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On Observer-Based Control of Nonlinear Systems

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Automatic Control

On Observer-Based Control of Nonlinear Systems

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Anders Robertsson

Lund 1999

Till min kära familj

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Contents

Preface

The theory presented in this thesis relates to observer design and output feedback control of nonlinear systems. Robot manipulators constitute one important class of nonlinear systems which has been considered.

The work has been conducted within the program "Mobile Autonomous Systems" connected to the Robotics Lab at the Department of Automatic Control, Lund Institute of Technology. Robotics is indeed a multidisciplinary topic and the constitution of the project group has lead to a fruitful interaction with a lot of people with different views on robotics. The Open Robot System which is used as the platform for the experimental research in the robotics laboratory has been one important corner stone. Through this concept it has been possible to implement dedicated control laws and to perform experiments on an industrial robot manipulator.

The variety in each part and the need of all the links to form a complete chain, not necessarily a kinematic one, is really the main reason for why I find control engineering being such an exciting and fascinating area. The requirement to cover, at least partially, the whole span from modeling, the analysis and theoretical design, re-iterated via simulations, the real-time aspects of implementation, ending up with running experiments in the laboratory, and then starting it all over again, has been challenging. To summarize, it has been a very interesting and most rewarding path to follow, giving me insight into both practical and theoretical aspects of robotics.

The work presented in this thesis is mainly based on the following publications:

Johansson, R. and A. Robertsson (1999): "Extension of the Yakubovich-Kalman-Popov lemma for stability analysis of dynamic output feedback systems." In *Proceedings of IFAC'99*, vol. F, pp. 393–398. Beijing, China.

Johansson, R., A. Robertsson, and R. Lozano-Leal (1999): “Stability analysis of adaptive output feedback control.” In *Proceedings of the 38th IEEE Conference on Decision and Control (CDC’99)*, pp. 3796–3801. Phoenix, Arizona.

Lefeber, E., A. Robertsson, and H. Nijmeijer (1999): “Linear controllers for tracking chained-form systems.” In Aeyels *et al*, Eds., *Stability and Stabilization of Nonlinear Systems*, vol. 246 of *Lecture Notes in Control and Information Sciences*, pp. 183–197. Springer-Verlag, Heidelberg. ISBN 1-85233-638-2.

Lefeber, E., A. Robertsson, and H. Nijmeijer (2000): “Linear controllers for exponential tracking of systems in chained form.” *International Journal of Robust and Nonlinear Control: Special issue on Control of Underactuated Nonlinear Systems*, **10:4**. In press.

Robertsson, A. and R. Johansson (1998a): “Comments on ‘Nonlinear output feedback control of dynamically positioned ships using vectorial observer backstepping’.” *IEEE Transactions on Control Systems Technology*, **6:3**, pp. 439–441.

Robertsson, A. and R. Johansson (1998b): “Nonlinear observers and output feedback control with application to dynamically positioned ships.” In *4th IFAC Nonlinear Control Systems Design Symposium (NOLCOS’98)*, vol. 3, pp. 817–822. Enschede, Netherlands.

Robertsson, A. and R. Johansson (1998c): “Observer backstepping and control design of linear systems.” In *Proceedings of the 37th IEEE Conference Decision and Control*, pp. 4592–4593.

Robertsson, A. and R. Johansson (1999): “Observer backstepping for a class of nonminimum-phase systems.” In *Proceedings of the 38th IEEE Conference on Decision and Control (CDC’99)*, pp. 4866–4871. Phoenix, Arizona.

A more complete list of the author’s publications within the Lund Program on “Mobile Autonomous Systems” is found in a separate section of the Bibliography.

Acknowledgements

This thesis could not have been completed without the help and continuous support from colleagues, friends, and family to whom I am most grateful. First and foremost, I would like to express my sincere gratitude to my supervisor Professor Rolf Johansson. He has been a constant source of inspiration and ideas, given me great challenges and been very supportive with all kinds of help. Rolf's profound knowledge of numerous disciplines combined with his generous sharing of time has lead to many long-lasting, interesting and enjoyable discussions, which have contributed substantially to this thesis.

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Before I started as a graduate student in Lund, I got a grant from Lund University to spend the academic year 1992–93 at the Center for Control Engineering and Computations, UC Santa Barbara. In particular, I would like to thank Prof. Petar Kokotović, Prof. Alan Laub, and all the other good friends in the CCEC-group for a most rewarding and interesting stay that has influenced me a lot. Henrik Olsson's and Ulf Jönsson's advice and generous sharing of previous experiences from UCSB were all very much appreciated.

Dr. Klas Nilsson's genuine interest in robotics and his habit to spend many late hours in the laboratory, have been most contagious. He has given me a lot of valuable insights and new aspects on both practical and theoretical issues on robotics. Rolf Braun has been very helpful with hardware design, re-design, and all the practical garbage collection, as the deserving real-time implementation by Roger Henriksson and Anders Ive unfortunately only handles the software aspects of the latter problem. I have also come to respect Anders Blomdell's large knowledge ranging from software issues to his craftsmanship of mechanical design. He has provided invaluable help, often under hard real-time constraints. It has also been great fun working together with Johan Eker during the experiments with the inverted pendulum in the robotics laboratory.

During my time as a PhD student I have had the great opportunity to make several visits to Professor Kwakernaak's group at the University of Twente. In particular, it has been a privilege to cooperate with Professor Henk Nijmeijer and Erjen Lefeber, co-authors to papers presented in this thesis. Their knowledge and wonderful sense of humor have contributed to many joyful moments, which I strongly hope to share with them also in

the future. I also want to thank Robert van der Geest for his hospitality and all the fun we have had together.

In addition to Rolf and Klas, I would also like to thank Gunnar Bolmsjö, Magnus Olsson, Per Hedenborn, and Per Cederberg from the Division of Robotics, Gustaf Olsson and Gunnar Lindstedt from the Department of Industrial Electrical Engineering and Automation, and Mathias Haage from the Department of Computer Science, all colleagues in the Lund Program on Mobile Autonomous Systems. The cooperation with them has broadened my view on robotics considerably.

Lennart Andersson has come to be a very good friend of mine. The fear that his strive for reducing dynamics versus mine of adding it would end up in a prevailing *status quo* has fortunately been unfounded. I gratefully acknowledge his help to reduce a lot of unstructured uncertainties in my work.

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Anders

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1

Introduction

1.1 Background and Motivation

In control engineering, the objective is to achieve a feasible control signal, which based on measurements, affects the controlled process to behave in a desired way, despite disturbances acting from the environment. Robot manipulators constitute good examples of nonlinear systems, which are used in numerous applications in industry. Still, they raise several challenging theoretical questions that remain to be answered. Although robot control is an area in its own right, it also serves well as an illustrative application for examples throughout the thesis.

For the robot manipulator, we are able to control its movements with the torques from the drives and calculate its configuration in space from measurements of the joint angles. In most industrial robot systems, there are no sensors for measuring the velocities or the accelerations. For certain applications, extra sensors can be added to the robot system to measure, for instance, contact forces when interacting with the environment. Typical disturbances and uncertainties are unknown load weights, stiffness constants, or inaccurate descriptions of the environment regarding the exact shape or location of an obstacle or object within the working range.

The inertia of the manipulator varies considerably with respect to the configuration, and for fast movements the influence of the centrifugal and Coriolis forces increases drastically. Nevertheless, with high-gain feedback much of the effects from the nonlinearities can be overcome and large gear-ratios may help to decouple the effects among the robot links. Despite all the nonlinearities in the equations of motion, control laws based on linearization—i. e., a locally valid approximation of the equations—perform well in a lot of applications. On the other hand, there are systems where a linear approximation would provide very little help, if any, in the controller design. Examples of this type of systems are the mobile robots in chained-form considered in the thesis. The linearization of this class of system around any equilibrium is not asymptotically stabilizable.

The performance can often be improved significantly if the knowledge of the nonlinearities are taken into consideration. Even though all gravity forces are hard to compensate for exactly, due to uncertainties in load, etc., the gravity forces acting on the manipulator itself are well known and can be taken into account *a priori* in feed-forward terms. Feedback is then used for stabilization and to compensate for inaccuracies in the model and the effects of external disturbances. The idea behind *feedback linearization* is to compensate for known nonlinearities and if possible transform the system, without any approximations, into a linear system for which a lot of control design methods apply. This concept has been used for long in robotics under the name of *computed torque* or the method of *inverse dynamics*.

For the trajectory tracking problem it is also of interest to let the control signal consist of one feedforward part based on the desired behavior and one part consisting of feedback with respect to the deviations from the desired trajectory. A reference trajectory for the manipulator does not only consist of the reference positions at every time, but also of the consistent velocities and accelerations.

The compensation for nonlinear terms and the trajectory tracking problem mentioned above imply the need for measuring or estimating the states in a system. The title of this thesis, “observer-based control of nonlinear systems”, should be read in contrast to “state feedback” control. The feedback principle is an important concept in control theory and many different control strategies are based on the assumption that all internal states are available for feedback. In most cases, however, only a few of the states or some functions of the states can be measured. This circumstance raises the need for techniques, which makes it possible not only to estimate states, but also to derive control laws that guarantee stability when using the estimated states instead of the true ones. In general, the combination of separately designed observers and state feedback controllers does not preserve performance, robustness, or even stability of each of the separate designs. A fundamental difference in properties of linear and nonlinear systems is found in the effects of bounded disturbances over a finite time horizon. Consider a linear system, for which there is a stabilizing state-feedback law. If the feedback law is fed with estimations of the actual states , the closed loop system will still be stable under the assumption that the observer errors converge to zero. For nonlinear systems stability is not guaranteed by exchanging measured states for estimated ones, even if we have exponential convergence in the observer. One obstacle is the *finite escape time phenomenon* where a solution may grow unbounded before the estimated states have converged.

In this thesis the question of observer design is addressed. The stability problem for the combination of observers and state feedback controllers is also investigated. For a special class of nonholonomic systems a separation principle is shown, which guarantees the stability for the combination of independently designed state feedback controllers and state observers. As mentioned before, few systems have this strong property, which justifies controller designs considering the effects from estimated states. We present an extension to the output-feedback design of *observer-based backstepping*. The extension applies to a class of nonlinear systems with unstable zero-dynamics, which was not previously comprised.

1.2 Outline and Summary of Contributions

This thesis consists of two major parts. The first part includes preliminary material and a short survey of related work. The second part consists of five appended published papers containing the main results of this thesis. The introductory Chapters 2, 3, and 4 aim towards a sufficiently complete overview of the dynamic output feedback problem and the observer design problem to present the contributions of the five papers in their appropriate context. To this purpose, some examples illustrating our results are also provided. Finally, Chapter 5 concludes the thesis by a summary of the results and a short discussion on open issues.

Below, the contents and main contributions of the papers are summarized. References to related publications are also given.

Paper A and B

Observer-based controllers for the output tracking problem of nonholonomic systems in chained form are presented in the two papers

Lefeber, E., A. Robertsson, and H. Nijmeijer (1999a): "Linear controllers for tracking chained-form systems." In Aeyels *et al*, Eds., *Stability and Stabilization of Nonlinear Systems*, vol. 246 of *Lecture Notes in Control and Information Sciences*. ©1999 Springer-Verlag, Heidelberg. ISBN 1-85233-638-2.

Lefeber, E., A. Robertsson, and H. Nijmeijer (2000): "Linear controllers for exponential tracking of systems in chained form." *International Journal of Robust and Nonlinear Control: Special issue on Control of Underactuated Nonlinear Systems*, **10:4**. In press. ©1999 John Wiley & Sons, Ltd.

Contributions New time-varying state feedback controllers and observers for the output tracking problem of nonholonomic systems in chained form are presented. A global stability result for the combination of controllers and observers in a "certainty equivalence" way is given, using theory from time-varying cascaded systems. Furthermore, in Paper B we present a stability result for linear time-varying systems. The state feedback and the output feedback control problem are considered also under partial input saturation constraints.

Related publications are [Lefeber *et al*, 1999b; Lefeber *et al*, 1999c].

Paper C

Stability analysis related to the Positive Real Lemma is presented in

Johansson, R. and A. Robertsson (1999): "Extension of the Yakubovich-Kalman-Popov lemma for stability analysis of dynamic output feedback systems." In *Proceedings of IFAC'99*, vol. F, pp. 393–398. Beijing, China. ©1999 IFAC

Contributions Relaxations of the minimality conditions in the Positive Real lemma, also known as the Yakubovich-Kalman-Popov lemma, with relevance to observerbased feedback control are presented.

Related publications are [Johansson and Robertsson, 1998; Johansson *et al*, 1999].

Paper D

A generalization of the design method *observer-based backstepping* is considered in an extended version of the paper

Robertsson, A. and R. Johansson (1999c): "Observer Backstepping for a Class of Nonminimum-Phase Systems." In *Proceedings of the 38th IEEE Conference on Decision and Control (CDC'99)*. Phoenix, Arizona. ©1999 IEEE

Contributions The design-method "observer-based backstepping" by Kanellakopoulos *et al* (1992) is extended to cover also a class of nonlinear systems in output-feedback form with linear *unstable* zero-dynamics and an algorithm for the observer-based controller is presented.

A related publication is [Robertsson and Johansson, 1998c].

Paper E

An observer design for the purpose of output feedback control is presented in

Robertsson, A. and R. Johansson (1998a): "Comments on 'Nonlinear Output Feedback Control of Dynamically Positioned Ships using Vectorial Observer Backstepping'." *IEEE Transactions on Control Systems Technology*, **6:3**, pp. 439–441. ©1999 IEEE

Contributions The paper presents a globally exponentially stable observer design for the purpose of output feedback control of ship dynamics [Fossen and Grøvlen, 1998]. The Lyapunov-based design extends previous results to ship with unstable sway-yaw dynamics.

Related publications are [Robertsson and Johansson, 1997; Robertsson and Johansson, 1998b].

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2

Nonlinear Feedback Control

2.1 Introduction

Many phenomena in nature and society can be described or approximated by mathematical models. The use of ordinary differential equations is one way of describing dynamic processes where one or more variables depend continuously on time. The description of a process in a mathematical terminology allows for a uniform framework of analysis and synthesis, the system theory, despite the fact that the original problems may come from widely differing areas such as robotics, biochemistry, economics, or telecommunication.

In the scope of linear systems, a vast collection of methods have been developed. Their use is found in both analysis and systematic design for continuous as well as for discrete-time systems, with system representation in the time domain or in the frequency domain, and in a deterministic or a stochastic setting. An important property distinguishing linear systems from nonlinear systems is that of the superposition principle, where the output response to a sum of different input signals is the sum of their individual responses. This allows for a simplified stability analysis. The influence of additive disturbances, such as measurement noise or load disturbances, can be considered separately in the linear case. Even for the simplest first-order nonlinear systems the questions of uniqueness and existence of solutions indicate the difficulties that we may encounter when we are leaving the linear framework [Khalil, 1996]. For general nonlinear systems very little can be said about specific properties and thus few general methods apply. The characterization of nonlinear systems with respect to special structures and particular properties is therefore a standard approach.

Robot dynamics represent a class of nonlinear systems which play an important role for industrial production and at the same time raise challenging theoretical questions. This has inspired and driven a lot of the development of nonlinear control theory during the last decades. An industrial robot consists of links connected by joints into a kinematic chain. Furthermore, there are typically some actuators and an end-effector with some application-specific tool attached. The kinematic chain can be described by trigonometric functions and the dynamics for the manipulator are often derived via classical mechanics using the Euler-Lagrange equations or the Newton-Euler formulation [Spong and Vidyasagar, 1989; Sciavicco and Siciliano, 1996]. Within the working range of a robot, the application often puts extra constraints, not only on the desired, but also on the feasible motion. In force control applications such as grinding, the interaction between the robot and a stiff environment can be modeled by *holonomic constraints*. The combined motion/force-control is restricted to a surface in space which can be described by the manipulator dynamics projected on a (sub-)manifold of the position coordinates [Goldstein, 1980]. The degrees of freedom, i. e., the number of independent state variables for the system, are reduced accordingly.

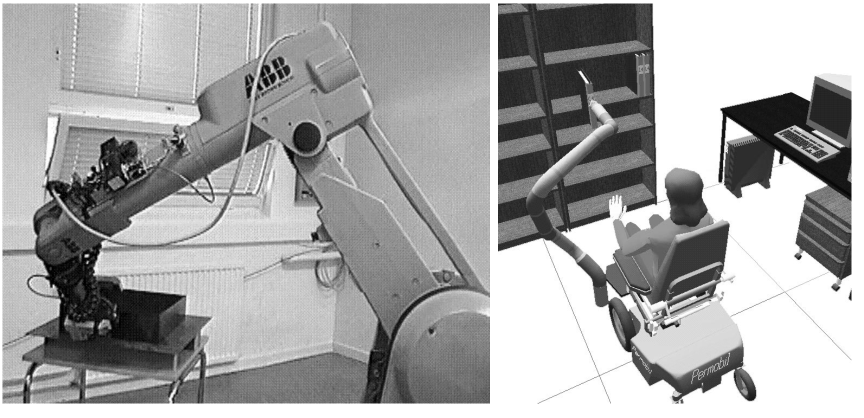

Figure 2.1 *Left:* Industrial robot with *holonomic* constraints in a path following operation. *Right:* Redundant robot arm and mobile robot with *nonholonomic* constraints (rolling without slipping).

Nonholonomic constraints, however, do not reduce the order of a system and such constraints can not be given in integrated form without actually solving the problem for a feasible trajectory [Goldstein, 1980]. An often-used example to illustrate nonholonomic constraints is a tire rolling on a surface without slipping. The constraints for not sliding are given as algebraic constraints involving the velocities, that is, for the derivatives of the states rather than for the states themselves. Both underactuated manipulators and redundant robot arms under inverse kinematic mappings can be subject to nonholonomic constraints [De Luca and Oriolo, 1994].

Although performance is an important criterion in control applications, a prerequisite, if not the primary goal, is asymptotic stability or stabilization of the system along a desired trajectory or to an equilibrium point. The possibility to determine stability without explicitly solving the system equations is a crucial part in nonlinear analysis and here the Lyapunov theory of Lyapunov, LaSalle, Krasovsky, Kalman *et al* plays a very important role [Khalil, 1996]. Although physical insight and energy-like functions often provide good guesses for Lyapunov functions, there is still a lack of general constructive methods.

The first two sections of this chapter review some important stability concepts and an overview of design methods for nonlinear systems is given. This introduction is intended both to present a foundation for some of the results presented in this and forthcoming chapters, and also to give an overview of the present state-of-the-art. In that perspective, two contributions are presented: Firstly, a new matrix formulation of the Kalman-Yakubovich-Popov Lemma is provided. Secondly, linear time-varying control for output feedback tracking of systems in chained form is presented.

2.2 Stability Theory

A comprehensive survey on general conditions on *existence*, *uniqueness*, and *finite-escape time* of solutions to ordinary differential equations is found in [Khalil, 1996]. In this section we recapitulate some central results and definitions in stability analysis, which will be used later on.

Lyapunov Stability Theory

DEFINITION 2.1—STABILITY [LYAPUNOV, 1892]
Assume that there is an autonomous system

$$\mathcal{S}: \qquad \frac{dx}{dt} = f(x), \qquad x \in \mathbb{R}^n \tag{2.1}$$

with an equilibrium x_e. The point x_e is a *stable* equilibrium if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for $\|x(t_0) - x_e\| \leq \delta$ it holds that $\|x(t) - x_e\| \leq \varepsilon$ for all $t > t_0$. □

DEFINITION 2.2—ATTRACTIVITY
The equilibrium x_e is said to be *attractive* if, for each $t_0 \in \mathbb{R}_+$, there is a $\delta(t_0) > 0$ such that for $\|x(t_0) - x_e\| < \delta(t_0)$ and $t > t_0$

$$\|x(t) - x_e\| \to 0, \quad t \to \infty \tag{2.2}$$

□

DEFINITION 2.3—ASYMPTOTIC STABILITY
Assume that there is an autonomous system

$$\mathcal{S}: \qquad \frac{dx}{dt} = f(x), \qquad x \in \mathbb{R}^n \tag{2.3}$$

with an equilibrium x_e. The equilibrium x_e is *asymptotically stable* if it is stable and if, in addition,

$$\lim_{t\to\infty} \|x(t) - x_e\| = 0 \tag{2.4}$$

□

An equivalent definition of asymptotic stability can be formulated by saying that an equilibrium is asymptotically stable if it is both stable and *attractive*.

THEOREM 2.1—LYAPUNOV STABILITY THEOREM [LYAPUNOV, 1892]
Let $x = 0$ be an equilibrium point for $\dot{x} = f(x)$ and let $V: \quad D \to \mathbb{R}$ be a continuously differentiable scalar function on a neighborhood D of $x = 0$ such that

$$\begin{aligned} V(0) &= 0 \text{ and } V(x) > 0 \text{ in } D - \{0\} && (2.5)\\ \dot{V}(x) &\leq 0 \text{ in } D && (2.6) \end{aligned}$$

Then, $x = 0$ is stable. Moreover, if

$$\dot{V}(x) < 0 \text{ in } D \tag{2.7}$$

then $x = 0$ is asymptotically stable in D. □

The growth rate of a nonlinear function is an important characterization in analysis.

DEFINITION 2.4—LIPSCHITZ
A vector valued nonlinearity $f : I\!R^n \times I\!R^m \times I\!R \to I\!R^n$ is said to be globally *Lipschitz* with respect to x with a Lipschitz constant γ if for all $x_1, x_2 \in I\!R^n$, all $u \in I\!R^m$, and uniformly in t—i. e., independently of the initial value t_0

$$|f(x_2, u, t) - f(x_1, u, t)| \le \gamma |x_2 - x_1| \tag{2.8}$$

□

The global property is, however, restrictive and many nonlinearities can be regarded as locally Lipschitz in some bounded, often physically well-motivated, region of the state-space.

The following two function classes are often used as lower or upper bounds on growth condition of Lyapunov function candidates and their derivatives.

DEFINITION 2.5—CLASS $\mathcal{K}$ FUNCTIONS [KHALIL, 1996]
A continuous function $\alpha : [0, a) \to I\!R_+$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\lim_{r\to\infty} \alpha(r) = \infty$. □

DEFINITION 2.6—CLASS $\mathcal{KL}$ FUNCTIONS [KHALIL, 1996]
A continuous function $\beta : [0, a) \times I\!R_+ \to I\!R_+$ is said to belong to class $\mathcal{KL}$ if for each fixed s the mapping $\beta(r, s)$ is a class $\mathcal{K}$ function with respect to r, and for each fixed r the mapping $\beta(r, s)$ is decreasing with respect to s and $\lim_{s\to\infty} \beta(r, s) = 0$. The function $\beta(\cdot, \cdot)$ is said to belong to class $\mathcal{KL}_\infty$ if for each fixed s, $\beta(r, s)$ belongs to class $\mathcal{K}_\infty$ with respect to r. □

For time-invariant systems, the LaSalle invariance theorem is one of the main tools for convergence analysis [LaSalle, 1967; Khalil, 1996]. For time-varying systems the following extension is useful.

LEMMA 2.1—LASALLE-YOSHIZAWA
Consider the time-varying system

$$\dot{x} = f(x, t);\; x \in I\!R^n,\; t \in I\!R \tag{2.9}$$

Let $x = 0$ be an equilibrium point of (2.9) and suppose that f is locally Lipschitz in x, uniformly in t. Let $V : I\!R^n \times I\!R_+ \to R_+$ be a continuously differentiable function such that $\forall t \ge 0, \forall x \in I\!R^n$

$$\alpha_1(\|x\|) \le V(x, t) \le \alpha_2(\|x\|) \tag{2.10}$$

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x, t) \le W(x) \le 0 \tag{2.11}$$

where α_1 and α_2 are class $\mathcal{K}_\infty$ functions and W is a continuous function. Then, all solutions to 2.9 are globally uniformly bounded and satisfy

$$\lim_{t\to\infty} W(x(t)) = 0.$$

In addition, if $W(x)$ is positive definite, then the equilibrium $x = 0$ is globally uniformly asymptotically stable. □

Passivity

Passivity theory is a very powerful branch of system theory that contains many intuitively appealing results regarding physical systems, but also stretches far beyond that [Aizerman and Gantmacher, 1964; Willems, 1972; Hill and Moylan, 1976; Sepulchre *et al*, 1997]. One of the main concepts is the dissipation of power.

In contrast to the Lyapunov theory, where state variables are considered, passivity theory is based on the input-output properties of a system. Many results in passivity theory originate from circuit theory and several basic concepts are generalizations from that context. For instance, the storage function $S(\cdot)$ corresponds to the energy in the system, the supply rate $w(\cdot,\cdot)$ corresponds to the input power, and the available storage $S_a(\cdot)$ corresponds to the largest amount of energy that can be extracted from the system for a certain initial condition.

Definition 2.7—Dissipativity
Consider a dynamical system Σ with equal input and output dimensions

$$\Sigma : \quad \begin{cases} \dot{x} &= f(x,u), \quad x \in I\!R^n, \quad u \in I\!R^p \\ y &= h(x,u), \quad y \in I\!R^p \end{cases} \tag{2.12}$$

If there exist a function $w(u,y)$, *the supply rate*, and a positive function $S(x) \geq 0$, *the storage function*, such that

$$\underbrace{S(x(t))}_{\text{storage fcn}} - S(x(0)) \leq \int_0^t \underbrace{w(u(\tau),y(\tau))}_{\text{supply rate}}\, d\tau$$

for all admissible inputs u and all $t \geq 0$, then the system is called *dissipative*. □

DEFINITION 2.8—PASSIVITY AND INPUT-OUTPUT PASSIVITY
A system Σ is said to be *passive* if it is dissipative and the supply rate $w(u, y) = u^T y$. The system (2.12) is passive if

$$w(u, y) \geq 0.$$

The system (2.12) is *input strictly passive (ISP)* if $\exists \varepsilon > 0$ such that $w(u, y) \geq \varepsilon \parallel u \parallel^2$. The system (2.12) is *output strictly passive (OSP)* if $\exists \varepsilon > 0$ such that $w(u, y) \geq \varepsilon \parallel y \parallel^2$. □

Whereas input strictly passive systems allow a certain amount of feedforward and still preserve the passivity, output strictly passive systems allow a certain amount of feedback and still preserve the passivity [Hill and Moylan, 1976].

The following theorem states the important interconnection property for passive systems.

THEOREM 2.2—PASSIVE INTERCONNECTION [POPOV, 1961; POPOV, 1973]
Assume that Σ_1 and Σ_2 are passive, then the well-posed feedback interconnections in Figure 2.2 are also passive from r to y.

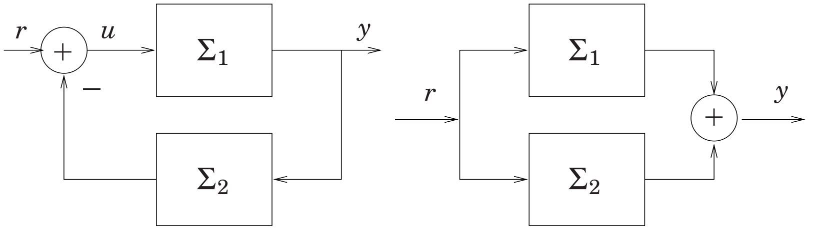

Figure 2.2 Passive interconnections of passive sub-systems.

□

Stability investigations using passivity theory require that the system be dissipative with $w = y^T u$ and $S(0) = 0$ and

$$S(x) - S(x_0) \leq \int_0^t y(\tau)^T u(\tau) d\tau$$

such that

- S is decreasing if $u = 0$;
- S is decreasing if $y = 0$ implies stable zero dynamics.

There are several results on systems rendered passive via feedback with several extensions of passivity, *feedback equivalence* and global stabilization of minimum-phase nonlinear systems [Byrnes *et al*, 1991; Kokotović and Sussmann, 1989].

EXAMPLE 2.1—PASSIVITY IN ROBOTICS [TAKEGAKI AND ARIMOTO, 1981]
The rigid robot manipulator described by the equations

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau \tag{2.13}$$

has a *passive* mapping from the input torque τ to the angular velocity $\dot{q}$.

To verify this, we want to show that there exist a β such that

$$\int_0^t \tau^T \dot{q} ds \geq -\beta,\ \forall t > 0 \tag{2.14}$$

Consider the total mechanical energy described by the Hamiltonian

$$H(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + \mathcal{U}(q).$$

$\mathcal{U}(q)$ is the potential energy due to gravity and $\partial\mathcal{U}/\partial q = G(q)$.

$$\begin{aligned}\frac{dH}{dt} &= \dot{q}^T M(q)\ddot{q} + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + \frac{\partial V}{\partial q}^T \dot{q} \\ &= (M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q))^T \dot{q} \\ &= \tau^T \dot{q}\end{aligned} \tag{2.15}$$

where we used the property that $\dot{M}(q) - 2C$ is a skew-symmetric matrix [Craig, 1988]. This implies that

$$\int_0^t \tau^T \dot{q} ds = H(q(t),\dot{q}(t)) - H(q(0),\dot{q}(0)) \geq -H(q(0),\dot{q}(0))$$

which fulfills the dissipativity condition of Eq. (2.14). □

This passivity property has been extensively used for control of robot manipulators and general *Euler-Lagrange systems* [Takegaki and Arimoto, 1981; Berghuis, 1993; Loría, 1996; Battilotti *et al*, 1997; Ortega *et al*, 1998].

Stability and Positive Real Transfer Functions

The existence of Lyapunov functions as well as storage functions of dissipative systems relies upon the Positive Real lemma:

Lemma 2.3—Positive Real Lemma (Yakubovich-Kalman-Popov)
Let $G(s) = C(sI - A)^{-1}B + D$ be a $p \times p$ transfer function matrix, where A is Hurwitz, (A, B) is controllable, and (A, C) is observable. Then G is strictly positive real if and only if there exist a symmetric positive definite matrix P, matrices L, R and a positive constant ε satisfying

$$\begin{aligned} PA + A^T P &= -LL^T - \varepsilon P \\ PB - C^T &= -LR^T \\ D + D^T &= RR^T \end{aligned} \tag{2.16}$$

□

Moylan (1974) relates the input-output property of passivity for square, i. e., the dimension of the control inputs equals the dimension of the outputs, nonlinear systems, affine in the control, with state dependent equations, which can be viewed as a generalization or a nonlinear extension of the Positive Real lemma. In this context, the passivity notion does not postulate an internal storage function as in [Willems, 1970; Moylan, 1974; Hill and Moylan, 1976].

