2i}T \in \mathbf{H}^\infty$, $T^{-1}P_{2o} \in \mathbf{H}^\infty$, $\tilde{\Phi}_\delta^* \in \mathbf{H}^\infty + \mathcal{C}$ and $BS^{-1} \in \mathbf{H}^\infty$, then

$$\begin{pmatrix} u_1 & & \star \\ & \ddots & \\ \star & & u_r \end{pmatrix} = (C\tilde{P}_{2i}T)T^{-1}P_{2o}\tilde{\Phi}_\delta^*BS^{-1} \in \mathbf{H}^\infty + \mathcal{C}.$$

Every $u_j \in \mathbf{H}^\infty + \mathcal{C}$. By Theorem 1.5.8 in [18] we have $\mathbb{P}_- u_j \in VMO$ and $u_j f_j = \bar{f}_j \bar{z} \bar{\theta}_j \in \mathbf{H}^2_-$. It means that $f_j \in \operatorname{Ker} \mathrm{T}_{u_j}$ and $\operatorname{Ker} \mathrm{T}^*_{u_j} = \{0\}$. It follows that the Toeplitz operator T_{u_j} has dense range in $\mathbf{H}^2$. Indeed, suppose that $f \perp \mathrm{T}_{u_j} \mathbf{H}^2$ and $f \neq 0$. Then $0 = (f, u_j g) = (\overline{u_j} f, g)$ for any $g \in \mathbf{H}^2$ and $f \in \operatorname{Ker} \mathrm{T}^*_{u_j}$, which is impossible since $\operatorname{Ker} \mathrm{T}^*_{u_j} = \{0\}$.

By Theorem 7.1.3 in [18] $\mathbb{P}_+ u_j \in VMO$ and $u_j \in VMO$, it means that u_j is a quasicontinuous function.

As $u_j \in \mathcal{QC}$, then the Toeplitz operator T_{u_j} is Fredholm by Theorem 3.3.2 in [18]. Therefore $\dim \operatorname{Ker} \mathrm{T}_{u_j} < \infty$ and $\dim \operatorname{Ker} \mathrm{T}^*_{u_j} < \infty$. Recall that $\operatorname{Ker} \mathrm{T}^*_{u_j} = \{0\}$. It means that $\operatorname{ind} \mathrm{T}_{u_j} = \dim \operatorname{Ker} \mathrm{T}_{u_j} - \dim \operatorname{Ker} \mathrm{T}^*_{u_j} > 0$ and we get that $-\infty < \operatorname{wno} u_j < 0$ since $\operatorname{wno} u_j = -\operatorname{ind} \mathrm{T}_{u_j}$.

We have
$$(CP_{2i}T)^* \bar{z} S^{-1} B^{-1} = (C\tilde{P}_{2i}T)^* U_1^* B^{-1},$$
where U_1 is an unitar matrix function such that $r < \operatorname{wno} U_1 < \infty$.

Finally,

$\operatorname{wno} \det(CP_{2i}T)^* \bar{z} S^{-1} B^{-1} = \operatorname{wno} \det T^* + \operatorname{wno} \det S^{-1} + \operatorname{wno} \det P_{2i}^* C^* \bar{z} B^{-1} = \operatorname{wno} \det(C\tilde{P}_{2i}T)^* + \operatorname{wno} \det U_1^* + \operatorname{wno} \det B^{-1}.$

Since $\operatorname{wno} \det T^* + \operatorname{wno} \det S^{-1} = 0$ and $\operatorname{wno} \det(C\tilde{P}_{2i}T)^* + \operatorname{wno} \det B^{-1} = 0$, we have $\operatorname{wno} \det P_{2i}^* C^* \bar{z} B^{-1} = \operatorname{wno} \det U_1^*$ and
$$\tilde{\Phi}_\delta = \tilde{U}\tilde{\Psi},$$
where $\tilde{U} = B^{-*} z C P_{2i}$ is an unitar matrix function such that $r < \operatorname{wno} \det \tilde{U} < \infty$ and $\tilde{\Psi} = P_{2o}$ is an outer function.

