Minimal Time Problem for an Inverted Pendulum. Maximun Principle and Phase Plane Discussion

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MINIMAL TIME PROBLEM FOR AN INVERTED PENDULUM.
MAXIMUM PRINCIPLE AND PHASE PLANE DISCUSSION. †

Per Hagander

ABSTRACT

The problem of raising a mathematical pendulum from the stable equilibrium to the unstable equilibrium using horizontal acceleration of the pivot point is discussed. A minimum time strategy is investigated both by means of the Maximum Principle and by phase plane argumentation.

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1. INTRODUCTION - SYSTEM EQUATIONS AND PROBLEM FORMULATION

Consider a mathematical pendulum which is controlled by the horizontal acceleration of the pivot point:

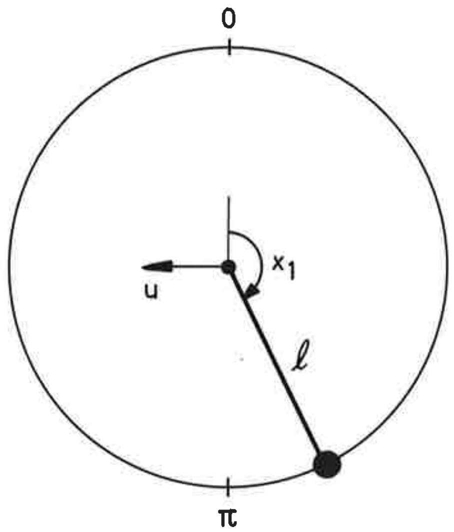


Fig. 1

The system is described by these equations:

$$\begin{cases} \frac{d}{dt} x_1 = x_2 \\ \frac{d}{dt} x_2 = \sin x_1 + u \cdot \cos x_2 \end{cases} \qquad (1)$$

where "time" t is normalized by $\omega = \sqrt{g/\ell}$, ℓ is the length of the pendulum, thus $t = \omega \times$ real time. The acceleration of the pivot point, u, is normalized by g. x_1 is the angle to the unstable equilibrium point.

The control is supposed to be done by a servo, and it is therefore natural to have u constrained:

$$|u| < u_m \qquad (2)$$

(with $u_m = 3$, for instance, i.e. maximum acceleration $\approx 30\ m/s^2$).

Determine now the strategy $u = u(t)$, to go from

$$x(0) = \begin{pmatrix} \pi \\ 0 \end{pmatrix} \quad \text{(the stable equilibrium point)}$$

to

$$x(T) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{(the unstable equilibrium point)}$$

in shortest possible time under the constraint:

$|u| < u_m$!

Choose first $u_m = 3$.

The solution to this problem is the control law (intuitively derived in [4]):

$$u^* = \begin{cases} +3 & x_1 \in (0, \alpha) \text{ or } x_1 \in (\pi/2, \pi) \\ -3 & x_1 \in (\alpha, \pi/2) \end{cases} \tag{3}$$

with $\alpha = \arcsin 2/3$.

In the next section the formulation of the Maximum Principle is written down and applied to the problem. The necessary conditions on an optimal strategy are stated and simplified. By means of some theorems on second order differential equations it is then found that the proposed u^* fulfils the conditions.

In the second section the problem is discussed from another point of view. By arguments in the phase plane it is possible to obtain (3) strictly, and it is also possible to generalize to other values of u_m.

2. THE MAXIMUM PRINCIPLE ATTACK.

2.1. Determination of the necessary conditions.

Introduce the Hamiltonian H and the adjoint vector p for the problem to bring the state vector of the system:

$$\begin{cases} \dot{x} = f(x,y) \\ x(0) = c \end{cases}$$

to a target set S, minimizing the functional:

$$\int_0^T L(x,u)\, dt .$$

With

$$f(x,u) = \begin{pmatrix} x_2 \\ \sin x_1 + u \cos x_1 \end{pmatrix}, \quad L(x,u) = 1$$

and

$$S = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

we get

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \sin x_1 + u \cdot \cos x_1 \end{cases} \qquad (4)$$

$$\begin{cases} \dot{p}_1 = - p_2(\cos x_1 - u \sin x_1) \\ \dot{p}_2 = - p_1 \end{cases} \qquad (5)$$

$$\begin{cases} x_1(0) = \pi \\ x_2(0) = 0 \end{cases} \qquad (6)$$

$$\begin{cases} x_1(T) = 0 \\ x_2(T) = 0 \end{cases} \tag{7}$$

$$H = 1 + p_1 x_2 + p_2(\sin x_1 + u \cos x_1) \tag{8}$$

The admissible controls are restricted by:

$$U = \{|u| \leq 3\} \tag{2}$$

u piecewise continuous.

Define:

$$M(x,p) = \min_{u \in U} H(x,p,u)$$

The Pontryagin maximum principle now says:

"Let $u = u(t)$, $0 \leq t \leq T$ be an admissible control that transforms the state from $x(0) = c$ to a point in the target set S, and let $x = x(t)$ be the associated trajectory! A necessary condition for $u = u(t)$ to be an optimal strategy is, that there exists a to $u = u(t)$ and $x = x(t)$ belonging costate vector $p = p(t)$, not identically zero, which satisfies:

A) $H(x,p,u) = M(x,p)$ $\qquad t \in (0,T)$

B) $H(x,p,u) = 0$ $\qquad t \in (0,T)$

C) $x(T)$ is on the boundary of the target set S, and $P(T)$, in this point normal to S. If S consists of only one point, no restrictions are made on the costate vector from this reason (Athans & Falb [1]).

The implications of these conditions on the particular case will now be discussed.

Apply A) to (8):

$$u = -3 \text{ sign } (p_2 \cdot \cos x_1) \tag{9}$$

Condition B) results in:

$$1 + p_1x_2 + p_2(\sin x_1 + u \cos x_1) = 0 \qquad (10)$$

It is easy to prove (ref. Leondes, [5]) that if A) is fulfilled and (10) holds for t = T then it holds for all $t\varepsilon(0,T)$. Thus

$$1 + p_1(T)x_2(T) + p_2(T)\Big[\sin x_1(T) + U(T) \cos x_1(T)\Big] = 0 \qquad (10')$$

As mentioned above, C) does not introduce any restriction in this case.

2.2. The system forced by the proposed control.

The next step is to prove that the suggested control u^* (defined by (3)) fulfils the condition (9) and (10').

From the construction of u^* it is clear that the associated state variable $x = x^*(t)$ must follow the equation (4) from the point (6) to the point (7). The function $x = x^*(t)$ is easy to obtain for instance by Runge Kutta integration of (4):

Time optimal Trajectory

Startpoint 3.1416 0.0000 Timestep 0.0001

Time	Position		Used U
0.2000	3.08180	-0.59579	3.0
0.4000	2.90522	-1.16166	3.0
0.6000	2.62242	-1.64724	3.0
0.8000	2.25668	-1.97734	3.0
1.0000	1.84690	-2.07797	3.0
1.1347	1.57095	-2.00007	3.0
1.1348	1.57075	-1.99998	3.0
1.1349	1.57055	-1.99988	-3.0
1.2000	1.44220	-1.94758	-3.0
1.4000	1.05675	-1.94731	-3.0
1.5616	0.72994	-2.12335	-3.0
1.5617	0.72973	-2.12351	-3.0
1.5618	0.72951	-2.12322	3.0
1.6000	0.65055	-2.01053	3.0
1.8000	0.31015	-1.38803	3.0
2.0000	0.09542	-0.76208	3.0
2.2000	0.00399	-0.15485	3.0
2.2500	-0.00000	-0.00478	3.0

Table I

Of special interest are the switching points t_α and $t_{\pi/2}$, the points where u^* changes the sign:

$t_{\pi/2} = 1.1348$ $\quad\quad$ $\left(x_1^*(t_{\pi/2}) = \pi/2\right)$

$t_\alpha = 1.5617$ $\quad\quad$ $\left(x_1^*(t_\alpha) = \alpha\right)$

It remains to prove that there exists a function $p = p^*(t)$ following (5), and that (9) and (10') are satisfied by the functions $u^*(t)$, $x^*(t)$ and $p^*(t)$.

Equation (5) can be rewritten:

$$\ddot{p}_2^*(t) + p_2^*(t)\underbrace{\left[u^*(t)\sin x_1^*(t) - \cos x_1^*(t)\right]}_{a(t)} = 0 \qquad (11)$$

where $a(t)$ can be obtained from $u^*(t)$ and $x_1^*(t)$.

Condition (9) results in:

$$u^*(t) = -\,3 \cdot \operatorname{sign}\left[p_2^*(t) \cdot \cos x_1^*(t)\right] \qquad (12)$$

and condition (10'):

$$1 + p_1^*(T)x_2^*(T) + p_2^*(T)\left[\sin x_1^*(T) + u^*(T) \cdot \cos x_1^*(T)\right] = 0$$

or simplified with account to $x_1^*(T) = x_2^*(T) = 0$, $u^*(T^-) = 3$:

$$1 + p_2^*(T) \cdot 3 = 0 \qquad (13)$$

It is convenient to convert (12) into a condition only on p_2^* by taking the sign of $\cos x_1$ into consideration:

$$\begin{cases} \cos x_1 > 0 & x_1 \varepsilon (0,\pi/2) \\ \cos x_1 < 0 & x_1 \varepsilon (\pi/2,\pi) \end{cases}$$

Thus

$$\begin{cases} p_2^* > 0 & x_1 \varepsilon (\alpha, \pi) \quad (\sin \alpha = 2/3) \\ p_2^* = 0 & x_1 = \alpha \\ p_2^* < 0 & x_1 \varepsilon (0, \alpha) \end{cases} \tag{14}$$

is required if p_2 should belong to an optimal control u^*.

Summing up:

We want to show that there exists boundary conditions to (11) consistent with (13) and (14).

