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1972

Document Version:
Publisher's PDF, also known as Version of record

[Link to publication](#)

Total number of authors:
1

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Linear Filtering With Unknown Initial Values

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Abstract: Linear filtering with unknown initial values is the dual
of the linear quadratic control problem with fixed endpoint.

The Kalman filter would require infinite initial gain and finite
gain would give infinite error covariance. It is here shown both
for continuous and discrete time that an optimal estimate with finite
error covariance can be obtained after a nonzero time interval giving
starting values for the Kalman filter. The requirements on observa-
bility are also specified. The structure of the optimal solution is
similar to the smoothing estimate.

Introduction: At the start up of a Kalman filter it is assumed that the statistics of the initial state are known. It is often unrealistic to rely upon that information. If no second order moment of the probability distribution of the initial state is assumed to exist, this paper gives the necessary modification for an optimal solution. It is usually suggested that the inverse of the covariance matrix should be used to start up the covariance equation, e.g. [5] , but this does not solve the problem with the filter. A finite gain will not give the correct start up, and the error covariance will never exist!

Statement of the problem: Regard the system

$$\begin{cases} dx = Axdt + dv \\ dy = Cxdt + de \end{cases} \qquad \begin{matrix} x(0) = x_o \\ y(0) = 0 \end{matrix}$$

where $Ex_o = 0$, and where v, e wiener processes with incremental covariance $R_1 dt$, $R_2 dt$, independent of each other and of x_o. Assume also that the probability distribution of the initial state is degenerated, so that there exists no finite second order moment in some linear subspace of the state space, while it is finite in the rest of the state space. Let the projections of x_o into the two subspaces be given by the projection matrix N_1

$$x_o = x_o^1 + x_o^2 = N_1 x_o + (I - N_1) x_o$$

where x_o^1 has the finite covariance R_o^1.

Determine the best linear estimate $\hat{x}(t_1)$ of $x(t_1)$ based on $y(t)$ during $[0, t_1]$.

First estimate linear combinations of $x(t_1)$ say $a^T x(t_1)$ and let best mean minimizing the mean square error

$$J = E(a^T x(t_1) - a^T \hat{x}(t_1))^2 \tag{1}$$

Admissible estimators are

$$a^T \hat{x}(t_1) = - \int_o^{t_1} u^T(t) dy(t)$$

The notation $a^T \hat{x}(t_1)$ will be justified later. There is really a $\hat{x}(t_1)$ such that $a^T \hat{x}(t_1)$ is the best estimate of $a^T x(t_1)$ for all a.

Solution by duality: The problem will be solved using the duality with linear quadratic control problems [1,4].

Rewrite (1) using the vector z(t) defined by

$$\begin{cases} -\dot{z} = A^T z + C^T u \\ z(t_1) = a \end{cases} \tag{2}$$

then

$$a^T x(t_1) - a^T \hat{x}(t_1) = z^T(t_0)x(t_0) + \int_0^{t_1} [z^T(t)dv(t) + u^T(t)de(t)]$$

In order to get a finite J, i.e. to have a finite covariance of $z^T(0)x(0)$, z(0) has to be orthogonal to all x_0^2, requiring

$$Nz(0) = (I-N_1)^T z(0) = 0 \tag{3}$$

so that

$$J = z^T(0)R_0^1 z(0) + \int_0^{t_1} (z^T(t)R_1 z(t) + u^T(t)R_2 u(t))dt \tag{4}$$

J should now be minimized with respect to u under the constraints (2) and (3) giving, e.g. [2] or [3],

$$u(t) = -R_2^{-1}C\Pi(t)z(t) - R_2^{-1}C\psi(t,0)N^T(NM(0,t_1)N^T)^{-1}N\psi^T(t_1,0)a \tag{5}$$

where

$$\begin{cases} \dot{\Pi} = A\Pi + \Pi A^T + R_1 - \Pi C^T R_2^{-1} C\Pi \\ \Pi(0) = R_0^1 \end{cases} \tag{6}$$

and where the fundamental matrix ψ satisfies

$$\begin{cases} \dfrac{d}{dt}\,\psi(t,s) = (A-\Pi(t)C^T R_2^{-1}C)\psi(t,s) \\ \psi(s,s) = I \end{cases}$$

and

$$M(t,t_1) = \int_t^{t_1} \psi^T(s,t)C^T R_2^{-1}C\psi(s,t)ds \qquad (7)$$

the observability Gramian.

