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Abstract

This note shows that there are square circles, at least in the same sense that there are round circles.

The "round square", or the "square circle" (I will take these to be equivalent, so I will take "round" to be the same as "circular") are well-worn examples of concepts that allegedly cannot be instantiated. Contemporary philosophers probably associate the notion with Meinong on basis of Russell's [1905] criticism of him in, or with Quine's [1905] invocation of the round square cupola of Berkeley College in his criticism of Russell's previous acceptence of Meinong. Its use as an example of *impossibility* or *inconceivability* goes back at least to Hobbes, who asserts that "(...) the word Round Quadrangle signifies nothing, but is a meere sound" [Hobbes, 1651, ch. IV]. Already Aristotle uses it as an example of a *non-entity*; in the *Categories*, we find him stating "(...) the square is no more a circle than the rectangle, for to neither is the definition of the circle appropriate."[Aristotle, 1984, ch. VIII]. Other famous philosophers invoking the round square as an example of an impossibility include Kant [Kant, 1783, §52], Berkeley [Berkeley, 1735, §XLVIII] Leibniz Leibniz [1966] and Spinoza [Spinoza, 1677, book I, prop. XI].

But start up any image manipulation program, such as Paintbrush, GIMP, or Photoshop, and draw a circle with a one pixels radius, making sure that any antialiasing is off. The result will be something like this, where we have marked the center of each pixel with an x.

On one interpretation, this is the *closest approximation* to a circle of radius one on a square grid. But we can also see it in another way: a computer screen has its own type of geometry, in which the distance between points is the Euclidean one, but rounded to the nearest integer. When the radius is a single pixel, the concepts of square and circle coincide.

This, admittedly, involves a lot of hand-waving, such as assuming certain tacit interpretations of circularity and squareness. The obvious counter-argument to the argument is that these interpretations are *wrong*; sqaureness and circularity, correctly construed, *are* incompatible.

We can, of course, easily conceive of an interpretation of the constituent concepts which makes the impossibility a logical truth. But this does no more than show that the nonexistence of round squares does not *contradict* the definitions of roundness and squareness. Compare with the parallel axiom: while the notions of "straight line" and "parallel" do not contradict there being parallel straight lines that intersect, it does not follow from them either.

So which concepts *do* "circle" and "square" entail? The usual definition of a *circle* is the following.

Definition 1. *A* circle *is the set of all points that have the same distance r from a point x called its* center.

This means that circles are definable in any two-dimensional space X in which distances are defined. In mathematical terms, such distances are determined by a *metric*, i.e. a function $d : X \times X \to \mathbb{R}^+ \cup \{0\}$ that satisfies the following axioms.

Reflexivity $d(a, b) = 0$ iff $a = b$.

Symmetry $d(a, b) = d(b, a)$.

Triangle inequality $d(a, c) \geqslant d(a, b) + d(b, c)$.

The most well-known metric is the Euclidean metric on $\mathbb{R}^2$, defined by

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

It is, as is also well-known, not the only one, and it is also not the one that describes distances in all planes in physical space. By definition 1, each metric on X gives rise to a specification of which sets of points are its circles.

What is a *square* then? Its definition is unfortunately somewhat more complicated than that of a circle, but a common, fairly straightforward one is the following.

Definition 2. *A* square *is a quadrilateral with equal sides which meet at right angles.*

Definition 3. *A* quadrilateral *is a closed figure consisting of four straight line segments.*

The concept of a *straight line segment* is commonly defined in terms of that of *geodesic*, which requires X to have differentiable structure. But straight line segments are also definable in any metric space through their property of being the images of paths of least length connecting two points.