Another view of the problem:

Vary the initial condition to (5) and consider the state equation (4) and its initial condition (6) [i.e. drop the end point condition (7)]. Then (9) and (10') generate a family of controls u(t) and associated trajectories x(t), which are extremals leading different end points in the state space. Now pick out of these controls the one leading to the end point specified by (7), and prove that it is possible to find one! This is necessarily a rather involved procedure. The method to circumvent the difficulties can from this point of view be formulated:

Guess a control $u = u^*$. Then prove that for certain boundary conditions to (5) the control generated by (9) is the suggested $u = u^*$. By the way is then obtained that the extremal leads to the desired end point.

2.3. Verification of stated conditions.

Considering the above relations it is now by means of some rather simple theorems (ref. Burkhill, [2]) on the second order linear differential equation:

$$\frac{d^2}{dt^2} x(t) + q(t)x(t) = 0$$

possible to show that there exists a solution to (11) fulfilling (13) and (14).

For the given u^* $x_1^*(t)$ is a monotonous function of t. Thus t can be represented as a function of x_1^*. The function a(t) is hard to determine explicitely, but $a\left[t(x_1^*)\right]$ is much easier to obtain:

$$a = \pm 3 \cdot \sin x_1^* - \cos x_1^*$$

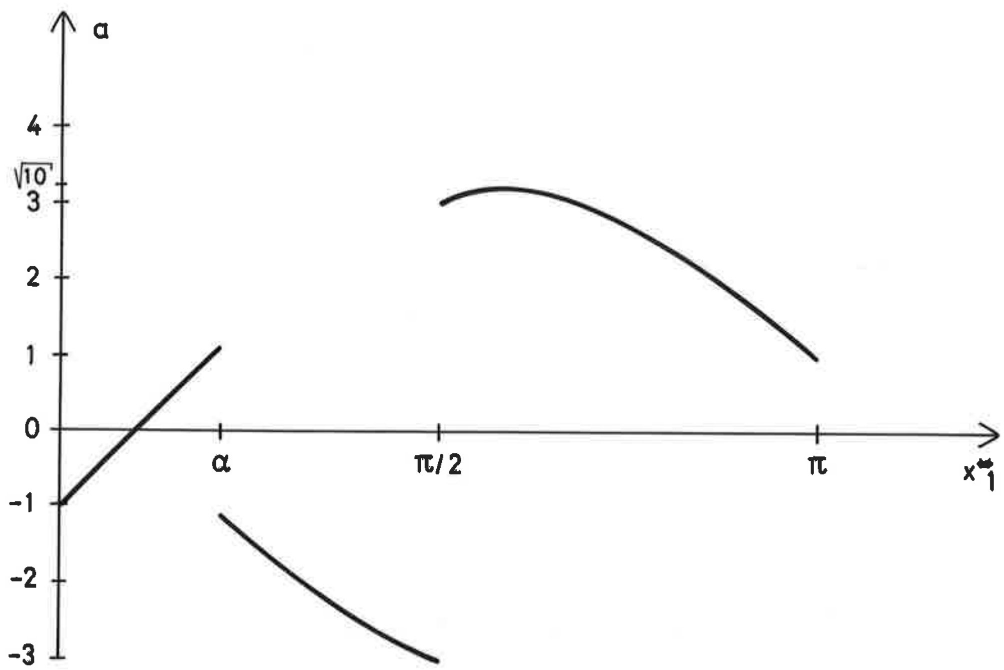


Fig. 2 — a as a function of x_1^*, for $u_m = 3$.

a is a piecewise continuous function (of both x_1^* and t), so there exists a solution to the second order, linear differential equation (11), if its two boundary values are specified at one instant of time (initial value problem). Considering (14) choose this point to be t_α, i.e.

$$\begin{cases} p_2^*(t_\alpha) = 0 \\ \frac{d}{dt} p_2^*(t_\alpha) = d \qquad ! \end{cases}$$

We now have to prove that $p_2^*(t)$ has no other zeroes, that the sign of p_2^* is correct, i.e. d must be < 0, and that it is possible to choose d so that (13) is fulfilled, i.e. $p_2^*(T) = -1/3$.

It is natural to do this investigation separately in the three x_1^* intervals: $(0,\alpha)$, $(\alpha,\pi/2)$, $(\pi/2,\pi)$.

(i) $t_\alpha < t < T$

In this interval (the last one) $a(t) < 1.3$. Compare (11) with the equation:

$$\begin{cases} q'' + 1.3\, q = 0 \\ q(t_\alpha) = 0 \\ q'(t_\alpha) = d \end{cases}$$

that has the solution:

$$q(t) = \frac{c}{\sqrt{1.3}} \sin \sqrt{1.3}(t-t_\alpha) \qquad (15)$$

Because of the continuity and because we suppose that d is < 0, there exists $\varepsilon > 0$ for which $p_2^*(t) < 0$ in $(t_\alpha, t_\alpha+\varepsilon)$.

A theorem about the oscillator equation now states (Burkhill, [2]):

$p_2^*(t) < q(t) < 0$, for $t > t_\alpha$ as long as $q(t) < 0$

Equation (15) implies that $q(t_\alpha) = 0$ and $q(t) < 0$ for $t\varepsilon(t_\alpha, t_\gamma)$, where $t_\gamma = t_\alpha + \pi/\sqrt{1.3} > t_\alpha + 2.5$.

According to Table I: $T = t_\alpha + 0.69$.

We have thus proved that:

$$p_2^*(t) < 0 \text{ in the whole interval } (t_\alpha, T),$$

and especially that:

$$p_2^*(T) < q(T) = \frac{d}{\sqrt{1.3}} \sin\sqrt{1.3} \cdot 0.69 \approx d \cdot 0.6$$

Note, that $d = 0$ gives the solution:

$$p_2^*(t) \equiv 0$$

to equation (11), and also:

$$p_2^*(T) = 0$$

As the solution $p_2^*(t)$ to (11) depends continuously on the initial value $\frac{d}{dt} p_2^*(t_\alpha) = d$, it must be possible to find $d < 0$ so that $p_2^*(T) = -1/3$. And condition (13) is fulfilled.

<u>(ii) $t_{\pi/2} < t < t_\alpha$</u>

$\frac{d}{dt} p_2^*(t)$ is continuous and $c < 0$. Thus there exists $\varepsilon > 0$ so that

$$\frac{d}{dt} p_2^*(t) < 0 \text{ for } t < (t_\alpha - \varepsilon, t_\alpha)$$

consequently, since

$$\frac{d^2}{dt^2} p_2^*(t) = -a(t)\, p_2^*(t),$$

$$\frac{d^2}{dt^2} p_2^*(t) > 0 \text{ for } t \varepsilon (t_\alpha - \varepsilon, t_\alpha)$$

and ε can be extended as long as $a(t) < 0$.

Conclusion:

$$\left. \begin{array}{r} p_2^*(t) > 0 \\ \frac{d}{dt} p_2^*(t) < 0 \\ \frac{d^2}{dt^2} p_2^*(t) > 0 \end{array} \right\} \quad \forall t \; \varepsilon (t_{\pi/2}, t_\alpha)$$

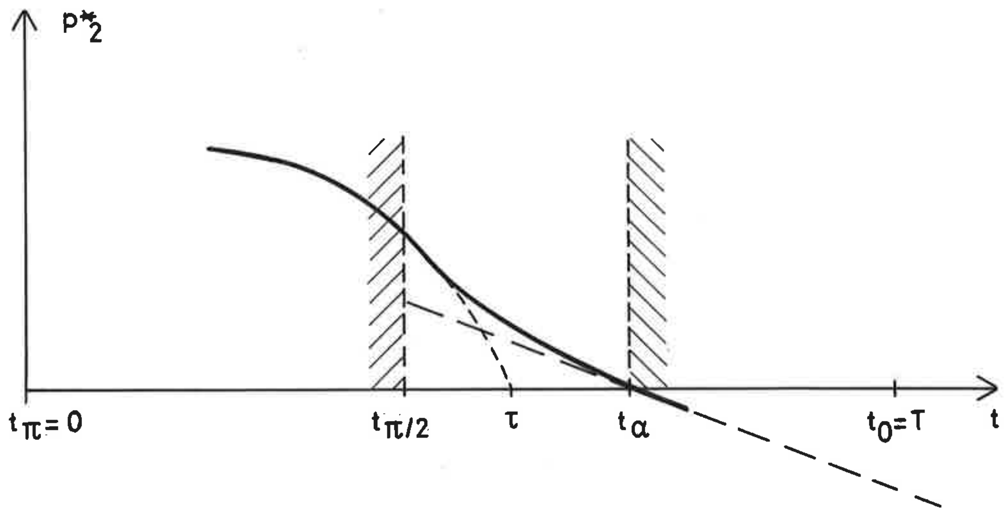


Fig. 3

(iii) $0 = t_\pi < t < t_{\pi/2}$

$a(t) > 0$ in this interval.

If during the interval $(t_{\pi/2}, t_\alpha)$ $a(t)$ were positive, then p_2^* (with the same boundary values at $t_{\pi/2}$) would be zero for some time τ, with $\tau \varepsilon (t_{\pi/2}, t_\alpha)$, because

$$\frac{d^2}{dt^2} p_2^* < 0$$

Consider now the equation:

$$\frac{d^2}{dt^2} q + \tilde{a}(t)\, q = 0$$

with boundary values at $t_{\pi/2}$ given.

In the interval $t \varepsilon (0,\tau)$ we have

$$0 < \tilde{a}(t) < \sqrt{10}$$

and

$$q(\tau) = 0$$

The properties of the oscillator equation says that the distance in time between two zeroes of q is greater than:

$$\frac{\pi}{\sqrt{\max \tilde{a}(t)}} = \frac{\pi}{\sqrt[4]{10}} = 1.767$$

According to Table I $t_\alpha = 1.5617$ (< 1.767), which proves that $q(t) > 0$ in $(0,\tau)$ and since $p_2^*(t) = q(t)$ in $(0,t_{\pi/2})$:

$$p_2^*(t) > 0 \qquad t \varepsilon (0,t_{\pi/2})$$

and this completes the verification of the conditions (13) and (14).