By solving (2),(5) expressions for z(t) and u(t) are obtained

$$z(t) = \psi^T(t_1,t)a - \int_t^{t_1} \psi^T(s,t)C^T R_2^{-1}C\psi(s,0)ds\,N^T(NMN^T)^{-1}N\psi^T(t_1,0)a =$$

$$= \psi^T(t_1,t)a - M(t,t_1)\psi(t,0)N^T(NMN^T)^{-1}N\psi^T(t_1,0)a$$

and

$$u(t) = [-R_2^{-1}C\Pi(t)\psi^T(t_1,t) + R_2^{-1}C\Pi(t)M(t,t_1)\psi(t,0)N^T(NMN^T)N\psi^T(t_1,0) -$$

$$- R_2^{-1}C\psi(t,0)N^T(NMN^T)^{-1}N\psi^T(t_1,0)]a$$

Thus

$$x(t_1) = \int_o^{t_1} \{\psi(t_1,t)\Pi(t) - \psi(t_1,0)N^T(NM(0,t_1)N^T)^{-1}N\psi^T(t,0)M(t,t_1)\Pi(t) +$$

$$+ \psi(t_1,0)N^T(NM(0,t_1)N^T)^{-1}N\psi^T(t,0)\}C^T R_2^{-1}dy(t) \qquad (8)$$

independent of a.

Note also that the minimal J is

$$J = a^T\Pi(t_1)a + a^T\psi(t_1,0)N^T(NM(0,t_1)N^T)^{-1}N\psi^T(t_1,0)a$$

The covariance of the estimation error, $\tilde{x}(t_1)$, is thus

$$P(t_1) = \Pi(t_1)+\psi(t_1,0)N^T(NM(0,t_1)N^T)^{-1}N\psi^T(t_1,0) = \qquad (9)$$

$$= \Pi(t_1) + \Sigma(t_1)$$

Further differentiate (8) to obtain a recursive formula for $\hat{x}(t_1)$ and use

$$\frac{d}{dt_1}(NM(0,t_1)N^T)^{-1} = -(NMN^T)^{-1}\{N\psi(t_1,0)C^TR_2^{-1}C\psi^T(t_1,0)N^T\}(NMN)^{-1}$$

so that

$$d\hat{x}(t_1) = A\hat{x}(t_1)dt_1+\{\Pi(t_1)+\psi(t_1,0)N^T(NMN^T)^{-1}N\psi(t_1,0)\}C^TR_2^{-1}[dy(t_1)-C\hat{x}(t_1)dt_1]=$$

$$= A\hat{x}(t_1)dt_1+P(t_1)C^TR_2^{-1}\{dy(t_1)-C\hat{x}(t_1)dt_1\} \qquad (10)$$

This is the usual Kalman Bucy filter.

However (10) could not be applied from t=0 since $\Sigma(t)$ and thus P(t) are infinite for t=0.

It is possible to rewrite $\hat{x}(t_1)$ using $\hat{x}_{\Pi}(t_1)$:

$$\begin{cases} d\hat{x}_{\Pi}(t) = A\hat{x}(t)dt+\Pi(t)C^TR_2^{-1}(dy(t)-C\hat{x}_{\Pi}(t)dt) \\ \hat{x}_{\Pi}(0) = 0 \end{cases} \qquad (11)$$

or

$$\hat{x}_{\Pi}(t_1) = \int_0^{t_1} \psi(t_1,t)\Pi(t)C^TR_2^{-1}dy(t)$$

then

$$\hat{x}(t_1) = \hat{x}_\pi(t_1) + \psi(t_1,0)N^T(NMN^T)^{-1}N[\int_0^{t_1} \psi^T(t,0)C^TR_2^{-1}dy(t) -$$

$$- \int_0^{t_1} \psi^T(t,0)(\int_t^{t_1} \psi^T(s,t)C^TR_2^{-1}C\psi(s,t)\pi(t)C^TR_2^{-1}dy(t))ds$$

$$= \hat{x}_\pi(t_1) + \psi(t_1,0)N^T(NMN^T)^{-1}N[\int_0^{t_1} \psi^T(t,0)C^TR_2^{-1}dy(t) -$$

$$- \int_0^{t_1} \psi^T(s,0)C^TR_2^{-1}C[\int_0^s \psi(s,t)\pi(t)C^TR_2^{-1}dy(t)]ds =$$

$$= \hat{x}_\pi(t_1) + \psi(t_1,0)N^T(NMN^T)^{-1} \int_0^{t_1} \psi^T(t,0)C^TR_2^{-1}[dy(t)-C\hat{x}_\pi(t)dt]$$

$$= \hat{x}_\pi(t_1) + \psi(t_1,0)N^T(N\ M\ N^T)^{-1}\lambda(0) \tag{12}$$

where

$$\begin{cases} -d\lambda(t) = (A-\pi C^TR_2^{-1})^T\lambda(t)dt + C^TR_2^{-1}[dy(t) - C\hat{x}_\pi(t)dt] \\ \lambda(t_1) = 0 \end{cases} \tag{13}$$