A *path* on X is a continuous function $p : [0, 1] \to X$, where $[0, 1]$ is the closed unit interval. We define the *length* of paths on X as a function $\ell : X^{[0,1]} \to \mathbb{R} \cup \{\infty\}$ that satisfies

$$\ell(p) = \sup_{\sigma \in S} \sum_{k=1}^{|\sigma|+1} d(p(\sigma_{k-1}), p(\sigma_k))$$

where S is the set of all $n+1$-tuples $(t_0, \ldots, t_n)$ of real numbers in $[0,1]$ such that $t_0 = 0$, $t_n = 1$, and $t_k \leqslant t_{k+1}$ for all k. This definition gives the length of a path p from a to b as the supremum of the total lengths of all piecewise linear approximations $p(t_0), \ldots, p(t_n)$ of p:[1]
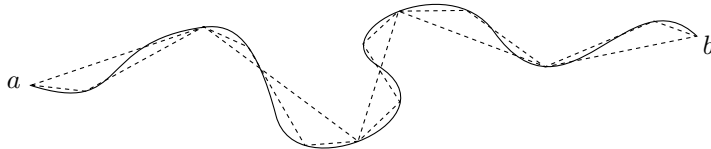
A *line segment* or *arc* is the image of a path. For any arc A with endpoints[2] a, b we say that a path p is a *parametrisation* of A if $p(0) = a$, $p(1) = b$, and p is injective except for possibly at its endpoints. Since it can be proved that $\ell(p) = \ell(q) = x$ for any parametrisations p, q of A, we say that A's length $\ell(A)$ is x.

The exact minimal distance definition of a straight line requires only that it is an arc that is of *locally* minimal length, i.e. no small change in the arc's shape gives a shorter arc. However, for the purposes of this note, we will not need to consider non-global minima, so it is enough for us to note that any arc A between a and b of minimal length is a straight line. Because of the triangle inequality, this is equivalent to to $\ell(A) = d(a,b)$ being a sufficient condition for A to be a straight line.

The other concept we need for the definition of quadrilateral is that of *closed figure*, which, classically, means one that can be drawn without lifting the pen, and which begins and ends at the same point. In our terms, this is simply an arc whose endpoints coincide. Since the parametrisation of an arc is arbitrary, we can thus define a quadrilateral as the image of a closed loop whose images of the intervals $[0, \frac{1}{4}]$, $[\frac{1}{4}, \frac{1}{2}]$, $[\frac{1}{2}, \frac{3}{4}]$ and $[\frac{3}{4}, 1]$ are all straight lines.

This completes the definition of quadrilateral. For defining what a square is, we also need to know what an *angle* is. This was a hotly debated topic in antiquity, and is still the subject of some confusion in students, as witnessed by our tendency to talk of angles as having a dimension (e.g. degrees or radians) that they mathematically do not. We will follow tradition and define angles in terms of circular arcs:

Definition 4. *A* circular arc *with center x and radius r is an arc that is a subset of a circle with centre x and radius r.*

The following is a very general definition of angle:

Definition 5. *The* angle *between two arcs A, B intersecting at x is the limit value*

$$\lim_{r \to 0} \inf_{C \in \Gamma_x^r} \frac{\ell(C)}{r}$$

where Γ_x^r is the set of circular arcs with center x and radius r whose endpoints lie on A and B.

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[1] Have we incurred a circularity in talking of *piecewise linear* approximations here, before we have define linearity? No; the concept does not appear in the definition given, but merely in our informal description of it.

[2] The *endpoints* of an arc are definable as the only two boundary points on that arc; it does not matter which one we denote by a or by b.

This definition assigns a unique angle between two arcs in most cases. It is based on the idea that an angle is simply the length of the section of the unit circle spanning two lines:
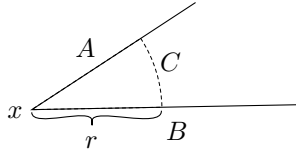
A *right* angle is one that is $\frac{1}{4}$ the length of a full circle. This definition means that such an angle will not, in every geometry, be equal to $\frac{\pi}{2}$, but is does ensure that the sum of angles of any square is four right angles. A different definition of right angle would require us to use a slightly more general definition of square as a quadrilateral with four *equal* angles, although our argument will go through in either case.

We are now ready to show that there are square circles. For simplicity, assume that X is a *topological 2-manifold*, which means that for each point $a \in X$, there is a neighbourhood Y such that Y is homeomorphic to an open subset of $\mathbb{R}^2$. This, in turns, means that there is a continuous injection *coord* : $X \to \mathbb{R} \times \mathbb{R}$ whose inverse is also continuous, which gives *coordinates* for each point. For the purposes of this note, it is enough to consider very simple (or trivial) such manifolds, in which a single coordinate function suffices for the whole of X. Fixing a specific a one-to-one homeomorphic bijection between pairs of real numbers (x, y) and points of X then allows us to refer to any point a by its unique coordinates (x, y), and we will therefore in the sequel not distinguish between a point and its coordinates.