3) We will show that $\Upsilon \in \mathbf{H}^\infty$. Recall that $(\ V \quad V_\perp\ )$ is a square unitary matrix. It means that $(\ V \quad V_\perp\ )^* (\ V \quad V_\perp\ ) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$ and

$$(\ V \quad V_\perp\ )(\ V \quad V_\perp\ )^* = VV^* + V_\perp V_\perp^* = I.$$

As $F + \delta^\top G = V((BB^*)^{-1} z P_1)) + V_\perp \Upsilon$, then we get

$$\delta^\top = \left((\ V \quad V_\perp\ ) \begin{pmatrix} \tilde{U}\tilde{\Psi} \\ \Upsilon \end{pmatrix} - F \right) G^\dagger =$$

$$\left((\ V \quad V_\perp\ ) \begin{pmatrix} \tilde{U}\tilde{\Psi} \\ \Upsilon \end{pmatrix} - (\ V \quad V_\perp\ ) \begin{pmatrix} V^* \\ V_\perp^* \end{pmatrix} F \right) G^\dagger =$$

$$(\ V \quad V_\perp\ ) \begin{pmatrix} \tilde{U}\tilde{\Psi} - V^* F \\ \Upsilon - V_\perp^* F \end{pmatrix} G^\dagger.$$

As (V $V_\perp$) is an unitary matrix, then we get

$$\|\delta_{opt}\|_\infty = \inf_{V,V_\perp,\tilde{U},\tilde{\Psi},\Upsilon} \left\| (V \quad V_\perp) \begin{pmatrix} \tilde{U}\tilde{\Psi} - V^*F \\ \Upsilon - V_\perp^* F \end{pmatrix} G^\dagger \right\|_\infty =$$

$$\inf_{V,V_\perp,\tilde{U},\tilde{\Psi},\Upsilon} \left\| \begin{pmatrix} \tilde{U}\tilde{\Psi} - V^*F \\ \Upsilon - V_\perp^* F \end{pmatrix} G^\dagger \right\|_\infty = \inf_{V,\tilde{U},\tilde{\Psi}} \left\| \begin{pmatrix} \tilde{U}\tilde{\Psi} - V^*F \\ 0 \end{pmatrix} G^\dagger \right\|_\infty$$

and $\Upsilon_{opt} = V_{\perp opt}^* F$. Since $V_\perp^* \in \mathbf{H}^\infty$ and $F \in \mathbf{H}^\infty$, we have that $\Upsilon \in \mathbf{H}^\infty$ as well.

4) Now we will get the parametrization $\Phi_\delta = U\Psi$, where $U \in \mathcal{U}$ and $\Psi \in \mathbf{H}^\infty$ is outer.

We have

$$\Phi_\delta = (V \quad V_\perp) \begin{pmatrix} \tilde{U}\tilde{\Psi} \\ \Upsilon \end{pmatrix} = (V \quad V_\perp) \begin{pmatrix} \tilde{U} & 0 \\ 0 & \Upsilon_i \end{pmatrix} \begin{pmatrix} \tilde{\Psi} \\ \Upsilon_o \end{pmatrix},$$

where Υ_i, Υ_o inner and outer functions such that $\Upsilon = \Upsilon_i \Upsilon_o$. Denote $U = (V \quad V_\perp) \begin{pmatrix} \tilde{U} & 0 \\ 0 & \Upsilon_i \end{pmatrix}$ and $\Psi = \begin{pmatrix} \tilde{\Psi} \\ \Upsilon_o \end{pmatrix}$.