Theorem 2.2—[Hill and Moylan, 1976]
Let

$$\begin{aligned} \dot{x} &= f(x) + G(x)u \\ y &= h(x) + J(x)u \end{aligned} \tag{2.17}$$

A necessary and sufficient condition for the system of Eq. (2.17) to be passive is that there exist real functions $V(\cdot)$, $l(\cdot)$, and $W(\cdot)$, where $V(x)$ is continuous and

$$V(x) \geq 0,\ \forall x \in \mathbb{R}^n,\ V(0) = 0 \tag{2.18}$$

such that

$$\begin{aligned} \nabla' V(x) f(x) &= -l'(x) l(x) \\ \frac{1}{2} G'(x) \nabla' V(x) &= h(x) - W'(x) l(x) \\ J(x) + J'(x) &= W'(x) W(x) \end{aligned} \tag{2.19}$$

□

Dissipativity and zero-state detectability of the system in Eq.(2.17) imply Lyapunov stability.

Input-to-State Stability (ISS)

The *input-to-state stability* concept by Sontag takes into account the effects from initial conditions $x(0)$ as well as from input signals for nonlinear systems [Sontag, 1988]. From the superposition property it follows that initial values do not affect the stability for linear systems whereas such effects may be crucial for the stability of nonlinear systems. A good overview to input-to-state stability is given in [Sontag, 1995].

DEFINITION 2.9—INPUT-TO-STATE STABILITY (ISS) [SONTAG, 1988]
A system

$$\dot{x} = f(x, u)$$

is said to be *Input-to-State Stable (ISS)* with respect to an input signal u if for any initial condition $x(0)$ and any $u(\cdot)$ continuous and bounded on $[0, \infty)$ the solution exists for all $t \geq 0$ and satisfies

$$|x(t)| \leq \beta(|x(0)|, 0) + \gamma(\sup_{0 \leq \tau \leq t} |u(\tau)|), \quad \forall t \geq 0$$

where $\beta(r, t)$ is a class $\mathcal{KL}$ function and $\gamma(t)$ is a class $\mathcal{K}$ function. □

Applications of input-to-state-stability to observer-based control have been reported in [Tsinias, 1993]. In cases where exact feedback linearization does not apply, a variety of extensions have been reported—e.g., partially linearizable systems with application to underactuated mechanical systems [Spong and Praly, 1996], and approximate linearization [Krener *et al*, 1988; Hauser *et al*, 1992a].

DEFINITION 2.10—COMPLETE CONTROLLABILITY [NIJMEIJER AND VAN DER SCHAFT, 1990]
A system is *completely controllable* if for every two finite states there exists an admissible control which drives the system from the one to the other in finite time. □

2.3 Obstacles and Complexity Issues

In this section we will study some phenomena and obstacles which have to be considered for control as well as for observer design.

Peaking Nonminimum phase systems are inherently difficult to control and it is well known that right-half plane zeros put an upper bound on the achievable bandwidth [Freudenberg and Looze, 1985; Åström, 1997; Goodwin and Seron, 1997]. However, the peaking phenomenon may have far worse consequences and may amount to finite-escape phenomena [Sussmann and Kokotović, 1991].

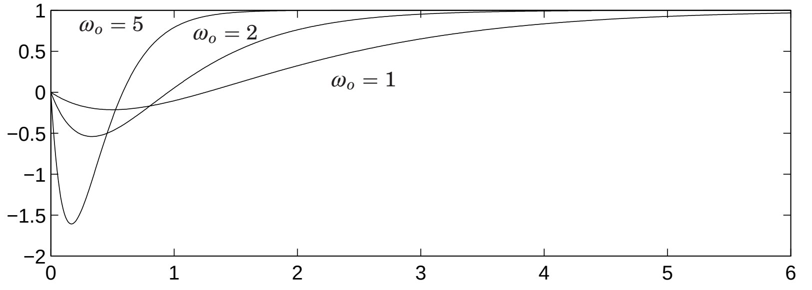

Figure 2.3 Step responses for the system in Eq. (2.20), $\omega_o = 1, 2$, and 5. Faster poles gives shorter settling times, but the transients grow significantly in amplitude, so called *peaking*.

EXAMPLE 2.2
Consider a controllable second order linear system with a zero in the right half-plane at $s = 1$. By state-feedback the closed-loop poles can be made arbitrarily fast, while the zero is fixed under the assumption that no unstable pole-zero cancellation takes place. The closed-loop system with both poles placed at $s = -\omega_o < 0$ will be

$$G_{cl}(s) = \frac{(-s+1)\omega_o^2}{s^2 + 2\omega_o s + \omega_o^2} \tag{2.20}$$

A step response will reveal a transient which grows in amplitude for faster closed loop poles (Fig.2.3). □

For linear systems large overshoots in states or outputs may be devastating for the performance, but it does not effect the overall stability of the system, while for nonlinear systems transients may drive a system out of a stable region, with trajectories possibly escaping to infinity in finite time.

In [Mita, 1977] the effect of peaking was studied for linear systems with observers and [Francis and Glover, 1978] studied trajectory boundedness with respect to linear quadratic cost criteria. In [Sussmann and Kokotović, 1989; Sussmann and Kokotović, 1991] the problem of globally stabilizing a cascade of one linear and one nonlinear subsystems, as in Fig. 2.4, is addressed. This class of cascaded systems can be interpreted as that of partially feedback linearizable systems, with applications to under-actuated mechanical systems [Spong and Praly, 1996]. The system can be written

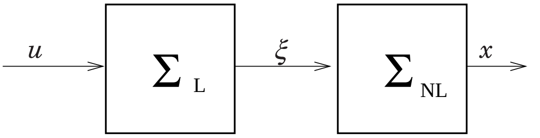

Figure 2.4 Cascade of a linear and a nonlinear subsystem.

in the following form:

$$\begin{aligned} \dot{x} &= f_0(x) + \sum_i \xi_i f_i(x, \xi) \\ \dot{\xi} &= A\xi + Bu \end{aligned} \tag{2.21}$$

The structure and properties of the coupling-terms $f_i(x, \xi)$, consisting of the "driving" linear states, ξ, and the "driven" nonlinear states, x, play a crucial role for the risk of peaking and the possibility of stabilizing the cascade [Sepulchre *et al*, 1997]. The stabilization of a cascaded system has received a lot of attention and has become a widespread and powerful tool in many designs [Mazenc and Praly, 1996; Sepulchre *et al*, 1997; Panteley and Loría, 1998a; Gronard *et al*, 1999].

Relative degree Relative degree is a complexity measure which answers the question: "How many times do you have to take the time derivative of the output before the input appears explicitly?" For single-input single-output linear systems it coincides with the difference between the number of poles and the number of zeros. The relative-degree notion has become a means to characterize the complexity of a control problem. Furthermore, it is a system property which is invariant under coordinate changes. For a nonlinear system with relative degree d

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{2.22}$$

we have

$$
\begin{aligned}
\dot{y} &= \frac{d}{dt}h(x) = \frac{\partial h(x)}{\partial x}\dot{x} = \frac{\partial h}{\partial x}f(x) + \frac{\partial h}{\partial x}g(x)u \\
&= L_f h(x) + \underbrace{L_g h(x)}_{=0\ if\ d>1}\ u \\
&\vdots \\
y^{(k)} &= L_f^k h(x) \qquad \text{if } k < d \\
&\vdots \\
y^{(d)} &= L_f^d h(x) + L_g L_f^{(d-1)} h(x) u
\end{aligned} \tag{2.23}
$$

Using the same kind of coordinate transformations as for the feedback linearizable systems above, we can introduce new state space variables, ξ, where the first d coordinates are chosen as

$$
\begin{cases}
\xi_1 &= h(x) \\
\xi_2 &= L_f h(x) \\
\vdots \\
\xi_d &= L_f^{(d-1)} h(x)
\end{cases} \tag{2.24}
$$

Under some conditions on involutivity, the Frobenius theorem guarantees the existence of another $(n-d)$ functions to provide a local state transformation of full rank [Nijmeijer and van der Schaft, 1990; Isidori, 1995]. Such a coordinate change transforms the system to the *normal form*

$$
\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
&\vdots \\
\dot{\xi}_{d-1} &= \xi_d \\
\dot{\xi}_d &= L_f^d h(\xi, z) + L_g L_f^{d-1} h(\xi, z) u \\
\dot{z} &= \psi(\xi, z) \\
y &= \xi_1
\end{aligned} \tag{2.25}
$$

where $\dot{z} = \psi(\xi, z)$ represent the zero dynamics [Byrnes and Isidori, 1991]. Note that the relative degree is *not* a robust property in the sense that it may change as a result of very small parametric variations in the system equation.

EXAMPLE 2.3—SLIDING BEAD ON ROD [HAUSER *et al*, 1992B]
Consider a sliding bead on a rod. Lagrangian mechanics provide the equations of motion

$$\begin{aligned} 0 &= (\frac{J_b}{R^2} + M)\ddot{r} + Mg\sin(\theta) - Mr\dot{\theta}^2 \\ \tau &= (Mr^2 + J + J_b)\ddot{\theta} + 2Mr\dot{r}\dot{\theta} + Mgr\cos(\theta) \end{aligned}$$

Reformulation to state-space form and normalization gives

$$\begin{aligned} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} &= \begin{pmatrix} x_2 \\ k(x_1x_4^2 - g\sin(x_3)) \\ x_4 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u \\ y &= x_1 \end{aligned}$$

where

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}^T = \begin{pmatrix} r & \dot{r} & \theta & \dot{\theta} \end{pmatrix}^T$$

Differentiate until the input appears

$$\begin{aligned} \dot{y} &= x_2 \\ &\vdots \\ y^{(3)} &= \underbrace{kx_2x_4^2 - kgx_4\cos(x_3)}_{L_f^3 h} + \underbrace{2kx_1x_4}_{L_g L_f^2 h} u \end{aligned}$$

In this case we see that $L_g L_f^2 h$ will vanish when $x_1 = r = 0$ or $x_4 = \dot{\theta} = 0$. The relative degree is thus not uniquely defined for this system. □

Zero dynamics For the case of exact linearization of the system in Eq. (2.25), a new input signal v will be chosen as

$$v = L_f^d h(\xi, z) + L_g L_f^{d-1} h(\xi, z) u \tag{2.26}$$

The resulting dynamics will be a chain of integrators of length d from the new input v to the output y. For linear systems the transformation and the change of input corresponds to a pole placement where d poles are placed at $s = 0$ and the remaining $n - d$ poles align with the system zeros, i. e., cancellation of all the zeros in the transfer function. The dynamics we get if we try to keep the output identically zero

$$\dot{z} = \psi(0, z) \tag{2.27}$$

is called the *zero dynamics* of the system [Byrnes and Isidori, 1991].

A system is called *minimum phase* if the zero dynamics $\dot{z} = \psi(0, z)$ are asymptotically stable. The converse is called a non-minimum phase system. This property can not be affected by feedback, as is familiar from linear systems where feedback does not affect the zeros. To find the zero dynamics, there is, however, no need to transform the system to normal form, which the next example will illustrate.

EXAMPLE 2.4—ZERO DYNAMICS FOR LINEAR SYSTEMS
Consider the linear system

$$y = \frac{s-1}{s^2+2s+1}u \tag{2.28}$$

with the following state-space description

$$\begin{cases} \dot{x}_1 &= -2x_1 + x_2 \quad +u \\ \dot{x}_2 &= -x_1 \qquad\qquad -u \\ y &= x_1 \end{cases} \tag{2.29}$$

To find the zero-dynamics, we assign $y \equiv 0$.

$$\begin{aligned} &\Rightarrow x_1 \equiv 0 \;\Rightarrow\; \dot{x}_1 \equiv 0 \;\Rightarrow x_2 + u = 0 \\ &\Rightarrow \dot{x}_2 = -u = x_2 \end{aligned} \tag{2.30}$$

The remaining dynamics is an unstable system corresponding to the zero $s = 1$ in the transfer function (2.28). □

A general conclusion is that feedback linearization can be interpreted as a nonlinear version of pole-zero cancellations which not can be used if the zero-dynamics are unstable, i. e., for *nonminimum-phase system*. In algebraic terms, we are faced with a model inversion problem or an operator inversion problem.

Obstacles for output-feedback control A fundamental difference between linear and nonlinear systems is the effect of bounded disturbances over a finite time horizon. Consider a linear system and assume that we have a stabilizing state-feedback law. If we instead base the feedback law on estimated states, the closed loop system will still be stable under the assumption that the observer error converges to zero. For nonlinear systems this is not the case even if we have exponential convergence in the observer. The obstacle is the problem with *finite escape time*. Separations principles for nonlinear systems will be discussed in more detail in Chapter 4.

In [Mazenc *et al*, 1994] it is shown that global complete observability and global stabilizability are not sufficient to guarantee global stabilizability by dynamic output feedback, i. e., no observer based design, whatever convergence properties for the observer, will solve the general global stabilization problem. The class of systems with dynamics of the form

$$
\begin{aligned}
\dot{z} &= H(z, x_1, \ldots, x_r) \\
\dot{x}_1 &= x_2 \\
&\vdots \\
\dot{x}_{r-1} &= x_r \\
\dot{x}_r &= x_r^k + F(z,x) + G(z,x)u \\
y &= x_1
\end{aligned}
\tag{2.31}
$$

is not globally asymptotically stabilizable by continuous dynamic output feedback and does not satisfy the "unboundedness observability property" if $k \geq r/(r-1)$ [Mazenc *et al*, 1994]. The following conjecture was formulated: "This shows that for global asymptotic stabilization by output feedback, we cannot go very far beyond linearity for relative degrees $r > 2$."

2.4 Control and Stabilization

Lyapunov stability theory as well as passivity can be used as instruments for stabilization:

Feedback Passivation

From Sec. 2.2 we know that passive systems are intrinsically easy to stabilize. Without any aspects of performance so far, negative feedback from the passive output to the input will do the job. One route to use this concept in design is first to look for an output function and a feedback transformation to render the system passive. This is the concept of *feedback passivation*, originating from results in [Molander and Willems, 1980; Kokotović and Sussmann, 1989; Byrnes *et al*, 1991].

One important question to be ask is "When can a nonlinear system be rendered passive via feedback?"

THEOREM 2.3—FEEDBACK PASSIVATION [KOKOTOVIĆ AND SUSSMANN, 1989]
Consider the affine nonlinear system

$$
\begin{aligned}
\dot{x} &= f(x) + g(x)u \\
y &= h(x)
\end{aligned}
\tag{2.32}
$$

The input affine nonlinear system of Eq. (2.32) is feedback passifiable if and only if it has relative degree one and the zero dynamics are weakly nonminimum phase. □

Departing from purely linear systems the first extension is a feedback connection of a linear system and a static nonlinear function.

As for nonlinear electro-mechanical systems, the passivity-based approach strives to exploit the specific structure in Euler-Lagrange systems and, in particular, its inherent passivity properties.

LEMMA 2.4—LAGRANGIAN MECHANICS
An Euler-Lagrange system has a stable equilibrium where its potential function has a minimum.

Proof See [Goldstein, 1980]. □

Based on this fundamental lemma, Takegaki and Arimoto (1981) proposed a control-law for mechanical manipulators which reshapes the potential function to have a minimum at the desired set-point, so called *energy shaping*. Asymptotic stability is achieved by *damping injection* [Takegaki and Arimoto, 1981]. The controller is basically a PD-controller with gravity compensation. If position measurements only are available, the question of stability from a certainty equivalence point occurs naturally when the estimated velocity is to be used in the derivative part of the controller. Local results for the flexible robot was reported in [Nicosia and Tomei, 1990]. These kind of "derivative filtering" controllers have the benefit that they are easy to implement as they do not require any calculation or inversion of the inertia matrix [Paden and Panja, 1988; Lefeber and Nijmeijer, 1997].

As for the regulation or set-point control, the LaSalle theorem is instrumental for proving asymptotic stability [LaSalle, 1960; Khalil, 1996]. One reason for this is that when choosing the energy of a system as a Lyapunov function candidate, it often turns out that its time derivative along the equations of dynamics is negative *semi*-definite only. For the tracking problem the LaSalle-Yoshizawa lemma (Lemma 2.1) or the Matrosov theorem have to be used instead due to the time-varying dynamics imposed by the reference trajectory [Hahn, 1967, p.263].

Lyapunov analysis and design

In control theory various aspects of stability are used; *Lyapunov stability* and input-output stability. Here we will mainly consider the Lyapunov stability concept which has played a fundamental role in system theory. It was introduced as a method for stability analysis in the seminal work by A. M. Lyapunov over a century ago and has evolved through many important contributions into a very powerful tool for analysis as well as for synthesis and design [Lyapunov, 1892]. It is now the foundation for many design methods for stability and control. A common approach starts with some chosen positive valued function, a Lyapunov function candidate, often to be interpreted as a generalized energy function. A control law is sought for which would render the function to decrease along all trajectories of the system, implying stability around a desired motion or around an equilibrium. The main obstacle for both the analysis and the synthesis problem is the lack of general methods for finding a suitable Lyapunov function, or a way of proving that there does indeed not exist any. Nevertheless, for some classes of systems with imposed structural properties there exist efficient methods to numerically find Lyapunov functions or to analytically derive them.

The problem of stabilization can be approached using the concept of *control Lyapunov functions* (CLF).

DEFINITION 2.11—CONTROL LYAPUNOV FUNCTIONS (CLF) [ARTSTEIN, 1983]
A smooth positive definite and radially unbounded function $V : I\!R^n \to I\!R_+$ is called a control Lyapunov function (CLF) for the time-invariant system

$$\dot{x} = f(x) + g(x)u, \quad x \in I\!R^n, \quad u \in I\!R, \quad f(0) = 0 \tag{2.33}$$

if

$$\inf_{u \in R} \left\{ \frac{\partial V}{\partial x}(x) \cdot [f(x) + g(x)u] \right\} < 0, \quad \forall x \neq 0.$$

□

Artstein's results about CLFs generalized the results in [Jurdjevič and Quinn, 1978; Jacobson, 1977] and showed the equivalence of asymptotic stabilizability and the necessary and sufficient condition for the existence of a control Lyapunov function [Artstein, 1983].

THEOREM 2.5—[ARTSTEIN, 1983]
The existence of a control Lyapunov function for a system is equivalent to global asymptotic stabilizability. □

Given a CLF $V(x)$, a stabilizing controller is provided by the Sontag formula

$$u_s(x) = \begin{cases} -\dfrac{\dfrac{\partial V}{\partial x} f + \sqrt{(\dfrac{\partial V}{\partial x})^2 + (\dfrac{\partial V}{\partial x} g)^4}}{\dfrac{\partial V}{\partial x} g}, & \dfrac{\partial V}{\partial x} g \neq 0; \\ 0, & \dfrac{\partial V}{\partial x} g = 0 \end{cases} \tag{2.34}$$

The stabilizing control law $u_s(x)$ for system (2.33), derived by the Sontag formula, is continuous at $x = 0$ if $V(x)$ satisfies the *small control property*.

DEFINITION 2.12—SMALL CONTROL PROPERTY
The control Lyapunov function $V(x)$ satisfies the *small control property* if for each $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that for all states $\|x\| < \delta, x \neq 0$ the inequality

$$\frac{\partial V}{\partial x}[f(x) + g(x)u(x)] < 0 \tag{2.35}$$

is satisfied with control action $\|u(x)\| < \varepsilon$.

□

Still, the general construction of an appropriate CLF is usually a very hard problem, as in a sense it is equivalent to the stabilizability problem. If successful, the CLF construction provides a sufficient condition for stability. For some subclasses of nonlinear systems the backstepping procedure offers a constructive methodology for these problems.

State Feedback and Exact Feedback Linearization

Under certain conditions a nonlinear system may have a linear representation via a nonlinear change of coordinates and the cancellation or inversion of remaining nonlinear terms. Exact linearization can be seen as a strive for reusing the design methods for linear systems by making the design in the converted coordinates. The basic idea can be described by the following example.

EXAMPLE 2.5—EXACT LINEARIZATION
Consider the nonlinear first-order system

$$\dot{x}_1 = f(x_1) + g(x_1)u \tag{2.36}$$

By using the linearizing control law

$$u = (v - f(x_1))/g(x_1)$$

the system is converted to the linear system

$$\dot{x}_1 = v$$

where v is a new input signal. □

Problems of stabilization associate with exact cancellation of the nonlinear state dependent term $f(\cdot)$ and the inversion of $g(\cdot)$. It may sometimes be unrealistic to think that exact cancellation of nonlinearities can be performed to give a linear system. All imperfections such as parameter uncertainties, state-estimation errors, etc., will contribute to errors which will show up as disturbances in the feedback-linearized equations

$$\begin{aligned} \dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= f(x) + g(x)u = v \end{aligned} \tag{2.37}$$

where $u = g(x)^{-1}(v - f(x))$ is the original control signal. An important observation which can be made is that for the system in Eq. (2.37) disturbance terms due to inexact cancellation enter at the same place as the control signal, that is, they satisfy the so-called *matching condition* [Khalil, 1996, p.548].

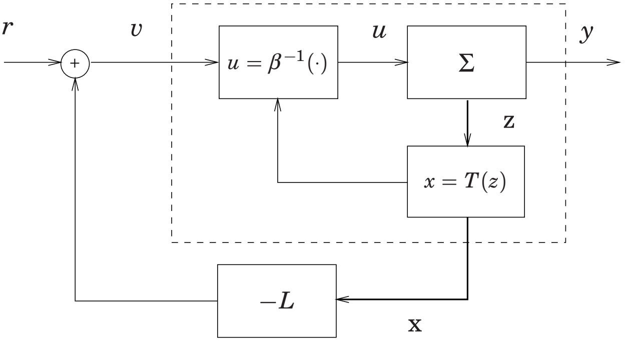

Figure 2.5 Inner *feedback linearization* and outer *linear* feedback control

For general nonlinear systems feedback linearization comprises

- state transformation
- inversion of nonlinearities

- linear feedback

with no approximations involved (Fig. 2.5). Feedback linearization is said to be exact in the case where the state transformation and the inversion are valid. Conditions for feedback linearization can be found in [Isidori, 1995; Nijmeijer and van der Schaft, 1990].

For systems which do not admit exact feedback linearization, approximate linearization techniques have been proposed [Krener and Isidori, 1984; Krener *et al*, 1988; Hauser *et al*, 1992a]. One of the drawbacks with feedback linearization, as pointed out in the section above, is that exact cancellation of nonlinear terms may not be possible due to e. g., parameter uncertainties. A suggested solution to this problem is a two-step procedure where the first step regards stabilization via feedback linearization around a nominal model. Considering known bounds on the uncertainties, the second step, so called *Lyapunov redesign*, will provide an additional term for stabilization, taking into account the effects of the disturbances [Khalil, 1996].

Even if a system is feedback linearizable, some of its nonlinear terms may act in a stabilizing way and thus be beneficial to keep for the purpose of regulation, whereas it may be desirable to compensate for them in the tracking problem.

The method of exact linearization described above, linearizes the state equations, which does not necessarily imply a linear input-output mapping. For the purpose of output tracking, input-output linearization may be considered.

Backstepping

The problem of finding a Lyapunov function has a simple solution in the control-affine one-dimensional case. *Integrator backstepping* provides a systematic method to stabilization of nonlinear systems which are transformable into *strict feedback form*. The main idea is to start out with the stabilization problem for a first-order subsystem and step-by-step increase the order of the subsystem considered. The method is constructive for generating Lyapunov functions [Kanellakopoulos *et al*, 1992; Krstić *et al*, 1995]. Backstepping can also be used to relax the *matching condition* on where disturbances may enter the system equations—see the discussion on the Lyapunov-redesign method in the previous section.

EXAMPLE 2.6—BACKSTEPPING
Consider the one-dimensional system with input u_1

$$\dot{x}_1 = f_1(x_1) + u_1 \tag{2.38}$$

where f is a known functions and $f(0) = 0$. In this case it is easy to find a control Lyapunov function, and any positive quadratic function $V_1(x_1) = c_1 x_1^2/2$, $c_1 > 0$, will do. We want to choose $u_1 = u_1(x_1)$ such that V_1 has a negative derivative along the solutions of Eq. (2.38):

$$\begin{aligned}\dot{V}_1 &= \frac{\partial V_1}{\partial x_1}\left[f_1(x_1) + u_1\right] = c_1 x_1 \left[f_1(x_1) + u_1\right] \\ &= -c_1 x_1^2 \le 0, \quad \text{for } u_1 \triangleq -x_1 - f_1(x_1)\end{aligned} \tag{2.39}$$

Extending the system with an integrator at the input

$$\begin{aligned}\dot{x}_1 &= f_1(x_1) + x_2 \\ \dot{x}_2 &= u\end{aligned} \tag{2.40}$$

the same approach as above can be used, if we first consider x_2 as a virtual control signal. If we could choose $x_2 = -x_1 - f_1(x_1)$, then the first state would be stabilized. To this purpose, introduce the *error state* $z_2 = x_2 - (-x_1 - f_1(x_1)) = x_2 + x_1 + f_1(x_1)$. Consider the Lyapunov function candidate

$$V_2 = V_1 + \frac{c_2}{2} z_2^2 \tag{2.41}$$

whose derivative is

$$\begin{aligned}\frac{dV_2}{dt} &= -c_1 x_1^2 + c_1 x_1 z_2 + z_2[\dot{x}_2 - \dot{x}_1] \\ &= -c_1 x_1^2 + z_2[u + x_1 - z_1 + c_1 x_1] \\ &= -c_1 x_1^2 - c_2 z_2^2 < 0, \quad (x_1, z_2) \neq (0, 0)\end{aligned} \tag{2.42}$$

if u is chosen as $u = -x_1 + z_1 - c_1 x_1 - c_2 z_2$. Transformation back to the original coordinates (x_1, x_2) does not effect the stability. □

Backstepping provides a systematic methodology to stabilize systems in *strict feedback form*:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1) + x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) + x_3 \\ &\ \vdots \\ \dot{x}_{n-1} &= f_{n-1}(x_1, x_2, \ldots, x_{n-1}) + x_n \\ \dot{x}_n &= f_n(x_1, x_2, \ldots, x_n) + u\end{aligned} \tag{2.43}$$

For the geometric conditions under which nonlinear systems can be transformed into strict feedback form, see [Krstić *et al*, 1995].

In the backstepping procedure, properties of nonlinear terms may be utilized to avoid cancellations of stabilizing terms, possibly reducing the control effort and avoiding the introduction of "unnecessary" high-gain control. Note, however, that by imposing the Lyapunov function derivative to the "standard quadratic choice" of

$$\frac{dV}{dt} = -\sum_i c_i z_i^2, \quad c_i > 0 \tag{2.44}$$

a lot of the freedom is lost. The backstepping procedure in this case can be interpreted as an exact linearization of dV/dt since all the nonlinearities in the derivative will be cancelled [Krener, 1999].

Output Feedback

In many applications only a part of the state vector is possible to measure. This restricts the use of state feedback methods. The output feedback problem is well motivated to consider, from practical aspects as well as from the large number of theoretical questions which still remain unanswered in this area. For the output feedback problem, the introduction of extra dynamics in the control law may be necessary for stabilization. In many cases the extra dynamics have the interpretation of observer dynamics. The shortcomings of static output feedback is illustrated in the following example on global stabilization of a double integrator.

EXAMPLE 2.7—[SONTAG, 1990, EX.6.2.1]
Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \\ y &= x_1. \end{aligned} \tag{2.45}$$

The system is easily shown to be both controllable and observable, but there does not exist any continuous function $u = k(y)$ which stabilizes the system in the sense that the trajectories from all possible initial points converge to the origin, $(x_1, x_2) = (0, 0)$.
Suppose such a $k(\cdot)$ would exist. Then consider the function

$$H(x_1, x_2) = x_2^2 - \int_0^{x_1} k(s)ds \tag{2.46}$$

As $\dot{H} \equiv 0$ along the trajectories of the system, which by the assumption all converge to the origin, it implies that H would be constant and equal to $H(0,0) = 0$. As $H(0,1) = 1$ is a contradiction, there does not exist such a $k(\cdot)$. □

Whereas some simple examples and classes of systems, which constitute important counter-examples for global stabilizability by means of output feedback have been presented in [Mazenc *et al*, 1994], Teel and Praly have shown that global stabilizability and observability imply semi-global stabilizability by output feedback [Teel and Praly, 1994].

2.5 New Results on Output Feedback Stabilization

In this section we present two contributions on feedback stabilization and tracking control with relevance to observer-based feedback control. Firstly, a new interpretation of the Yakubovich-Kalman-Popov matrix equation is given [Johansson and Robertsson, 1999]. Secondly, results on tracking control of systems in chained form are presented [Lefeber *et al*, 1999a; Lefeber *et al*, 2000].

The Yakubovich-Kalman-Popov (YKP) Lemma and Observer Feedback

This important lemma supports results of stability theory of nonlinear feedback (circle criterion; Popov criterion) and adaptive system theory and for extensions of Lyapunov theory such as passivity theory [Khalil, 1996; Ortega *et al*, 1995a]. Although the lemma is very powerful in its predictions, the strictly positive real (SPR) conditions imposed on the transfer function are, unfortunately, rather restrictive for application. The minimality requirement also preclude application to stabilization of observer-based feedback. As the Lyapunov functions generated by backstepping techniques apparently cannot be designed using the YKP lemma, there might exist other relevant classes of non-SPR systems for which extended regions of stability might exist. Relevant such problem classes to investigate include non-SPR transfer functions and dynamic output feedback or observer-based design violating the controllability condition of the YKP lemma.

A key observation is that the YKP matrix equation may then be formulated as

$$
\begin{aligned}
-Q &= \begin{pmatrix} PA + A^T P & PB - C^T \\ B^T P - C & -(D + D^T) \end{pmatrix} \\
&= \begin{pmatrix} P & 0 \\ 0 & I_m \end{pmatrix} \begin{pmatrix} A & B \\ -C & -D \end{pmatrix} + \begin{pmatrix} A & B \\ -C & -D \end{pmatrix}^T \begin{pmatrix} P & 0 \\ 0 & I_m \end{pmatrix} \qquad (2.47)
\end{aligned}
$$

which is recognized as a special Lyapunov equation

$$
\mathcal{P}_o \mathcal{A} + \mathcal{A}^T \mathcal{P}_o = -Q \qquad (2.48)
$$

Using this observation, the YKP matrix equation may be used as a means to generate CLFs suitable for observer-supported output feedback.