Impose the following metric on X:

$$d(a, b) = ||a - b|| = \max(|x_a - x_b|, |y_a - y_b|)$$

This is known as the *Chebyshev distance*, or the *chessboard distance* since, in discrete spaces, it describes the minimum number of moves it takes to get a king from one square on a chessboard to another.

Now consider the points $a = (-1, 1)$, $b = (1, 1)$, $c = (1, -1)$ and $d = (-1, -1)$, and let SC be the union of the straight lines ab, bc, cd and da (these are uniquely determined since we have assumed X to be homeomorphic to $\mathbb{R}^2$).
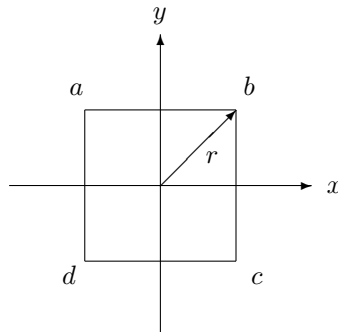
Given our definitions, the coordinate system, and the metric we have imposed, it is now easy to prove the following.

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Lemma 1. *SC is a circle with radius r.*

Lemma 2. *SC is a square with side $2r$.*

Corollary 1. *SC is a square circle.*

This shows conclusively that there are square circles, at least in the abstract, mathematical sense in which there are *round* circles. Rather than being an impossibility, the square circle implies no contradiction at all. I do not, of course, claim that any of this would be news to a mathematician educated after the 19th century, but the round square remains a staple of philosophers' impossibilia. A search in JSTOR's philosophy section results in 874 books or articles mentioning of the "round square" or "square circle" published between 2000 and 2013.

I do not mean to insinuate that all of these philosophers are unaware of the mathematical fact that there are round squares. Sometimes examples get carried along even when we know that they are not strictly true, or at least no longer have strong reasons to believe them. The pseudo-Aristotelian definition of man as a rational animal is an example of this. In that case, however, the presupposition that rationality by necessity is something reserved for humans alone arguably held back theorising about rationality as well as about nonhuman behaviour until the 20th century.

What we should ask, therefore, is if the existence of square circles is a mere mathematical curiosity, or whether recognising that the square circle implies no contradiction points to a deeper lesson about philosophical prejudices that we may have carried with us. I think that there are in fact several such lessons, but at least one of them concerns the stability and determinacy of our so-called common concepts.

Let us imagine a typical philosopher using the round square concept as an example of impossibility being presented with our proof of its existence. As with the pixelated round square, perhaps he would say something like "that is not at all what we mean with 'round' and 'square' when we say that the round square is impossible." But what do "we" mean then? How does one determine that?

It should be clear that an appeal to Platonism would be of no help. We have shown that there are perfectly viable and natural concepts of roundness and squareness that they imply no contradiction, so for a Platonist, this round square should be as much an existing universal as the round circle and the square square. Invoking Platonism does not help the philosopher determine which of the purported universals (the Chebyshev round square, or the Euclidean round square) is meant.

More useful, perhaps, would be an appeal to Kantian intuition. According to this view, concepts such as circlularity and squareness are features of our innate abilities to experience the world as subject to spatial organisation. Since these abilities are taken to be the same for all humans, one might postulate that the philosopher's use of *square* and *circle* would express *these* concepts, and no others.

The problem with this is that squareness and circularity are, first and foremost, *geometrical* concepts. Furthermore, as was made plain during the 19th century, the Kantian view does not capture at all what geometry is like. Valid

arguments for geometrical theses proceed exactly like arguments for other mathematical theses, and *never* invoke spatial intuition. Much of contemporary geometry rather concerns things about which most people, arguably, have no spatial intuitions at all.