Recall that $\det(V \quad V_\perp)$ is a constant function. Then

$$\text{wno} \det U = \text{wno} \det(V \quad V_\perp) + \text{wno} \det \begin{pmatrix} \tilde{U} & 0 \\ 0 & \Upsilon_i \end{pmatrix} =$$

$$0 + \text{wno} \det \tilde{U} + \text{wno} \det \Upsilon_i > 0.$$

It remains to show that $\text{wno} \det \Upsilon_i < \infty$. Since $\delta \in \overline{\mathbf{H}^\infty}$, $F, G \in \mathbf{A}$, then $\Phi_\delta^* \in \mathbf{H}^\infty + \mathcal{C}$ and $(V_\perp \Upsilon)^* = \Upsilon^* V_\perp^* \in \mathbf{H}^\infty + \mathcal{C}$. But $\Upsilon \in \mathbf{H}^\infty$ and $V_\perp^* \in \mathbf{H}^\infty$, then Υ have to be in a disk-algebra $\mathbf{A}$. Since Υ is continuous, Υ_i is a finite Blashke product and $\text{wno} \det \Upsilon_i < \infty$.

It's clear that U is unitar matrix. We have shown that the winding number of U is finite and $\text{wno} \det U > 0$. Then $U \in \mathcal{U}$ and $\Psi \in \mathbf{H}^\infty$ is outer. We get the parametrization $\Phi_\delta = U\Psi$.

$\square$

5.6 The classes of equivalences and the distance between the classes.

In the scalar case we have introduced the class of equivalence and distance δ between them. It was shown that $\delta-$metric has a close connection with

the $\nu-$gap metric. In this section we will generalize the result to the matrix case.

Consider the dual parametrization

$$F + \delta^\top G = U\Psi, \ U \in \mathcal{U}, \Psi \in \mathbf{H}^\infty.$$

We have seen that for the optimal δ we have wno $\det U > 0$, i.e. on the boundary of the stability region, we have a non-stabilizable plant. This gives an idea to define the classes of equivalence in the factor space and to introduce a distance between them.

Let $\mathcal{U}_0$ be the class of all unitar matrix functions with well-defined winding numbers such that

$$\mathcal{U}_0 = \{U \in \mathbf{L}^\infty : U^*U = I, \ \text{wno} \det U = 0\}.$$

Suppose that we have two plants P_1 and P_2. Let $F_1 = \begin{bmatrix} -M_1 & N_1 \end{bmatrix}$ be a left coprime factorization of P_1 and $F_2 = \begin{bmatrix} -M_2 & N_2 \end{bmatrix}$ be a left coprime factorization of P_2. We will call two factorizations *equivalent* $F_1 \sim F_2$ if we can find Q such that $Q \in \mathcal{U}_0$ and $F_2 = QF_1$.

Define the distance between equivalent classes as follows

$$\delta(F_1, F_2) = \inf\{\|\delta\|_\infty \ : \ F_1 + \delta^\top G \in \text{class}(F_2)\}.$$

Lemma 8. δ *is a metric on the factor space of the plant factorizations.*

Proof. A function $\delta(\cdot, \cdot)$ is a metric on the space of factorization if, for all factorizations F_1, F_2 and F_3

1) $\delta(F_1, F_2) \geq 0$, with $\delta(F_1, F_2) = 0$ if and only if $F_1 \sim F_2$
2) $\delta(F_1, F_2) = \delta(F_2, F_1)$
3) $\delta(F_1, F_2) \leq \delta(F_1, F_3) + \delta(F_3, F_2)$.

1) It's clear that $\delta(F_1, F_2) = \inf \|\delta\|_\infty \geq 0$. Next if $F_1 \sim F_2$ then $F_1 \in \text{class}(F_2)$ so $\delta(F_1, F_2) = 0$.

If now $\delta(F_1, F_2) = 0$ so we can find Q with the property from above so $F_1 + \delta^\top G = QF_2$, where δ is such that $\inf \|\delta\|_\infty = 0$. It means that $F_1 \in \text{class}(F_2)$ and $F_1 \sim F_2$.