2.4. Conclusions

We have now found such boundary conditions to (5)

$$\begin{cases} p_2(t_\alpha) = 0 \\ p_2(T) = -1/3 \end{cases}$$

that among the controls generated by (9) and (10') u^* is the one leading to the end point

$$\begin{cases} x_1(T) = 0 \\ x_2(T) = 0 \end{cases}$$

We can therefore state that u^* fulfils the Pontryagin necessary conditions for optimality.

3. THE PENDULUM PROBLEM ATTACKED WITH PHASE PLANE ANALYSIS.

Consider again our problem stated on p. 1, and let u_m be arbitrary.

From the Maximum Principle (chapter 2) we know that an optimal control must be "bang-bang", i.e.

$$u(t) = \pm u_m \qquad 0 < t < T$$

This immense reduction of suitable controls, only the switching times and the sign remains to be determined, makes it possible to that the problem analytically by arguments in the phase plane.

Another fundamental property of the system equations is, that for the two values of u in question ($u = \pm u_m$) the phase plane curves can be obtained explicitly.

3.1. Analytical expressions for the phase plane trajectories.

Consider:

$$\dot{x}_1 \ddot{x}_1 = [\sin x_1 \pm u_m \cos x_1]\dot{x}_1 \quad !$$

Define:

$$\begin{cases} k = \sqrt{1 + u_m^2} \\ \gamma = \text{arc tg } u_m \quad ! \end{cases}$$

This leads to

$$\frac{d}{dt}(\dot{x}_1)^2 = 2[k \cdot \sin(x_1 \pm \gamma)] \cdot \dot{x}_1$$

or

$$x_2 = \pm \sqrt{-2k \cos(x_1 \pm \gamma) + 2c} \tag{16}$$

i.e. a field of phase plane trajectories with one arbitrary constant c, which can be specified by one point on the trajectory.

Our problem is to combine these (16) trajectories from $(\pi,0)$ to $(0,0)$ in an optimal way.

3.2. Two switching points.

For large values of u_m a strategy with two switching points (rather near each other) must be time optimal. The phase plane trajectory then consists of three parts, of which two have a fixed equation (through the start point and the end point). There only remains to calculate the switching points or the constant of the intermediate part.

The starting curve has the equation:

$$x_2 = - \sqrt{2\{- k \cos (x + \gamma) - 1\}} \qquad \text{(I)}$$

the final curve:

$$x_2 = - \sqrt{2\{- k \cos (x + \gamma) + 1\}} \qquad \text{(III)}$$

and the intermediate one:

$$x_2 = - \sqrt{2\{- k \cos (x - \gamma) + c\}} \qquad \text{(II)}$$

where c is not yet determined.

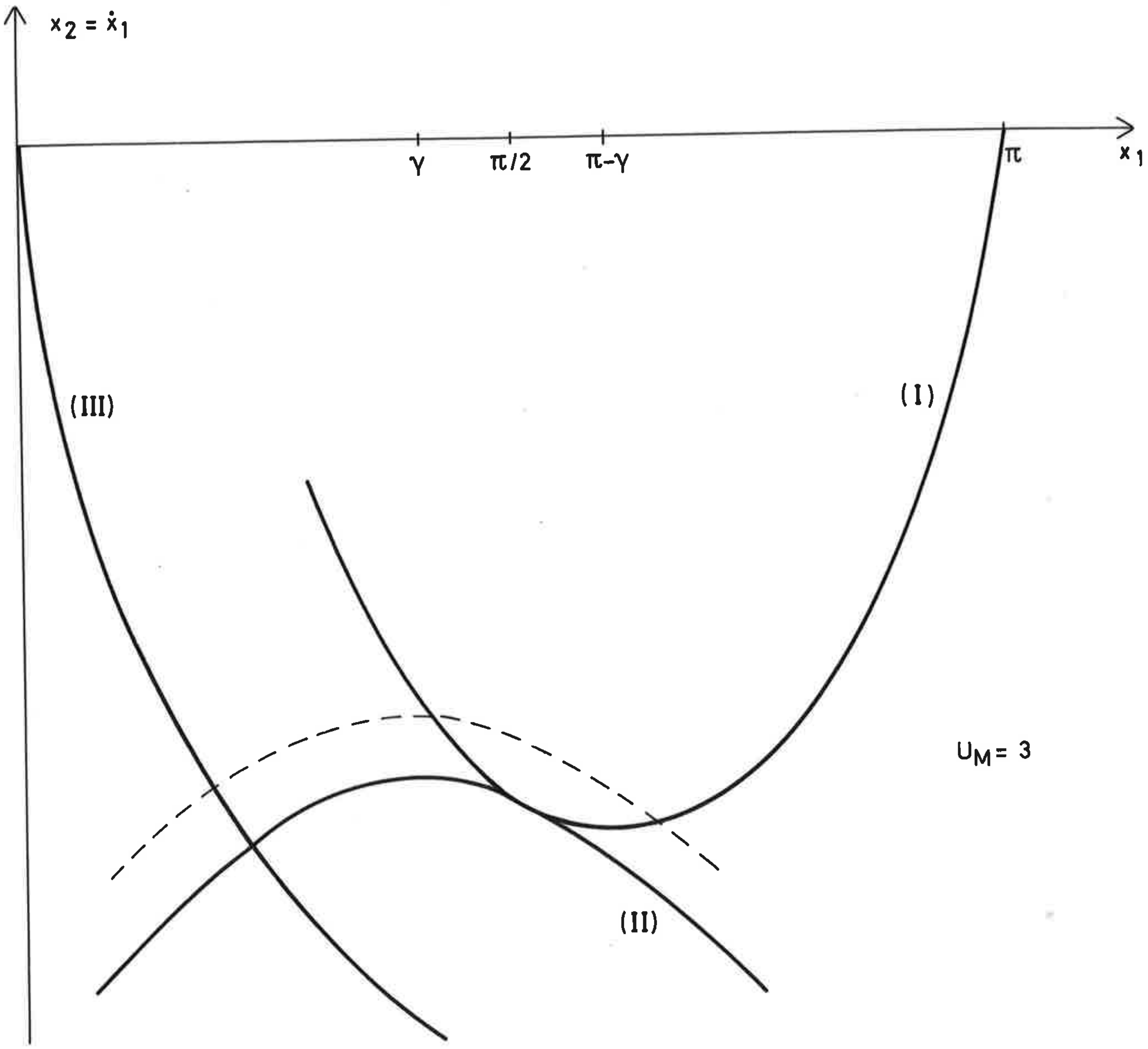


Fig. 4

Curve II has, independent of c, maximum for $x_1 = \gamma$ and an increase in c causes, for every x_1, an increase in $|x_2|$.

Consider:

$$x_2 = \frac{dx_1}{dt} \quad , \quad t = \int_0^{x_1(t)} \frac{dx_1}{x_2}$$

In order to minimize time we shall subsequently choose c as large as possible, i.e. so that curve I is tangent to curve II.

Thus we have:

$$\begin{cases} \sqrt{2\{-k\cos(x_1+\gamma)-1\}} = \sqrt{2\{-k\cos(x_1-\gamma)+c\}} \\ -k\sin(x_1+\gamma) = k\sin(x_1-\gamma) \end{cases}$$

and

$$\begin{cases} x_1 = \pi/2 \\ c = 2u_m - 1 \end{cases}$$

We have proved that it is optimal to choose the first switching point at $x_1 = \pi/2$. The second switching point can be calculated from the intersection between the two curves I and II:

$$x_1 = \text{arc sin}\,(1 - 1/u_m) \qquad (= \alpha)$$

For more examples of this way of solving time optimal trajectories see Tsien [6].

3.3. Limited validity.

In order that curve II should bring the pendulum point $x_1 = \pi/2$ to the point $x_1 = \arcsin(1 - \frac{1}{u_m})$ we must in the equation for curve II stipulate that the expression under the root sign is positive. This gives the condition:

$c > k$ or

$u_m > 4/3$

If $u_m = 4/3$ the x_1 axis is tangent to curve II at $x_1 = \gamma$. The pendulum would stop in this unstable equilibrium point! For lower u_m values the pendulum never reaches $x_1 = \gamma$. It turns back towards $x_1 = \pi$ again. For those u_m values we must start in the "wrong" direction and give the pendulum more energy before it goes up. It will be necessary to use three or more switching points. For u_m values just above 4/3 it is probable that the optimal strategy has three switching points, although it is possible to control with only two. In fig. 5 the phase plane for $u_m = 4/3$ is shown with both 2 and 3 switchings.

3.4. Three switching points.

The four parts of the trajectory are called from the start to the end: 0, I, II, III.

The switching between I and II is also in this case at $x_1 = \pi/2$ according to an argumentation analogue to that of two switchings in chapter 3.2. It is still optimal to choose part I to be tangent to part II.