Paper C presents theory for extension of the Yakubovich-Kalman-Popov lemma for stability analysis relevant for observer-based feedback control systems. We show that minimality is not necessary for existence of Lur'e-Lyapunov functions. Relaxation of the controllability and observability conditions imposed in the YKP lemma can be made for a class of nonlinear systems described by a linear time-invariant system (LTI) with a feedback-connected cone-bounded nonlinear element. Implications for positivity, factorization and passivity are given in Paper C. A number of interpretations may also be relevant:

Negative integral output feedback: Let the linear system

$$
\begin{pmatrix} \dot{x} \\ y \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad y \in \mathbb{R}^p \qquad (2.49)
$$

be controlled with the negative integral output feedback control law

$$
\dot{u} = -(y - r), \quad r \in \mathbb{R}^p \qquad (2.50)
$$

A state vector for the feedback-connected system is provided by

$$
X = \begin{pmatrix} x \\ u \end{pmatrix} \qquad (2.51)
$$

with the derivative

$$
\dot{X} = \begin{pmatrix} \dot{x} \\ \dot{u} \end{pmatrix} = \begin{pmatrix} A & B \\ -C & -D \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} + \begin{pmatrix} 0 \\ r \end{pmatrix} \qquad (2.52)
$$

or

$$\dot{X} = \mathcal{A}X + \begin{pmatrix} 0 \\ r \end{pmatrix} \tag{2.53}$$

Thus, the stability condition for the feedback interconnected system is precisely that the eigenvalues of $\mathcal{A}$ be in the open left half plane.
Then, the autonomous system $\dot{X} = \mathcal{A}X$ is stable with a Lyapunov function $V(X) = X^T\mathcal{P}X$ for the feedback-interconnected system where $\mathcal{P}$ solves the Lyapunov equation $\mathcal{P}\mathcal{A} + \mathcal{A}^T\mathcal{P} = -\mathcal{Q}$ for $\mathcal{Q} = \mathcal{Q}^T > 0$.
Thus, the class of systems satisfying the property of eigenvalues of $\mathcal{A}$ in the open left half plane is that of output feedback systems stable under negative integral output feedback. It is well known from passivity theory that stable feedback interconnection can be made of one strictly passive subsystem Σ_1 and of another passive but not necessarily strictly passive subsystem Σ_2 [Popov, 1973], [Krstić *et al*, 1995, p.508]. As an integrator is an example of a passive subsystem, it follows from this result that there is stable feedback connection of a strictly passive subsystem with an integrator subsystem. Obviously, the class of linear systems with stable $\mathcal{A}$ includes the class of linear strictly passive systems.

Secondly, from the construction of the YKP equation and the associated Lyapunov function, it follows that the class of SPR systems are stable under negative integral feedback.

Thirdly, another class of systems is that of feedback positive real (FPR) systems where there is an L such that the transfer function $Z(s) = C(sI - A + BL)^{-1}B$ is positive real and $A - BL$ is stable [Kokotović and Sussmann, 1989].

Molander and Willems (1980) made a characterization of the conditions for stability of feedback systems with a high gain margin

$$\dot{x} = Ax + Bu, \quad z = Lx, \quad u = -f(Lx, t) \tag{2.54}$$

with $f(\cdot)$ enclosed in a sector $[K_1, K_2]$, see [Molander and Willems, 1980]. These authors suggested the following procedure to find a state-feedback vector L such that the closed-loop system will tolerate any $f(\cdot)$ enclosed in a sector $[K_1, K_2]$. Synthesis of a state-feedback vector L with a robustness sector $[K_1, \infty)$ follows from their Theorem 3:

- Pick a matrix $Q = Q^T > 0$ such that (A, Q) is observable;
- Solve Riccati equation $PA + A^TP - 2K_1PBB^TP + Q = 0$;
- Take $L = B^TP$.

The Molander-Willems equations may be summarized as a YKP matrix equation

$$\begin{pmatrix} P & 0 \\ 0 & I_m \end{pmatrix} \begin{pmatrix} A - K_1 B L & B \\ -L & 0 \end{pmatrix} + \begin{pmatrix} A - K_1 B L & B \\ -L & 0 \end{pmatrix}^T \begin{pmatrix} P & 0 \\ 0 & I_m \end{pmatrix} + \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix} = 0$$

which can be recognized as an FPR condition—i.e., the stability condition will be that of an SPR condition on $L(sI - A + K_1 B L)^{-1} B$.

In the special case when $K_1 = 0$, the Riccati equation among the Molander-Willems equations will specialize to a Lyapunov equation and the stability condition for $f(\cdot)$ anywhere in the first and third quadrants will be that of an SPR condition on $L(sI - A)^{-1} B$.

The YKP procedure suggests a constructive means to provide a positive real transfer function. The method includes one state feedback transformation and one transformation which provides an output variable from linear combinations of the outputs to render the system transfer function positive real. To that purpose, a required but not measured state may be replaced by a reconstructed state provided that appropriate observer dynamics are included.

Tracking Control of Systems in Chained Form

In recent years the control, and in particular the stabilization, of nonholonomic dynamic systems have received considerable attention. Typically, nonholonomic constraints arise from physical laws as the Newton law of conservation of momentum, showing up as constraints on accelerations rather than (holonomic) constraints on positions. These types of second order nonholonomic constraints are present in for example the control of under-actuated mechanical manipulators. The trailer with a cart, Fig. 2.6, is subject to nonholonomic velocity constraints, where the tires are rolling along the surface without slipping. For these systems it should be noted that there does not exist any smooth stabilizing static state-feedback control law, since the Brockett necessary condition for smooth stabilization is not met [Brockett, 1983]. For an overview of nonholonomic systems we refer to the survey paper [Kolmanovsky and McClamroch, 1995] and references cited therein.

Although the stabilization problem for nonholonomic control systems is now well understood, the tracking control problem has received less attention. In fact, it is unclear how the stabilization techniques available can be extended directly to tracking problems for nonholonomic systems.

In Paper A and B we study the output tracking problem for the class of nonholonomic systems in *chained form* with two input signals [Micaelli

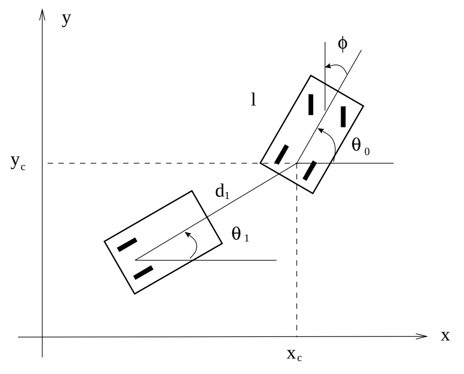

Figure 2.6 Car with a trailer, see [Micaelli and Samson, 1993].

and Samson, 1993]

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= u_1 x_2 \\ &\vdots \\ \dot{x}_n &= u_1 x_{n-1} \end{aligned} \tag{2.55}$$

with the first and the last state as our output signals

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_n \end{pmatrix} \tag{2.56}$$

It is well known that many mechanical systems with nonholonomic constraints can be locally or globally converted, under coordinate change and state feedback, into the chained form or into a generalization of it [Murray and Sastry, 1993; Tilbury *et al*, 1995].

The purpose is to develop *simple* tracking controllers for this class of systems. Based on a result for (time-varying) cascaded systems [Panteley and Loría, 1998b] we divide the tracking error dynamics into a cascade of two linear sub-systems which we can stabilize independently of each other with linear time-varying controllers. Using the same approach we also consider the tracking problem for chained form systems by means of dynamic output-feedback.

Furthermore, we partially deal with the tracking control problem under input constraints. The only results on saturated tracking control of nonholonomic systems that we are aware of are [Jiang *et al*, 1998] which deals with this problem for a mobile robot with two degrees of freedom, and [Jiang and Nijmeijer, 1999] that deals with general chained form systems.

3

Observers

3.1 Introduction

Filtering and reconstruction of signals are used in numerous different types of applications and play a fundamental role in modern signal processing, telecommunications, and control theory. Diagnosis and supervision of critical processes are of major importance for reliability and safety in industry today. The application of observers in fault detection and isolation provide one means to these problems [Alcorta Garcia and Frank, 1997; Hammouri *et al*, 1999].

The evaluation of estimation techniques can be traced via the least-squares methods of Gauss, Fischer's maximum likelihood approach, the seminal work on optimal filtering by Kolmogorov and Wiener, and the important results on recursive filtering by Kalman and Bucy [Kolmogorov, 1939; Wiener, 1949; Kalman and Bucy, 1961]. In this Section we will mainly consider non-stochastic methods for nonlinear estimation. Linear systems constitute an important subclass for which the observer problem is well known. The class of systems which can be transformed into a linear part and a nonlinear part depending only on measured states and inputs was characterized in [Krener and Isidori, 1983; Bestle and Zeitz, 1983]. This class of systems allows for linear error dynamics via *output injection*. New methods in nonlinear system theory have been applied not only to the control problem, but also to the observer design problem. An overview of some different methods on observer design can be found in [Misawa and Hedrick, 1989; Besançon, 1996; Nijmeijer and Fossen, 1999].

The observability problem can be stated as: "For a given dynamical system, when and how is it possible to reconstruct the internal states from output measurements of the system?". Usually we have unknown initial conditions, but the measurements and the system equations are supposed to be known, possibly disturbed by some noise or structured disturbances. A similar problem appears for *synchronization* of dynamical systems [Nijmeijer and Mareels, 1997].

For linear systems the observability and detectability properties are closely connected to the existence of observers with strong properties, such as for instance exponential convergence of the errors. Known statistical properties of measurement and process noise allow us to find an optimal observer, *the Kalman Filter* [Kalman and Bucy, 1961].

The Kalman Filter

The linear system

$$\begin{aligned} \dot{x} &= Ax + v, \\ y &= Cx + e, \quad \mathcal{E}\{\begin{pmatrix} v \\ e \end{pmatrix}\} = 0, \quad \mathcal{E}\{\begin{pmatrix} v \\ e \end{pmatrix}\begin{pmatrix} v \\ e \end{pmatrix}^T\} = Q \end{aligned} \tag{3.1}$$

has an innovations-form realization

$$\begin{aligned} \dot{x} &= Ax + Kw \\ y &= Cx + w \end{aligned} \tag{3.2}$$

with $s > 0$ and K fulfilling the Riccati equation

$$\begin{pmatrix} I & K \\ 0 & I \end{pmatrix}\begin{pmatrix} S & 0 \\ 0 & R \end{pmatrix}\begin{pmatrix} I & K \\ 0 & I \end{pmatrix}^T = \begin{pmatrix} A & 0 \\ C & I \end{pmatrix}\begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A & 0 \\ C & I \end{pmatrix}^T + Q \tag{3.3}$$

Its transfer function

$$Y(s) = H(s)W(s), \quad H(s) = C(sI - A)^{-1}K + I \tag{3.4}$$

$$H^{-1}(s) = -C(sI - (A - KC))^{-1}K + I \tag{3.5}$$

and its inverse suggest the Kalman filter realization

$$\begin{aligned} \dot{\widehat{x}} &= (A - KC)\widehat{x} + Ky \\ \widehat{y} &= C\widehat{x} \\ \varepsilon &= y - \widehat{y} = -C\widehat{x} + y \end{aligned} \tag{3.6}$$

That the prediction error $\varepsilon(t)$ reconstructs the innovation $w(t)$ is verified by the transfer function relationship

$$\varepsilon(s) = H^{-1}(s)Y(s) = H^{-1}(s)(H(s)W(s)) = W(s) \tag{3.7}$$

EXAMPLE 3.1—KALMAN FILTER & EXACT LINEARIZATION
Consider the feedback-linearized system

$$\begin{aligned}\dot{x} &= Ax + Bu_1 + v\\ y &= Cx + Du + e\\ u_1(x) &= \beta^{-1}(x)(u - \alpha x)\end{aligned} \tag{3.8}$$

with the innovations model

$$\begin{aligned}\dot{x} &= Ax + B\beta^{-1}(x)(u - \alpha(x)) + Kw\\ y &= Cx + Du + w\end{aligned} \tag{3.9}$$

or

$$\begin{aligned}\dot{x} &= Ax - B\beta^{-1}(x)\alpha(x) + B\beta^{-1}(x)u + Kw\\ y &= Cx + Du + w\end{aligned} \tag{3.10}$$

The formulation of the Kalman filter is straightforward as

$$\begin{aligned}\dot{\widehat{x}} &= (A - KC)\widehat{x} + (B - KD)u_1(x) + Ky\\ \widehat{y} &= C\widehat{x} + Du_1(x)\\ \varepsilon &= y - \widehat{y} = -C\widehat{x} - Du_1 + y\end{aligned} \tag{3.11}$$

Since x and thus $u_1(x)$ are not available to measurement, the context of application of such a Kalman filter is very limited. The use of so called *pseudo-linearization*, where $u_1(\hat{x})$ is used instead of $u_1(x)$, will be discussed later on. □

For linear systems, there is no distinction between local and global stability results. In the case of nonlinear dynamical systems, however, this is not the case where rigorous proofs often rely on Lyapunov techniques with local regions of validity only. Another important difference is that for general nonlinear systems, the observability properties depend upon the input signal. Even though the states of a system may be fully observable for most input signals, some inputs, so called *singular inputs*, may render the system unobservable.

EXAMPLE 3.2—SINGULAR INPUTS
Consider the system

$$\begin{aligned}\dot{x}_1 &= (u - 1)\cdot x_2\\ \dot{x}_2 &= x_2\\ y &= x_1\end{aligned} \tag{3.12}$$

The system in Eq. (3.12) is clearly observable for all $u \neq 1$. The signal $u(t) = 1$ is a *singular input* for the system, for which it loses observability.
$\square$

In this thesis the main focus will be on the deterministic observer for the purpose of output-feedback control of nonlinear continuous time systems. The rest of the Chapter is organized in the following way: First some preliminaries and definitions are given in Section 3.2. Thereafter an overview of traditional and recent observer design techniques are given in Section 3.3, including some new results on observers from Papers A, B, and E.

3.2 Preliminaries

Observability for linear systems is characterized by the Kalman rank condition [Kalman *et al*, 1969]. The linear time-invariant system

$$\begin{aligned} \frac{dx}{dt} &= Ax + Bu, \ t \geq 0, \ x \in I\!R^n, \ u \in I\!R^m \\ y &= Cx, \ y \in I\!R^p, \ x(0) = x_0 \end{aligned} \tag{3.13}$$

is said to be *observable* if the observability matrix

$$W_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{3.14}$$

has full rank, i. e., *rank* $W_o = n$. For linear systems the input signal does not influence the observability property. Furthermore, observability of the system ensures the existence of a state observer. Such properties are not valid for nonlinear systems [Nijmeijer and van der Schaft, 1990; Isidori, 1995; Besançon and Bornard, 1997].

Consider the smooth nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + \sum_{j=1}^{m} g_j(x) u_j \\ y_i &= h_i(x) \quad i \in [1 \ldots p] \end{aligned} \tag{3.15}$$

where $x \in M \subset I\!R^n$, f, g_j and h_i are smooth on M.

DEFINITION 3.1—INDISTINGUISHABLE STATES [NIJMEIJER AND VAN DER SCHAFT, 1990]
Two states $x_1, x_2 \in M$ are said to be *indistinguishable*, $x_1 I x_2$, for the system (3.15) if for every admissible input function u the output function $t \mapsto y(t, 0, x_1, u)$, $t \geq 0$, of the system for initial state $x(0) = x_1$, and the output function $t \mapsto y(t, 0, x_2, u)$, $t \geq 0$ of the system for initial state $x(0) = x_2$, are identical on their common domain of definition. The system is called observable if $x_1 I x_2 \Rightarrow x_1 = x_2$. □

Alternative notation is presented in [Sontag, 1990; Hermann and Krener, 1977].

DEFINITION 3.2—OBSERVATION SPACE [NIJMEIJER AND VAN DER SCHAFT, 1990]
The observation space, O, is the linear space (over $I\!R$)

$$O = span\{L_{X_1} L_{X_2} \dots L_{X_k} h_j | X_i \in f, g_1, \dots, g_m, j \in [1..m], k \in N\}$$ □

REMARK 3.1—[GAUTHIER *et al*, 1992]
Consider a series expansion of the output of the system of Eq. (3.15)

$$y(t) = \sum_{k=0}^{+\infty} L_f^k h(x_0) t^k / k!$$

For the linear case this expression is specialized to the standard form

$$y(t) = \sum_{k=0}^{+\infty} C(At)^k x_0 / k!$$

□

DEFINITION 3.3—LOCAL OBSERVABILITY [NIJMEIJER AND VAN DER SCHAFT, 1990]
The system (3.15) is said to be *locally observable at* x_0 if there exists a neighborhood W of x_0 such that for every neighborhood $V \subset W$ of x_0 the relation $x_0 I^V x_1$ implies that $x_1 = x_0$. If the system is locally observable at each x_0 then it is called *locally observable*. □

DEFINITION 3.4—WEAK DETECTABILITY
The system (3.15) is said to be *weakly detectable* if there exists a continuous function

$$g : I\!R^n \times I\!R^p \times I\!R^m \to I\!R^n,\ g(0,0,0) = 0$$

and a C^1-function

$$W : I\!R^n \times I\!R^n \to I\!R_+$$

such that

$$f(x,u) = g(x,h(x),u) \tag{3.16}$$
$$\alpha_1(\| x - z \|) \leq W(x,z) \leq \alpha_2(\| x - z \|) \tag{3.17}$$
$$\frac{\partial W}{\partial x} f(x,u) + \frac{\partial W}{\partial z} g(z,h(x),u) \leq -\alpha_3(\| x - z \|) \tag{3.18}$$

where α_i, $(i = 1..3)$, are *class $\mathcal{K}$ functions*. □

DEFINITION 3.5—ZERO-STATE DETECTABILITY [ISIDORI, 1999]
The system of Eq. (3.15) is said to be *Zero-State Detectable* if, for any initial condition $x(0)$ and zero input $u = [u_1 \ldots u_p]^T \equiv 0$, the condition of identical zero output $y = [h_1(x) \ldots h_p(x)] = 0$, $t \geq 0$ implies that the state converges to zero, $\lim_{t\to+\infty} x(t) \to 0$. □

DEFINITION 3.6—OBSERVER LINEARIZATION PROBLEM [ISIDORI, 1995]
Given the autonomous, nonlinear system described by

$$\begin{aligned} \dot{x} &= f(x), \quad x \in I\!R^n \\ y &= h(x), \quad y \in I\!R, \\ x(0) &= x_0 \end{aligned} \tag{3.19}$$

the solution to the *observer linearization problem* involves finding

- a neighborhood U_0 of x_0
- a coordinate transformation $z = \Phi(x)$ on U_0
- and a mapping $k : h(U_0) \to I\!R^n$

such that

$$\begin{aligned} \dot{z} &= L_f\Phi(x)_{|x=\Phi^{-1}(z)} = Az + k(Cz) \\ y &= h(\Phi^{-1}(z)) = Cz \end{aligned} \tag{3.20}$$

where the pair $[C, A]$ is observable. □

THEOREM 3.1—[ISIDORI, 1995, THEOREM 4.9.4]
The *observer linearization problem* is solvable if and only if

(i) $\dim(\text{span}\{dh(x_0), dL_f h(x_0), \ldots, dL_f^{n-1}h(x_0)\}) = n$

(ii) the unique vector field τ which is the solution of

$$\begin{aligned} L_\tau h = L_\tau L_f h = \cdots = L_\tau L_f^{n-2} h &= 0 \\ L_\tau L_f^{n-1} h &= 1 \end{aligned} \tag{3.21}$$

satisfies

$$[\tau, ad_f^k f] = 0 \text{ for } k = 1, 3, \ldots, 2n-1. \tag{3.22}$$

where $[\cdot,\cdot]$ denotes the Lie bracket of two vector fields and ad denotes the repeated Lie bracket $ad_f^k g \triangleq [f, ad_f^{k-1} g], k \geq 1$, with $ad_f^0 g = g$.
□

DEFINITION 3.7—UNBOUNDEDNESS OBSERVABILITY [MAZENC *et al*, 1994]
A system

$$\begin{aligned} \dot{x} &= f(x,u), \quad x \in I\!R^n, u \in I\!R^p \\ y &= h(x), \quad y \in I\!R^m \end{aligned} \tag{3.23}$$

is said to have *the unboundedness observability property* if, for any solution, $x(\cdot)$, right maximally defined on $[0,T)$, with T finite, corresponding to a bounded input function $u(\cdot)$ in $L^\infty([0,T);R)$, we have

$$\limsup_{t\to T} |h(x(t))| = +\infty,$$

i. e., all possible finite-escape time phenomena should be observable from the output. □

Saturations in the output mapping $h(x)$, will prevent any possibility to observe unbounded solutions and restrict systems with respect to the above defined *unboundedness observability property*. However, saturations constitute only one obstacle and the case is more general than that, which the following example, taken from [Mazenc *et al*, 1994], will illustrate.

EXAMPLE 3.3—[MAZENC *et al*, 1994]

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_2^k + u \\ y &= x_1 \end{aligned} \tag{3.24}$$

Suppose that $u \equiv 0$, and that $x_2(0) = x_{20} > 0$. For $t \in [0,T)$, where $T = 1/((k-1)x_{20}^{k-1})$ we have the solution

$$x_2(t) = \frac{x_{20}}{(1 - x_{20}^{k-1}(k-1)t)^{1/(k-1)}}$$

$$x_1(t) = \begin{cases} x_1(0) - \log(1 - x_{20}t), & k = 2 \\ x_1(0) + x_{20}^{2-k}\frac{1}{k-2}[1 - (1 - x_{20}^{k-1}(k-1)t)^{(k-2)/(k-1)}], & k > 2 \end{cases}$$

So even though $\lim_{t \to T} x_2(t) = +\infty$ we have different behavior of the output $y = x_1$ for $k = 2$ and $k > 2$. Whereas for $k = 2$ the finite escape phenomenon is observed at the output when the time $t \to T$, this is not the case for $k > 2$. In Eq. (3.24) we have a system which is completely observable, see [Gauthier and Bornard, 1981]. Still, it does not satisfy the unboundedness observability property. Hence, the hierarchy among different definitions regarding the observability problem for nonlinear systems has to be paid careful attention. □

3.3 Observer Structure and Design

For linear systems the observability condition implies existence of exponentially converging observers. For general nonlinear systems the different definitions and properties on observability described in the previous section are fundamental, but the relation to observers and the observer design is far more complex then for the linear case.

A standard approach to solve the state reconstruction problem is to use a copy of the observed system and to add some correction terms attenuating the difference of the outputs, i. e., using a full-state model-based observer and *output injection*. The order of the observer can be reduced by just reconstructing the complementary part of the state vector which is not directly accessible from the output signal or from some linear or nonlinear combination of the measurements. This is called a reduced-order observer.

It should be noted that there is an inconsistency in the terminology regarding observers, in particular what a *Luenberger observer* denotes. In some contexts it is used to distinguish the deterministic observer from the stochastic Kalman-filter [Luenberger, 1971], whereas in other contexts it is used to represent a reduced-order observer [Åström and Wittenmark, 1997].

Derivative filters Despite the rapid development of modern control theory, the PID controller is still the most commonly used control scheme in industry today [Yamamoto and Hashimoto, 1991]. The introduction of the derivative part is often very well motivated for stability reasons and may in electro-mechanical applications often be interpreted as additionally added damping to the system. In most realizations, the derivative part is implemented as a ”low-pass filtered” derivative,

$$\frac{dx}{dt} \approx \frac{p}{(pT+1)^n} x(t), \quad n \geq 1 \tag{3.25}$$

where p is the differential operator. For frequencies well below $1/T$ the filter provides a good approximation of the derivative, while for higher frequencies it behaves like a constant gain when $n = 1$ and attenuates for $n > 1$. High-frequency noise is often present in measurements and without roll-off in the filter it will be amplified and corrupt the 'derivative estimate' considerably. There is of course a trade-off in the achievable band-width of the closed loop system and the sensitivity to measurement noise for the choice of T.

EXAMPLE 3.4—'DERIVATIVE FILTER'
Consider a linear system with transfer function

$$G_{u\to y}(s) = \frac{1}{s^3 + 3s^2 - 5s + 1} \tag{3.26}$$

and the state space realization

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -x_1 + 5x_2 - 3x_3 + u \\ y &= x_1 \end{aligned} \tag{3.27}$$

The state feedback $u = -8x_2 + v$ would stabilize the system with a closed-loop triple pole in $s = -1$. From the separation principle for linear systems we know that choosing $u = -8\hat{x}_2 + v$, where $\hat{x}_2$ comes from a linear asymptotically stable observer, will give the same closed loop-poles as the state feedback controller, and the observer poles will be canceled in the transfer function. The performance and transient behavior will suffer significantly, but the stability is guaranteed for arbitrarily slow observer poles. This separation property is however not met using the standard approach of 'derivative filters'. Let

$$u(t) = -8\frac{p}{(p/a+1)}y(t) + v(t)$$

where p is the differential operator. This control law gives the closed loop transfer function

$$G_{v\to y}(s) = \frac{s+a}{s^4 + (a+3)s^3 + (-5+3a)s^2 + (3a+1)s + a}$$

which is stable for $a > 2.78$. The filter matches the "true" derivative for a larger frequency range with increasing value of a and in the limit we get

$$\lim_{a\to+\infty} G_{v\to y}(s) = \frac{s+a}{(s+a)(s+1)^3}$$

which recovers the pole-pattern of the closed loop transfer function as expected. □

Observers for linear systems

Consider the linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{3.28}$$

Full-order observers Under observability/detectability assumptions on the pair $[A, C]$ a full-order observer for the system of Eq. (3.28) is readily constructed as

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + K(y - \hat{y}) \\ \hat{y} &= C\hat{x} \end{aligned} \tag{3.29}$$

where the term $K(y - \hat{y})$ represents linear *output injection*. For asymptotically stable error dynamics the gain vector K should be chosen such that $(A - KC)$ is Hurwitz—i. e., the eigenvalues of $(A - KC)$ lie strictly in the left half plane.

Reduced-order observers Full-order observers reconstruct the whole state vector even though some of the elements in it or combinations of them are known through the output signals. One of the key observations in Luenberger's seminal paper on reduced-order observers was that even though a state-space transformation does not change the spectrum of the system matrix A, it can change the spectrum for a sub-matrix of A [Luenberger, 1964]. Here we will however use a slightly different point of view to explain the reduced-order observer, which generalizes to reduced order observers for nonlinear systems [Kailath, 1980, p.311].

Consider an observable (detectable) linear system. There exists a non-unique state representation where the system can be written as

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \\ y &= \begin{bmatrix} I & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned} \tag{3.30}$$

As x_1 can be measured, there is only need to reconstruct x_2. As direct feedback of the output $y = x_1$ does not affect the system matrix A_{22}, we will temporarily assume that we have access to the derivative of the output

$$\dot{y} = \dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1 u \tag{3.31}$$

An observer of the form

$$\begin{aligned}\dot{\hat{x}}_2 &= A_{21}x_1 + A_{22}\hat{x}_2 + B_2 u + K(\dot{y} - \dot{\hat{y}}) \\ &= A_{21}x_1 + A_{22}\hat{x}_2 + B_2 u + K(A_{12}x_2 - A_{12}\hat{x}_2)\end{aligned} \tag{3.32}$$

would give the following dynamics for the estimation error $\tilde{x}_2 = x_2 - \hat{x}_2$

$$\dot{\tilde{x}}_2 = (A_{22} - KA_{12})\tilde{x}_2 \tag{3.33}$$

Note that if the system in Eq. (3.30) is detectable, the pair $\{A_{12}, A_{22}\}$ is also detectable. Whereas this observer would allow for exponentially convergent error dynamics, difficulties remain in implementation. As the derivative of the output is to be used, the implementation properties are not clear.
However, Eq. (3.32) can be re-written as

$$\begin{aligned}\frac{d}{dt}(\hat{x}_2 - Ky) &= A_{21}x_1 + A_{22}\hat{x}_2 + B_2 u - K\dot{\hat{y}} \\ &= A_{21}x_1 + A_{22}\hat{x}_2 + B_2 u - K(A_{11}x_1 + A_{12}\hat{x}_2 + B_1)u\end{aligned} \tag{3.34}$$

By introducing the extra state vector $z = \hat{x}_2 - Ky$, we have an observer realizable as

$$\begin{aligned}\frac{dz}{dt} &= (A_{21} - KA_{11})x_1 + (A_{22} - KA_{12})\hat{x}_2 + (B_2 - KB_1 u) \\ \hat{x}_2 &= z + Ky\end{aligned} \tag{3.35}$$

For stabilization and control via linear state feedback, $u = -Lx$, it would suffice to reconstruct a linear combination of the states, $\hat{u} = -L\hat{x}$, rather than the full state vector or a reduced state vector which together with the measurements span the whole state space [Fortmann and Williamson, 1972; Sirisena, 1979].

Open-loop observers For some systems it is trivial to design an observer and to achieve a converging state estimate. For asymptotically stable linear systems, it is enough to make a direct copy of the system (3.28) without any output injection. The error dynamics will be

$$\dot{\tilde{x}} = A\tilde{x}$$

which is exponentially stable for A being Hurwitz. There are of course obvious drawbacks with this approach. There is no freedom in affecting the convergence rate of the estimates, as it solely depends upon the eigenvalues of A.