Note the ressemblance with the smoothing estimate formulas e.g. [3]. Thus if x_0^2 is observable at $t=\tau$, i.e. $(NM(0,\tau)N^T)^{-1}$ exist, $\hat{x}(\tau)$ and $P(\tau)$ could be obtained using (11), (12), (13) and (9). What approximately happens as $\tau \to 0$ is demonstrated in Example 1.

Discrete time results: The discrete time case

$$\begin{cases} x(t+1) = \phi(t+1,t)x(t) + v(t) \\ y(t) = \theta x(t) + e(t) \end{cases}$$

with analog v, e and x(0), can be treated in the same way. The best one step predictor $\hat{x}(t_1|t_1-1)$ is

$$\hat{x}(t_1|t_1-1) = \hat{x}_\Pi(t_1|t_1-1) + \psi(t_1,t_o)N^T(NM(t_o,t_1)N^T)^{-1}N\lambda(t_o-1)$$

where

$$\begin{cases} \hat{x}_\Pi(t+1|t) = \phi(t+1,t)\hat{x}_\Pi(t|t-1) + K(t)[y(t)-\theta\hat{x}_\Pi(t|t-1)] \\ \hat{x}_\Pi(t_o|t_o-1) = 0 \end{cases}$$

$$K(t) = \phi(t+1,t)\Pi(t)\theta^T[\theta\Pi\theta^T+R_2]^{-1}$$

$$\psi(t+1,t) = \phi(t+1,t) - K(t)\theta(t)$$

$$\begin{cases} \Pi(t+1) = \phi(t+1,t)\Pi(t)\phi^T(t+1,t)+R_1-K(t)\theta\Pi(t)\phi^T(t+1,t) \\ \Pi(t_o) = R_o^1 \end{cases}$$

$$M(t_o,t_1) = \sum_{t=t_o}^{t_1-1} \psi^T(t,t_o)\theta^T(R_2+\theta\Pi(t)\theta^T)^{-1}\theta\psi(t,t_o)$$

$$\begin{cases} \lambda(t-1) = \psi^T(t+1,t)\lambda(t)+\theta^T[R_2+\theta\Pi\theta^T]^{-1}[y(t)-\theta\hat{x}_\Pi(t|t-1)] \\ \lambda(t_1-1) = 0 \end{cases}$$

The covariance of the error $\tilde{x}(t_1|t_1-1)$ is

$$P(t_1)= \Pi(t_1)+\Sigma(t_1) = \Pi(t_1)+\psi(t_1,t_o)N^T(NM(t_o,t_1)N^T)^{-1}N\psi^T(t_1,t_o)$$

which is finite provided that the part of the initial state that does not have any finite second moment, is observable for the interval $[t_o,t_1]$. This assures that the inverse exists. The proof is done in the same way using duality but is rather lengthy.

Example 1 (Continuous time)

$$\begin{cases} \dot{x} = -ax + v \\ y = x + e \end{cases} \qquad Ex_0 = 0, \; Ex_0^2 = 1, \; R_2 = 1, \; R_1 = r_1$$

$$\begin{cases} \dot{\Pi} = -2a\Pi + r_1 - \Pi^2 \\ \Pi(0) = 0 \end{cases}$$

$$\Pi(t) = [\Lambda_{21}(t,0) + \Lambda_{22}(t,0)\Pi(0)] / [\Lambda_{11}(t,0) + \Lambda_{12}(t,0)\Pi(0)]$$

$$\frac{d}{dt}\Lambda(t,0) = \begin{bmatrix} a & 1 \\ r_1 & -a \end{bmatrix} \Lambda(t,0), \; \Lambda(0,0) = I, \; \lambda = \sqrt{r_1 + a^2}$$

$$\Lambda(t,0) = \begin{bmatrix} \cosh \lambda t + \frac{a}{\lambda} \sinh \lambda t & \frac{1}{\lambda} \sinh \lambda t \\ \frac{r_1}{\lambda} \sinh \lambda t & \cosh \lambda t - \frac{a}{\lambda} \sinh \lambda t \end{bmatrix}$$