The equations of the curves are:

0: $x_2 = + \sqrt{2(- k \cos(x_1 - \gamma) - 1)}$

I: $x_2 = \pm \sqrt{2(-k \cos(x_1 + \gamma) + C_I)}$

II: $x_2 = - \sqrt{2(- k \cos(x_1 - \gamma) + C_{II})}$

III: $x_2 = - \sqrt{2(- k \cos(x_1 + \gamma) + 1)}$

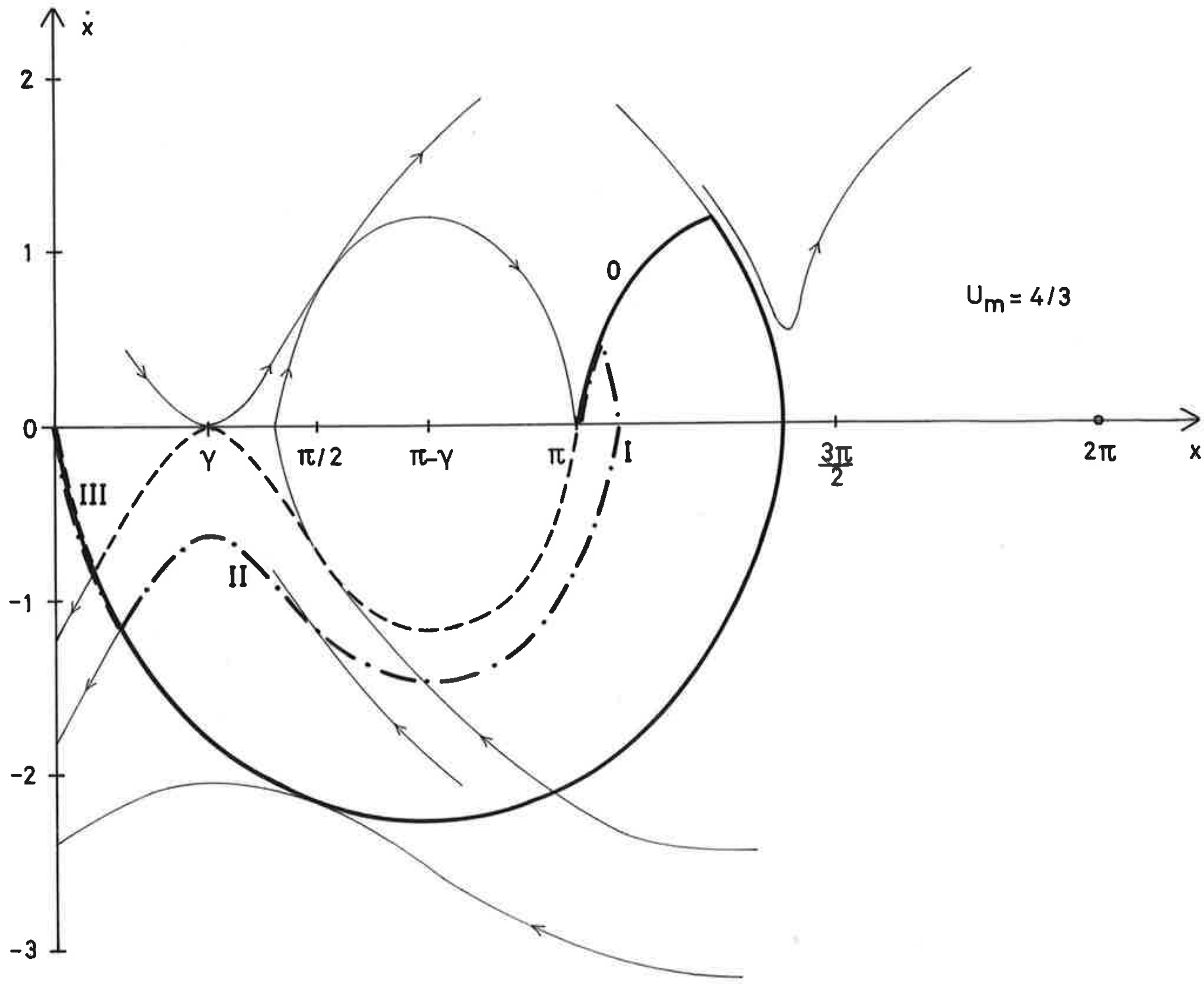


Fig. 5 - Theoretical trajectories for $u_m = 4/3$.

If we have the first switching for $x_1 = \pi + \varepsilon$, we get:

$$\sqrt{2\left(-k\cos(\pi+\varepsilon-\gamma)-1\right)} = \sqrt{2\left(-k\cos(\pi+\varepsilon+\gamma)+C_I\right)}$$

The second switching at $x_1 = \pi/2$ gives:

$$-\sqrt{2\left(-k\cos(\frac{\pi}{2}+\gamma)+C_I\right)} = -\sqrt{2\left(-k\cos(\frac{\pi}{2}-\gamma)+C_{II}\right)}$$

and

$$C_{II} = -1 + 2u_m(1+\sin\varepsilon)$$

The third switching point is the intersection between curves II and III:

$$x_1 = (C_{II}-1)/2u_m = \arcsin(1-\frac{1}{u_m}+\sin\varepsilon) \tag{17}$$

The realizability of part II gives the condition $C_{II} > k$, or

$$u_m > \frac{4+4\sin\varepsilon}{3+8\sin\varepsilon+4\sin^2\varepsilon} \tag{18}$$

A corresponding condition on part I gives an upper limit of ε. The velocity in the wrong direction must not become too large, or it cannot be retardated and no change of direction can be possible. It implies $C_I \leqslant 1$ or

$$\sin\varepsilon \leqslant 1/u_m \tag{19}$$

This condition can also be seen from (17). Equality in (19) makes the last switching point be $x_1 = \pi/2$, the same as the second one and thus there would be only one for the whole trajectory.

The choice of an optimal ε in the interval

$$0 \leqslant \varepsilon \leqslant \arcsin 1/u_m$$

is not easy, since x_1 no longer varies monotonously with time and

and the integral

$$\int \frac{dx_1}{x_2}$$

thus is not geometrically obvious. It is possible to determine expressions for the time in the different parts of the trajectory and then to take the derivative with respect to ε. The resulting equations are, however, not solvable analytically. One possible way to evaluate the optimal ε easier is to simulate the equations for some values of ε.

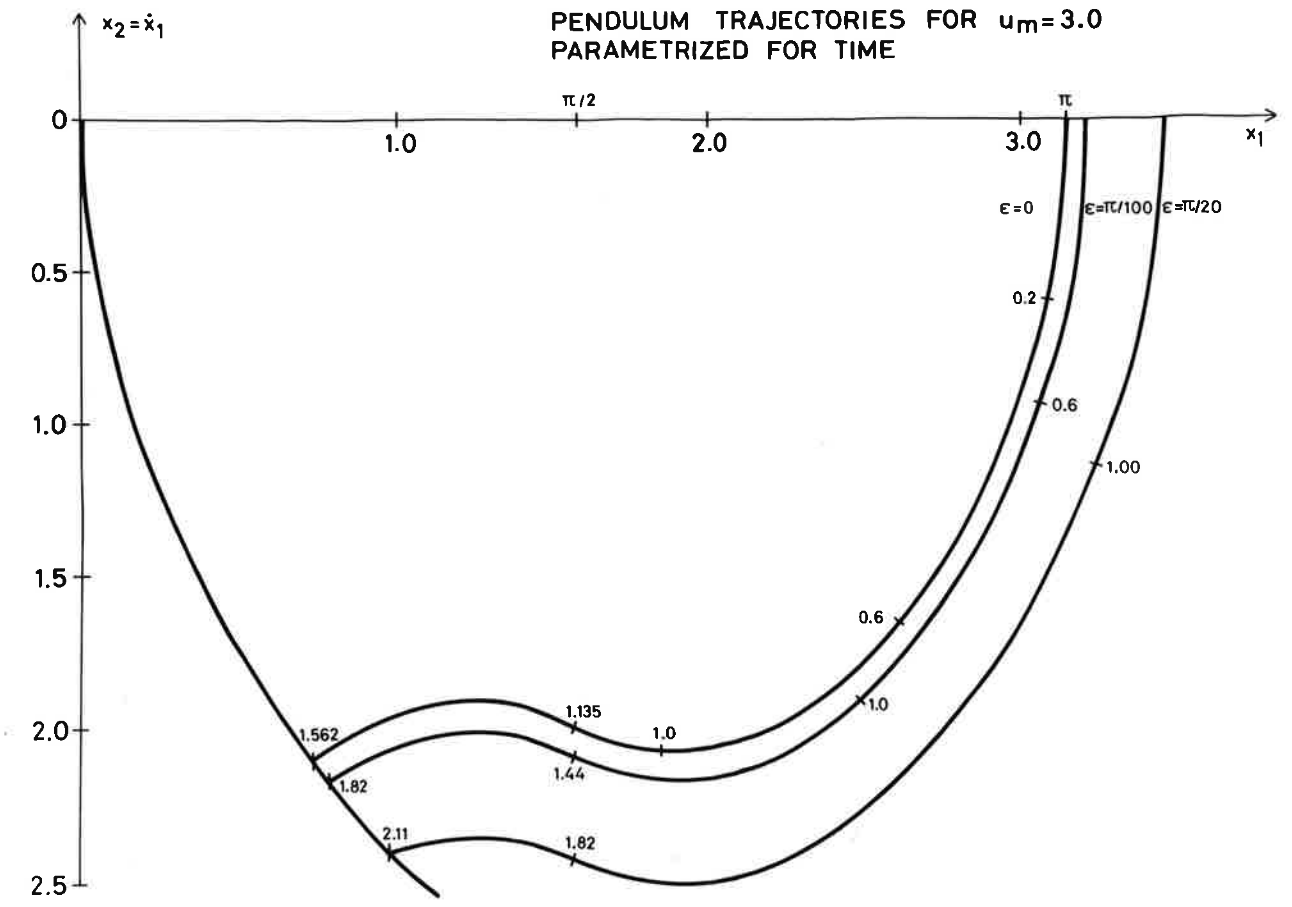
3.5. Simulation

The equations above are easy to code for Runge-Kutta integration. Table II shows the result for a few values of u_m both for 2 switching points, i.e. $\varepsilon = 0$, and 3 switchings and various ε values. The shortest end point times T and corresponding ε is noted for every u_m.

The trajectories, for $u_m = 3$ and $u_m = 4/3$, are drawn in fig. 6 and fig. 7, and the most relevant output from the program is reprinted in appendix. It is notable how difficult it is to come exactly to the end point (0,0). Small variations in the last switching point, and that is a necessity since we have to use finite step length in the Runge-Kutta integration, induces rather large errors in the final velocity (and the final time).