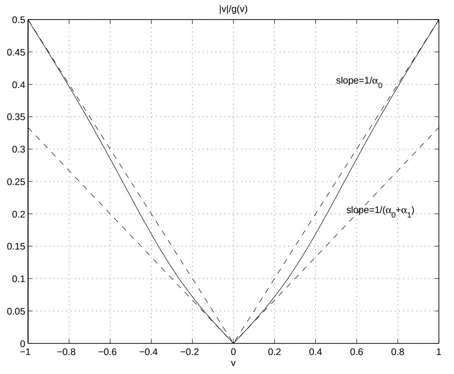

Figure 3.1 The characteristics of the nonlinearity $|v|/g(v)$ in the LuGre-friction model

EXAMPLE 3.5—FRICTION OBSERVER [CANUDAS DE WIT *et al*, 1993]
The LuGre model for friction can be described by the system

$$\begin{aligned} \frac{dz}{dt} &= v - \sigma_0 \frac{|v|}{g(v)} z \\ F &= \sigma_0 z + \sigma_1(v) \frac{dz}{dt} + F_v v \\ g(v) &= \alpha_0 + \alpha_1 e^{-(v/v_o)^2} \end{aligned} \tag{3.36}$$

where F is the friction force, v is the velocity, and σ_0, α_0, α_1 are positive parameters [Canudas de Wit *et al*, 1993; Canudas de Wit *et al*, 1995; Olsson, 1996; Gäfvert *et al*, 1999]. The function $|v|/g(v)$ is positive for non-zero velocities (Fig. 3.1). Since the dynamics in Eq. (3.36) are stable, it is possible to use

$$\begin{aligned} \frac{d\hat{z}}{dt} &= v - \sigma_0 \frac{|v|}{g(v)} \hat{z} \\ \hat{F} &= \sigma_0 \hat{z} + \sigma_1(v) \frac{d\hat{z}}{dt} + F_v v \end{aligned} \tag{3.37}$$

as a friction observer. The state observation error $\tilde{z} = z - \hat{z}$ will have the stable dynamics

$$\dot{\tilde{z}} = -\sigma_0 \frac{|v|}{g(v)} \tilde{z} \tag{3.38}$$

under the assumption that the velocity v is measurable. Hence, the open-loop observer above results in a converging friction estimate for non-zero velocities. □

EXAMPLE 3.6
Consider the position signal from a resolver mounted in a robot servo, (Fig. 3.2). Resolvers and tracking resolver-to-digital converters are considered to give very accurate angle measurements. However, amplitude imbalance and imperfect quadrature of the resolver will cause small disturbances [Hanselman, 1990]. Amplitude imbalance give rise to an additive error and the measured angle signal, q_{meas}, can be approximated by

$$q_{meas}(t) \approx q(t) + \alpha \sin(k \cdot q(t) + \beta) \tag{3.39}$$

where $q(t)$ is the true motor angle and α, β, and k are constants. Even though the disturbance can be neglected in the position measurement as α is very small, Figure 3.2 shows the effect of reconstructing the velocity by using a derivative filter with and without compensating for the disturbance. Note that the amplitude and the frequency of the disturbance increase with increased velocity. An approximate inverse of the nonlinearity in Eq. (3.39) before the filtering, reduces the disturbance in the velocity estimate significantly.

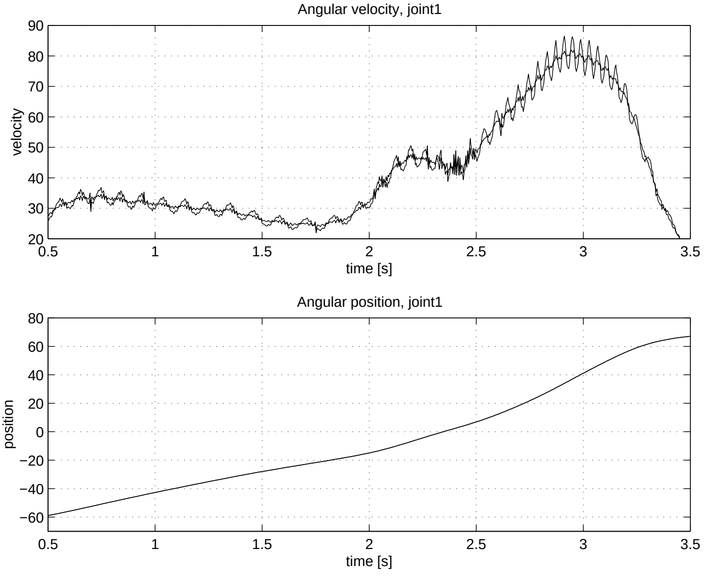

Figure 3.2 Compensated and uncompensated signals from resolver measurements of a robot joint. The differences in the position measurements are very small but the effect in the velocity estimates is apparent.

□

In [Gauthier *et al*, 1992] design of an *extended Luenberger observer* is discussed with related work in [Zeitz, 1987; Birk and Zeitz, 1988], and [Tsinias, 1990].

Extended Kalman filters

The extended Kalman filter (EKF) is a commonly used method for estimating the state of a nonlinear system. The method consists of designing an observer for a linearization of the true system along an estimated trajectory [Gelb, 1974; Ljung, 1979; Cruz and Nijmeijer, 1999]. The state estimation for the nonlinear system

$$\begin{aligned} \dot{x} &= f(x,t) + e(t), \quad e \sim \mathcal{N}(0, Q) \\ y &= h(x,t) + v(t), \quad v \sim \mathcal{N}(0, R) \end{aligned} \tag{3.40}$$

will be

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x},t) + K(t)\,(y - h(\hat{x},t)) \\ \hat{y} &= h(\hat{x},t) \end{aligned} \tag{3.41}$$

where the gain vector $K(t)$ and the estimation error covariance matrix are updated with respect to the linearizations

$$\frac{\partial f(x,t)}{\partial x}\Big|_{x=\hat{x}(t)} \text{ and } \frac{\partial h(x,t)}{\partial x}\Big|_{x=\hat{x}(t)}$$

In contrast to the linear Kalman filter where the the gain vector $K(t)$ can be pre-calculated, the gain for the extended Kalman filter has to be updated on-line via the solution of a Riccati differential solution. A linearized Kalman filter algorithm where a predefined trajectory should be followed allows for off-line computations. The extended Kalman filter is based on a first order Taylor series expansion of the nonlinearities in order to estimate the covariance matrix. A standard extension is to use more terms in the Taylor series expansion to estimate higher-order moments.

The observer design along the methods described above may although being straight-forward from an theoretical point of view, lead to lengthy and tedious calculations, not the least from numerically motivated implementation aspects. In the Ph.D. thesis by Sørlie, a computer-aided design tool for symbolic derivation of extended Kalman filters is presented as a remedy to the implementation problem [Sørli, 1996].

Observers for Bilinear Systems

Bilinear systems constitute an important subclass of nonlinear systems and many processes can be described by bilinear models [Mohler and

Kolodziej, 1980]. In the same way an approximative linearization or an exact feedback linearization is used to get a 'linear system' from a nonlinear one, a possible reduction to a bilinear systems often allows for a richer tool-box with respect to analysis as well as synthesis. The observer problem for bilinear systems has been extensively studied [Hara and Furuta, 1976; Funahashi, 1979; Gauthier *et al*, 1992; Lin and Byrnes, 1994; Lin, 1995]. The observer design in [Funahashi, 1979] considers exponential convergence of the estimation error irrespective of the input signal, and extends the results from [Hara and Furuta, 1976]. The sufficient conditions for the existence of an observer, as stated in [Funahashi, 1979], are expressed as linear matrix inequalities and allows for efficient numerical computations [Boyd *et al*, 1994].

Observers for Nonlinear systems

The observer design for linear systems was extended via the notion of *output injection* by [Krener and Isidori, 1983] and [Bestle and Zeitz, 1983] to a class of nonlinear systems of the form

$$\begin{aligned} \dot{x} &= Ax + f(y,u) \\ y &= Cx. \end{aligned} \tag{3.42}$$

As the nonlinearity f only depends on the measurable output and the known control signal, an observer for the system in Eq. (3.42) can be realized as

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + f(y,u) + K(y-\hat{y}) \\ \hat{y} &= C\hat{x}. \end{aligned} \tag{3.43}$$

When the pair $[A, C]$ is detectable, a proper choice of the K-matrix, namely that the matrix $A - KC$ is Hurwitz, renders the linear observer error $e = x - \hat{x}$ to be (globally) exponentially stable.

Even if a system description is not directly in the form of Eq. (3.42), there might exist an invertible state transformation $\chi = S(x)$ which allows for a observer design with linear error dynamics in the new variables. The convergence $\hat{\chi} \to \chi$ then implies $\hat{x} = S^{-1}(\hat{\chi}) \to x$. In [Krener and Isidori, 1983; Krener and Respondek, 1985; Marino and Tomei, 1991] geometric conditions characterizing the class of nonlinear systems for which the transformation is possible are presented, see also [Nijmeijer and van der Schaft, 1990; Isidori, 1995].

Lipschitz systems Many standard nonlinearities, as for instance the trigonometric functions in robot kinematics, can be bounded by linear

functions satisfying a Lipschitz condition.
Consider the system

$$\begin{aligned} \dot{x} &= Ax + f(x,u,t) + \phi(y,u,t) \\ y &= Cx \end{aligned} \tag{3.44}$$

where the nonlinearity $f(x,u,t)$ is Lipschitz with respect to the state x. For the model-based observer

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + f(\hat{x},u,t) + \phi(y,u,t) + L(y - \hat{y}) \\ \hat{y} &= C\hat{x} \end{aligned} \tag{3.45}$$

the following theorem can be used for analysis.

THEOREM 3.1—[THAU, 1973]
Given the system in Eq. (3.44) and the corresponding observer in Eq. (3.45) with the gain matrix L. If the Lyapunov equation

$$(A - LC)^T P + P(A - LC) = -Q, \quad P = P^T > 0, Q = Q^T > 0 \tag{3.46}$$

is satisfied with

$$\gamma < \frac{\lambda_{min}(Q)}{2\lambda_{max}(P)} \tag{3.47}$$

then the observer error $\tilde{x} = x - \hat{x}$ is asymptotically stable. □

The theorem of Thau assures asymptotical stability of the estimates, but unfortunately, Eq. (3.45) provides very little insight how the observer gain L can be found. For any observable system (A, C) the eigenvalues of the system matrix $(A - LC)$ can be placed arbitrarily, but the crucial part is the relation between these eigenvalues and the spectral radius of the matrix P. The ratio in Eq. (3.47) can be shown to be maximized for $Q = I$ [Patel and Toda, 1980].

Raghavan and Hedrick have proposed a procedure how to construct the observer gain L, based on theory for quadratic stabilization of uncertain systems [Raghavan and Hedrick, 1994]. Rajamani has given a good overview of the problem relating Eq. (3.46) and Eq. (3.47) and presents an algorithm for computation of the observer gain [Rajamani, 1998]. However, the structure of the nonlinearities are not fully utilized which makes the results somewhat conservative as the observer gain, if found, will give an asymptotically stable observer for all nonlinearities satisfying the Lipschitz condition.

Arcak and Kokotović have recently suggested an observer-based design for control of systems which include monotonic nonlinearities in the unmeasured states [Arcak and Kokotović, 1999]. An important ingredient in the control design is the observer design for systems of the form

$$\begin{aligned} \dot{x} &= A_o x - G\phi(Hx) + \gamma(y,u) \\ y &= C_o x \end{aligned} \tag{3.48}$$

where A_o and C_o are in *observer canonical form*. The vector ϕ have the components

$$\phi_i = \phi_i(H_i x) \tag{3.49}$$

which are all either zero or monotonically increasing nonlinearities. Extra freedom in the design is introduced by using $G\phi(H\hat{x}+K(y-C_o\hat{x}))$ in the observer instead of $G\phi(H\hat{x})$. In short, the observer design decomposes the error dynamics into a linear system in feedback with a multivariable sector nonlinearity. Linear matrix inequalities (LMIs) are used to state the conditions for the existence of stable observer error dynamics with respect to the imposed observer structure. Efficient numerical solvers for LMIs can give an answer to the question if feasible solutions exist for a particular system and if so provide corresponding observer gains [Willems, 1970; Boyd *et al*, 1994]. Note that this design covers the cases of sector bounded nonlinearities for systems with global Lipschitz constants and also incorporates the structure of the nonlinearities which is a shortcoming of the Lipschitz methods in the previous section.

EXAMPLE 3.7—PENDULUM OBSERVER

After an appropriate choice of time scale, the equation of motion for an inverted pendulum can be written as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \sin(x_1) + u\cos(x_1) \\ y &= x_1 \end{aligned} \tag{3.50}$$

where u is the normalized acceleration of the pivot, x_1 the pendulum angle and x_2 the angular velocity. An observer for the pendulum may be suggested along the ideas presented in [Arcak and Kokotović, 1999]:

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + k_1\tilde{x}_1 \\ \dot{\hat{x}}_2 &= \sin(\hat{x}_1 + l_1\tilde{x}_1) + u\cos(\hat{x}_1 + l_2\tilde{x}_1) + k_2\tilde{x}_1 \\ \hat{y} &= \hat{x}_1 \\ \tilde{x}_1 &\triangleq y - \hat{y} \end{aligned} \tag{3.51}$$

In the sequel we will use $l_2 = l_1$ for simplicity.
By using the following standard trigonometric relations

$$\begin{aligned}
\sin(x_1) - \sin(\hat{x}_1 + l_1\tilde{x}_1) &= 2\sin\left(\frac{(1-l_1)\tilde{x}_1}{2}\right) \cdot \gamma_1(t) \\
\gamma_1 \triangleq \cos\left(x_1 + \frac{(l_1-1)\tilde{x}_1}{2}\right)&, \quad |\gamma_1| \le 1 \\
\cos(x_1) - \cos(\hat{x}_1 + l_1\tilde{x}_1) &= 2\sin\left(\frac{(1-l_1)\tilde{x}_1}{2}\right) \cdot \gamma_2(t) \\
\gamma_2 \triangleq -\sin\left(x_1 + \frac{(l_1-1)\tilde{x}_1}{2}\right)&, \quad |\gamma_2| \le 1
\end{aligned} \tag{3.52}$$

the dynamics for the observation error $\tilde{x} = x - \hat{x}$ may be expressed as

$$\begin{aligned}
\dot{\tilde{x}}_1 &= -k_1\tilde{x}_1 + \tilde{x}_2 \\
\dot{\tilde{x}}_2 &= -k_2\tilde{x}_1 + 2\sin\left(\frac{(1-l_1)\tilde{x}_1}{2}\right) \cdot (\gamma_1(t) + u \cdot \gamma_2(t))
\end{aligned} \tag{3.53}$$

The system in (3.53) can be partitioned into a feedback connection of a linear system and a sector-bounded time-varying nonlinearity

$$\begin{aligned}
\frac{d}{dt}\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} &= \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix}\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v \\
z &= \begin{bmatrix} (1-l_1) & 0 \end{bmatrix} \tilde{x} \\
v &\triangleq 2\sin(\frac{z}{2}) \cdot (\gamma_1(t) + u \cdot \gamma_2(t))
\end{aligned} \tag{3.54}$$

Assumption 3.2—Bounded acceleration
Assume that we have bounded acceleration $|u| < u_{max}$, which implies

$$|\gamma_1(t) + u \cdot \gamma_2(t)| \le \sqrt{1 + u_{max}^2} \triangleq \beta \tag{3.55}$$

and gives the following bound on the time-varying nonlinearity:

$$|v(z,t)| \le |\gamma_1(t) + u \cdot \gamma_2(t)| \cdot |2\sin(\frac{z}{2})| \le \beta|z| \tag{3.56}$$

□

The linear part of the system in Eq. (3.54) has the transfer function

$$G_{v\to z}(s) = \frac{1-l_1}{s^2 + sk_1 + k_2} \tag{3.57}$$

It is obvious that for any given pair of strictly positive constants (k_1, k_2) the linear system is stable and furthermore the parameter l_1 can be chosen such that $|G_{v \to z}| < 1/\beta$ which implies stability of the closed loop system in Eq. (3.54) from the small gain theorem.

In the limit $l_1 \to 1$, there is an exact cancellation of the nonlinear term in the error dynamics, whereas for $l = 0$ we have the observer proposed in [Eker and Åström, 1996].

It should be noted that the stability analysis above is made for the deterministic case, without any measurement noise. Simulations show, however, fairly good behavior also for values of l_1 close to 1, see Fig. 3.3. They also indicate that the main benefit of the additional feedback term $l_1\tilde{x}_1$ is for slow observer poles and low values of k_1 and k_2, which can be expected. □

Observers for systems in chained form The observability property for the chained-form systems in Eqs. (2.55,2.56) was considered in [Astolfi, 1995]. Under the assumption that the first control signal was chosen as $u_1(t) = -c_1 x_1(t)$, a change of coordinates was proposed which allows for a locally stable observer.

In Paper A and Paper B we present two observers for the chained-form system with the first and the last state as outputs. Firstly, we present a globally exponentially stable observer under an observability condition which is related to the persistence of excitation with respect to the first component of the state, x_1. In the second observer design a new theorem for linear time-varying systems is used.

PROPOSITION 3.3
Consider the chained-form system in Eq. (2.55) with outputs (2.56).

Define

$$w(t,t_0) = \begin{bmatrix} 1 \\ \int_{t_0}^{t} u_1(\tau)d\tau \\ \left(\int_{t_0}^{t} u_1(\tau)d\tau\right)^2 \\ \vdots \\ \left(\int_{t_0}^{t} u_1(\tau)d\tau\right)^{n-2} \end{bmatrix} = \begin{bmatrix} 1 \\ x_1(t) - x_1(t_0) \\ (x_1(t) - x_1(t_0))^2 \\ \vdots \\ (x_1(t) - x_1(t_0))^{n-2} \end{bmatrix} \tag{3.58}$$

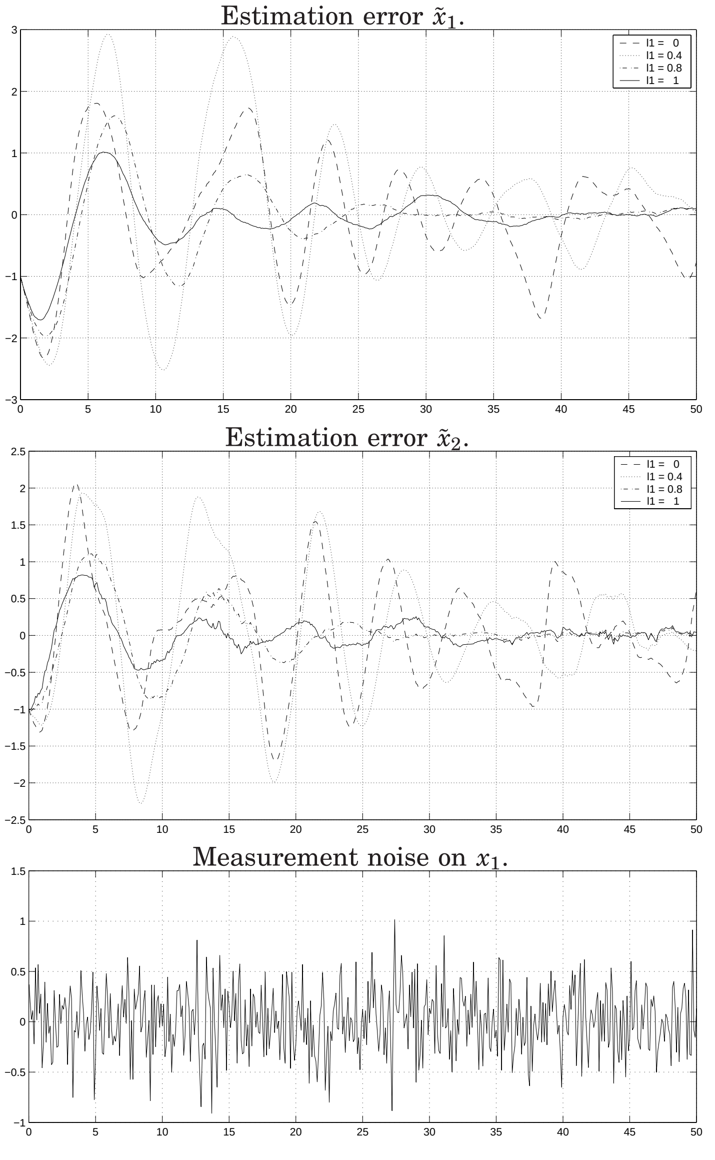

Figure 3.3 Simulations of error dynamics for the observer in Example 3.7 for various values of l_1 (0, 0.4, 0.8, 1).

Assume that there exist $\delta, \varepsilon_1, \varepsilon_2 > 0$ such that for all $t > 0$:

$$\varepsilon_1 I \leq \int_t^{t+\delta} w(t,\tau)w(t,\tau)^T d\tau \leq \varepsilon_2 I. \tag{3.59}$$

Then the observer

$$\begin{bmatrix} \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \\ \dot{\hat{x}}_4 \\ \vdots \\ \dot{\hat{x}}_n \end{bmatrix} = \begin{bmatrix} 0 & \dots & \dots & \dots & 0 \\ u_1 & \ddots & & & \vdots \\ 0 & u_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u_1 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \\ \vdots \\ \hat{x}_n \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u_2 + H(t)\tilde{x}_n \tag{3.60}$$

where $\tilde{x}_n = x_n - \hat{x}_n$ and

$$\begin{aligned} M_\alpha(t-\delta,t) &= \int_{t-\delta}^{t} 2e^{4\alpha(\tau-t)}\Phi^T(\tau,t-\delta)C^T C\Phi(\tau,t-\delta)d\tau \\ H(t) &= \left[\Phi^T(t-\delta,t)M_\alpha(t-\delta,t)\Phi(t-\delta,t)\right]^{-1} C^T \quad (\alpha > 0) \end{aligned} \tag{3.61}$$

guarantees that the observation error $\tilde{x} = x - \hat{x}$ converge to zero exponentially. □

Proof See Paper A.

THEOREM 3.2
Consider the chained form system in Eq. (2.55) with outputs (2.56).

The estimates $[\hat{x}_{2,e}, \dots, \hat{x}_{n,e}]^T$ generated by the observer

$$\begin{aligned} \begin{bmatrix} \dot{\hat{x}}_{2,e} \\ \dot{\hat{x}}_{3,e} \\ \vdots \\ \vdots \\ \dot{\hat{x}}_{n,e} \end{bmatrix} &= \begin{bmatrix} -k_2 & -k_3 u_{1,r}(t) & \dots & \dots & \dots \\ u_{1,r}(t) & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u_{1,r}(t) & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_{2,e} \\ \hat{x}_{3,e} \\ \vdots \\ \vdots \\ \hat{x}_{n,e} \end{bmatrix} \\ &+ \begin{bmatrix} \dots & l_5 \cdot u_{1,r}(t) & l_4 & l_3 \cdot u_{1,r}(t) & l_2 \end{bmatrix}^T (x_{n,e} - \hat{x}_{n,e}) \end{aligned} \tag{3.62}$$

converge uniformly asymptotically stable ($\mathcal{K}$-exponential stability) towards the true states of the system in Eq. (2.55) provided that k_i, l_i

$(i = 2, \ldots, n)$ are such that the polynomials

$$\lambda^{n-1} + k_2\lambda^{n-2} + \cdots + k_{n-1}\lambda + k_n \tag{3.63}$$

$$\lambda^{n-1} + l_2\lambda^{n-2} + \cdots + l_{n-1}\lambda + l_n \tag{3.64}$$

are Hurwitz and $u_{1,r}$ satisfies the conditions of Assumption (B.6) in Paper B.

Proof See Paper B. □

Remark: The conditions on the reference trajectory $u_{1,r}(t)$ in Assumption (B.6) concerns amongst others differentiability and boundedness of $u_{1,r}$. Furthermore, singular inputs, like for instance $u = u_{1,r} \equiv 0$, for which the chained form system is no longer controllable or observable, are excluded.

The dynamic-output control of chained form systems will be considered in Chapter 4 for which we will use the estimated states from the above proposed model-based observers.

Observers for Interconnected Systems The theory for interconnected systems can be used for analysis as well as synthesis of observers [Besançon and Hammouri, 1998; Besançon, 1999]. In general, however, the combination of asymptotically stable observers for separate subsystems, does not guarantee stable state estimation of the full interconnected system.

One the main concepts for the observer designs in Papers B and A is the decoupling of the error dynamics into a cascaded form. We use similar ideas in the following example which treats the velocity reconstruction problem for an inverted pendulum application.

EXAMPLE 3.8—ROBOT PENDULUM

Consider the configuration of a rotational pendulum held in a robot gripper, see Fig 3.4. Different strategies for balancing the pendulum can be considered by controlling one or more robot joints. By moving the base joint of the robot, joint 1, the whole manipulator will rotate in the horizontal plane and by neglecting the rotational effects in the Furuta pendulum model, we get the following normalized model:

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \sin(x_1) + L(-D\ x_4 + \tau) \cdot \cos(x_1) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -D\ x_4 + \tau \\
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x
\end{aligned}
\tag{3.65}
$$

where L is the distance from the rotational axis of joint 1 to the pendulum pivot and Dx_4 is damping in the robot servo. The variables x_1 and x_2 are the pendulum angle and velocity, respectively. The control signal τ is the torque applied to the robot joint, x_3 is the joint angle, and x_4 the joint velocity, the term $L(-Dx_4 + \tau)$ being the acceleration of the pivot. The system in Eq. (3.65) can be viewed as a cascade of two systems—the pendulum and the robot—with the connection term $-LDx_4 \cos(x_1)$. As for the observer design, two separate observers can be designed where we re-use the observer from Example 3.7 for the pendulum and assign an ordinary linear observer for the linear robot joint dynamics:

$$
\begin{aligned}
\dot{\hat{x}}_3 &= -k_3\tilde{x}_3 + x_4 \\
\dot{\hat{x}}_4 &= -k_4\tilde{x}_3 - D\hat{x}_4 + \tau
\end{aligned}
\tag{3.66}
$$

For the stability analysis, we use similar ideas as presented in Paper A and Paper B, by decoupling the observer error dynamics into two systems in cascade with a time-varying coupling term—see also [Besançon, 1999] for observer design of interconnected systems. With the choice $l_2 = 1$ in Eq. (3.51), we get the error dynamics

$$
\begin{aligned}
\Sigma_1 &: \begin{cases} \dot{\tilde{x}}_1 = -k_1\tilde{x}_1 + \tilde{x}_2 \\[2ex] \dot{\tilde{x}}_2 = -k_2\tilde{x}_1 + 2\sin\left(\dfrac{(1-l_1)\tilde{x}_1}{2}\right) \cdot \gamma_1(t) - LD\cos(x_1)\tilde{x}_4 \end{cases} \\
\Sigma_2 &: \begin{cases} \dot{\tilde{x}}_3 = -k_3\tilde{x}_3 + \tilde{x}_4 \\ \dot{\tilde{x}}_4 = -k_4\tilde{x}_3 - D\tilde{x}_4 \end{cases}
\end{aligned}
\tag{3.67}
$$

where γ_1 is defined in Eq. (3.52). In contrast to the results presented in Paper B, the coupling term $-LD\cos(x_1(t))\tilde{x}_4$ does not depend on the states $(\tilde{x}_1, \tilde{x}_2)$ in Σ_1 which permits simplified stability analysis. System Σ_2 can be exponentially stabilized for appropriate values of (k_3, k_4), which implies that the coupling term is exponentially vanishing. As the small gain analysis in Example 3.7 assures input-to-state stability for the Σ_1 system, stability for the error dynamics in Eq. (3.67) is concluded.

Figure 3.4 shows a sequence from an experiment of swing-up and balancing of an inverted pendulum. The arrows in the pictures indicate the rotation of the pendulum. Figure 3.5 shows the control signal and the estimated states. The swing-up strategy is the energy-based method proposed in [Wiklund *et al*, 1993]. After the swing-up there is a switch to a linear controller for keeping the pendulum in the up-right position. The observer presented above is used for the velocity estimation. The controller scheme is implemented in the real-time system PÅLSJÖ connected to the Open Robot Control System in the Robotics Lab of the Department of Automatic Control, Lund [Eker, 1997; Blomdell, 1997; Nilsson, 1996]. □

Backstepping designs of observers Song *et al* have recently proposed a backstepping approach for design of reduced-order nonlinear observers [Song *et al*, 1997]. A backstepping-like method is used to find a coordinate transformation between two canonical state-space representations. A different approach is taken in [Kang and Krener, 1998] where a locally convergent nonlinear observer is designed using backstepping.

In Paper E we present an observer for ship dynamics where the design makes use of Lyapunov theory in a recursive way. The structure of the problem allows for an approach similar to backstepping, where the output injection gains and some parameters in the Lyapunov function candidate are used to “linearize” the Lyapunov candidate derivative, leaving only negative quadratic terms left. The design extends the results from [Fossen and Grøvlen, 1998] to cover systems with unstable sway-yaw dynamics.

For the velocity estimation problem of robot manipulators in Section 3.4 we use similar ideas as outlined above and additional design freedom is introduced by a state-space transformation [Slotine and Li, 1987; Johansson, 1990].