$$\Pi(t) = r_1 \tanh \lambda t / (\lambda + a \tanh \lambda t)$$

$$\Sigma(t) = \psi^2(t,0) / \int_0^t \psi^2(s,0) ds$$

$$\psi(t,0) = \phi(t,0) - \phi(t,0)\Pi(t)(\Pi(t)+1)^{-1}$$

For $t = \varepsilon \to 0$: $\psi(\varepsilon,0) \approx 1$, $M(0,\varepsilon) \approx \varepsilon$, $\Sigma(\varepsilon) \approx 1/\varepsilon$

$$\lambda(0) \approx \int_0^\varepsilon y(s) ds, \; \hat{x}_\Pi(\varepsilon) \approx 0$$

and thus

$$\begin{cases} \hat{x}(\varepsilon) = \frac{1}{\varepsilon} \int_0^\varepsilon y(s) ds \approx y(0) \\ P(\varepsilon) = 1/\varepsilon \end{cases}$$

This gives the starting values for

$$\dot{\hat{x}}(t) = -a\hat{x}(t) + P(t)(y(t) - \hat{x}(t))$$

when started from $t = \varepsilon$.

Example 2 (Discrete time)

$$x(t+1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t) + v(t) \qquad N = I \quad R_1 = I \quad R_2 = 1$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + e(t)$$

$\Pi(0) = 0$, $K(0) = 0$, $\psi(1,0) = \phi$, $\Pi(1) = I$
$\hat{x}(1|0) = 0$

$$M(0,1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \lambda_1(-1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} y(0)$$

$$\hat{x}(1|0) = \begin{bmatrix} ? \\ 1 \end{bmatrix} y(0), \quad P(1) = \begin{bmatrix} \infty & 0 \\ 0 & 2 \end{bmatrix}$$

$$K(1) = \frac{1}{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \psi(2,1) = \phi \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1/2 & 0 \end{bmatrix}$$

$$\hat{x}_\Pi(2|1) = \frac{1}{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} y(1) \qquad \Pi(2) = \begin{bmatrix} 2 & 0 \\ 0 & 3/2 \end{bmatrix}$$

$$M(0,2) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{2} \phi^T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \phi = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} > 0$$

$$\lambda_2(0) = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} y(1)$$

$$\lambda_2(-1) = \phi^T \lambda_2(0) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} y(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} y(0) + \frac{1}{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} y(1)$$

$$\hat{x}(2|1) = \frac{1}{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} y(1) + \psi(2,1)\psi(1,0)M^{-1}(0,2)\lambda_2(-1) =$$

$$= \frac{1}{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} y(1) + \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}^{-1} \lambda_2(-1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} y(0) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} y(1)$$

$$P(2) = \begin{bmatrix} 2 & 0 \\ 0 & 3/2 \end{bmatrix} + \phi \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \phi \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}^{-1} \phi^T \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \phi^T =$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 3/2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

Remark: The expression can be obtained also by starting with $R_o = cI$ and $c \to \infty$ but for large systems the expressions of $P(t_1)$ and $x(t_1)$ in c would be extremely complicated. Formel manipulations on computer would be necessary. Numerical solution for large parameters c would give large round off errors.

Conclusions: The optimal filters for systems with unknown initial values
have been deduced both for continuous and discrete time. The error cova-
riance becomes finite as soon as the unknown initial modes become observ-
able. Using a formula similar to smoothing the estimate and its error
covariance could then be obtained, and a usual Kalman filter started.
The error covariance still does not contain the actual measurements.

References:

[1] Åström, K.J.: Introduction to Stochastic Control Theory. Academic Press,
     New York 1970

[2] Brockett, R.W.: Finite Dimensional Linear Systems, John Wiley, New York
     1970

[3] Hagander, P: Linear Control and Estimation Using Operator Factorization,
     Report 7114, July 1971.
     Division of Automatic Control, Lund Institute of Technology, Lund, Sweden.

[4] Kalman, R.E. and R.S. Bucy: New Results in Linear Filtering and Predic-
     tion Theory, Trans. ASME, serD, J. Basic Eng. 83, 95 - 107 (Dec 1961)

[5] Sorenson, H.W.: Controllability and Observability of Linear, Stochastic,
     Time-Discrete Control Systems
     in C.T. Leondes Advances in Control Systems 6. pp 95 - 158. Academic
     Press, New York 1968