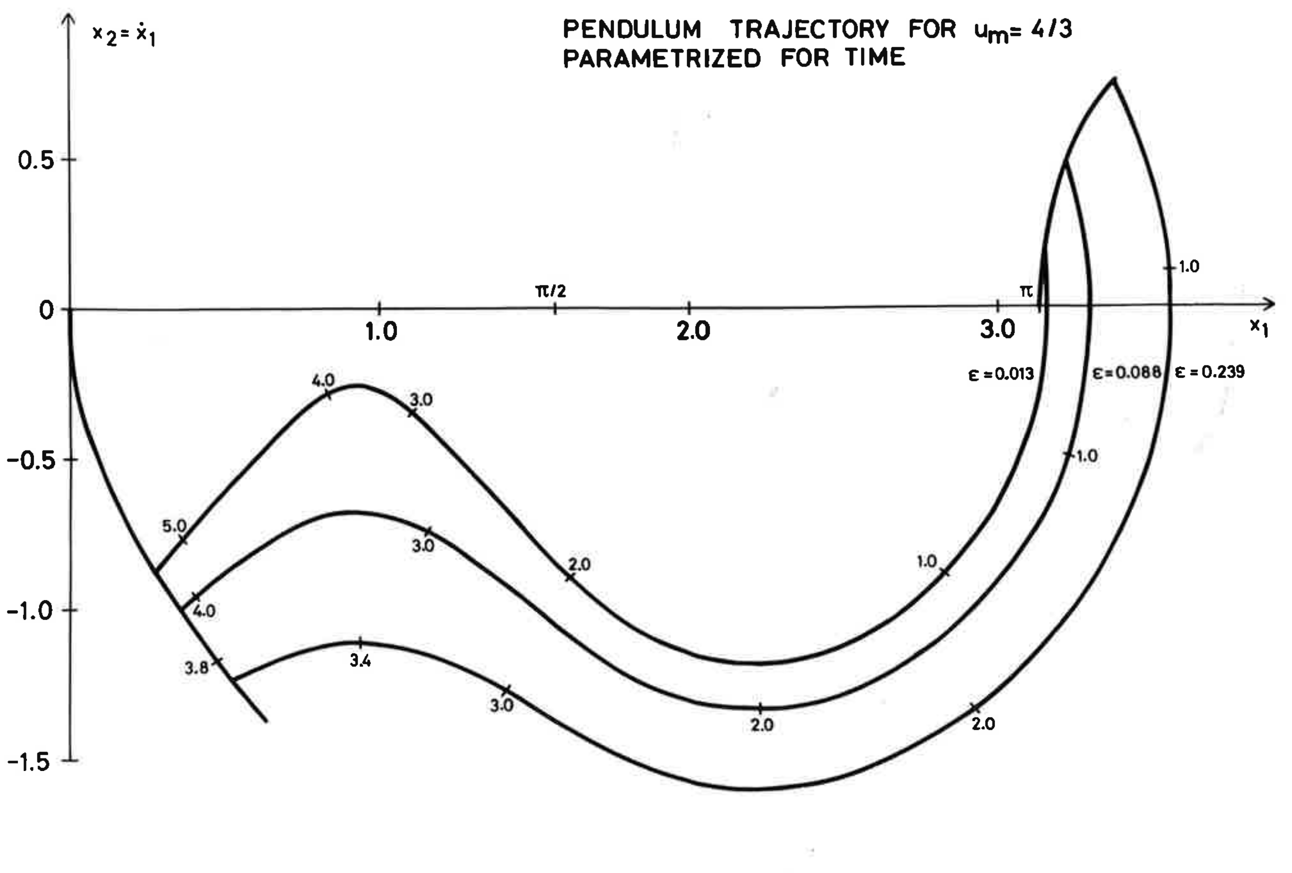
Table II:

u_M	ε_{min}	$\varepsilon_{rel.min.}$	$T_{\varepsilon=0}$	$T_{rel.min.}$
3.0	0	–	2.251	–
2.0	0	–	2.95	–
1.5	0	0.075	4.04	4.19
1.4	0.160		4.75	4.44
4/3	0.205		∞	4.61
1.2	0.300		∞	4.97



PENDULUM TRAJECTORIES FOR u_m= 3.0
PARAMETRIZED FOR TIME
$x_2 = \dot{x}_1$
x_1
$\pi/2$
π
0
1.0
2.0
3.0
0.5
1.5
2.5
ε=0
ε=π/100
ε=π/20
0.2
0.6
1.00
0.6
1.0
1.562
1.135
1.0
1.82
1.44
2.11
1.82

Fig. 6



PENDULUM TRAJECTORY FOR u_m= 4/3
PARAMETRIZED FOR TIME
$x_2 = \dot{x}_1$
x_1
$\pi/2$
π
0.5
0
-0.5
-1.0
-1.5
1.0
2.0
3.0
ε=0.013
ε=0.088
ε=0.239
1.0
1.0
4.0
3.0
5.0
3.0
2.0
1.0
4.0
3.8
3.4
3.0
2.0
2.0

Fig. 7

3.6. Final remarks

Notice the relative minimum for T in the simulations with $u_m = 1.5$. It is probably so that we have one for $u_m = 2.0$ and $u_m = 3.0$ as well, but less distinct and closer to $\varepsilon = 0$. When u_m decreases the well is moved towards larger ε values and grows deeper and deeper.

For still smaller u_m the alternative with four switching points is getting relevant. Condition (18) must not be violated and that makes it perhaps impossible with only three switchings. If u_m is very small it might be necessary with even more switchings, which means that the pendulum must be wagged for quite a while until it receives the required energy to be raised.

In these cases we have the freedom to vary 2 or more switching points, which makes it almost impossible to minimize with the simulation technique above. Other methods have to be applied.

A theoretically simpler variant of the pendulum problem, namely

$$\begin{cases} \ddot{x} + \sin x = u \\ |u| \leq U_m \end{cases}$$

(i.e. the applied control is the torque of the pendulum) has been treated independently by Flügge-Lotz [3]. It is then possible to show that the time optimal trajectory has at most two switchings, and the switching curves are rather easily established. She also introduces the concept of "indifference curves", the points in the phase plane from which it is possible to go to the end point along different trajectories at the same time.

It could be valuable for the problem of this report to use the indifference curves, but no attempt is done. The difficulty of very small u_m values would still remain.

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APPENDIX

Output from the simulations in 3.5.

EPS indicates first switching for $x_1 = \pi + EPS$

A second $x_1 = A$ $(= \pi/2)$

B third $x_1 = B$ $(= \arcsin (1-\frac{1}{u_m} + \sin \varepsilon))$

$u_m = 1.2$ pp A1 - A4

$u_m = 4/3$ pp A5 - A11

$u_m = 1.4$ pp A12 - A13

$u_m = 1.5$ pp A14 - A20

TRAJECTORY

STARTPOINT 3.1416 0.0000 TIMESTEP 0.0010 A = 1.570796 B = 0.244420
EPS= 0.075398

TIME	POSITION		USED U
0.147	3.15453	0.17576	-1.2
0.148	3.15471	0.17695	-1.2
0.200	3.16551	0.23839	-1.2
0.299	3.19483	0.35338	-1.2
0.300	3.19518	0.35452	-1.2
0.400	3.23408	0.36408	1.2
0.600	3.28081	0.10185	1.2
0.800	3.27459	-0.16380	1.2
1.000	3.21565	-0.42390	1.2
1.200	3.10607	-0.66838	1.2
1.400	2.95017	-0.88478	1.2
1.562	2.79469	-1.02936	1.2
1.563	2.79366	-1.03015	1.2
1.600	2.75502	-1.05831	1.2
1.800	2.53073	-1.17373	1.2
1.823	2.50363	-1.18268	1.2
1.824	2.50244	-1.18304	1.2
2.000	2.29022	-1.21898	1.2
2.200	2.04818	-1.18884	1.2
2.230	2.01268	-1.17793	1.2
2.231	2.01150	-1.17754	1.2
2.400	1.81949	-1.08683	1.2
2.600	1.61752	-0.92388	1.2
2.800	1.45192	-0.73655	-1.2
3.000	1.32038	-0.58481	-1.2
3.200	1.21580	-0.46607	-1.2
3.400	1.13212	-0.37486	-1.2
3.600	1.06434	-0.30638	-1.2
3.800	1.00831	-0.25678	-1.2
4.000	0.96056	-0.22317	-1.2
4.200	0.91812	-0.20351	-1.2
4.400	0.87831	-0.19663	-1.2
4.600	0.83864	-0.20209	-1.2
4.800	0.79663	-0.22024	-1.2
5.000	0.74963	-0.25217	-1.2
5.200	0.69471	-0.29981	-1.2
5.400	0.62848	-0.36595	-1.2
5.600	0.54686	-0.45433	-1.2
5.800	0.44495	-0.56965	-1.2
6.000	0.31684	-0.71732	-1.2
6.200	0.16818	-0.65804	1.2
6.400	0.06293	-0.39744	1.2
6.600	0.00823	-0.15130	1.2
6.680	-0.00002	-0.05502	1.2

TRAJECTORY

STARTPOINT 3.1416 0.0000 TIMESTEP 0.0010 A = 1.570796 B = 0.46865
EPS= 0.289027

TIME	POSITION		USED U
0.147	3.15453	0.17576	-1.2
0.148	3.15471	0.17695	-1.2
0.200	3.16551	0.23839	-1.2
0.299	3.19483	0.35338	-1.2
0.300	3.19518	0.35452	-1.2
0.400	3.23629	0.46688	-1.2
0.600	3.35088	0.67453	-1.2
0.800	3.49465	0.64451	1.2
1.000	3.59375	0.34501	1.2
1.200	3.63229	0.04003	1.2
1.400	3.60972	-0.26549	1.2
1.562	3.54683	-0.51034	1.2
1.563	3.54632	-0.51184	1.2
1.600	3.52635	-0.56705	1.2
1.800	3.38364	-0.85733	1.2
1.823	3.36355	-0.88948	1.2
1.824	3.36266	-0.89087	1.2
2.000	3.18501	-1.12363	1.2
2.200	2.93704	-1.34700	1.2
2.230	2.89620	-1.37540	1.2
2.231	2.89483	-1.37632	1.2
2.400	2.65059	-1.50464	1.2
2.600	2.34096	-1.57601	1.2
2.800	2.02668	-1.55048	1.2
3.000	1.72700	-1.43196	1.2
3.200	1.45866	-1.24894	-1.2
3.400	1.22388	-1.10976	-1.2
3.600	1.01041	-1.03568	-1.2
3.800	0.80533	-1.02582	-1.2
4.000	0.59582	-1.07998	-1.2
4.200	0.37687	-1.01261	1.2
4.400	0.20333	-0.72638	1.2
4.600	0.08491	-0.46102	1.2
4.800	0.01783	-0.21190	1.2
4.941	-0.00002	-0.04173	1.2