Passivity-based observers The ideas for passivity-based control in robotics has been used also for observer design and observer-based control [Ailon and Ortega, 1993; Berghuis, 1993; Berghuis and Nijmeijer, 1993b; Battilotti *et al*, 1997]. For the Lyapunov-based observer design in [Fossen and Grøvlen, 1998; Robertsson and Johansson, 1998a] global stability results were achieved. However, it should be noted that a simplified model for the ship dynamics was used. In real applications of ship positioning, disturbances from waves, currents, winds, etc., are needed to be taken care of. From a practical point of view it makes sense to compen-

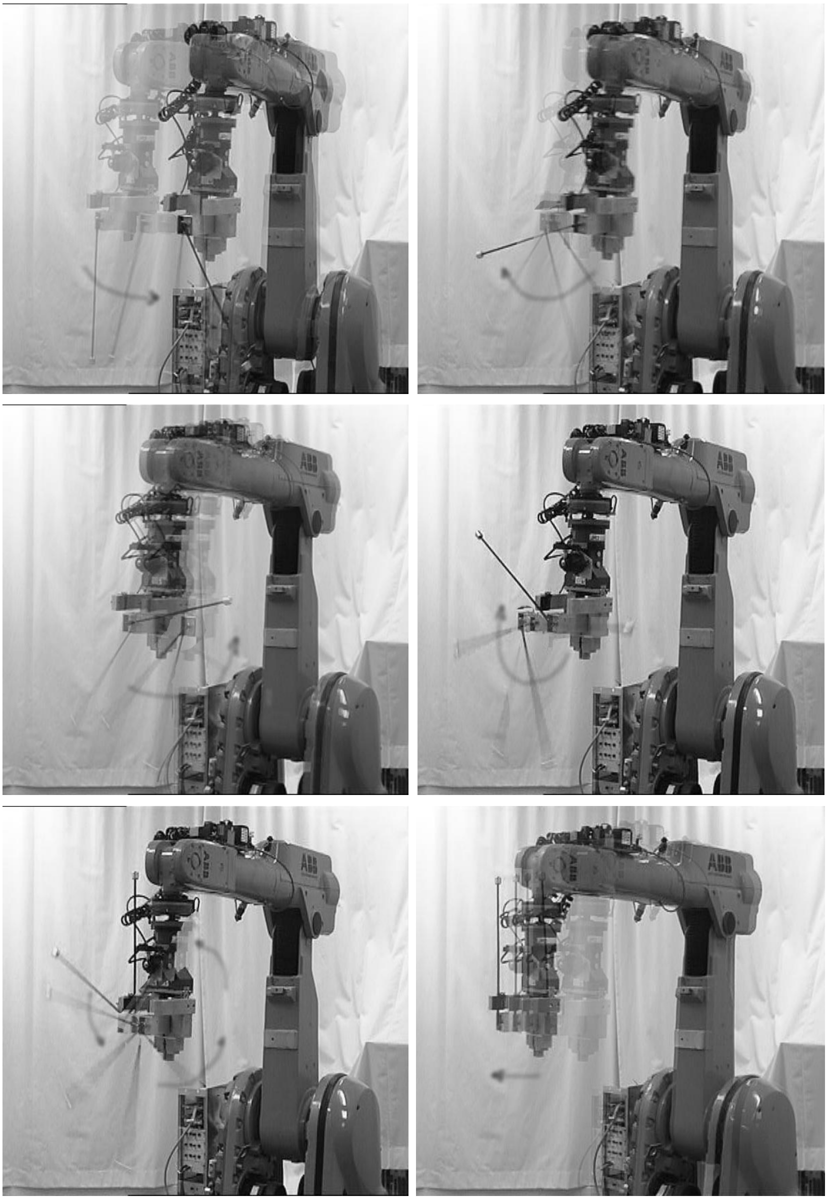

Figure 3.4 Successful energy-based swing-up and balancing of an inverted pendulum with velocity observer from Example 3.7. The arrows edited into the figures indicate the direction of motion of the pendulum. The total length of the sequence shown is about 6 seconds. The input signal, i. e., the torque to joint 1 and the estimated signals are found in Figure 3.5.

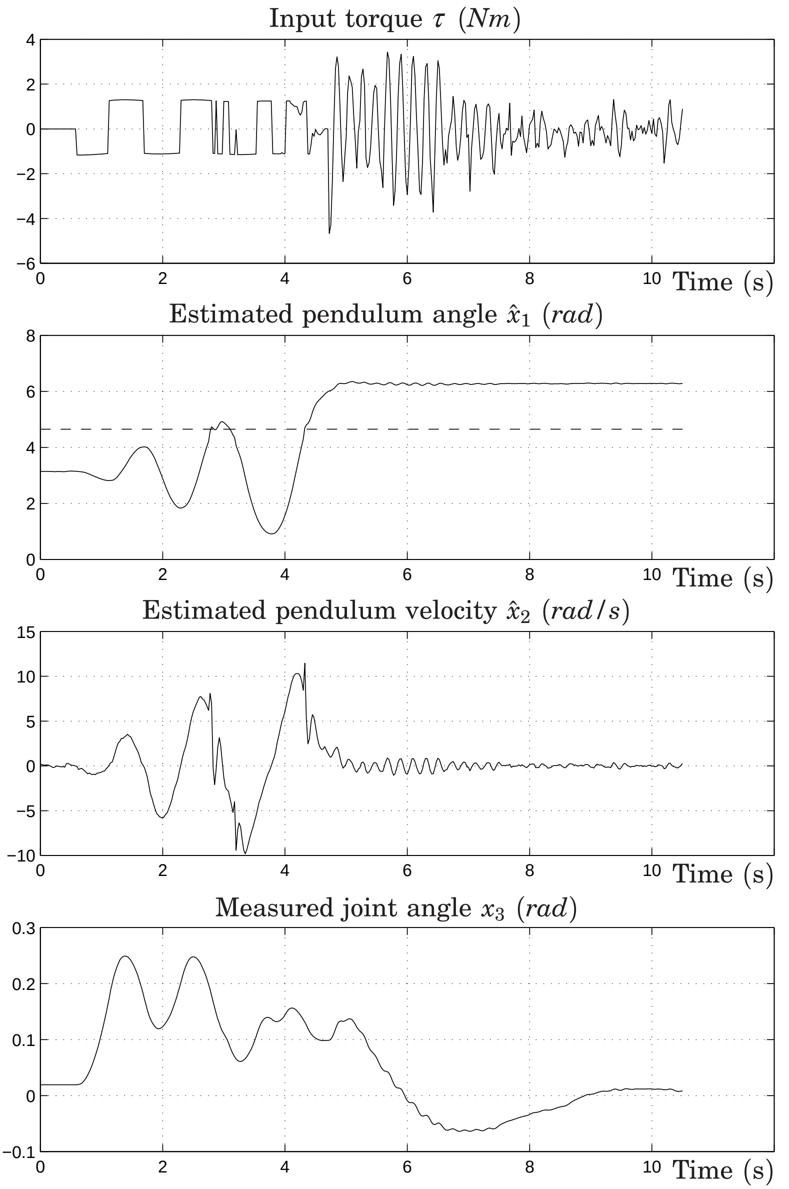

Figure 3.5 Control signal and estimated states for the swing-up and balancing of the pendulum in (Fig. 3.4). The observer from Example 3.7 is used for the velocity estimation. The sensor for the pendulum angle is a potentiometer which has a discontinuity at the angle marked with the dashed line in the second diagram.

sate for the slowly varying disturbances like drift and currents, while the faster oscillations caused by waves, often appearing within the controller bandwidth, should be filtered out and not tried to be compensated for. In [Fossen and Strand, 1999], a passivity based observer for the ship dynamics, including estimation of low frequency drift terms and the wave dynamics, is presented.

3.4 Velocity-Observers for Robot Manipulators

In many motion controllers for electro-mechanical systems the full state vector is assumed to be available for feedback. In reality very few industrial robots are equipped with tachometers for velocity measurements of the links or velocity measurements in the drives. PD and PID controllers have for long been known to perform well in the control of mechanical manipulators in industry, but it was only recently a theoretical proof for stability of the controlled system was presented [Arimoto and Miyazaki, 1986; Spong, 1987; Craig, 1989]. The derivative part in the PD controller implies that velocity measurements are important in the cause of stabilization. Another way of viewing it is via the use of a *computed torque-based controller*, from which it is obvious that the velocity feedback plays the role of damping in the system.

In AC-drives and induction machines the speed and flux estimates can greatly enhance the control performance without increasing the costs for additional sensors; in particular this is true for low-power machines where accurate speed sensors could cost in same same range as the motors themselves [Nicklasson, 1996]. Although position measurements often are of high quality, today's velocity measurements—e. g., from tachometers—still contain a lot of noise.

For flexible-joint manipulators, there is a difference in angles and angular velocities between the actuator and the link side. The most common configuration with the sensor on the same side as the actuator is called *co-located measurements*. If the sensor measures the angle on the link-side it is said to be *non-co-located* [Craig, 1989, p.289]. In the absence of measurement devices on both sides of a flexible joint, observers are necessary for assessment of the added dynamics. In the literature, most observer-based control designs for flexible links have considered measurements on the link-side, and a few from an application point of view the more motivated problem of co-located sensors [Janković, 1995; Ailon and Ortega, 1993]. An obvious attempt to cope with the problem of lacking speed measurements is to numerically differentiate the accurate position signal or using some derivate-filter with 'roll-off' for high frequencies. This method, also known as ‘dirty derivatives’ is conceptually very simple and has been used extensively in applications. In spite of its simplicity, the stability properties for the set-points and tracking following problem was only recently shown in the robotics setting [Arimoto and Miyazaki, 1986; Kelly, 1993; Berghuis and Nijmeijer, 1993a; Kelly *et al*, 1994; Loría, 1996]. However, for very low and very high frequencies this approach may not be adequate from a performance perspective [Belanger, 1992]. This motivates us to have a look at alternative methods.

Observers for Robot Manipulators

In this section we propose two velocity observers for robot manipulators, one reduced-order observer and one full-order observer for which semi-global stability results are shown.

PROPERTY 3.1—RIGID ROBOTS
Consider the n-linked rigid robot manipulator with revolute joints which can be described by the equations of motion

$$\begin{aligned} &M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau \\ &q,\, \dot{q} \in I\!R^n \end{aligned} \tag{3.68}$$

where q is the vector of joint angles, M is the mass inertia matrix, C represents the centrifugal and Coriolis forces, G consists of the gravity dependent terms, and τ is the vector of input torques. The matrices occurring in Eq. (3.68) have some important properties, useful for control and observer design [Craig, 1988]:

(M1) $M(q) = M(q)^T > 0$, $\forall q \in I\!R^n$.

(M2) $0 < M_m <\| M(q) \|< M_M$, where M_m and M_M are positive constants.

(C1) $C(q,\dot{q})\dot{q}$ is uniquely determined, but C can be decomposed in many ways. Here we define C using the Christoffel symbols. $C(q,\dot{q})$ then can be written as

$$C(q,\dot{q}) = \begin{pmatrix} \dot{q}^T C_1(q) \\ \vdots \\ \dot{q}^T C_n(q) \end{pmatrix}$$

where $C_i(q) = C_i(q)^T \in R^{n\times n}$ and bounded.

(C2) $C(q,\dot{q}_1)\dot{q}_2 = C(q,\dot{q}_2)\dot{q}_1, \quad \forall q, \dot{q}_1, \dot{q}_2 \in I\!R^n$.

(C3) $\| C(q,\dot{q}) \|< C_M \| \dot{q} \|, \quad \forall q, \dot{q} \in I\!R^n$, where C_M is a positive constant.

The vector norm and the matrix norm are the ordinary Euclidean norm and the matrix-two-norm, respectively:

$$\begin{aligned} \|q\| &= \|q\|_2 = \sqrt{q^T q}, && q \in I\!R^m \\ \|A\| &= \|A\|_2 = \sqrt{\rho(A^T A)}, && A \in I\!R^{m\times m} \end{aligned} \tag{3.69}$$

where $\rho(A)$ is the spectral radius of A. □

ASSUMPTION 3.4—BOUNDED VELOCITY

The angular velocity of the robot, $\dot{q}$, is bounded by a known constant ω_{max} such that

$$\|\dot{q}(t)\| \le \omega_{max}, \quad \forall t \in I\!R \tag{3.70}$$

□

Introduce the joint angles and joint velocities as the Lagrangian coordinates

$$x = \begin{pmatrix} \dot{q} \\ q \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{3.71}$$

The Lagrangian state space representation is

$$\begin{aligned} \dot{x} &= \begin{pmatrix} \ddot{q} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} M^{-1}u \\ \dot{q} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} \begin{pmatrix} \dot{q} \\ q \end{pmatrix} + \begin{pmatrix} M^{-1} \\ 0 \end{pmatrix} u \\ u &\triangleq \tau - G(q) - C(q,\dot{q})\dot{q} \end{aligned} \tag{3.72}$$

where we introduced u for a more compact notation.

Reduced-order robot observer

Consider a reduced-order observer for estimating the angular velocity, $\dot{q} = x_1$, when the angle q is measurable:

$$\dot{\hat{x}}_1 = M^{-1}[\tau - C(q,\hat{x}_1)\hat{x}_1 - G(q)] + K(x_1 - \hat{x}_1) \tag{3.73}$$

The dynamics for the observation error, $\tilde{x}_1 = x_1 - \hat{x}_1$, will be

$$\begin{aligned} \dot{\tilde{x}}_1 &= M^{-1}[-C(q,x_1)x_1 + C(q,\hat{x}_1)\hat{x}_1] - K(x_1 - \hat{x}_1) \\ &= M^{-1}[-2C(q,x_1)\tilde{x}_1 + C(q,\tilde{x}_1)\tilde{x}_1] - K\tilde{x}_1 \end{aligned} \tag{3.74}$$

where we used the following property

$$\begin{aligned} C(q,\hat{x}_1)\hat{x}_1 &= C(q,x_1-\tilde{x}_1)(x_1-\tilde{x}_1) \\ &= C(q,x_1)x_1 - 2C(q,x_1)\tilde{x}_1 + C(q,\tilde{x}_1)\tilde{x}_1 \end{aligned} \tag{3.75}$$

following from **(C2)**.
Consider the quadratic Lyapunov function candidate

$$V(\tilde{x}_1) = \frac{1}{2}\tilde{x}_1^T\tilde{x}_1 \tag{3.76}$$

which is positive definite and decrescent.
The time derivative of V along the solutions of Eq. (3.74) is

$$\begin{aligned} \frac{dV}{dt} &= \tilde{x}_1^T\dot{\tilde{x}}_1 = \tilde{x}_1^T M^{-1}[-2C(q,x_1)\tilde{x}_1 + C(q,\tilde{x}_1)\tilde{x}_1] - \tilde{x}_1^T K\tilde{x}_1 \\ &\le \Big(\frac{C_M}{M_m}(2\|\tilde{x}_1\| + \omega_{max})\cdot I - K\Big)\|\tilde{x}_1\|^2 \end{aligned} \tag{3.77}$$

where we have used Assumption (3.4) and the properties from (3.1).
From Eq. (3.77) we can conclude that if

$$0 < \|x_1\| < (KM_m/C_M - \omega_{max}\cdot I)/2 \tag{3.78}$$

then there exists a constant $\alpha > 0$ such that

$$\dot{V} < -\frac{\alpha}{2}\|\tilde{x}_1\|^2 = -\alpha V, \quad \forall \tilde{x}_1 \neq 0 \tag{3.79}$$

For any given bound on the velocity ω_{max}, we are free to choose the observer gain K such that we can guarantee exponential convergence for all initial values satisfying Eq. (3.78). As this region can be arbitrarily increased by the gain, we have *semi-global exponential stability*.

In particular starting with zero initial conditions for the velocity estimate Assumption (3.4) assures that $\|\tilde{x}_1(0)\| \le \omega_{max}$ and the observer with $K > 3C_M\omega_{max}/M_m$ will give exponential stability.

REMARK 3.2
The observer in Eq. (3.73) can not be implemented directly as the unmeasurable term Kx_1 appears in the differential equation. This may be overcome in standard way by introducing the variable

$$z = \hat{x}_1 - Kq$$

The observer in Eq. (3.73) can then be implemented as

$$\begin{aligned} \frac{d}{dt}(\hat{x}_1 - Kq) &= \dot{z} = M^{-1}[\tau - C(q, \hat{x}_1)\hat{x}_1 - G(q)] - K\hat{x}_1 \\ \hat{x}_1 &= z + Kq \end{aligned} \tag{3.80}$$

□

A Lyapunov Approach to Velocity Observer Design for Robotic Systems

The following full-order model based observer for system (3.72) is proposed

$$\begin{aligned} \dot{\hat{x}} &= \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} \hat{x} + \begin{pmatrix} M^{-1} \\ 0 \end{pmatrix} (\hat{u} + v) + \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} (q - H\hat{x}) \\ H &\triangleq \begin{pmatrix} 0 & I \end{pmatrix}, \qquad \hat{C} \triangleq C(q, \dot{\hat{q}}) \\ \hat{u} &\triangleq \hat{u}(\dot{\hat{q}}, q) = \tau - G(q) - C(q, \dot{\hat{q}})\dot{\hat{q}} = \tau - G - \hat{C}\dot{\hat{q}} \end{aligned} \tag{3.81}$$

for some observer gains K_1 and K_2 and a term v which is to be defined later on. The dynamics for the estimation error $\tilde{x} = \hat{x} - x$ are

$$\begin{aligned} \dot{\tilde{x}} &= \left(\begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} - KH \right) \tilde{x} + \begin{pmatrix} M^{-1} \\ 0 \end{pmatrix} (\hat{u} - u + v) \\ &= \underbrace{\begin{pmatrix} 0 & -K_1 \\ I & -K_2 \end{pmatrix}}_{A_L} \tilde{x} + \begin{pmatrix} M^{-1} \\ 0 \end{pmatrix} (\tilde{u} + v) \end{aligned} \tag{3.82}$$

where

$$\begin{aligned} \tilde{u} = \hat{u} - u &= -\hat{C}\dot{\hat{q}} + C\dot{q} = -C\dot{\tilde{q}} + (C - \hat{C})\dot{\hat{q}} \\ &= -C\dot{\tilde{q}} - \tilde{C}\dot{\hat{q}} = -\underbrace{\begin{pmatrix} C & 0 \end{pmatrix}}_{F} \tilde{x} - \underbrace{\begin{pmatrix} \tilde{C} & 0 \end{pmatrix}}_{\tilde{F}} \hat{x} \end{aligned} \tag{3.83}$$

As a Lyapunov function candidate we propose $V(\tilde{x}, t)$ defined as

$$\begin{aligned} V(\tilde{x}, t) &= \frac{1}{2}(T\tilde{x})^T P (T\tilde{x}) \\ P &= \begin{pmatrix} M(q(t)) & 0 \\ 0 & I \end{pmatrix}, \quad T = \begin{pmatrix} I & T_1 \\ 0 & T_2 \end{pmatrix} \end{aligned} \tag{3.84}$$

For any nonsingular matrix T_2, V is positive definite and decrescent, by Property **(M2)**:

$$\frac{M_m}{2}\|\tilde{x}\|^2 \leq V(\tilde{x},t) \leq \frac{M_M}{2}\|\tilde{x}\|^2 \tag{3.85}$$

The time derivate of V is

$$\begin{aligned}
\frac{dV}{dt} &= \frac{1}{2}\dot{\tilde{x}}^T T^T P T \tilde{x} + \frac{1}{2}\tilde{x}^T T^T P T \dot{\tilde{x}} + \frac{1}{2}\tilde{x}^T T^T \dot{P} T \tilde{x} \\
&= \frac{1}{2}\tilde{x}^T T^T \left(A^T P + PA + \begin{pmatrix} \dot{M} - C - C^T & C T_1 T_2^{-1} \\ T_2^{-T} T_1^T C^T & 0 \end{pmatrix} \right) T\tilde{x} \\
&\quad + \underbrace{\frac{1}{2}(\tilde{x}^T T^T \begin{pmatrix} I \\ 0 \end{pmatrix} v - \tilde{x}^T T^T \begin{pmatrix} \tilde{C} & 0 \\ 0 & 0 \end{pmatrix} \hat{x})}_{\phi} + \phi^T
\end{aligned} \tag{3.86}$$

$$A \triangleq T A_L T^{-1}$$

Having defined C as the Christoffel symbols, we now use the skew-symmetric property (C2) of the robot equations to get

$$\begin{aligned}
\frac{dV}{dt} &= \frac{1}{2}\tilde{x}^T T^T \Big(A^T P + PA \\
&\quad + \begin{pmatrix} \dot{M} - (\frac{1}{2}\dot{M} - N) - (\frac{1}{2}\dot{M} - N)^T & C T_1 T_2^{-1} \\ T_2^{-T} T_1^T C^T & 0 \end{pmatrix} \Big) T\tilde{x} + \phi + \phi^T \\
&= \frac{1}{2}\tilde{x}^T T^T \left((A^T P + PA) + \begin{pmatrix} 0 & 2C T_1 T_2^{-1} \\ 0 & 0 \end{pmatrix} \right) T\tilde{x} \\
&\quad + \tilde{x}^T T^T \begin{pmatrix} v - \tilde{C} \cdot \hat{x}_1 \\ 0 \end{pmatrix}
\end{aligned} \tag{3.87}$$

To take care of the term $C \cdot \hat{x}_1$ we use the freedom in v

$$\begin{aligned}
v &= -C(x_2, \hat{x}_1) T_1 \tilde{x}_2 = -C(x_2, x_1) T_1 \tilde{x}_2 - C(x_2, \tilde{x}_1) T_1 \tilde{x}_2 \\
&= -C \cdot T_1 \tilde{x}_2 - \tilde{C} \cdot T_1 \tilde{x}_2
\end{aligned} \tag{3.88}$$

The last term of Eq. (3.87) can then be written

$$\begin{aligned}
v - \tilde{C} \cdot \hat{x}_1 &= v - \tilde{C} \cdot (x_1 + \tilde{x}_1) \\
&= v - C \cdot \tilde{x}_1 - \tilde{C} \cdot \tilde{x}_1 = -C \cdot (\tilde{x}_1 + T_1 \tilde{x}_2) - \tilde{C} \cdot (\tilde{x}_1 + T_1 \tilde{x}_2) \\
&= \begin{pmatrix} -C - \tilde{C} & 0 \\ 0 & 0 \end{pmatrix} T_1 \tilde{x}
\end{aligned} \tag{3.89}$$

which gives us the symmetric Lyapunov candidate derivative

$$\dot{V} \quad = \quad \frac{1}{2}\tilde{x}^T T^T Q T \tilde{x} \tag{3.90}$$

where[1]

$$Q \triangleq \begin{pmatrix} MT_1 + T_1^T M - C - \tilde{C} & -M(K_1 + T_1^2 + T_1 K_2)T_2^{-1} + T_2^T + C T_1 T_2^{-1} \\ * & -T_2(T_1 + K_2)T_2^{-1} - T_2^{-T}(T_1 + K_2)^T T_2^T \end{pmatrix}$$

Due to the quadratic velocity dependence in the Coriolis and centrifugal terms, we have introduced third order terms in the Lyapunov derivative. This is a well-known obstacle for global stability of the stand-alone observer [Berghuis and Nijmeijer, 1993b]. However, using Assumption (3.4), we can achieve local stability results. To finish the stability analysis, we make the following choices of the design parameters.

<u>Element $(2,2)$:</u> For stability it is necessary to have

$$-T_2(T_1 + K_2)T_2^{-1} - T_2^{-T}(T_1 + K_2)^T T_2^T < 0$$

Take any $Q_{22} = Q_{22}^T > 0$ and let

$$K_2 \triangleq -T_1 + T_2^{-1} Q_{22} T_2 \tag{3.91}$$

<u>Element $(1,2)$:</u> Cancel the off-diagonal elements which do not depend on the velocity by choosing

$$K_1 \triangleq -T_1^2 - T_1 K_2 + M^{-1} T_2^T T_2 \tag{3.92}$$

and for simplicity

$$T_2 \triangleq T_1 \tag{3.93}$$

<u>Element $(1,1)$:</u> It is necessary to have $MT_1 + T_1^T M < 0$.
This is satisfied if we choose $T_1 = T_1^T < 0$, in particular we can take $T_1 = -\kappa_1 I, \kappa_1 > 0$.

This gives us

$$Q = \begin{pmatrix} -2\kappa_1 M - C - \tilde{C} & C \\ C^T & -2Q_{22} \end{pmatrix}$$

[1] Q is symmetric and '$*$' marks the transpose of the $(1,2)$-element.

For Q to be negative definite, we use the property of $Q_{22} > 0$ which implies the following condition on the Schur complement

$$\begin{aligned}
&-2\kappa_1 M - C - \tilde{C} + \frac{1}{2} C Q_{22}^{-1} C^T \\
&\quad \leq -2\kappa_1 M_m + C_M(\|x_1\| + \|\tilde{x}_1\|) + \frac{C_M^2}{2}\|x_1\|^2 \|Q_{22}^{-1}\| \\
&\quad \leq -2\kappa_1 M_m + C_M(\|\tilde{x}_1\| + \omega_{max}) + \frac{C_M^2 \omega_{max}^2}{2}\|Q_{22}^{-1}\| < 0
\end{aligned} \tag{3.94}$$

For a choice of κ_1 as

$$\kappa_1 > (C_M(\|\tilde{x}_1\| + \omega_{max}) + \frac{C_M^2}{2}\omega_{max}^2 \|Q_{22}^{-1}\|)/2M_m \tag{3.95}$$

Q is negative. Furthermore $\|\tilde{x}_1\| < \|T^{-1}\| \cdot \|z\|$, so for a κ_1 satisfying Eq. (3.95) the observer will converge exponentially for all initial values $\tilde{x}_1$ such that

$$\|\tilde{x}_1\| \leq (2\kappa_1 M_m - \frac{C_M^2 \omega_{max}^2}{2}\|Q_{22}^{-1}\|)/C_M - \omega_{max} \tag{3.96}$$

Furthermore, for all initial conditions $\|\tilde{x}_1(0)\|$ satisfying Eq. (3.96) there exist a positive constant $\alpha < \|Q\|/M_M$ such that

$$\frac{dV}{dt} = -(T\tilde{x})^T Q T\tilde{x} \leq -\alpha(T\tilde{x})^T M_m T\tilde{x} \leq -\alpha V$$

As T is a static, nonsingular matrix,

$$\lim_{t\to+\infty} T\tilde{x}(t) = 0 \Longrightarrow \lim_{t\to+\infty} \tilde{x}(t) = 0$$

For any given bound on the velocity ω_{max}, we are free to choose observer gains such that we can guarantee exponential convergence for all initial values satisfying Eq. (3.96). As this region can be increased arbitrarily much for large enough value of κ_1, we have *semi-global exponential stability*. Without any restrictions in the analysis above, we can choose $K_1 = diag(k_{1i}) > 0$, $i = 1 \ldots n$, where $\kappa_1 = \max(k_{1i})$. The resulting observer will be

$$\begin{aligned}
\dot{\hat{x}} &= \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} \hat{x} + \begin{pmatrix} M^{-1} \\ 0 \end{pmatrix} (\hat{u} - C(q, \hat{x}_1)T_1\tilde{q}) + \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} (q - \hat{q}) \\
\hat{u} &\stackrel{\triangle}{=} \hat{u}(\dot{\hat{q}}, q) = \tau - G(q) - C(q, \dot{\hat{q}})\dot{\hat{q}}
\end{aligned} \tag{3.97}$$

Figure 3.6 Furuta pendulum, Department of Automatic Control, Lund.

The quadratic dependence on the velocity terms $C(q,\dot{q})\dot{q}$ could not be globally bounded in the Lyapunov derivative but the semi-global result assures that given any bounded region for the robot angles and velocities, which always will be the case in an application, observer gains can be found to assure exponentially converging estimates.

The similarities between the resulting observer above and the observer presented in [Berghuis and Nijmeijer, 1993b] are obvious, but the Lyapunov-based design method above relates closer to the method used in [Robertsson and Johansson, 1998a] than to the passivity based design.

Simulation study The model-based velocity observer from Section 3.4 has been simulated with application to the Furuta pendulum, (Fig. 3.6). The control signal is the torque τ applied to the arm turning in the horizontal plane. The rotational angle of the arm is denoted ϕ. The pendulum is attached to the end of the arm and its angle measured from the vertical stand-up position is denoted θ. The angles but not the angular velocities are measured.

The following equations of motion and the identified physical parameters for the process in Fig. (3.6) are taken from [Gäfvert, 1998]:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tau, \quad q = \begin{bmatrix} \phi \\ \theta \end{bmatrix} \tag{3.98}$$

where

$$\begin{aligned} M &= \begin{pmatrix} \alpha + \beta \sin^2\theta & \gamma\cos\theta \\ \gamma\cos\theta & \beta \end{pmatrix} \\ C(\phi,\theta,\dot{\phi},\dot{\theta}) &= \begin{pmatrix} \beta\cos\theta\sin\theta\dot{\theta} & \beta\cos\theta\sin\theta\dot{\phi} - \gamma\sin\theta\dot{\theta} \\ -\beta\cos\theta\sin\theta\dot{\phi} & 0 \end{pmatrix} \\ G(\phi,\theta) &= \begin{pmatrix} 0 \\ -\delta\sin\theta \end{pmatrix} \end{aligned} \tag{3.99}$$

For the simulation we use the physical parameter values

$\alpha = 0.00335$ [kg·m^2],

$\beta = 0.00389$ [kg·m^2],

$\gamma = 0.00249$ [kg·m^2],

$\delta = 0.0976$ [kg^2·m^2/s^2]. '

For the observer gains we use

$$T_1 = \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix}, \quad T_2 = \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix}, \quad Q_{22} = \begin{bmatrix} -20 & 0 \\ 0 & -20 \end{bmatrix}, \tag{3.100}$$

and the corresponding gain matrices from Eqs. (3.91–3.93) in the observer of Eq. (3.97). Simulation results are found in Figures 3.7 and 3.8.

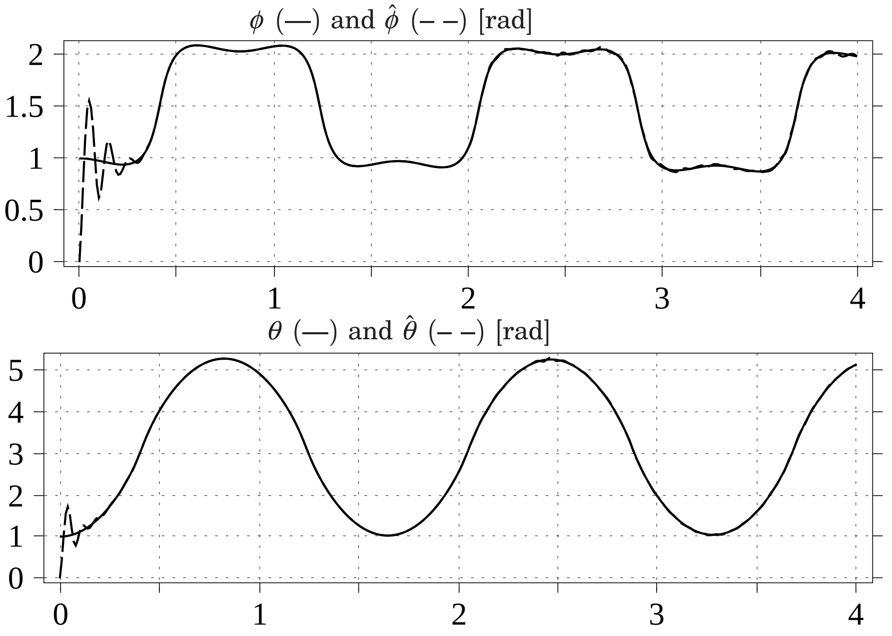

Figure 3.7 Real and estimated angles: **a)** (ϕ, $\hat{\phi}$), **b)** (θ, $\hat{\theta}$). The start values for the angles in the estimator are (0,0), which gives the initial observer errors (1,1) [rad]. The first 2 seconds of the simulation are noise free to study the transient behavior and after time $t = 2$ seconds, noise with variance $(\frac{\pi}{180})^2$ is added to the measurements.