Now, perhaps the philosopher only meant to say "when I have imagined a circle, I have never also imagined it as a square; in fact, I do not even think I *can* imagine a square circle". This is hard to argue against, but neither the impossibility nor the non-existence of round squares follows from it. As Berkeley pointed out, we cannot imagine a triangle which is "neither oblique nor rectangle, equilateral, equicrural nor scalenon" [Berkeley, 1734, §13], but not even the esteemed Bishop of Cloyne drew from this the conclusion that there are no triangles.

Most of our abilities to imagine forms have likely developed or been learnt in close connection with our abilities to orient ourselves in physical space, and, especially, around physical objects. We may therefore finally consider the answer that what is meant by "square" or "circle" should be actual physical squares or circles, or some kind of idealisations thereof. This is right in the sense that there *are* physical things that are shaped in a roughly circular manner, and physical things that are shaped in a roughly square manner, and these classes of things, as far as we have found, do not overlap. But this does not even show that round squares are a *physical* impossiblility; perhaps space-time is so warped somewhere that a two-dimensional section of it actually instantiates something very close to the Chebyshev metric, or the Taxicab metric, which also allows round squares. In such a part of space-time, one could perhaps even *draw* a round square, at least in the same idealised sense that one can draw a round circle.

The important point is that it is not the *whole* physical world that plays into what our concepts are like. We largely learn about circles and squares by ostension, and the examples of them we are given are medium-sized objects in a part of space that is close enough to be Euclidean that it is very hard to spot the difference. "There is no circular square between 1 mm and 10 m in radius in my vicinity" is a statement for which we have much stronger grounds than "there are no circular squares", or even "there can be no circular squares". We have a reasonable ability to identify circles and squares of this kind.

The main conclusion I would like to draw from the existence of square circles in more exotic spaces, however, is that concepts such as squareness and circularity are only partially defined, and even which parts are defined is fairly vague. For the moment interpreting concepts as *individual* abilities of classification, these are evolved in specific contexts. In such contexts, it is likely that we get significant levels of interpersonal agreement. But outside a concept's "natural habitat", it become vague, indeterminate, and sometimes simply inapplicable. This is the case with our pre-formal versions of "circle" and "square", whose ordinary use in classifying surfaces of middle-sized objects in a Euclidean plane does not determine their interpretations in other kinds of spaces.

This does not, of course, mean that we cannot *extend* concepts as we wish, and most of the time we do not even notice that we do so. Thus, when the round square is taken as an example of an impossibility, the principle of charity makes us extend the concepts of "round" and "square" so that the example holds true. The *conscious* extension of geometrical concepts is largely the business of mathematics, however.

—saying that no circle is square is an explicit part of a new definition of the concept; it doesn't follow from anything.

As so eminently illustrated by Lakatos [1976], mathematical concepts, like any others, are in general not crystallised, definite, or exact. Even when explicitly defined, they tend to rest on other concepts in which vagueness remains. The standard antidote to this—expression in a formal language—does not solve the problem completely. Very few theories or concepts can be reduced to *purely* logical ones, and therefore we will always need undefined predicates, whose interpretation will remain open outside the narrow confines in which they can have been learned or introduced.

This insight has, I believe, importance for the methodology of philosophy. It indicates that the use of most thought experiments in conceptual analysis are far less useful than what it commonly presupposed. If only the "centre" of a concept is somewhat stable, since it has been determined by the contexts it has evolved in or been learnt in, then it is meaningless to try to draw conclusions from intuitions about what it would be applicable to in situations very different from these.

A consequence of this is that arguments against, say, utilitarianism, which make use of extreme or uncommon situations, such as ones in which one can only stem a riot by sentencing an innocent man, will always be invalid. Gettier arguments have no relevance for the analysis of knowledge, since these involve events that are very unlikely and also very much unlike typical uses of locutions such as "he *knew* that *p*". And deriving a counter-intuitive or even absurd consequence from a metaphysical thesis does not constitute a counterargument to that thesis, since metaphysics as a whole arguably lies far from our usual experience. After all, if even the concepts *square* or *circle* are so vague that it is not determinate whether they are contradictory or not, how much more so must this not hold of concepts such as *knowledge, meaning, right, justice* or *reality*?

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