TRAJECTORY

STARTPOINT 3.1416 0.0000 TIMESTEP 0.0010 A = 1.570796 B = 0.48217
EPS= 0.301593

TIME	POSITION		USED U
0.147	3.15453	0.17576	-1.2
0.148	3.15471	0.17695	-1.2
0.200	3.16551	0.23839	-1.2
0.299	3.19483	0.35338	-1.2
0.300	3.19518	0.35452	-1.2
0.400	3.23629	0.46688	-1.2
0.600	3.35088	0.67453	-1.2
0.800	3.49766	0.68118	1.2
1.000	3.60404	0.38112	1.2
1.200	3.64973	0.07523	1.2
1.400	3.63407	-0.23159	1.2
1.562	3.57655	-0.47808	1.2
1.563	3.57607	-0.47959	1.2
1.600	3.55729	-0.53530	1.2
1.800	3.42059	-0.82933	1.2
1.823	3.40114	-0.86206	1.2
1.824	3.40027	-0.86347	1.2
2.000	3.22698	-1.10186	1.2
2.200	2.98249	-1.33456	1.2
2.230	2.94200	-1.36463	1.2
2.231	2.94063	-1.36561	1.2
2.400	2.69734	-1.50446	1.2
2.600	2.38636	-1.58978	1.2
2.800	2.06795	-1.57778	1.2
3.000	1.76165	-1.47033	1.2
3.200	1.48501	-1.28978	-1.2
3.400	1.24271	-1.14438	-1.2
3.600	1.02281	-1.06562	-1.2
3.800	0.81207	-1.05278	-1.2
4.000	0.59732	-1.10568	-1.2
4.200	0.37578	-1.01059	1.2
4.400	0.20263	-0.72447	1.2
4.600	0.08458	-0.45920	1.2
4.800	0.01786	-0.21010	1.2
4.949	-0.00002	-0.03031	1.2

TRAJECTORY

STARTPOINT 3.1416 0.0000 TIMESTEP 0.0010 A = 1.570796 B = 0.49574
EPS= 0.314159

TIME	POSITION		USED U
0.147	3.15453	0.17576	-1.2
0.148	3.15471	0.17695	-1.2
0.200	3.16551	0.23839	-1.2
0.299	3.19483	0.35338	-1.2
0.300	3.19518	0.35452	-1.2
0.400	3.23629	0.46688	-1.2
0.600	3.35088	0.67453	-1.2
0.800	3.50007	0.71772	1.2
1.000	3.61372	0.41717	1.2
1.200	3.66654	0.11048	1.2
1.400	3.65783	-0.19748	1.2
1.562	3.60572	-0.44540	1.2
1.563	3.60528	-0.44692	1.2
1.600	3.58770	-0.50306	1.2
1.800	3.45714	-0.80047	1.2
1.823	3.43835	-0.83372	1.2
1.824	3.43751	-0.83515	1.2
2.000	3.26878	-1.07877	1.2
2.200	3.02807	-1.32034	1.2
2.230	2.98798	-1.35203	1.2
2.231	2.98663	-1.35306	1.2
2.400	2.74462	-1.50223	1.2
2.600	2.43270	-1.60152	1.2
2.800	2.11054	-1.60342	1.2
3.000	1.79788	-1.50764	1.2
3.200	1.51301	-1.33147	-1.2
3.400	1.26305	-1.17951	-1.2
3.600	1.03667	-1.09563	-1.2
3.800	0.82029	-1.07938	-1.2
4.000	0.60042	-1.13055	-1.2
4.200	0.37636	-1.01185	1.2
4.400	0.20297	-0.72567	1.2
4.600	0.08469	-0.46036	1.2
4.800	0.01774	-0.21126	1.2
4.941	-0.00003	-0.04110	1.2

TRAJECTORY

STARTPOINT 3.1416 0.0000 TIMESTEP 0.0010 A = 1.570796 B = 0.26568
EPS= 0.012566

TIME	POSITION		USED U
0.147	3.15586	0.17129	1.3
0.148	3.15604	0.16995	1.3
0.200	3.16305	0.09968	1.3
0.299	3.16627	-0.03468	1.3
0.300	3.16623	-0.03604	1.3
0.400	3.15585	-0.17140	1.3
0.600	3.09494	-0.43565	1.3
0.800	2.98283	-0.68120	1.3
1.000	2.82464	-0.89392	1.3
1.200	2.62852	-1.05785	1.3
1.400	2.40579	-1.15759	1.3
1.562	2.21547	-1.18361	1.3
1.563	2.21428	-1.18361	1.3
1.600	2.17051	-1.18226	1.3
1.800	1.93811	-1.12899	1.3
1.823	1.91226	-1.11806	1.3
1.824	1.91115	-1.11756	1.3
2.000	1.72372	-1.00389	1.3
2.200	1.54046	-0.82148	-1.3
2.230	1.51624	-0.79321	-1.3
2.231	1.51545	-0.79228	-1.3
2.400	1.39390	-0.65110	-1.3
2.600	1.27743	-0.51947	-1.3
2.800	1.18394	-0.42045	-1.3
3.000	1.10744	-0.34863	-1.3
3.200	1.04296	-0.29979	-1.3
3.400	0.98620	-0.27089	-1.3
3.600	0.93339	-0.26012	-1.3
3.800	0.88099	-0.26679	-1.3
4.000	0.82549	-0.29131	-1.3
4.200	0.76318	-0.33525	-1.3
4.400	0.68991	-0.40135	-1.3
4.600	0.60090	-0.49355	-1.3
4.800	0.49042	-0.61699	-1.3
5.000	0.35162	-0.77773	-1.3
5.200	0.18838	-0.73189	1.3
5.400	0.07126	-0.44253	1.3
5.600	0.01031	-0.16884	1.3
5.704	-0.00000	-0.02977	1.3

. TRAJECTORY

STARTPOINT 3.1416 0.0000 TIMESTEP 0.0010 A = 1.570796 B = 0.30494

EPS= 0.050265

TIME	POSITION		USED U
0.147	3.15597	0.19529	-1.3
0.148	3.15617	0.19661	-1.3
0.200	3.16817	0.26487	-1.3
0.299	3.20004	0.33138	1.3
0.300	3.20037	0.32999	1.3
0.400	3.22638	0.18973	1.3
0.600	3.23592	-0.09459	1.3
0.800	3.18876	-0.37554	1.3
1.000	3.08665	-0.64220	1.3
1.200	2.93373	-0.88106	1.3
1.400	2.73717	-1.07562	1.3
1.562	2.55316	-1.18860	1.3
1.563	2.55198	-1.18915	1.3
1.600	2.50761	-1.20829	1.3
1.800	2.25897	-1.26452	1.3
1.823	2.22987	-1.26567	1.3
1.824	2.22861	-1.26570	1.3
2.000	2.00739	-1.23743	1.3
2.200	1.76935	-1.13039	1.3
2.230	1.73577	-1.10811	1.3
2.231	1.73466	-1.10734	1.3
2.400	1.55975	-0.95589	-1.3
2.600	1.38659	-0.78387	-1.3
2.800	1.24308	-0.65865	-1.3
3.000	1.12035	-0.57526	-1.3
3.200	1.01046	-0.52964	-1.3
3.400	0.90614	-0.51937	-1.3
3.600	0.80040	-0.54387	-1.3
3.800	0.68619	-0.60449	-1.3
4.000	0.55599	-0.70438	-1.3
4.200	0.40152	-0.84806	-1.3
4.400	0.22437	-0.80330	1.3
4.600	0.09345	-0.50932	1.3
4.800	0.01948	-0.23271	1.3
4.942	-0.00002	-0.04232	1.3

TRAJECTORY

STARTPOINT 3.1416 0.0000 TIMESTEP 0.0010 A = 1.570796 B = 0.38475

EPS= 0.125604

TIME	POSITION		USED U
0.147	3.15597	0.19529	-1.3
0.148	3.15617	0.19661	-1.3
0.200	3.16817	0.26487	-1.3
0.299	3.20074	0.39261	-1.3
0.300	3.20114	0.39389	-1.3
0.400	3.24681	0.51864	-1.3
0.600	3.33957	0.32496	1.3
0.800	3.37424	0.02104	1.3
1.000	3.34794	-0.28346	1.3
1.200	3.26140	-0.57973	1.3
1.400	3.11736	-0.85626	1.3
1.562	2.96217	-1.05497	1.3
1.563	2.96111	-1.05610	1.3
1.600	2.92127	-1.09708	1.3
1.800	2.68223	-1.28225	1.3
1.823	2.65255	-1.29903	1.3
1.824	2.65125	-1.29973	1.3
2.000	2.41344	-1.39162	1.3
2.200	2.13161	-1.41106	1.3
2.230	2.08935	-1.40588	1.3
2.231	2.08794	-1.40567	1.3
2.400	1.85522	-1.33787	1.3
2.600	1.60200	-1.18191	1.3
2.800	1.38418	-1.00551	-1.3
3.000	1.19580	-0.88803	-1.3
3.200	1.02516	-0.82771	-1.3
3.400	0.86105	-0.82247	-1.3
3.600	0.69251	-0.87212	-1.3
3.800	0.50843	-0.97843	-1.3
4.000	0.30497	-0.94569	1.3
4.200	0.14656	-0.64203	1.3
4.400	0.04682	-0.35824	1.3
4.600	0.00238	-0.08763	1.3
4.639	-0.00002	-0.03559	1.3