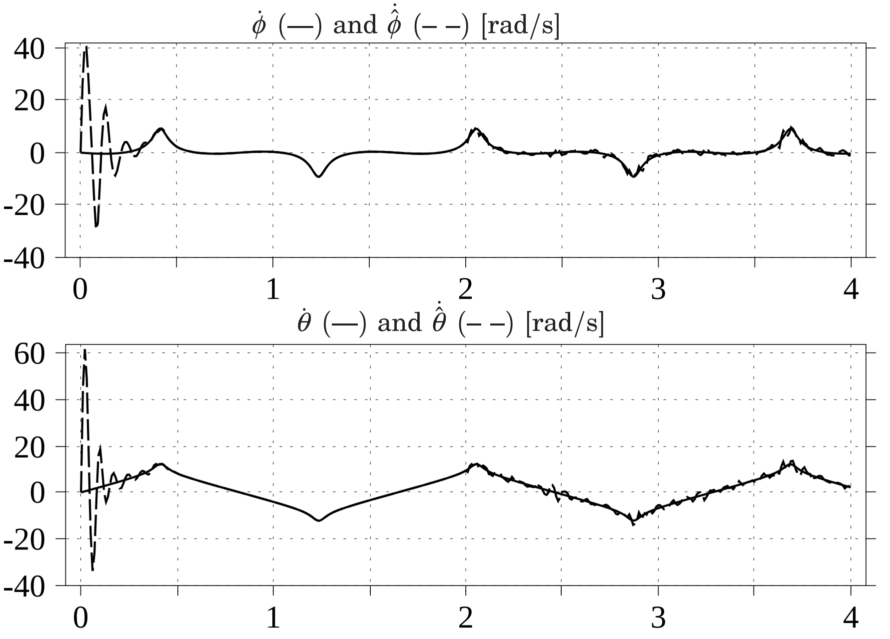

Figure 3.8 Real and estimated velocities for the arm and the pendulum : **a)** $(\dot{\phi}, \dot{\hat{\phi}})$, **b)** $(\dot{\theta}, \dot{\hat{\theta}})$. After time $t = 2$ seconds, noise with variance $(\frac{\pi}{180})^2$ is added to the measurements.

4

Observer-Based Control

4.1 Introduction

The output feedback problem for nonlinear systems has received a lot of attention during the last decades and a rich variety of solutions has been proposed [Krener and Isidori, 1983; Cebuhar *et al*, 1991; Marino and Tomei, 1991; Praly, 1992; Esfandiari and Khalil, 1992; Krstić *et al*, 1993; Busawon *et al*, 1993; Pomet *et al*, 1993; Lin and Byrnes, 1994; Mazenc *et al*, 1994; Teel and Praly, 1994; Nicosia and Tomei, 1995; Loría, 1996; Battilotti, 1996; Isidori, 1998]. The output control problem generally implies a restriction in the possibility to use all the states directly for feedback. For linear systems the well known *separation principle* allows the problem to be split into two sub-problems which can be solved independently: the design of a state-feedback controller and the design of a state observer, see e. g., [Åström, 1970; Kwakernaak and Sivan, 1972]. The separation principle does not apply for nonlinear system in general. However, for some particular classes of systems, such as bilinear systems, or systems with certain structural interconnection properties, separation principles have been reported in the literature [Gauthier and Kupka, 1992; Lin, 1995; Lefeber *et al*, 1999a; Loría *et al*, 1999].

Extensions of the linear separation principle and an overview of classes of systems for which the separation principle holds are described in the next section. The rest of the chapter discusses the topic of control with respect to estimated states, when a 'certainty equivalence methodology' does not apply.

Nonlinear separation principle

A fundamental difference in properties of linear and nonlinear systems is found in the effects of bounded disturbances over a finite time horizon.

Consider some linear system for which there is a stabilizing state-feedback law. If instead the feedback law is provided by estimated states only, then the closed loop system will still be stable under the assumption that the observer errors converge to zero. For nonlinear systems this may not be the case even if we have exponential convergence in the observer. The obstacle is the problem with *finite escape time*, which allows solutions to grow unbounded before the estimated states have converged.

Conditions for a nonlinear separation principle is discussed in [Safonov and Athans, 1978; Vidyasagar, 1980; Glad, 1987b; Tsinias, 1993] where the combination of a stabilizing state-feedback controller and a converging state estimator is considered. Observability as a prerequisite to observer convergence has been extended to nonlinear systems using the notion of *weak state detectability* [Tsinias, 1993]. In that context, Vidyasagar and Tsinias consider nonlinear systems of the form

$$\begin{aligned} \dot{x} &= f(x,u), \quad && x \in \mathbb{R}^n,\, u \in U \subset \mathbb{R}^m \\ y &= h(x), && y \in \mathbb{R}^p \end{aligned} \tag{4.1}$$

To the purpose of observer convergence, it is necessary that the system of Eq. (4.1) be *weakly detectable*, see Def. (3.4). Teel and Praly showed in their seminal paper that "Global stabilizability and observability imply semi-global stabilizability by output feedback" [Teel and Praly, 1994].

For nonlinear systems, affine in the control signal, Sontag showed that global asymptotical stabilizability via continuous state feedback implies global input-to-state stabilizability with respect to actuator disturbances [Sontag, 1988]. There are however fundamental differences how actuator and sensor disturbances affect the stabilizability property. Even though some classes of systems have been shown to be input-to-state stabilizable with respect to sensor noise [Freeman and Kokotović, 1993], Freeman (1995) presented a counter example which shows that the analogous statement to Sontag's theorem for sensor disturbances in general is false [Freeman, 1995]. One consequence of the counter example presented, is that there is no global "separation principle" for nonlinear systems, as the sensor disturbances can be interpreted as transients in converging state estimates. It should be noted that the controllers considered above are restricted to memoryless state feedback controllers.

For the output feedback control of nonlinear systems, Freeman (1995) stated and discussed the following different questions regarding observer and controller separation:

a) (Separation) Given an observer and a stabilizing state feedback control law $u(x)$, will the "certainty equivalence" feedback $u = u(\hat{x})$

provide global asymptotic stability?

b) (Observer separation) Given an observer, does there exist a stabilizing state feedback control law $u(x)$, such that $u = u(\hat{x})$ provide global asymptotic stability?

c) (Controller separation) Given a stabilizing controller $u(x)$, does there exist an observer, such that $u = u(\hat{x})$ provide global asymptotic stability?

d) (No separation) Do there exist an observer and a stabilizing controller $u(x)$, such that $u = u(\hat{x})$ provide global asymptotic stability?

It is widely known that the answer to question a) is false for general nonlinear systems, see Example 4.1. Freeman showed that the answer to b) also is false in the general case. It should be noted though, that the answer to question b) does not give an answer to question c). The sensitivity of feedback control to observer errors is relevant in the context of robot control using position measurement only [Nicosia and Tomei, 1995]. The same paper provides a discussion of the separation principle in robotics.

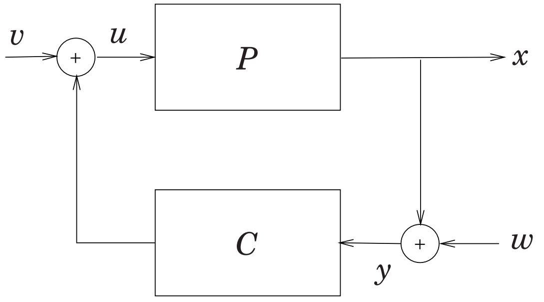

Figure 4.1 Actuator disturbances v and sensor disturbances w in the feedback connection of the plant P and the controller C [Freeman, 1995].

In [Mazenc *et al*, 1994], it is shown that global complete observability and global stabilizability are not sufficient to guarantee global stabilizability by dynamic output feedback—i. e., no observer based design, whatever convergence properties for the observer, will solve the general global stabilization problem.

The following class of systems with dynamics of the form

$$\begin{aligned}\dot{z} &= H(z, x_1, \ldots, x_r)\\ \dot{x}_1 &= x_2\\ &\vdots\\ \dot{x}_{r-1} &= x_r\\ \dot{x}_r &= x_r^k + F(z,x) + G(z,x)u\\ y &= x_1\end{aligned} \tag{4.2}$$

is not globally asymptotically stabilizable by continuous dynamic output feedback and does not satisfy the "unboundedness observability property" if $k \geq r/(r-1)$. Other problems and aspects of systems that are globally asymptotically stabilizable by continuous dynamic output feedback are to be found in [Byrnes and Isidori, 1991] and [Angeli and Sontag, 1999]. As for counter examples to stabilization, the following example demonstrates a finite escape phenomenon despite exponential estimation convergence [Krstić *et al*, 1995, p. 285].

EXAMPLE 4.1—FINITE ESCAPE TIME [KRSTIĆ *et al*, 1995]
Consider the system

$$\begin{aligned}\dot{x} &= -x + x^4 + x^2\xi\\ \dot{\xi} &= -k\xi + u, \quad k > 0\end{aligned} \tag{4.3}$$

By using backstepping and introducing the "error state" $z = \xi + x^2$, a stabilizing state feedback law $u(x, \xi)$ can be found:

$$u = -c(\xi + x^2) - x^3 + k\xi - 2x(-x + x^2(\xi + x^2)), \quad c > 0 \tag{4.4}$$

The asymptotically stable closed-loop dynamics expressed in the (x, z) variables will be

$$\begin{aligned}\dot{x} &= -x + x^2 z\\ \dot{z} &= -cz - x^3\end{aligned} \tag{4.5}$$

For the case of only x being measured, ξ can be estimated by the observer

$$\dot{\hat{\xi}} = -k\hat{\xi} + u \tag{4.6}$$

The estimation error, $\tilde{\xi} = \xi - \hat{\xi}$, converges exponentially to zero:

$$\dot{\tilde{\xi}} = -k\tilde{\xi} \quad \Rightarrow \quad \tilde{\xi}(t) = \tilde{\xi}(0)e^{-kt}$$

By using the estimated state $\hat{\xi}$ in the control law (4.4) we get the dynamics

$$\begin{aligned}\dot{x} &= -x + x^2 z + x^2\tilde{\xi} \\ \dot{z} &= -cz - x^3 + 2x^3\tilde{\xi} \\ \dot{\tilde{\xi}} &= -k\tilde{\xi}\end{aligned} \tag{4.7}$$

Although we have exponential convergence of $\tilde{\xi}$ to zero, the presence of the "disturbance terms" $x^2\tilde{\xi}$ and $2x^3\tilde{\xi}$ will cause unbounded solutions in finite time for certain initial conditions. To see this, consider the case of $z \equiv 0$:

$$\begin{aligned}\dot{x} &= -x + x^2\tilde{\xi} \\ \tilde{\xi}(t) &= \tilde{\xi}(0)e^{-kt}\end{aligned} \tag{4.8}$$

which has the solution

$$x(t) = \frac{x(0)(1+k)}{[1+k-\tilde{\xi}(0)x(0)]e^{t} + \tilde{\xi}(0)x(0)e^{-kt}} \tag{4.9}$$

The state x will grow to infinity in finite time for all initial conditions $\tilde{\xi}(0)x(0) > 1 + k$. For further details, see [Krstić *et al*, 1995]. □

4.2 Classes of Systems with a Separation Property

The use of state detection as a means to accomplish local stabilizability was treated in [Vidyasagar, 1980; Tsinias, 1991]. Some restricted classes of systems admit a global separation principle between controller and state estimator design. Below follows a survey of some continuous-time systems regarding this property.

Linear systems

Considering a linear time-invariant system

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned} \tag{4.10}$$

which is controllable and observable. For such a system it is straightforward to show that the separation principle apply for both full order and reduced order observers. The lack of finite escape-time phenomena allows for large transients without jeopardizing the stability.

The solution to the Linear Quadratic Gaussian (LQG) optimization problem consists of the combination of the state-feedback controller which is the optimal solution to the LQ-problem and the separately designed optimal Kalman-filter. As was shown in the famous paper by Doyle (1978), it is, however, interesting to note that even for the linear case the classical separation principle does not preserve any robustness from the LQ-design and do not guarantee any gain or phase margins whatsoever [Doyle, 1978]. Various suggestions to improve the robustness can be found in [Doyle and Stein, 1979; Petersen and McFarlane, 1994].

Infinite dimensional LTI systems For the case of infinite-dimensional systems, the strong stabilizability results in [Bounit and Hammouri, 1997] show a global separation principle based on state-estimates from a full-order model-based observer structure.

Bilinear systems

The controllability and observability properties for bilinear systems have been studied in [Hara and Furuta, 1976; Gauthier and Kupka, 1994]. For bilinear systems with a dissipative drift term, a separation principle has been stated in [Gauthier and Kupka, 1992] and further generalized in [Lin, 1995] to systems of the form

$$\dot{x} = \Sigma_{i=0}^{k} f_i(x)u^i.$$

This generalization is important in the sense that it can be viewed as a kth order Taylor expansion of any nonlinear system, not necessarily affine in the control signal u. Lin's results have been generalized to multi-input systems in [Ghulchak and Shirjaev, 1995].

Nonlinear Systems

In [Busawon *et al*, 1993], systems which are uniformly observable, i. e., observable for every input signal, are studied. In [Gauthier and Bornard, 1981], it is shown that these systems can be written in a canonical form consisting of a chain of integrators plus a nonlinear term acting on the first state derivative, via a state transformation. Similar results have been shown for multivariable systems [Bornard and Hammouri, 1991]. Under a Lipschitz condition of the nonlinearity, a high-gain observer is suggested which used together with a stabilizing state-feedback control scheme in a certainty equivalence sense guarantees the stability of the output feedback controlled system.

The Input-to-State Stability (ISS) is a strong property for nonlinear systems. Whenever this property can be assured for disturbances entering additively to the states in a stabilizing state feedback law, estimates from a converging state observer can be used in a *certainty equivalence* approach [Tsinias, 1993]. Similar ideas can be considered in the context of stabilization of cascaded systems. Given a stabilizing state feedback law and an asymptotically stable state observer, the observer error dynamics converging to zero could be interpreted as the "driving connection" between two systems in cascade. As remarked in Chapter 2, the structure and growth order of the interconnection terms are crucial for the stability of the cascade. For systems where the unmeasured states appear affine in the dynamics, the global regulation problem has been considered in [Battilotti, 1996].

Nonholonomic systems in chained form In Paper A and Paper B contributions to the output feedback tracking problem for the class of nonholonomic systems in chained form are presented. The overall control law uses a *certainty equivalence* combination of a linear time-varying state-feedback controller and a separately designed state observer. One of the key observations in the design is the possibility to decouple the system into a cascaded form, where the order of the cross terms is of major importance in the analysis. In contrast to the regulation problem, the tracking problem requires results on time-varying cascaded systems [Panteley and Loría, 1998b]. The resulting closed loop system is globally $\mathcal{K}$-exponentially stable, which implies global uniform asymptotic stability.

Adaptive Control Adaptive control for systems with unknown constant parameters, θ, can also be put in the observer framework, by increasing the state-space with the equations

$$\frac{d\theta}{dt} = 0$$

Structured uncertainties and unknown parameter in models of mechanical manipulators allow for a unified approach in state and parameter estimation [Craig, 1988; Nicosia and Tornambè, 1989].

Adaptive schemes which allow a separation of the controller and the parameter update law—i. e., *the controller/update law modularity*—are discussed in [Middleton and Goodwin, 1988; Krstić and Kokotović, 1995; Krstić *et al*, 1995; de Querioz *et al*, 1999].

4.3 Observer-Based Control

"Dirty derivatives" The use of a derivative filter has recently regained large interest and been used instrumentally in a passivity framework for stabilizing control of, e. g., robots and rotating machines [Berghuis, 1993; Berghuis and Nijmeijer, 1993a; Kelly *et al*, 1994; Loría *et al*, 1997; Ortega *et al*, 1998]. In many of these applications, only the positions coordinates are measured, while the velocities, needed for damping injection, are not. The interconnection properties of passive systems is an important tool for the stability analysis and is also used in a constructive way in the synthesis, see Theorem (2.2). An attempt to illustrate the idea is found in Figure 4.2. The derivative filters are in most cases equivalent to the standard implementation of the derivative part in an ordinary PID-controller with bounded high-frequency gain, $dy/dt \approx p/(p+a)y(t)$. An important

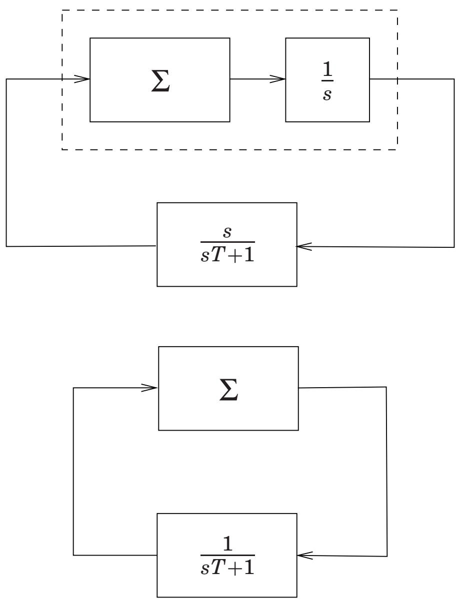

Figure 4.2 Stabilization with derivative filter. (Σ is assumed to be a passive operator).

class of nonlinear systems to which a lot of electro-mechanical systems belong is the class of *Euler-Lagrange systems*, i. e., system whose dynamics are derived from the Euler-Lagrange equations [Goldstein, 1980]. The interconnection properties of the Euler-Lagrange systems have been used for controller design [Ortega *et al*, 1995b; Loría, 1996]. The controllers used, are themselves chosen to have Euler-Lagrange properties which are preserved for the closed-loop interconnection. An extensive overview for the passivity-based control of Euler-Lagrange system is given in [Ortega *et al*, 1998].

Exact linearization

The concept of *feedback linearization*, also known as *exact linearization,* has a long tradition in robotics control under the name of *computed torque methods* or *dynamic inverse methods*. When combining model-based observers and feedback linearization methods there are two main alternative routes to follow. Either the observer is based on the original (nonlinear) model or it tries to reconstruct the states of the resulting linear system after the coordinate transformation.

Feedback linearization with observer In order to achieve a feedback linearized system, access to the full state-vector is often necessary for the coordinate transformations [Isidori, 1995; Nijmeijer and van der Schaft, 1990]. For the output feedback case this is in general not possible and methods based on *pseudo-linearization* and Taylor expansions of nonlinearities have been proposed [Krener and Isidori, 1983; Nicosia *et al*, 1986; Wang and Rugh, 1989]. The *certainty equivalence* approach using estimates from an observer in the linearizing feedback law as well as in the linear controller has been studied in [Etchechoury *et al*, 1996]. Under (local) Lipschitz conditions on the nonlinearities in the state transformations, (local) stability results have been stated.

In [Berghuis, 1993] a passivity-based observer for control of robot dynamics is used. An interesting property is shown in the fact that using a computed torque controller, the model-based observer dynamics are rendered linear through the control law's feedback linearizing property, see also [Song *et al*, 1996]. It is instructive to take the example in [Glad, 1987b] and compare various extensions with respect to the properties of exact linearization and observer dynamics:

EXAMPLE 4.1—CONTROL AND STABLE OBSERVER [GLAD, 1987B, EX. 1]
The system

$$\dot{x} = x^3 + u \tag{4.11}$$

with the nominal control law

$$u = -x^3 - x \tag{4.12}$$

gives rise to the globally asymptotically stable system

$$\dot{x} = -x \tag{4.13}$$

With any estimate $\hat{x}$ which converges exponentially towards x the resulting closed loop system will be

$$\begin{aligned} \dot{x} &= -x^3 + (-\hat{x}^3 - \hat{x}) = 3x^2\tilde{x} - 3x\tilde{x}^2 + 3\tilde{x}^3 - (x - \tilde{x}) \\ \dot{\tilde{x}} &= -\alpha\tilde{x}, \quad \alpha > 0 \end{aligned} \tag{4.14}$$

where $\tilde{x} = x - \hat{x}$ denotes the observer error. That only local asymptotical stability can be obtained is seen from the dynamics of the quantity $z = x\tilde{x}$

$$\dot{z} = \underbrace{(3z - 3\tilde{x}^2 - 1 - \alpha)}\, z + \tilde{x}^4 + \tilde{x}^2 \tag{4.15}$$

No matter how fast the dynamics for the estimator can be made, and no matter how small a non-zero initial observer error is chosen, an initial value of

$$z(0) > (1 + \alpha)/3 + \tilde{x}^2$$

will cause instability. □

The system in the example above can not be globally output feedback linearized via the estimated state $\hat{x}$. It should be noted that high values of the observer gain α increase the region of attraction, but the influence of measurement noise is also increased. In the following two examples we consider a second order system to illustrate what can be achieved for reduced versus full-order observers.

EXAMPLE 4.2—REDUCED-ORDER OBSERVER

Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1^3 + u \\ y &= x_1 \end{aligned} \tag{4.16}$$

with the state feedback law

$$u = -a_1 x_1 - a_2 x_2 - x_1^3 \tag{4.17}$$

which for any positive values of (a_1, a_2), will result in the exponentially stable linear system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a_1 x_1 - a_2 x_2 \\ y &= x_1 \end{aligned} \tag{4.18}$$

For simplicity, we choose $a_1 = 1,\ a_2 = 1$.

A reduced-order observer for the (feedback linearized) system of Eq. (4.18) can be written as

$$\begin{aligned} \hat{x}_2 &= z + Kx_1 \\ \dot{z} &= -x_1 + (-1 - K)\hat{x}_2 \end{aligned} \tag{4.19}$$

which will give an exponentially convergent estimate of x_2 for any $K > 0$. Using the estimated state $\hat{x}_2$ in the control law (4.17), the closed loop system will be

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x - \tilde{x}_2 \\ \dot{\tilde{x}}_2 &= (-1 - K)\tilde{x}_2 \end{aligned} \tag{4.20}$$

where $\tilde{x}_2 = x_2 - \hat{x}_2$ denotes the observation error. Due to exact cancellation of the nonlinear term, the closed loop system will be globally exponentially stable as we can use the separation principle from linear systems. However, there will be no robustness against measurement noise. □

EXAMPLE 4.3—FULL-ORDER OBSERVER

A full-order observer for the system of Eq. (4.18) may be written as

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} (y - \hat{y}). \tag{4.21}$$

The characteristic equation for the error dynamics will be

$$s^2 + (k_1 + 1)s + (k_2 + 1)$$

An appropriate choice of (k_1, k_2), can make the estimation errors converge to zero arbitrarily fast. The closed loop dynamics can be written as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1^3 - \hat{x}_1 - \hat{x}_2 - \hat{x}_1^3 \\ &= 3x_1^2\tilde{x}_1 - 3x_1\tilde{x}_1^2 + \tilde{x}_1^3 - (x_1 - \tilde{x}_1) - (x_2 - \tilde{x}_2) \\ \begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} &= \begin{bmatrix} -k_1 & 1 \\ -k_2 - 1 & -1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} \end{aligned} \tag{4.22}$$

The same type of quadratic destabilizing term as in Example 4.1 show up in Eq. (4.22), namely $3x_1^2\tilde{x}_1$. This term is an obstacle for global stability. The properties of the high-gain observer design, in combination with *bounded* state feedback control, will be discussed in the next section. □

A third alternative, to the examples above, is to implement the full order observer, but only use some of the estimated states in combination with the direct measurements. Similar ideas were used in [Berghuis *et al*, 1992] with a linear observer for the regulation problem of robot manipulators.

Bounded control and high-gain designs

As the discussion and the examples in the introduction show, peaking and finite escape time phenomena are major obstacles for the stabilization problem. When using a control law based on a stabilizing state feedback controller in combination with a stable observer, instability may occur due to initial errors and transients in the state estimates, causing the system to evolve outside the region of convergence. Due to the peaking phenomenon, observer design with solely faster convergence rates for the estimation errors is not a systematic solution to the problem.

In system theory limited control action has often been considered as a major obstacle to achievable performance. In particular during the last decade, another aspect of saturated control action has gained large interest, namely that of using bounded control as a means of stabilization. Among the pioneers in this area, Teel with his work on nested saturated control should be mentioned [Teel, 1992; Teel and Praly, 1995].

In [Esfandiari and Khalil, 1992], the combination of a high-gain observers and a globally bounded state-feedback controller was introduced to overcome the above mentioned stability problem. This technique has been extensively used in many contexts. In high-gain design, a common tool for stability analysis is the singular perturbation approach with generalizations of the Tichonov theorem [Kokotović *et al*, 1986; Esfandiari and Khalil, 1992].

Recently Atassi and Khalil presented a separation principle for stabilization based on high-gain observers and saturated control [Atassi and Khalil, 1997; Atassi and Khalil, 1999]. For (very) high observer gains, not only the region of attraction is shown to be recovered, but also the performance and the trajectories of the system under state feedback.
Their results relate to the following class of input-output linearizable systems

$$\begin{aligned}
\dot{x} &= Ax + B\big(f(x,z) + G(x,z)u\big)\\
\dot{z} &= \psi(x,z,u)\\
y &= Cx\\
\zeta &= q(x,z)
\end{aligned} \tag{4.23}$$

and the following assumptions are made:

Assumption 4.1 The functions f, ψ, and G are locally Lipschitz in their argument. $f(0,0) = \psi(0,0,0) = 0$ and G is nonsingular over the domain of interest.

Assumption 4.2 There exists a stabilizing state feedback controller $u = \phi(x, \zeta)$ which satisfies

(i) ϕ is locally Lipschitz over the domain of interest, and $\phi(0,0) = 0$.

(ii) ϕ is a globally bounded function of x.

(iii) The origin is an asymptotically stable equilibrium of the closed loop system with $u = \phi(x, \zeta)$.

Under Assumptions 4:1–2 it can be shown that the output feedback controller $u = \phi(\hat{x}, \zeta)$ recovers the performance of the state feedback controller $u = \phi(x, \zeta)$ locally, where the state estimates $\hat{x}$ are obtained using the high-gain observer

$$\dot{\hat{x}} = A\hat{x} + B[f(\hat{x}, \zeta) + G(\hat{x}, \zeta)\psi(\hat{x}, \zeta, u)] + H(y - C\hat{x}) \tag{4.24}$$

where

$$H = \begin{pmatrix} \alpha_1/\varepsilon \\ \alpha_2/\varepsilon^2 \\ \vdots \\ \alpha_n/\varepsilon^n \end{pmatrix}, \quad H_0 = \begin{pmatrix} -\alpha_1 & -\alpha_2 & \dots & -\alpha_{n-1} & -\alpha_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

with the positive constants $\{\alpha_i\}$ chosen such that H_0 be Hurwitz.

Thus, control based on the high-gain observer of Eq. (4.24) can be used to provide semi-global stabilization.

EXAMPLE 4.4—HIGH-GAIN OBSERVER [ATASSI AND KHALIL, 1997]
Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1^3 + u \\ y &= x_1 \end{aligned} \tag{4.25}$$

In the notation above we have

$$\begin{aligned} &A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad f(x) = x_1^3, \\ &C = [1 \quad 0], \quad G = 1, \qquad \psi = u \end{aligned} \tag{4.26}$$

A high-gain observer designed with respect to the stable characteristic polynomial $(s+k)^2,\ k>0$, will be

$$\begin{aligned}\dot{\hat{x}}_1 &= \hat{x}_2 + \frac{2k}{\varepsilon}(y-\hat{x}_1)\\ \dot{\hat{x}}_2 &= \hat{x}_1^3 + u + \frac{k^2}{\varepsilon^2}(y-\hat{x}_1)\\ \hat{y} &= \hat{x}_1\end{aligned} \tag{4.27}$$

with the corresponding observer error $\tilde{x} = x - \hat{x}$

$$\begin{aligned}\dot{\tilde{x}}_1 &= -\frac{2k}{\varepsilon}\tilde{x}_1 + \tilde{x}_2\\ \dot{\tilde{x}}_2 &= -\frac{k^2}{\varepsilon^2}\tilde{x}_1 + {x_1}^3 - \hat{x}_1^3\end{aligned} \tag{4.28}$$

Introducing $\beta = k/\varepsilon$ we see that we get the same observer and error equations as if we directly would have designed the model based observer with respect to the characteristic polynomial $(s+\beta)^2$, that is, with observer poles becoming infinitely fast as $\varepsilon \to 0^+$. □

In [Janković, 1997], a combination of high-gain observers and adaptive backstepping is used to achieve output feedback tracking. As high-gain observers are sensitive to measurement noise, Janković use a reduced order high-gain observer to estimate those states only, which enter the dynamics in a nonlinear fashion. The tracking controller is a hybrid of state-feedback control and observer-based feedback control. Similar to the method described in [Krstić *et al*, 1993], states not available to measurement are estimated by means of adaptive backstepping.

Lyapunov-based Methods and Passivity-based Output Feedback

Systems linear in the non-measurable states Nonlinear systems which are linear with respect to the non-measurable states are considered for the regulation problem in [Cebuhar *et al*, 1991; Praly, 1992; Pomet *et al*, 1993; Battilotti, 1996] and for tracking in [Freeman and Kokotović, 1996]. The methods assume knowledge of a stabilizing state feedback controller and in most of the cases a relating Lyapunov function. In [Freeman and Kokotović, 1996] single-input-single-output systems in *extended strict feedback form* are considered. This class of systems can be decomposed into three subsystems, where the dynamics for the unmeasured states η constitute one subsystem. The states of the ζ subsystem are the tracking variables subsystems of the following structure, see also Fig (4.3).