To prove 2) take δ_1 such that $F_1 + \delta_1^\top G = Q_1 F_2$ and δ_2 such that $F_2 + \delta_2^\top G = Q_2 F_1$. We will show that $\inf \|\delta_1\|_\infty = \inf \|\delta_2\|_\infty$. Multiplying the first equation by Q_2 we get

$$Q_2 F_1 + Q_2 \delta_1^\top G = F_2 + \delta_2^\top G + Q_2 \delta_1^\top G = F_2 + (\delta_2^\top + Q_2 \delta_1^\top) G = Q_2 Q_1 F_2.$$

It means that $\inf \|\delta_2^\top + Q_2 \delta_1^\top\|_\infty = 0$. Then

$$0 = \|\delta_2^\top + Q_2 \delta_1^\top\|_\infty \geq \|\delta_2\|_\infty - \|Q_2 \delta_1^\top\|_\infty$$

and $\|\delta_2\|_\infty \leq \|Q_2 \delta_1^\top\|_\infty = \|\delta_1\|_\infty$. Therefore $\inf \|\delta_2\|_\infty \leq \inf \|\delta_1\|_\infty$ and $\delta(F_2, F_1) \leq \delta(F_1, F_2)$. By multiplying the second equation by Q_1 we can show in the same way as above that $\delta(F_1, F_2) \leq \delta(F_2, F_1)$. Then $\delta(F_2, F_1) = \delta(F_1, F_2)$.

To prove 3) take δ_{13} such that we can find Q_1 and $F_1 + \delta_{13}^\top G = Q_1 F_3$. Take δ_{23} such that we can find Q_2 and $F_3 + \delta_{23}^\top G = Q_2 F_2$. Then by the straightforward calculations we get

$$F_1 + \delta_{13}^\top G = Q_1 Q_2 F_2 - Q_1 \delta_{23}^\top G,$$

$$F_1 + (\delta_{13}^\top + Q_1 \delta_{23}^\top) G = Q_1 Q_2 F_2,$$

where wno det $Q_1 Q_2 =$ wno det $Q_1 +$ wno det $Q_2 = 0$. It means that $F_1 \in \text{class}(F_2)$.

Then $\delta(F_1, F_2) \leq \|\delta_{13}^\top + Q \delta_{23}^\top\|_\infty \leq (\|\delta_{13}\|_\infty + \|Q \delta_{23}^\top\|_\infty) = (\|\delta_{13}\|_\infty + \|\delta_{23}\|_\infty)$. By calculating the infimum over δ_{13} and δ_{23} we get $\delta(F_1, F_2) \leq \delta(F_1, F_3) + \delta(F_3, F_2)$. $\qquad\square$

Now we can get an expression for our metric $\delta(F_1, F_2)$. By the definition of $\delta(F_1, F_2)$ we are looking for a δ (we apologize for the abuse of notation) such that $F_1 + \delta^\top G \in \text{class}(F_2)$. It means that there exists $Q \in \mathcal{U}_0$ such that $F_1 + \delta^\top G = Q F_2$. We have to take infimum over all such Q to find the optimal δ, i.e. the distance between equivalent classes, so we get that

$$\delta(F_1, F_2) = \inf_{Q \in \mathcal{U}_0} \|F_1 G^\dagger - Q F_2 G^\dagger\|_\infty$$

In [13] the ν-gap metric between two plants P_1 and P_2 was introduced as

$$\delta_\nu(P_1, P_2) = \inf_{\substack{Q, Q^{-1} \in \mathbf{L}_\infty \\ \text{wno det}(Q) = 0}} \|G_1 - G_2 Q\|_\infty,$$

where G_1 and G_2 are the *normalized* right coprime factorizations of P_1 respectively P_2.

In particular, if $G = I$ and F_1 is the normalized coprime factorization of the nominal plant P_1, it is clear that $\delta \geq \delta_\nu$. Furthermore, the distance from P_1 to the set of all non-stabilizable plants is the same ν_{opt} in both metrics. In all other cases, the ν-gap metric does not generally provide the optimal stability margin, whereas the δ-metric does. In this sense, the δ-metric is more appropriate to robust stabilization of parametrically uncertain plants.