TRAJECTORY

STARTPOINT 3.1416 0.0000 TIMESTEP 0.0010 A = 1.570796 B = 0.452685

EPS= 0.188496

TIME	POSITION		USED U
0.147	3.15597	0.19529	-1.3
0.148	3.15617	0.19661	-1.3
0.200	3.16817	0.26487	-1.3
0.299	3.20074	0.39261	-1.3
0.300	3.20114	0.39389	-1.3
0.400	3.24681	0.51864	-1.3
0.600	3.36919	0.58968	1.3
0.800	3.45620	0.27867	1.3
1.000	3.48025	-0.03861	1.3
1.200	3.44081	-0.35507	1.3
1.400	3.33877	-0.66322	1.3
1.562	3.21199	-0.89916	1.3
1.563	3.21109	-0.90056	1.3
1.600	3.17682	-0.95176	1.3
1.800	2.96050	-1.20357	1.3
1.823	2.93252	-1.22920	1.3
1.824	2.93129	-1.23030	1.3
2.000	2.69932	-1.39638	1.3
2.200	2.40741	-1.50744	1.3
2.230	2.36205	-1.51592	1.3
2.231	2.36054	-1.51617	1.3
2.400	2.10287	-1.52107	1.3
2.600	1.80565	-1.43522	1.3
2.800	1.53459	-1.26373	-1.3
3.000	1.29855	-1.10820	-1.3
3.200	1.08672	-1.02141	-1.3
3.400	0.88550	-1.00196	-1.3
3.600	0.68149	-1.04938	-1.3
3.800	0.46122	-1.16470	-1.3
4.000	0.25674	-0.86498	1.3
4.200	0.11389	-0.56701	1.3
4.400	0.02868	-0.28771	1.3
4.559	-0.00001	-0.07390	1.3

TIMEOPTIMAL TRAJECTORY

STARTPOINT 3.1416 0.0000 TIMESTEP 0.0010 A = 1.570796 B = 0.466441

EPS= 0.201062

TIME	POSITION		USED U
0.147	3.15597	0.19529	-1.3
0.148	3.15617	0.19661	-1.3
0.200	3.16817	0.26487	-1.3
0.299	3.20074	0.39261	-1.3
0.300	3.20114	0.39389	-1.3
0.400	3.24681	0.51864	-1.3
0.600	3.37164	0.63673	1.3
0.800	3.46799	0.32489	1.3
1.000	3.50113	0.00599	1.3
1.200	3.47038	-0.31302	1.3
1.400	3.37637	-0.62528	1.3
1.562	3.25535	-0.86635	1.3
1.563	3.25448	-0.86778	1.3
1.600	3.22139	-0.92045	1.3
1.800	3.01039	-1.18230	1.3
1.823	2.98289	-1.20933	1.3
1.824	2.98168	-1.21049	1.3
2.000	2.75216	-1.38884	1.3
2.200	2.46014	-1.51634	1.3
2.230	2.41448	-1.52740	1.3
2.231	2.41295	-1.52773	1.3
2.400	2.15211	-1.54695	1.3
2.600	1.84817	-1.47600	1.3
2.800	1.56780	-1.31410	-1.3
3.000	1.32254	-1.15029	-1.3
3.200	1.10297	-1.05711	-1.3
3.400	0.89507	-1.03339	-1.3
3.600	0.68502	-1.07867	-1.3
3.800	0.45899	-1.17946	1.3
4.000	0.25529	-0.86076	1.3
4.200	0.11327	-0.56295	1.3
4.400	0.02886	-0.28368	1.3
4.573	-0.00003	-0.05112	1.3

4.611

TRAJECTORY

STARTPOINT 3.1416 0.0000 TIMESTEP 0.0010 A = 1.570796 B = 0.480257
EPS= 0.213628

TIME	POSITION		USED U
0.147	3.15597	0.19529	-1.3
0.148	3.15617	0.19661	-1.3
0.200	3.16817	0.26487	-1.3
0.299	3.20074	0.39261	-1.3
0.300	3.20114	0.39389	-1.3
0.400	3.24681	0.51864	-1.3
0.600	3.37324	0.68368	1.3
0.800	3.47892	0.37113	1.3
1.000	3.52118	0.05078	1.3
1.200	3.49917	-0.27051	1.3
1.400	3.41333	-0.58645	1.3
1.562	3.29824	-0.83218	1.3
1.563	3.29741	-0.83365	1.3
1.600	3.26556	-0.88767	1.3
1.800	3.06024	-1.15893	1.3
1.823	3.03326	-1.18728	1.3
1.824	3.03207	-1.18850	1.3
2.000	2.80542	-1.37868	1.3
2.200	2.51384	-1.52248	1.3
2.230	2.46796	-1.53613	1.3
2.231	2.46642	-1.53655	1.3
2.400	2.20285	-1.57040	1.3
2.600	1.89260	-1.51505	1.3
2.800	1.60317	-1.36494	1.3
3.000	1.34844	-1.19344	-1.3
3.200	1.12097	-1.09333	-1.3
3.400	0.90634	-1.06479	-1.3
3.600	0.69032	-1.10734	-1.3
3.800	0.45897	-1.18123	1.3
4.000	0.25491	-0.86255	1.3
4.200	0.11253	-0.56482	1.3
4.400	0.02773	-0.28573	1.3
4.551	-0.00004	-0.08271	1.3

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4.613

TRAJECTORY

STARTPOINT 3.1416 0.0000 TIMESTEP 0.0010 A = 1.570796 B = 0.494136
EPS= 0.226195

TIME	POSITION		USED U
0.147	3.15597	0.19529	-1.3
0.148	3.15617	0.19661	-1.3
0.200	3.16817	0.26487	-1.3
0.299	3.20074	0.39261	-1.3
0.300	3.20114	0.39389	-1.3
0.400	3.24681	0.51864	-1.3
0.600	3.37397	0.72791	1.3
0.800	3.48846	0.41480	1.3
1.000	3.53933	0.09321	1.3
1.200	3.52564	-0.22998	1.3
1.400	3.44761	-0.54901	1.3
1.562	3.33829	-0.79874	1.3
1.563	3.33749	-0.80024	1.3
1.600	3.30686	-0.85542	1.3
1.800	3.10722	-1.13498	1.3
1.823	3.08077	-1.16452	1.3
1.824	3.07961	-1.16579	1.3
2.000	2.85605	-1.36670	1.3
2.200	2.56539	-1.52568	1.3
2.230	2.51938	-1.54179	1.3
2.231	2.51784	-1.54229	1.3
2.400	2.25210	-1.59018	1.3
2.600	1.93633	-1.55019	1.3
2.800	1.63861	-1.41201	1.3
3.000	1.37474	-1.23525	-1.3
3.200	1.13964	-1.12808	-1.3
3.400	0.91860	-1.09446	-1.3
3.600	0.69698	-1.13390	-1.3
3.800	0.46100	-1.18298	1.3
4.000	0.25661	-0.86412	1.3
4.200	0.11393	-0.56617	1.3
4.400	0.02889	-0.28684	1.3
4.564	-0.00001	-0.06632	1.3

TRAJECTORY

STARTPOINT 3.1416 0.0000 TIMESTEP 0.0010 A = 1.570796 B = 0.28975

EPS=0

TIME	POSITION		USED U
0.200	3.11369	-0.27812	1.4
0.400	3.03112	-0.54451	1.4
0.600	2.89756	-0.78567	1.4
0.800	2.71960	-0.98558	1.4
1.000	2.50722	-1.12711	1.4
1.200	2.27365	-1.19549	1.4
1.400	2.03446	-1.18272	1.4
1.600	1.80589	-1.09034	1.4
1.800	1.60294	-0.92889	1.4
1.821	1.58365	-0.90855	1.4
1.822	1.58274	-0.90757	1.4
1.823	1.58183	-0.90659	1.4
1.824	1.58092	-0.90560	1.4
1.825	1.58002	-0.90461	1.4
1.826	1.57912	-0.90363	1.4
2.000	1.43610	-0.74618	-1.4
2.200	1.30146	-0.60726	-1.4
2.400	1.19058	-0.50766	-1.4
2.600	1.09613	-0.44212	-1.4
2.800	1.01172	-0.40682	-1.4
3.000	0.93153	-0.39961	-1.4
3.200	0.85003	-0.42004	-1.4
3.400	0.76159	-0.46938	-1.4
3.600	0.66016	-0.55057	-1.4
3.800	0.53894	-0.66817	-1.4
4.000	0.39008	-0.82795	-1.4
4.200	0.21455	-0.80226	1.4
4.400	0.08501	-0.49656	1.4
4.600	0.01478	-0.20789	1.4
4.720	-0.00003	-0.03922	1.4

TIMEOPTIMAL TRAJECTORY

STARTPOINT 3.1416 0.0000 TIMESTEP 0.0010 A = 1.570796 B = 0.464920
EPS= 0.163363

TIME	POSITION		USED U
0.147	3.15669	0.20505	-1.4
0.148	3.15690	0.20644	-1.4
0.200	3.16950	0.27812	-1.4
0.299	3.20370	0.41223	-1.4
0.300	3.20411	0.41357	-1.4
0.400	3.25206	0.54451	-1.4
0.600	3.36870	0.48174	1.4
0.800	3.43293	0.15924	1.4
1.000	3.43215	-0.16697	1.4
1.200	3.36640	-0.48928	1.4
1.400	3.23732	-0.79818	1.4
1.562	3.08903	-1.02870	1.4
1.563	3.08800	-1.03004	1.4
1.600	3.04898	-1.07900	1.4
1.800	2.80897	-1.31092	1.4
1.823	2.77856	-1.33336	1.4
1.824	2.77723	-1.33431	1.4
2.000	2.52951	-1.46956	1.4
2.200	2.22747	-1.53380	1.4
2.230	2.18144	-1.53449	1.4
2.231	2.17991	-1.53447	1.4
2.400	1.92296	-1.49396	1.4
2.600	1.63645	-1.35612	1.4
2.800	1.38376	-1.17933	-1.4
3.000	1.15989	-1.07148	-1.4
3.200	0.95035	-1.03583	-1.4
3.400	0.74080	-1.07162	-1.4
3.600	0.51689	-1.17960	-1.4
3.800	0.29544	-0.95304	1.4
4.000	0.13672	-0.63772	1.4
4.200	0.03901	-0.34223	1.4
4.386	-0.00005	-0.07902	1.4

TIMEOPTIMAL TRAJECTORY

STARTPOINT 3.1416 0.0000 TIMESTEP 0.0010 A = 1.570796 B = 0.339837
EPS = 0.