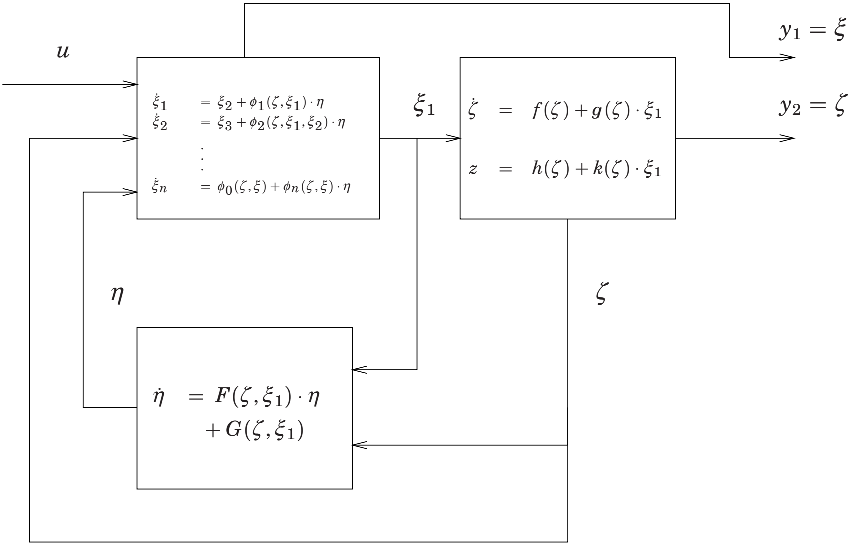

Figure 4.3 System in extended strict feedback form where η is the vector of unmeasured states [Freeman and Kokotović, 1996].

$$
\begin{aligned}
\Sigma_\eta :\quad & \dot{\eta} = F(\zeta,\xi)\eta + G(\zeta,\xi) \\
\Sigma_\zeta :\quad & \begin{cases} \dot{\zeta} = f(\zeta) + g(\zeta)\xi_1 \\ z = h(\zeta) + k(\zeta)\xi_1 \end{cases} \\
\Sigma_\xi :\quad & \begin{cases} \dot{\xi}_1 = \xi_2 + \phi_1(\zeta,\xi_1)\eta \\ \dot{\xi}_2 = \xi_3 + \phi_2(\zeta,\xi_1,\xi_2)\eta \\ \quad\vdots \\ \dot{\xi}_n = \phi_0(\zeta,\xi) + \phi_n(\zeta,\xi)\eta + \phi_u(\zeta,\xi)u \end{cases} \\
& y = \begin{bmatrix} \zeta \\ \xi \end{bmatrix}
\end{aligned}
\tag{4.29}
$$

Note that the unmeasured variables, η, do not enter the Σ_ζ-system and enters the Σ_ξ-system linearly. Another restriction is that the Σ_η-system is stable in the sense of Lyapunov for all values of ξ and ζ. The solution can be interpreted as a state-feedback controller in combination with a reduced order observer for the Σ_η-system.

For systems where the unmeasured states enter linearly, Battilotti has also proposed a solution to the global output feedback stabilization problem [Battilotti, 1996]. The method allows the solution to be divided into two separate subproblems considering stabilization via *full-state feedback* (SF) and the *output injection problem* (OI) respectively. This can be interpreted as a separation principle for this class of nonlinear systems, although the final controller consist of a nontrivial combination of the controllers and the Lyapunov functions for the subproblems.

Consider the system

$$\begin{aligned} \dot{x} &= f(x_m)x + g(x_m)u, \quad x = \begin{bmatrix} x_m \\ x_u \end{bmatrix} \in \mathbb{R}^n,\, u \in \mathbb{R}^m \\ y &= x_m, \quad y \in \mathbb{R}^p \end{aligned} \tag{4.30}$$

where x_m is the vector of the direct measurable states and x_u is the vector of unmeasured states not available for state feedback. The two subproblems are

SF Stabilize the system (4.30) in $(x_m, x_u) = (0, 0)$ with a state feedback law

$$u = u_{SF}(x_m, x_u), \qquad u_{SF}(0,0) = 0$$

OI Stabilize the system

$$\begin{aligned} \dot{x}_m &= f_1(x_m)x + g_1(x_m)u_{OI_1}(x_m) \\ \dot{x}_u &= f_2(x_m)x + g_2(x_m)u_{OI_2}(x_m) \\ y &= x_m \end{aligned} \tag{4.31}$$

with output injection

$$u_{OI}(x_m) = \begin{bmatrix} u_{OI_1}(x_m) \\ u_{OI_2}(x_m) \end{bmatrix}, \qquad u_{OI}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where f_i and g_i, (i=1,2), are the corresponding components of $f(x_m)$ and $g(x_m)$ in Eq. (4.30).

If one to each of the two subproblems can assign a smooth Lyapunov function of the form

$$\begin{aligned} V_{SF} &= \frac{1}{2}x_u^T P x_u + x_u^T \zeta(x_m) + \xi(x_m) > 0, \quad \forall (x_m, x_u) \neq 0 \\ V_{OI} &= \frac{1}{2}x_u^T \overline{P} x_u + x_u^T \overline{\zeta}(x_m) + \overline{\xi}(x_m) > 0, \quad \forall (x_m, x_u) \neq 0 \end{aligned} \tag{4.32}$$

where P and $\overline{P}$ are positive, symmetric matrices, then Theorem 3 in [Battilotti, 1996] guarantees the existence of a Lyapunov function for the output feedback problem and suggests a procedure to derive the corresponding control law.

EXAMPLE 4.5—DYNAMIC SHIP POSITIONING
Consider the ship dynamics from [Fossen and Grøvlen, 1998]

$$\begin{aligned}\dot{\eta} &= J(\eta)\nu \\ \dot{\nu} &= A_1\eta + A_2\nu + B\tau \\ y &= \eta\end{aligned} \tag{4.33}$$

where

$$\begin{aligned}J(\eta) &= \begin{bmatrix} cos(\psi) & -sin(\psi) & 0 \\ sin(\psi) & cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ J^{-1}(\eta) &= J^T(\eta), \qquad \det\{J(\eta)\} = 1, \,\forall\eta\end{aligned} \tag{4.34}$$

For the notation and the matrices appearing in Eq. (4.33) we refer to Paper E. The state feedback problem (SF) can be solved with exact linearization. Using

$$\ddot{\eta} = \dot{J}(\eta)\nu + J(\eta)\dot{\nu} \tag{4.35}$$

the dynamics in Eq. (4.33) can be rewritten as

$$J^T(\eta)\ddot{\eta} - J^T(\eta)\dot{J}(\eta)J^T(\eta)\dot{\eta} = A_1\eta + A_2J^T(\eta)\dot{\eta} + B\tau \tag{4.36}$$

which is globally stabilized by the control law

$$u_{SF} = B\tau_{SF} = -A_1\eta - A_2\nu - J^T(\eta)\dot{J}(\eta)\nu - J^T(\eta)\left(\Lambda_D\dot{\eta} + \Lambda_K\eta\right) \tag{4.37}$$

resulting in the asymptotically stable dynamics

$$\ddot{\eta} + \Lambda_D\dot{\eta} + \Lambda_K\eta = 0 \tag{4.38}$$

for some positive matrices Λ_D and Λ_K. A corresponding Lyapunov function is

$$V_{SF} = [\,\dot{\eta}^T \quad \eta^T\,]\underbrace{\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}}_{P}\begin{bmatrix} \dot{\eta} \\ \eta \end{bmatrix} \tag{4.39}$$

where P satisfies the linear matrix inequality

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}\begin{bmatrix} -\Lambda_D & -\Lambda_K \\ I & 0 \end{bmatrix} + \begin{bmatrix} -\Lambda_D^T & I \\ -\Lambda_K^T & 0 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} < 0 \tag{4.40}$$

In order to utilize the results in [Battilotti, 1996], however, we need to have a quadratic term in the unmeasured states, ν, with a positive *constant* weighting matrix. The freedom in Λ_D and Λ_K allows for the choice $P_{11} = pI$, where p is a positive constant. Via the state transformation

$$\begin{bmatrix} \dot{\eta} \\ \eta \end{bmatrix} = \begin{bmatrix} J(\eta) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \nu \\ \eta \end{bmatrix} \tag{4.41}$$

the Lyapunov function V_{SF} can be rewritten as

$$\begin{aligned} V_{SF} &= [\,\nu^T \quad \eta^T\,] \begin{bmatrix} J^T(\eta) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} pI & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \begin{bmatrix} J(\eta) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \nu \\ \eta \end{bmatrix} \\ &= [\,\nu^T \quad \eta^T\,] \begin{bmatrix} pI & J^T(\eta)P_{12} \\ P_{12}^T J(\eta) & P_{22} \end{bmatrix} \begin{bmatrix} \nu \\ \eta \end{bmatrix} \\ &= p\nu^T\nu + 2\nu^T J^T(\eta)P_{12}\eta + \eta^T P_{22}\eta \end{aligned} \tag{4.42}$$

The output injection problem is solved by the design of the globally convergent observer in Paper E where the weighting matrix $\overline{P}$ in the Lyapunov function $\overline{V}_{OI}$ is constant.

The solutions to the two subproblems satisfy the conditions for Theorem 3 and an output feedback controller can thus be designed following the guidelines in [Battilotti, 1996]. $\square$

Observer-based backstepping The main idea behind *observer-based backstepping* , or *observer backstepping* for short, is to apply the backstepping procedure to the error between the estimated states and the desired trajectory, instead of to the error between the true states and the desired trajectory [Kanellakopoulos *et al*, 1992]. First we show by an example that the observer-backstepping technique applied to a linear control object and combined with linear control system design gives rise to a non-standard composition of the control object, the observer, and the controller. The resultant system is characterized by a full-order observer and a reduced-order control system design which in its complexity does not go beyond the relative degree of the control object.

EXAMPLE 4.6—[ROBERTSSON AND JOHANSSON, 1998C]
Consider a third order linear system with relative degree two, where the zero lies strictly in the left half plane. The state-space realization in *observer canonical form* is

$$\begin{aligned} \dot{x} &= Ax + Bu = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ b_2 \\ b_3 \end{bmatrix} u \\ y &= Cx = [\,1 \quad 0 \quad 0\,]\, x \end{aligned} \tag{4.43}$$

For reconstruction of the states we use a full order observer

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + Bu + K(y - \hat{y}) \\ \hat{y} &= C\hat{x}\end{aligned} \tag{4.44}$$

To the purpose of tracking error analysis, introduce

$$z_r = \begin{bmatrix} y_r \\ \dot{y}_r \end{bmatrix}, \quad \hat{z} = \begin{bmatrix} \widehat{y} \\ \dot{\widehat{y}} \end{bmatrix}, \quad z = \hat{z} - z_r, \quad \tilde{y} = y - \hat{y} \tag{4.45}$$

where $y_r(t)$ is a given, twice differentiable reference trajectory. By the relative degree properties and standard model matching arguments, it can be justified that

$$\begin{aligned}\widehat{y} &= C\widehat{x} \\ \dot{\widehat{y}} &= C\dot{\widehat{x}} = CA\widehat{x} + \underbrace{CB}_{0}u + CKC\tilde{x} \\ \ddot{\widehat{y}} &= CA\dot{\widehat{x}} + CKC\dot{\tilde{x}} \\ &= CA^2\widehat{x} + \underbrace{CAB}_{\neq 0}u + CAK\tilde{y} + CKC(A - KC)\tilde{x}\end{aligned} \tag{4.46}$$

The tracking error dynamics will be

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= CA^2\widehat{x} + u + CAK\tilde{y} + CKC(A - KC)\tilde{x} - \ddot{y}_r\end{aligned} \tag{4.47}$$

Applying observer backstepping, we first introduce the error coordinates

$$\zeta = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \begin{bmatrix} z_1 \\ \hat{z}_2 - \alpha_2 \end{bmatrix} \tag{4.48}$$

where α_2 will be defined below.

Step 1. Let

$$\begin{aligned}V_1 &= \frac{1}{2}\zeta_1^2 \\ \dot{V}_1 &= \zeta_1\dot{\zeta}_1 = \zeta_1(\hat{z}_2 - \dot{y}_r) \\ &= \zeta_1(\zeta_2 + \alpha_2(\zeta_1, z_r) - \dot{y}_r) \\ &= -c_1{\zeta_1}^2 + \zeta_2\zeta_1\end{aligned} \tag{4.49}$$

where

$$\begin{aligned} \alpha_2 &= -c_1\zeta_1 + \dot{y}_r \\ \dot{\zeta}_1 &= -c_1\zeta_1 + \zeta_2 \end{aligned} \tag{4.50}$$

Step 2. Let

$$\begin{aligned} V_2 &= V_1 + \frac{1}{2}\zeta_2^2 \\ \dot{V}_2 &= -c_1{\zeta_1}^2 + \zeta_2\left[\zeta_1 + \dot{\zeta}_2\right] \\ &= -c_1\zeta_1^2 - c_2\zeta_2^2 + \zeta_2 CKC(A-KC)\tilde{x} \end{aligned} \tag{4.51}$$

where we choose

$$CABu = -\zeta_1 - c_2\zeta_2 - CA^2\widehat{x} - CAK\tilde{y} + \ddot{y}_r + c_1(-c_1\zeta_1 + \zeta_2) \tag{4.52}$$

Note that there is a remaining cross-term in the derivative of V_2. For any linear observer design which provides asymptotically converging state estimates, there exist positive definite, symmetric matrices P_o and Q_o satisfying the Lyapunov equation

$$(A-KC)^T P_o + P_o(A-KC) = -Q_o \tag{4.53}$$

The estimation error will be exponentially stable with the Lyapunov function properties

$$\begin{aligned} V_o(\tilde{x}) &= \tilde{x}^T P_o \tilde{x} > 0, \quad \|\tilde{x}\| \neq 0 \\ \frac{d}{dt}V_o &= -\tilde{x}^T Q_o \tilde{x} < 0, \quad \|\tilde{x}\| \neq 0 \end{aligned} \tag{4.54}$$

Let V be a Lyapunov function candidate for the error system $\{\tilde{x}, \zeta\}$:

$$\begin{aligned} V &= V_2 + \beta\tilde{x}^T P_o \tilde{x}, \quad \beta > 0 \\ \dot{V} &= -c_1\zeta_1^2 - c_2\zeta_2^2 - \beta\tilde{x}^T Q_o \tilde{x} + \zeta_2 CKC(A-KC)\tilde{x} \\ &= \begin{bmatrix} \zeta \\ \tilde{x} \end{bmatrix}^T \begin{bmatrix} \begin{pmatrix} -c_1 & 0 \\ 0 & -c_2 \end{pmatrix} & \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} CKC(A-KC) \\ \left(\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} CKC(A-KC)\right)^T & -\beta Q_o \end{bmatrix} \begin{bmatrix} \zeta \\ \tilde{x} \end{bmatrix} \end{aligned} \tag{4.55}$$

By the Schur complement of the weighting matrix in Eq. (4.55), we see that $\dot{V}$ can be negative definite for large enough β. As this parameter can be chosen independently of the design of the controller and the observer, we have the regular separation principle for linear systems. The closed-loop error dynamics then fulfill

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A - KC & 0 \\ A_{\tilde{x}z} & A_z \end{bmatrix} \begin{bmatrix} \tilde{x} \\ z \end{bmatrix} \tag{4.56}$$

Stability properties are determined by the subsystem stability properties associated with the matrices $A - KC$ and A_z without any critical influence with respect to stability from the matrix $A_{\tilde{x}z}$ describing the interaction. In addition, the stable eigenvalues of A_z depend on the choice of the positive parameters c_1 and c_2. So far we have only considered stable zero-dynamics for the reason of analysis described above. The closed loop error dynamics for the full system will be

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{z} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A - KC & 0 & 0 \\ A_{\tilde{x}z} & A_z & 0 \\ A_{\tilde{x}\eta} & A_{z\eta} & A_\eta \end{bmatrix} \begin{array}{c} \begin{bmatrix} \tilde{x} \\ z \\ \eta \end{bmatrix} \end{array} \begin{array}{l} \text{observer error} \\ \text{tracking error} \\ \text{zero-dynamics} \end{array} \tag{4.57}$$

and the stability of the full system is determined by the matrices $A - KC$, A_z, and A_η, while the cross-terms affect the transients and the tracking property.

The resultant system structure is interesting in that it provides the converse to the case of state feedback control supported by reduced-order observers. Whereas such feedback control object is based on a full-order representation of the control object and a reduced-order observer, we here find a full-order observer and a reduced-order model for the control object. □

The observer backstepping approach above is applicable to minimum-phase systems, which for the linear case implies that the zeros of the transfer function lie strictly in the left half-plane. The following section will consider nonlinear system which has linear but unstable zero-dynamics. Typical examples where this may be relevant is in the control of flexible structures such as weak robot arms or systems with weak couplings between rotating masses [Dewey and Devasia, 1996].

The following example will point out some problems when naively applying the backstepping procedure to a linear system with a zero in the right half-plane.

EXAMPLE 4.7—PAPER D [ROBERTSSON AND JOHANSSON, 1999]
Consider the linear system

$$Y(s) = \frac{s-1}{s^4} U(s) \tag{4.58}$$

which has a zero in the right half plane. The state-space realization in output-feedback form is for linear systems also known as *the observer canonical form* [Kailath, 1980]:

$$\begin{aligned} \dot{x} &= A_1 x + B_1 u = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} u \\ y &= C_1 x = [1 \quad 0 \quad 0 \quad 0]\, x \end{aligned} \tag{4.59}$$

The system is in strict feedback form, and applying the backstepping design, we will reach the control input u after three steps. Any stabilizing linear controller for the first three states will have the form

$$u_3 = -l_1 x_1 - l_2 x_2 - l_3 x_3, \quad l_i > 0,\ i = 1 \dots 3$$

However, the state x_4, which represents the zero-dynamics, will be unstable and we can not neglect it in the design as we could have done if the zero-dynamics were stable. Even worse, it is not even possible to re-use our “stabilizing” control law u_3 and extend it with additional feedback from the state in the zero-dynamics to stabilize the whole system, as shown below.

Using

$$u(x) = u_3(x_1, x_2, x_3) - l_4 x_4$$

the closed loop system has the characteristic polynomial

$$\lambda(s) = s^4 + (-l_4 + l_3)\, s^3 + (l_2 - l_3)\, s^2 + (-l_2 + l_1)\, s - l_1$$

which is clearly unstable. □

Thus, Example 4.7 shows that the straight-forward, and in this case naive, use of the backstepping method will fail to stabilize such a nonminimum-phase system. This will of course also be the case for observer-based backstepping, under the assumption that only the first state x_1 is measurable.

In Paper D, we discuss the topic of observer-based backstepping for a class of nonlinear systems with linear, unstable zero dynamics. Extensions of the observer backstepping method are made and a design algorithm for this class of nonminimum-phase systems is presented.

EXAMPLE 4.8—FLEXIBLE ROBOT ARM
This example aims at illustrating the notation and the transformations used in Paper D.

Consider the model for a flexible one-link robot arm [Marino and Tomei, 1995]. The dynamics are given by

$$\begin{aligned}
\dot{\chi}_1 &= \chi_2 \\
\dot{\chi}_2 &= -M_1 \sin(\chi_1) - K_1(\chi_1 - \chi_3) \\
\dot{\chi}_3 &= \chi_4 \\
\dot{\chi}_4 &= -B_1\chi_4 + K_2(\chi_1 - \chi_3) + \tau \\
y &= h(\chi) = \chi_3
\end{aligned} \tag{4.60}$$

where χ_1 is the angle of the arm, χ_2 is the angular velocity of the arm, χ_3 is the angle on the motor side, and χ_4 is the angular velocity on the motor side. The angle measurement is *co-located*, i. e., measuring on the motor side. The input signal τ is the driving torque from the motor and it is easy to see that it enters the equation for the second derivative of the output, which implies that the system has relative degree two. The constants M_1, B_1, K_1, and K_2 are all positive.

For observer-design it is natural to consider the transformation used in [Sanchis and Nijmeijer, 1998]:

$$\begin{gathered}
x = T\chi, \quad T = \begin{bmatrix} h(\chi) \\ L_f h(\chi) \\ L_f^2 h(\chi) \\ L_f^3 h(\chi) \end{bmatrix} \\
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ K_2 & 0 & -K_2 & -B_1 \\ -K_2B_1 & K_2 & K_2B_1 & B_1^2 - K_2 \end{bmatrix} \begin{bmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{bmatrix}
\end{gathered} \tag{4.61}$$

The dynamics expressed in the x-coordinates are

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -B_1K_1 & -(K_1+K_2) & -B_1 \end{bmatrix}}_{A_x} x + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \psi_4(x) \end{bmatrix}}_{\psi(x)} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ -B_1 \\ B_1^2 - K_2 \end{bmatrix}}_{B_x} \tau \tag{4.62}$$

$$y = [1 \quad 0 \quad 0 \quad 0]\, x$$

$$\psi_4(x) \triangleq -M_1K_2 \sin\left(\frac{x_3 + B_1x_2 + K_2x_1}{K_2}\right)$$

Both the χ-system of Eq. (4.60) and the x-system of Eq. (4.62) are in strict-feedback form. The structure of Eq. (4.62) is similar to the output-feedback form referred to in Paper D, except that the nonlinearity ψ_4 also depends on unmeasured states. This obstacle will be dealt with in the observer design below. It is also evident from the signs in the B_x-vector that the linearization of Eq. (4.62) will not have asymptotically stable zero-dynamics.

Observer design

In [Sanchis and Nijmeijer, 1998] a sliding-mode observer for the the system in Eq. (4.62) was derived. Here we propose an observer along the ideas presented in [Arcak and Kokotović, 1999]. System (4.62) can be written as

$$\begin{aligned} \dot{x} &= A_x x + B_x u + G\psi(Hx) \\ y &= C_x x \\ G &\triangleq [0 \quad 0 \quad 0 \quad 1]^T, \qquad H \triangleq \frac{1}{K_2}[K_2 \quad B_1 \quad 1 \quad 0] \end{aligned} \tag{4.63}$$

Following the same outline as for the pendulum observer in Example 3.7, we propose the observer

$$\begin{aligned} \dot{\hat{x}} &= A_x\hat{x} + B_x u + G\psi(H\hat{x} + L_2(y - C_x\hat{x})) + L_1(y - C_x\hat{x}) \\ \hat{y} &= C_x\hat{x} \end{aligned} \tag{4.64}$$

where $\tilde{x} = x - \hat{x}$ denotes the observer error. By rewriting the difference of the nonlinearities as

$$\begin{aligned} \sin(Hx) - \sin(H\hat{x} + L_2C_x\tilde{x}) &= 2\sin\left(\frac{(H - L_2C_x)\tilde{x}}{2}\right) \cdot \gamma(t) \\ \gamma \triangleq \cos\left(\frac{H(x+\hat{x}) + L_2C_x\tilde{x}}{2}\right)&, \quad |\gamma| \le 1 \end{aligned} \tag{4.65}$$

the error dynamics can be decomposed into a feedback connection of a linear system Σ and a sector-bounded time-varying nonlinearity, (Fig. 4.4):

$$\Sigma: \quad \begin{cases} \dot{\tilde{x}} &= (A_x - L_1 C_x)\tilde{x} + Gv \\ y_x &\triangleq \dfrac{(H - L_2 C_x)\tilde{x}}{2} \end{cases} \tag{4.66}$$
$$v \triangleq -2M_1 K_2\, \gamma(t) \cdot \sin\left(y_x\right)$$

Furthermore, we can use the (restrictive) sector bound

$$-(H - L_2 C_x)\tilde{x} \leq 2\sin\left(\frac{(H - L_2 C_x)\tilde{x}}{2}\right) \leq (H - L_2 C_x)\tilde{x}$$

to specify a sector condition, see Fig. 4.4. For the parameters used in

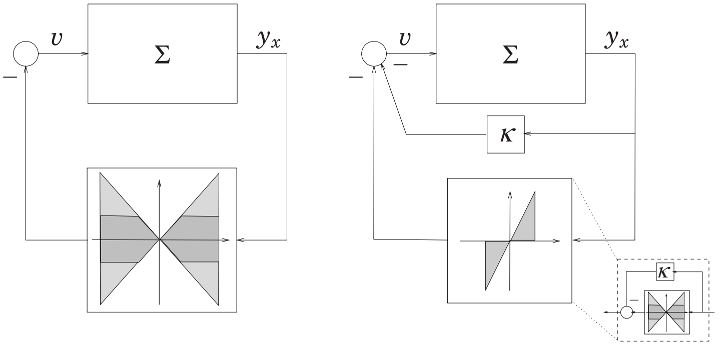

Figure 4.4 *Left:* Partitioning of the observer error dynamics into a linear system and a time-varying sector bounded nonlinearity. *Right:* Loop transfer of the system.

[Sanchis and Nijmeijer, 1998] observer gains L_1 and L_2 can be found which asymptotically stabilizes the error dynamics in Eq.(4.66).

Control design

Given the state estimates from the observer (4.64) and the transformation in Eq. (4.61) relating the x and the χ-coordinates, observer-based backstepping can be performed for the system in Eq. (4.60) along the algorithm proposed in Paper D. □

5

Concluding Remarks

In this thesis, the problem of observer design and observer-based control for nonlinear systems is addressed. The deterministic continuous-time systems have been in focus. The observer-based control strategies presented include separation results where the combination of independently designed observers and state-feedback controllers assures stability. In addition, the new results provide a generalization to the observer-backstepping method where the controller is designed with respect to estimated states, taking into account the effects of the estimation errors.

Results

The results in the thesis can be summarized as follows:

- New time-varying state feedback controllers and observers for the tracking problem of nonholonomic systems in chained form are presented. Furthermore, global stability results for the output tracking problem are shown, using the *certainty-equivalence* combination of the state controllers and the observers. A solution to the control problem under input saturation is also presented;
- Relaxation of the minimality conditions in the Yakubovich-Kalman-Popov lemma, with relevance to observer-based feedback control;
- The design method known as observer-based backstepping is extended to cover a class of nonlinear systems in output-feedback form, accommodating also linear *unstable* zero-dynamics.
  An observer-based control algorithm is provided;

- For the purpose of output-feedback control of Euler-Lagrange systems, a Lyapunov-based observer design is presented. In application to ship dynamics, a globally exponentially stable observer design extends previous results with application to ships with unstable sway-yaw dynamics. The similarity between the equations of motion for the ship model and more general mechanical manipulators allows for an extension to semi-global exponential stability results for the velocity estimation in rigid robot manipulators.

Open Issues and Future Work

The state-estimation problem is relevant also in many disciplines other than nonlinear control theory in the narrow sense. Combinations of different sensors for measuring the same or related quantities, e. g., the use of redundant sensor arrays, result in correlated measurements and raise the need for systematic methods to handle these signals in an optimal way. For linear systems the Kalman filter is a solution to the sensor fusion problem, but for nonlinear systems only partial answers are given. Furthermore, with increased process complexity and the use of safe-critical systems, the need for reliable diagnosis and supervision is obvious. Observer-based fault-detection and isolation are instrumental in such a context. Opportunities of state-estimation application are numerous.

Output feedback control is a challenging area and observer-based feedback control is one means to solve this problem. In the area of stochastic control and estimation for nonlinear systems, comparatively few results are reported are in the literature. Systematic observer design for nonlinear systems is still an open issue and there is also a lack of general methods for using observers in output feedback control schemes. For some important classes of systems such as Euler-Lagrange systems—e. g., robot manipulators and rotating machines—many important results regarding regulation and tracking have been presented during the last decades. Still, it is obvious that much remains to be done.

Bibliography

Ailon, A. and R. Ortega (1993): "An observer-based set-point controller for robot manipulators with flexible joints." *System & Control Letters*, **21**, pp. 329–335.

Aizerman, M. and F. Gantmacher (1964): *Absoulte Stability of Regulator Systems*. Holden-Day, San Francisco.

Alcorta Garcia, E. and P. M. Frank (1997): "Deterministic nonlinear observer-based approaches to fault analysis: A survey." *Control Engineering Practice*, **5:5**, pp. 663–670.

Angeli, D. and E. Sontag (1999): "Forward completeness, unboundedness observability, and their Lyapunov characterizations." `http://www.math.rutgers.edu/~sontag/FTP_DIR/uo.ps.gz`.

Arcak, M. and P. Kokotović (1999): "Nonlinear observers - a circle criterion design." In *Proceedings of the 38th IEEE Conf Decision and Control (CDC'99)*, pp. 4872–4877. Phoenix, Arizona.

Arimoto, S. and F. Miyazaki (1986): "Stability and robustness of PD feedback control with gravity compensation for robot manipulators." In Paul and Yacef Toumi, Eds., *Robotics: Theory and Applications*, vol. 3, pp. 67–72.

Artstein, Z. (1983): "Stabilization with relaxed controls." *Nonlinear Analysis*, **TMA-7**, pp. 1163–1173.

Astolfi, A. (1995): "Discontinuous output feedback control of nonholonomic chained systems." In *Proceedings of the 3rd European Control Conference*, pp. 2626–2629. Rome, Italy.

Åström, K. J. (1970): *Introduction to Stochastic Control Theory*. Academic Press, New York.

Åström, K. J. (1997): "Limitations on control system performance." In *Proceedings of the European Control Conference (ECC'97)*, vol. 1. Brussels, Belgium. TU-E-E4.

Åström, K. J. and B. Wittenmark (1997): *Computer-Controlled Systems*, third edition. Prentice Hall.

Atassi, A. and H. Khalil (1997): "A separation principle for the stabilization of a class of nonlinear systems." In *Proceedings of the European Control Conf.-97*, vol. 2. Brussels, Belgium. WE-A-A-4.

Atassi, A. and H. Khalil (1999): "A separation principle for the stabilization of a class of nonlinear systems." *IEEE Transactions on Automatic Control*, **44:9**, pp. 1672–1687.

Battilotti, S. (1996): "Global output regulation and disturbance attenuation with global stability via measurement feedback for a class of nonlinear systems." *IEEE Transactions on Automatic Control*, **41:3**, pp. 315–327.

Battilotti, S., L. Lanari, and R. Ortega (1997): "On the role of passivity and output injection in the output feedback stabilization problem: Application to robot control." *European Journal of Control*, **3:3**, pp. 92–103.

Belanger, P. (1992): "Estimation of angular velocity and acceleration from shaft encoder measurements." In *Proceedings of IEEE Conf. on Robotics and Automation*, vol. 1, pp. 585–592. Nice, France.

Berghuis, H. (1993): *Model-based Robot Control: From Theory to Practice*. PhD thesis, University of Twente, The Netherlands.

Berghuis, H. and H. Nijmeijer (1993a): "Global regulation of robots using only position measurements." *System & Control Letters*, **21**, pp. 289–293.