5.7 Conclusion

In this chapter we discuss the problem of finding the optimal stability margin for the system with matrix uncertainty. We introduce the dual problem. In the general case there is a gap between the solutions of the primal and dual problems. Therefore we cannot say that for every ν either the primal or the dual problem has a solution. But we show that, under the unstructured uncertainty condition, there is no duality gap between these two problems.

Our main result is to find a parametrization of the plant with the worst uncertainty. The regular part of the dual problem is complicated because it is stated in terms of sequences of functions. Our parametrization may help to deduce a simpler algorithm for calculation of the low bound of the optimal stability margin and for constructing of the suboptimal controller. The parametrization is not restricted to the class of rational functions but can be used for a large class of systems. We demand only that F and G are in the disk-algebra A.

We also introduce a metric in the space of factorizations and show that this metric has close connection to ν-gap metric.

Bibliography

[1] Ghulchak A. Duality in Robust Control: Uncertainty vs. Controller. *in Conference of Decision and Control*, 2001.

[2] Ghulchak A. Robust Stabilizability and Low-Order Controllers: Duality Principle in Case Studies. Conference on Decision and Control, Atlantis, Bahamas, December 14-17 2004.

[3] Ghulchak A. and Rantzer A. Duality in $\mathbf{H}^\infty$ cone optimization. *SIAM J.Control Optim.*, 2002.

[4] Ghulchak A. and Rantzer A. Robust Control Under Parametric Uncertainty Via Primal-Dual Convex Analysis. *IEEE Transaction on Automatic Control*, 47(4), 2002.

[5] Rantzer A. and Megretski A. A convex parametrization of robustly stabilizing controllers. *IEEE Transaction on Automatic Control*, 39(9):1802–1808, 1994.

[6] Ostrowski A.M. and Taussky O. On the Variation of the Determinant of a Positive Definite Matrix. *Nederl. Indag. Math.*, 13:383–385, 1951.

[7] Garnett John B. *Bounded analytic functions*. Academic Press,Inc., 1987.

[8] Francis B.A. *A Course in $\mathbf{H}^\infty$ Control Theory*. Springer-Verlag, 1987.

[9] Nirenberg L. Brzis H. and Stampacchia. A remark on Ky Fan's minimax principle. *Boll. Un. Mat. Ital.*, 6, 1972.

[10] Tannenbaum A. Foias C. and Zames G. Weighted Sensitivity Minimization for Delay Systems. *IEEE Transaction on Automatic Control*, AC-31(8):763–766, 1986.

[11] Gantmaher F.R. *Teoria matric*. Nauka, 1966.

[12] Luenberger D. G. *Optimization By Vector Space Methods*. John Wiley & Sons, Inc., 1969.

[13] Vinnicombe G. *Uncertainty And Feedback*. Imperial College Press, 2001.

[14] Helson H. and Lowdenslager D. Prediction theory and fourier series in several variables. II. *Acta Mathematica*.

[15] Robertson J.B. and Rosenberg M. The decomposition of matrix-valued measures. *Michigan Math. J.*, Volume 15:353–368, 1968.

[16] Vidyasagar M. *Control system synthesis : a factorization approach*. Cambridge,Mass.:MIT Press,, 1985.

[17] Chandrasekharan P.C. *Robust Control of Linear Dynamical Systems*. Academic Press, 1996.

[18] Peller V. *Hankel Operator and Their Applications*. Springer, 2003.

[19] Peller V. and Treil S. Approximation by analytic matrix functions: the four block problem. *J. Funct. Anal.*, 148(1):191–228, 1997.

[20] Rudin W. *Real and Complex Analysis*. McGraw-Hill,Inc., 1987.

[21] Jabr H.A. Youla D.C. and Bongiorno J.J. Modern *Wiener-H*opf design of optimal controllers: Part *II*. *IEEE Transaction on Automatic Control*, 21:319–338, 1976.

[22] Doyle J.C. Zhou K. *Essential of Robust Control*. Prentice Hall, 1998.