TIME	POSITION		USED U
0.200	3.11169	-0.29798	1.5
0.400	3.02324	-0.58330	1.5
0.600	2.88021	-0.84103	1.5
0.800	2.68987	-1.05306	1.5
1.000	2.46334	-1.19985	1.5
1.200	2.21540	-1.26482	1.5
1.400	1.96344	-1.23974	1.5
1.600	1.72532	-1.12786	1.5
1.800	1.51711	-0.94710	-1.5
1.821	1.49742	-0.92815	-1.5
1.822	1.49650	-0.92726	-1.5
1.823	1.49557	-0.92638	-1.5
1.824	1.49464	-0.92549	-1.5
1.825	1.49372	-0.92461	-1.5
1.826	1.49279	-0.92373	-1.5
2.000	1.34414	-0.79201	-1.5
2.200	1.19674	-0.69040	-1.5
2.400	1.06473	-0.63761	-1.5
2.600	0.93864	-0.63084	-1.5
2.800	0.80935	-0.66973	-1.5
3.000	0.66756	-0.75638	-1.5
3.200	0.50334	-0.89498	-1.5
3.400	0.30712	-1.00253	1.5
3.600	0.14056	-0.66673	1.5
3.800	0.03907	-0.35105	1.5
3.988	-0.00004	-0.06627	1.5

TRAJECTORY

STARTPOINT 3.1416 0.0000 TIMESTEP 0.0010 A = 1.570796 B = 0.366021
EPS= 0.025133

TIME	POSITION		USED U
0.147	3.15777	0.21970	-1.5
0.148	3.15799	0.22118	-1.5
0.200	3.17111	0.25000	1.5
0.299	3.18833	0.09772	1.5
0.300	3.18843	0.09617	1.5
0.400	3.19031	-0.05855	1.5
0.600	3.14783	-0.36489	1.5
0.800	3.04533	-0.65649	1.5
1.000	2.88725	-0.91782	1.5
1.200	2.68150	-1.12961	1.5
1.400	2.44010	-1.27112	1.5
1.562	2.22921	-1.32232	1.5
1.563	2.22789	-1.32244	1.5
1.600	2.17889	-1.32542	1.5
1.800	1.91626	-1.28527	1.5
1.823	1.88681	-1.27472	1.5
1.824	1.88554	-1.27424	1.5
2.000	1.67075	-1.15607	1.5
2.200	1.45802	-0.97256	-1.5
2.230	1.42921	-0.94852	-1.5
2.231	1.42826	-0.94774	-1.5
2.400	1.27796	-0.83802	-1.5
2.600	1.11895	-0.76139	-1.5
2.800	0.96977	-0.73936	-1.5
3.000	0.81964	-0.77086	-1.5
3.200	0.65776	-0.85740	-1.5
3.400	0.47277	-1.00260	-1.5
3.600	0.26615	-0.93063	1.5
3.800	0.11351	-0.59938	1.5
4.000	0.02511	-0.28728	1.5
4.137	-0.00004	-0.08040	1.5

TRAJECTORY

STARTPOINT 3.1416 0.0000 TIMESTEP 0.0010 A = 1.570796 B = 0.420986
EPS= 0.075398

TIME	POSITION		USED U
0.147	3.15777	0.21970	-1.5
0.148	3.15799	0.22118	-1.5
0.200	3.17149	0.29798	-1.5
0.299	3.20814	0.44165	-1.5
0.300	3.20858	0.44308	-1.5
0.400	3.25015	0.34166	1.5
0.600	3.28621	0.01797	1.5
0.800	3.25730	-0.30629	1.5
1.000	3.16437	-0.62023	1.5
1.200	3.01082	-0.90975	1.5
1.400	2.80336	-1.15571	1.5
1.562	2.60319	-1.30724	1.5
1.563	2.60188	-1.30802	1.5
1.600	2.55298	-1.33514	1.5
1.800	2.27521	-1.42642	1.5
1.823	2.24235	-1.43056	1.5
1.824	2.24092	-1.43071	1.5
2.000	1.98917	-1.41680	1.5
2.200	1.71515	-1.30768	1.5
2.230	1.67628	-1.28352	1.5
2.231	1.67499	-1.28269	1.5
2.400	1.47138	-1.12687	-1.5
2.600	1.26056	-0.99307	-1.5
2.800	1.06956	-0.92619	-1.5
3.000	0.88483	-0.93020	-1.5
3.200	0.69302	-0.99915	-1.5
3.400	0.48056	-1.13720	-1.5
3.600	0.26411	-0.92645	1.5
3.800	0.11228	-0.59543	1.5
4.000	0.02465	-0.28348	1.5
4.138	-0.00006	-0.07513	1.5

TRAJECTORY

STARTPOINT 3.1416 0.0000 TIMESTEP 0.0010 A = 1.570796 B = 0.562193
EPS= 0.201062

TIME	POSITION		USED U
0.147	3.15777	0.21970	-1.5
0.148	3.15799	0.22118	-1.5
0.200	3.17149	0.29798	-1.5
0.299	3.20814	0.44165	-1.5
0.300	3.20858	0.44308	-1.5
0.400	3.25995	0.58330	-1.5
0.600	3.39475	0.62216	1.5
0.800	3.48474	0.27618	1.5
1.000	3.50489	-0.07497	1.5
1.200	3.45484	-0.42481	1.5
1.400	3.33554	-0.76586	1.5
1.562	3.19007	-1.02689	1.5
1.563	3.18904	-1.02844	1.5
1.600	3.14994	-1.08495	1.5
1.800	2.90444	-1.36054	1.5
1.823	2.87283	-1.38818	1.5
1.824	2.87144	-1.38936	1.5
2.000	2.61053	-1.56401	1.5
2.200	2.28556	-1.66699	1.5
2.230	2.23546	-1.67248	1.5
2.231	2.23379	-1.67261	1.5
2.400	1.95161	-1.65247	1.5
2.600	1.63228	-1.52297	1.5
2.800	1.34582	-1.35339	-1.5
3.000	1.08491	-1.27100	-1.5
3.200	0.83134	-1.27982	-1.5
3.400	0.56689	-1.37987	-1.5
3.600	0.32470	-1.03427	1.5
3.800	0.15196	-0.69666	1.5
4.000	0.04465	-0.37953	1.5
4.191	-0.00003	-0.08972	1.5

TIMEOPTIMAL TRAJECTORY

STARTPOINT 3.1416 0.0000 TIMESTEP 0.0001 A = 1.570796 B = 0.72972

TIME	POSITION		USED U
0.2000	3.08130	-0.59579	3.0
0.4000	2.90522	-1.16166	3.0
0.6000	2.62242	-1.64724	3.0
0.8000	2.25668	-1.97734	3.0
1.0000	1.84690	-2.07797	3.0
1.1347	1.57095	-2.00007	3.0
1.1348	1.57075	-1.99998	3.0
1.1349	1.57055	-1.99988	-3.0
1.2000	1.44220	-1.94758	-3.0
1.4000	1.05675	-1.94731	-3.0
1.5616	0.72994	-2.12335	-3.0
1.5617	0.72973	-2.12351	-3.0
1.5618	0.72951	-2.12322	3.0
1.6000	0.65055	-2.01053	3.0
1.8000	0.31015	-1.38803	3.0
2.0000	0.09542	-0.76208	3.0
2.2000	0.00399	-0.15485	3.0
2.2500	-0.00098	-0.00478	3.0

TRAJECTORY

STARTPOINT 3.1416 0.0000 TIMESTEP 0.0010 A = 1.570796 B = 0.746717
EPS= 0.012566

TIME	POSITION		USED U
0.147	3.16488	0.10961	3.0
0.148	3.16498	0.10658	3.0
0.200	3.16644	-0.05066	3.0
0.299	3.14662	-0.34933	3.0
0.300	3.14627	-0.35234	3.0
0.400	3.09609	-0.65046	3.0
0.600	2.90831	-1.21866	3.0
0.800	2.61414	-1.70306	3.0
1.000	2.23787	-2.02570	3.0
1.200	1.81975	-2.11188	3.0
1.400	1.40956	-1.97711	-3.0
1.562	1.09152	-1.97636	-3.0
1.563	1.08955	-1.97686	-3.0
1.600	1.01601	-1.99966	-3.0
1.800	0.60430	-1.94308	3.0
1.823	0.56041	-1.87280	3.0
1.824	0.55854	-1.86973	3.0
2.000	0.27789	-1.31716	3.0
2.200	0.07716	-0.69331	3.0
2.230	0.05774	-0.60150	3.0
2.231	0.05714	-0.59844	3.0
2.393	-0.00006	-0.10908	3.0

TRAJECTORY

STARTPOINT 3.1416 0.0000 TIMESTEP 0.0010 A = 1.570796 B = 1.023718
EPS= 0 188496

TIME	POSITION		USED U
0.147	3.17395	0.43937	-3.0
0.148	3.17439	0.44233	-3.0
0.200	3.20139	0.59579	-3.0
0.299	3.27462	0.88212	-3.0
0.300	3.27550	0.88496	-3.0
0.400	3.37253	0.90949	3.0
0.600	3.49127	0.27756	3.0
0.800	3.48358	-0.35451	3.0
1.000	3.34946	-0.98607	3.0
1.200	3.09024	-1.59967	3.0
1.400	2.71469	-2.13506	3.0
1.562	2.34211	-2.43921	3.0
1.563	2.33967	-2.44059	3.0
1.600	2.24849	-2.48631	3.0
1.800	1.73951	-2.55035	3.0
1.823	1.68100	-2.53716	3.0
1.824	1.67846	-2.53650	3.0
2.000	1.24341	-2.43499	-3.0
2.200	0.76741	-2.17583	3.0
2.230	0.70344	-2.08897	3.0
2.231	0.70135	-2.08603	3.0
2.400	0.39222	-1.56630	3.0
2.600	0.14207	-0.93683	3.0
2.800	0.01622	-0.32489	3.0
2.379	-0.00006	-0.08738	3.0