Berghuis, H. and H. Nijmeijer (1993b): "A passivity approach to controller-observer design for robots." *IEEE Transactions on Robotics and Automation*, **9:6**, pp. 740–754.

Berghuis, H., H. Nijmeijer, and P. Löhnberg (1992): "Observer design in the tracking control problem of robots." In *Proc IFAC Sympos. NOLCOS Conference on Decision and Control*, pp. 588–593. Bordeaux, France.

Besançon, G. (1996): *Contributions à l'Etude et à Observation des Systèmes Non Linéaires avec Recours au Calcul Formel*. PhD thesis, Institute National Polytechnique de Grenoble, Grenoble, France.

Besançon, G. (1999): "A viewpoint on observability and observer design for nonlinear systems." In Nijmeijer and Fossen, Eds., *New Directions in Nonlinear Observer Design*, Lecture notes in Control and Information Sciences, pp. 3–22. Springer-Verlag, London.

Besançon, G. and G. Bornard (1997): "On characterizing classes of observer forms for nonlinear systems." In *Proceedings of the European Control Conference (ECC'97)*. Brussels, Belgium. TU-E-C2.

Besançon, G. and H. Hammouri (1998): "On observer design for interconnected systems." *Journal of Mathematical Systems, Estmation & Control*, **8:3**.

Bestle, D. and M. Zeitz (1983): "Canonical form observer design for nonlinear time-variable systems." *International Journal of Control*, **38**, pp. 419–431.

Birk, J. and M. Zeitz (1988): "Extended Luenberger observer for nonlinear multivariable systems." *International Journal of Control*, **47:6**, pp. 1823–1836.

Blomdell, A. (1997): "A real time control language for the Pålsjö environment." Master thesis ISRN LUTFD2/TFRT--5578--SE. Department of Automatic Control, Lund Institute of Technology, Lund, Sweden.

Bornard, G. and H. Hammouri (1991): "A high gain observer for a class of uniformly observable systems." In *Proc 30th IEEE Conference on Decision and Control (CDC'91)*, pp. 1494–1496.

Bounit, H. and H. Hammouri (1997): "Bounded feedback stabilization and global separation principle of distributed parameter systems." *IEEE Transactions on Automatic Control*, **42:3**, pp. 414–419.

Boyd, S., L. El Ghaoui, E. Feron, and V. Balakrishnan (1994): *Linear Matrix Inequalities in System and Control Theory*, vol. 15. SIAM, Philadelphia, PA.

Brockett, R. (1983): "Asymptotic stability and feedback stabilization." In Brockett *et al*, Eds., *Differential Geometric Control Theory*, pp. 181–191. Birkhauser, Boston, MA.

Busawon, K., A. El Assoudi, and H. Hammouri (1993): "Dynamical output feedback stabilization of a class of nonlinear systems." In *Proceedings of the 32nd IEEE Conference on Decision and Control*, pp. 1966–1971. IEEE, San Antonio, Texas.

Byrnes, C. and A. Isidori (1991): "Asymptotic stabilization of minimum phase nonlinear systems." *IEEE Transactions on Automatic Control*, **36**, pp. 1122–1137.

Byrnes, C., A. Isidori, and J. Willems (1991): "Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems." *IEEE Transactions on Automatic Control*, **36:11**, pp. 1228–1240.

Canudas de Wit, C., H. Olsson, K. J. Åström, and P. Lischinsky (1993): "Dynamic friction models and control design." In *Proceedings of the 1993 American Control Conference*, pp. 1920–1926. San Francisco, California.

Canudas de Wit, C., H. Olsson, K. J. Åström, and P. Lischinsky (1995): "A new model for control of systems with friction." *IEEE Transactions on Automatic Control*, **40:3**, pp. 419–425.

Cebuhar, W., R. Hirschhorn, and J. Pomet (1991): "Some results on dynamic output feedback regulation of nonlinear systems." In *Proceedings of the 30th Conference on Decision and Control*, pp. 1811–1812. Brighton, England.

Craig, J. J. (1988): *Adaptive Control of Mechanical Manipulators*. Addison-Wesley Publishing Company, Inc.

Craig, J. J. (1989): *Introduction to Robotics – Mechanics and Control*. Addison-Wesley Publishing Company, Inc.

Cruz, C. and H. Nijmeijer (1999): "Synchronization through extended Kalman filtering." In Nijmeijer and Fossen, Eds., *New Directions in Nonlinear Observer Design*, Lecture notes in Control and Information Sciences, pp. 469–490. Springer-Verlag, London.

De Luca, A. and G. Oriolo (1994): "Nonholonomy in redundant robots under kinematic inversion." In *Proceedings of the 4th IFAC Symposium on Robot Control (SYROCO'94)*, vol. 1, pp. 179–184. Capri, Italy.

de Querioz, M. S., D. M. Dawson, and M. Agarwal (1999): "Adaptive control of robot manipulators with controller/update law modularity." *Automatica*, **35:8**, pp. 1379–1390.

Dewey, J. S. and S. Devasia (1996): "Experimental and theoretical results in output-trajectory redesign for flexible structures." In *Proceedings of the 35th IEEE Conference on Decision and Control(CDC'96)*, vol. 4, pp. 4210–4215.

Doyle, J. (1978): "Guaranteed margins for LQG regulators." *IEEE Transactions on Automatic Control*, **23:4**, pp. 756–757.

Doyle, J. and G. Stein (1979): "Robustness with observers." *IEEE Transactions on Automatic Control*, **24:4**, pp. 607–611.

Eker, J. (1997): *A Framework for Dynamically Configurable Embedded Controllers*. Lic Tech thesis ISRN LUTFD2/TFRT--3218--SE, Department of Automatic Control, Lund Institute of Technology, Lund, Sweden.

Eker, J. and K. J. Åström (1996): "A nonlinear observer for the inverted pendulum." In *Proc. IEEE Conference on Control Applications*, pp. 332–337. Dearborn, Michigan.

Esfandiari, F. and H. Khalil (1992): "Output feedback stabilization of fully linearizable systems." *International Journal of Control*, **56**, pp. 1007–1037.

Etchechoury, M., J. Solsona, and C. Muravchik (1996): "On the stability of nonlinear plants that include an observer for their feedback linearization." *International Journal of Systems Science*, **27:12**, pp. 1461–1466.

Fortmann, T. and D. Williamson (1972): "Design of low-order observers for linear feedback control laws." *IEEE Transactions on Automatic Control*, **17:3**, pp. 301–308.

Fossen, T. I. and Å. Grøvlen (1998): "Nonlinear output feedback control of dynamically positioned ships using vectorial observer backstepping." *IEEE Transactions on Control System and Technology*, **6**, pp. 121–128.

Fossen, T. I. and J. Strand (1999): "Passive nonlinear observer design for ships using Lyapunov methods: Experimental results with a supply vessel." *Automatica*, **35:1**, pp. 3–16.

Francis, B. A. and K. Glover (1978): "Bounded peaking in the optimal linear regulator with cheap control." *IEEE Transactions on Automatic Control*, **23**, pp. 608–617.

Freeman, R. (1995): "Global internal stability does not imply global external stabilizability for small sensor disturbances." *IEEE Transactions on Automatic Control*, **40**, Dec, pp. 2119–2122.

Freeman, R. and P. Kokotović (1993): "Global robustness of nonlinear systems to state measurement disturbances." In *Proceedings of the 32nd IEEE Conference on Decision and Control*, vol. 2, pp. 1507–1512. San Antonio, Texas.

Freeman, R. and P. Kokotović (1996): "Tracking controllers for systems linear in the unmeasured states." *Automatica*, **32:5**, pp. 735–746.

Freudenberg, J. and D. Looze (1985): "Right half plane poles and zeros and design tradeoffs in feedback systems." *IEEE Transactions on Automatic Control*, **30**, pp. 555–565.

Funahashi, Y. (1979): "Stable state estimator for bilinear systems." *International Journal of Control*, **29:2**, pp. 181–188.

Gäfvert, M. (1998): "Derivation of Furuta pendulum dynamics." Technical Report. Department of Automatic Control, Lund Institute of Technology, Lund, Sweden.

Gäfvert, M., J. Svensson, and K. Åström (1999): "Friction and friction compensation in the Furuta pendulum." In *Proceedings of the 5th European Control Conference*. Karlsruhe, Germany. CM-4:3.

Gauthier, J. and G. Bornard (1981): "Observability for any $u(t)$ for a class of nonlinear systems." *IEEE Transactions on Automatic Control*, **26**, pp. 922–926.

Gauthier, J., H. Hammouri, and S. Othman (1992): "A simple observer for nonlinear systems applications to bioreactors." *IEEE Transactions on Automatic Control*, **37:6**, pp. 875–880.

Gauthier, J. and I. Kupka (1992): "A separation principle for bilinear systems with dissipative drift." *IEEE Transactions on Automatic Control*, **37:12**, pp. 1970–1974.

Gauthier, J. and I. Kupka (1994): "Observability and observers for nonlinear systems." *SIAM Journal on Control and Optimization*, **32**, pp. 974–994.

Gelb, A., Ed. (1974): *Applied Optimal Control*. The M.I.T. Press, Cambridge, Massachusetts.

Ghulchak, A. and A. Shirjaev (1995): "Comments on dynamic output stabilization of a class of multi-input discrete-time nonlinear systems." *Manuscript*.

Glad, T. (1987a): "Robustness of nonlinear state feedback—a survey." *Automatica*, **23:4**, pp. 425–435.

Glad, T. (1987b): "Stability of nonlinear systems with observers." Technical Report LiTH-ISY-0888. Division of Automatic Control, Linköping University, Sweden.

Goldstein, H. (1980): *Classical Mechanics*, 2nd edition. Addison-Wesley.

Goodwin, G. and M. Seron (1997): "Fundamental design tradeoffs in filtering, prediction, and control." *IEEE Transactions on Automatic Control*, **42:9**, pp. 1240–1251.

Gronard, F., R. Sepulchre, and G. Bastin (1999): "Global stabilization of feedforward systems with exponentially unstable Jacobian linearization." *Systems & Control Letters*, **37**, pp. 107–115.

Hahn, W. (1967): *Stability of Motion*. Springer-Verlag.

Hammouri, H., M. Kinnaert, and E. H. El Yaagoubi (1999): "Observer-based approach to fault detection and isolation for nonlinear systems." *IEEE Transactions on Automatic Control*, **44:10**, pp. 1879–1884.

Hanselman, D. (1990): "Signal processing techniques for improved resolver-to-digital conversion accuracy." In *16th Annual Conference of IEEE Industrial Electronics Society (IECON'90)*, vol. 1, pp. 6–10.

Hara, S. and K. Furuta (1976): "Minimal order state observers for bilinear systems." *International Journal of Control*, **24:5**, pp. 705–718.

Hauser, J., S. Sastry, and P. Kokotović (1992a): "Nonlinear control via approximate input-output linearization." *IEEE Transactions on Automatic Control*, **37**, pp. 392–398.

Hauser, J., S. Sastry, and G. Meyer (1992b): "Nonlinear control design for slightly nonminimum-phase systems: Applications to V/STOL aircraft." *Automatica*, **28:4**, pp. 665–679.

Hermann, R. and A. Krener (1977): "Nonlinear controllability and observability." *IEEE Transactions on Automatic Control*, **22:5**, pp. 728–740.

Hill, D. and P. Moylan (1976): "Stability results for nonlinear dissipative systems." *IEEE Transactions on Automatic Control*, **21**, pp. 708–711.

Isidori, A. (1995): *Nonlinear Control Systems*, 3rd edition. Springer-Verlag, Berlin.

Isidori, A. (1998): "Semiglobal practical stabilization of uncertain nonminimum-phase nonlinear systems via output feedback." In *Proceedings of the Fourth IFAC Symposium on Nonlinear Control Systems Design (NOLCOS'98)*, pp. 643–648. Enschede, The Netherlands.

Isidori, A. (1999): *Nonlinear Control Systems II*. Springer-Verlag, London.

Jacobson, D. (1977): *Extensions of Linear-Quadratic Control, Optimization and Matrix Theory*. Academic Press, New York.

Janković, M. (1995): "Observer based control for elastic joints." *IEEE Transactions on Automatic Control*, **11:4**, pp. 618–623.

Janković, M. (1997): "Adaptive nonlinear output feedback tracking with a partial high-gain observer and backstepping." *IEEE Transactions on Automatic Control*, **42:1**, pp. 106–113.

Jiang, Z.-P., E. Lefeber, and H. Nijmeijer (1998): "Stabilization and tracking of a nonholonomic mobile robot with saturating actuators." In *Proceedings of Controlo'98, Third Portuguese Conference on Automatic Control*, vol. 1, pp. 315–320. Coimbra, Portugal.

Jiang, Z.-P. and H. Nijmeijer (1999): "Observer-controller design for nonholonomic systems." In Nijmeijer and Fossen, Eds., *New Trends in Nonlinear Observer Design*, Lecture Notes in Control and Information Sciences, pp. 207–228. Springer-Verlag, London.

Johansson, R. (1990): "Quadratic optimization of motion coordination and control." *IEEE Trans. Automatic Control*, **35:11**, pp. 1197– 1208.

Johansson, R. and A. Robertsson (1998): "Stability analysis of adaptive output feedback systems." In *Proceedings of the 37th IEEE Conference on Decision and Control*, pp. 4008–4009.

Johansson, R. and A. Robertsson (1999): "Extension of the Yakubovich-Kalman-Popov lemma for stability analysis of dynamic output feedback systems." In *Proceedings of IFAC'99*, vol. F, pp. 393–398. Beijing, China.

Johansson, R., A. Robertsson, and R. Lozano-Leal (1999): "Stability analysis of adaptive output feedback control." In *Proceedings of the 38th IEEE Conference on Decision and Control (CDC'99)*. Phoenix, Arizona.

Jurdjevič, V. and J. Quinn (1978): "Controllability and stability." *Journal-of-Differential-Equations*, **28:3**, pp. 381–389.

Kailath, T. (1980): *Linear Systems*. Prentice-Hall, Inc, Englewood Cliffs, New Jersey.

Kalman, R. and R. Bucy (1961): "New results in linear filtering and prediction theory." *Trans ASME, J. Basic Eng., ser. D*, **83**, pp. 95–107.

Kalman, R. E., P. L. Falb, and M. A. Falb (1969): *Topics in Mathematical System Theory*. McGraw-Hill, New York.

Kanellakopoulos, I., P. Kokotović, and S. Morse (1992): "A toolkit for nonlinear feedback control." *System & Control Letters*, **18**, June, pp. 83–92.

Kang, W. and A. Krener (1998): "Nonlinear observer design, a backstepping approach." *Manuscript*.

Kelly, R. (1993): "A simple set-point controller by using only position measurements." In *Proc. 12th. IFAC World Congress*, vol. 6, pp. 173–176.

Kelly, R., R. Ortega, A. Ailon, and A. Loría (1994): "Global regulation of flexible joints robots using approximate differentiation." *IEEE Transactions on Automatic Control*, **39:6**, pp. 1222–1224.

Khalil, H. (1996): *Nonlinear Systems*, 2nd edition. Prentice Hall.

Kokotović, P., H. Khalil, and J. O'Reilly (1986): *Singular Perturbation Methods in Control: Analysis and Design*. Academic Press, New York.

Kokotović, P. and H. Sussmann (1989): "A positive real condition for global stabilization of nonlinear systems." *System & Control Letters*, **13**, pp. 125–133.

Kolmanovsky, I. and N. McClamroch (1995): "Developments in nonholonomic control problems." *IEEE Control Systems Magazine*, **16:6**, pp. 20–36.

Kolmogorov, A. (1939): "Sur l'interpolation et extrapolation des suites stationaires." *C.R. Acad. Sci.*, **208**, pp. 2043–2045.

Krener, A. (1999): "Feedback linearization." In Baillieul and Willems, Eds., *Mathematical Control Theory*, pp. 66–98. Springer-Verlag, New York.

Krener, A. and A. Isidori (1983): "Linearization by output injection and nonlinear observers." *System & Control Letters*, **3**, pp. 47–52.

Krener, A. and A. Isidori (1984): "Approximate linearization by state feedback and coordinate change." *System & Control Letters*, **5:3**, pp. 181–185.

Krener, A., S. Karahan, and M. Hubbard (1988): "Approximate normal forms of nonlinear systems." In *Proc, 27th IEEE Conference on Decision and Control*, pp. 1223–1229. Austin, Texas.

Krener, A. and W. Respondek (1985): "Nonlinear observers with linearizable error dynamics." *SIAM Journal on Control and Optimization*, **23:2**, pp. 197–216.

Krstić, M., I. Kanellakopoulos, and P. Kokotović (1995): *Nonlinear and Adaptive Control Design*. John Wiley & Sons, New York.

Krstić, M. and P. Kokotović (1995): "Adaptive nonlinear design with controller-identifier separation and swapping." *IEEE Transactions on Automatic Control*, **40:3**, pp. 426–440.

Krstić, M., P. Kokotović, and I. Kanellakopoulos (1993): "Adaptive nonlinear output-feedback control with an observer-based identifier." In *Proceedings of American Control Conference*, vol. 3, pp. 2821–2825. San Francisco, CA.

Kwakernaak, H. and R. Sivan (1972): *Linear Optimal Control Systems*. Wiley, New-York.

LaSalle, J. (1960): "Some extensions of Lyapunov's second method." *IRE Transactions on Circuit Theory*, **7:4**, pp. 520–527.

LaSalle, J. (1967): "An invariance principle in the theory of stability." In J.K.Hale and LaSalle, Eds., *Differential Equations and Dynamical Systems*. Academic Press, New York.

Lefeber, A. A. J. and H. Nijmeijer (1997): "Globally bounded tracking controllers for rigid robot systems." In *Proceedings of the European Control Conference (ECC'97)*. Brussels, Belgium. TH-E-G3.

Lefeber, E., A. Robertsson, and H. Nijmeijer (1999a): "Linear controllers for tracking chained-form systems." In Aeyels *et al*, Eds., *Stability and Stabilization of Nonlinear Systems*, vol. 246 of *Lecture Notes in Control and Information Sciences*, pp. 183–197. Springer-Verlag, Heidelberg. ISBN 1-85233-638-2.

Lefeber, E., A. Robertsson, and H. Nijmeijer (1999b): "Output feedback tracking of nonholonomic systems in chained form." In *Proceedings of the 5th European Control Conference (ECC'99)*. Karlsruhe, Germany. Paper 772 (AP-4).

Lefeber, E., A. Robertsson, and H. Nijmeijer (1999c): "Output feedback tracking of nonholonomic systems in chained form." In *18th Benelux meeting on System and Control*. Houthalen-Helchteren, Belgium.

Lefeber, E., A. Robertsson, and H. Nijmeijer (2000): "Linear controllers for exponential tracking of systems in chained form." *International Journal of Robust and Nonlinear Control: Special issue on Control of Underactuated Nonlinear Systems*, **10:4**. In press.

Lin, W. (1995): "Bounded smooth state feedback and a global principle for non-affine nonlinear systems." *System & Control Letters*, **26:1**, pp. 41–53.

Lin, W. and C. Byrnes (1994): "KYP lemma, state-feedback and dynamic output-feedback in discrete-time bilinear-systems." *System & Control Letters*, **23:2**, pp. 127–136.

Ljung, L. (1979): "Asymptotic behavior of the extended Kalman filter as a parameter estimator for linear systems." *IEEE Transactions on Automatic Control*, **24:1**, pp. 36–50.

Loría, A. (1996): *On Output Feedback Control of Euler-Lagrange Systems*. PhD thesis, Universite de Technologie de Compiegne, France.

Loría, A., T. Fossen, and E. Panteley (1999): "A cascaded approach to a separation principle for dynamic ship positioning." *to appear in IEEE Transactions on Control Systems Technology*.

Loría, A., R. Kelly, R. Ortega, and V. Santibañez (1997): "On global output feedback control of Euler-Lagrange systems under input constraints." *IEEE Transactions on Automatic Control*, **42:8**, pp. 1138–1142.

Luenberger, D. (1964): "Observing the state of a linear system." *IEEE Trans Mil. Electronics*, **8**, pp. 74–80.

Luenberger, D. (1971): "An introduction to observers." *IEEE Transactions on Automatic Control*, **16:6**, pp. 596–602.

Lyapunov, A. M. (1892): Общая Задача об Устойчивости Движения *(The general problem of the stability of motion)*. Kharkov Mathematical Society, Kharkov, Russia.

Marino, R. and P. Tomei (1991): "Dynamic output feedback linearization and global stabilization." *System & Control Letters*, **17**, pp. 115–121.

Marino, R. and P. Tomei (1995): *Nonlinear Control Design: Geometric, Adaptive, and Robust*. Prentice Hall information and system sciences series. Prentice-Hall, London.

Mazenc, F. and L. Praly (1996): "Adding integrations, saturated controls, and stabilization for feedforward systems." *IEEE Transactions on Automatic Control*, **41:11**, pp. 1559–1578.

Mazenc, F., L. Praly, and W. Dayawansa (1994): "Global stabilization of output feedback systems: Examples and counter-examples." *System & Control Letters*, **23**, pp. 119–125.

Micaelli, A. and C. Samson (1993): "Trajectory tracking for unicycle-type and two-steering-wheels mobile robots." Technical Report 2097. INRIA.

Middleton, R. H. and C. Goodwin (1988): "Adaptive computed torque control for rigid link manipulators." *System & Control Letters*, **10**, pp. 9–16.

Misawa, E. and J. Hedrick (1989): "Nonlinear observers, a state-of-the-art survey." *Trans. of ASME, Journal of Dynamic Systems, Measurements and Control*, **11**, pp. 344–352.

Mita, T. (1977): "On zeros and responses of linear regulators and linear observers." *IEEE Transactions on Automatic Control*, **22**, pp. 423–428.

Mohler, R. R. and W. J. Kolodziej (1980): "An overview of bilinear system theory and applications." *IEEE Transactions on Systems, Man and Cybernetics*, **10:10**, pp. 683–688.

Molander, P. and J. C. Willems (1980): "Synthesis of state feedback control laws with a specified gain and phase margin." *IEEE Transactions on Automatic Control*, **25:5**, pp. 928–931.

Moylan, P. (1974): "Implications of passivity in a class of nonlinear systems." *IEEE Transactions on Automatic Control*, **19**, pp. 373–381.

Murray, R. and S. Sastry (1993): "Nonholonomic motion planning: Steering using sinusoids." *IEEE Transactions on Automatic Control*, **38**, pp. 700–716.

Nicklasson, P. J. (1996): *Passivity-Based Control of Electric Machines*. PhD thesis 96-17-W, Dept. Engineering Cybernetics, Norwegian University of Science and Technology, Trondheim, Norway.

Nicosia, S. and P. Tomei (1990): "Robot control by using only joint position measurements." *IEEE Transactions on Automatic Control*, **35:9**, pp. 1058–1061.

Nicosia, S. and P. Tomei (1995): "A global output feedback controller for flexible joint robots." *Automatica*, **31:10**, pp. 1465–1469.

Nicosia, S., P. Tomei, and A. Tornambè (1986): "Feedback control of elastic robots by pseudo-linearization techniques." In *Proceedings of the 25th IEEE Conference on Decision and Control*, vol. 1, pp. 397–402. New York, NY.

Nicosia, S. and A. Tornambè (1989): "High-gain observers in the state and parameter estimation of robots having elastic joints." *System & Control Letters*, **13**, pp. 331–337.

Nijmeijer, H. and T. I. Fossen, Eds. (1999): *New Directions in Nonlinear Observer Design*, vol. 244 of *Lecture notes in Control and Information Sciences*. Springer-Verlag, London.

Nijmeijer, H. and I. Mareels (1997): "An observer looks at synchronization." *IEEE Transactions on Circuits and Systems–I: Fundamental Theory and Applications*, **44:10**, pp. 882–890.

Nijmeijer, H. and A. van der Schaft (1990): *Nonlinear Dynamical Control Systems*. Springer-Verlag, New York.

Nilsson, K. (1996): *Industrial Robot Programming*. PhD thesis ISRN LUTFD2/TFRT--1046--SE, Department of Automatic Control, Lund Institute of Technology, Lund, Sweden.

Olsson, H. (1996): *Control Systems with Friction*. PhD thesis ISRN LUTFD2/TFRT--1045--SE, Department of Automatic Control, Lund Institute of Technology, Lund, Sweden.

Ortega, R., A. Loría, R. Kelly, and L. Praly (1995a): "On output feedback global stabilization of Euler-Lagrange systems." *Int. J. of Robust and Nonlinear Control: Special Issue on Mechanical Systems*, **5:4**, pp. 313–324. Eds. H. Nijmeijer and A. van der Schaft.

Ortega, R., A. Loría, R. Kelly, and L. Praly (1995b): “On passivity-based output feedback global stabilization of Euler-Lagrange systems.” *International Journal of Robust and Nonlinear Control ; Special issue on Control of Nonlinear Mechanical Systems*, **5:4**, pp. 313–325.

Ortega, R., A. Loría, P. Nicklasson, and H. Sira-Ramírez (1998): *Passivity-based Control of Euler-Lagrange Systems: Mechanical, Electrical and Electromechanical Applications*. Springer-Verlag London Berlin Heidelberg.

Paden, B. and R. Panja (1988): “Globally asymptotically stable ’PD+’ controller for robot manipulators.” *International Journal of Control*, **47**, pp. 1697–1712.

Panteley, E. and A. Loría (1998a): “On global uniform asymptotic stability of nonlinear time-varying systems in cascade.” *Systems & Control Letters*, **33:2**, pp. 131–138.

Panteley, E. and A. Loría (1998b): “On global uniform asymptotic stability of nonlinear time-varying systems in cascade.” *Systems & Control Letters*, **33:2**, pp. 131–138.

Patel, R. and M. Toda (1980): “Quantitative measures of robustness in multivariable systems.” In *Proc. of American Control Conference*, vol. 2. San Francisco, CA. TP8-A.

Petersen, I. and D. McFarlane (1994): “Optimal guaranteed cost control and filtering for uncertain linear systems.” *IEEE Transactions on Automatic Control*, **39:9**, pp. 1971–1977.

Pomet, J., R. Hirschhorn, and W. Cebuhar (1993): “Dynamic output feedback regulation for a class of nonlinear systems.” *Math of Control, Signals and Systems*, **6**, pp. 106–124.

Popov, V. M. (1961): “Absolute stability of nonlinear systems of automatic control.” *Avtomatika i Telemekhanika*, **22:8**.

Popov, V. M. (1973): *Hyperstability of Control Systems*. Springer-Verlag, New York.

Praly, L. (1992): “Lyapunov design of a dynamic output feedback for system linear in their unmeasured state components.” In *Proc IFAC Nonlinear Control Systems Design Symposium*, pp. 31–36. Bordeaux, France.

Raghavan, S. and J. Hedrick (1994): “Observer design for a class of nonlinear systems.” *International Journal of Control*, **59:2**, pp. 515–528.

Rajamani, R. (1998): "Observers for Lipschitz nonlinear systems." *IEEE Transactions on Automatic Control*, **43:3**, pp. 397–401.

Robertsson, A. and R. Johansson (1997): "A note on 'Nonlinear control using only position feedback: An observer backstepping approach'." In *Proceedings of "Robotikdagarna 1997"*, pp. 1–6. Linköping, Sweden.

Robertsson, A. and R. Johansson (1998a): "Comments on 'Nonlinear output feedback control of dynamically positioned ships using vectorial observer backstepping'." *IEEE Transactions on Control Systems Technology*, **6:3**, pp. 439–441.

Robertsson, A. and R. Johansson (1998b): "Nonlinear observers and output feedback control with application to dynamically positioned ships." In *4th IFAC Nonlinear Control Systems Design Symposium (NOLCOS'98)*, vol. 3, pp. 817–822. Enschede, Netherlands.

Robertsson, A. and R. Johansson (1998c): "Observer backstepping and control design of linear systems." In *Proceedings of the 37th IEEE Conference Decision and Control*, pp. 4592–4593.

Robertsson, A. and R. Johansson (1999): "Observer backstepping for a class of nonminimum-phase systems." In *Proceedings of the 38th IEEE Conference on Decision and Control (CDC'99)*, pp. 4866–4871. Phoenix, Arizona.

Safonov, M. and M. Athans (1978): "Robustness and computational aspects of nonlinear stochastic estimators and regulators." *IEEE Transactions on Automatic Control*, **23**, pp. 717–725.

Sanchis, R. and H. Nijmeijer (1998): "Sliding controller—sliding observer design for nonlinear systems." *European Journal of Control*, **4:3**, pp. 197–208.

Sciavicco, L. and B. Siciliano (1996): *Modeling and Control of Robot Manipulators*. McGraw-Hill, New York.

Sepulchre, R., M. Janković, and P. Kokotović (1997): *Constructive Nonlinear Control*. Springer-Verlag, New York.

Sirisena, H. (1979): "Minimal-orders for linear functions of a state vector." *International Journal of Control*, **29**, pp. 235–254.

Slotine, J. and W. Li (1987): "On the adaptive control of robot manipulators." *International Journal of Robotics Research*, pp. 49–59.

Song, R.-Y., S. Ishijima, and A. Kojima (1996): "Observer based linearization of I/O linearizable nonlinear systems." In *Proc. 13th IFAC World Congress*, vol. E, pp. 269–274. San Francisco, CA.

Song, R.-Y., S. Ishijima, and A. Kojima (1997): "Design of nonlinear observer by a backstepping approach." *Electrical Engineering in Japan*, **121:3**, pp. 53–59.

Sontag, E. (1988): "Smooth stabilization implies coprime factorization." *IEEE Transactions on Automatic Control*, **34**, pp. 435–443.

Sontag, E. (1990): *Mathematical Control Theory: Deterministic Finite Dimensional Systems*. Number 6 in Texts in Applied Mathematics. Springer-Verlag, New York.

Sontag, E. (1995): "On the input-to-state stability property." *European Journal of Control*, **1:1**, pp. 24–36.

Sørli, J. (1996): *On Grey-Box Model Definition and Symbolic Derivation of Extended Kalman Filters*. PhD thesis ISSN 0347-1071, Royal Institute of Technology, Stockholm, Sweden. Dep. of Signals, Sensors, and Systems.

Spong, M. (1987): "Modeling and control of elastic joint robots." *Trans. of ASME, Journal of Dynamic Systems, Measurements and Control*, **109**, pp. 310–319.

Spong, M. and L. Praly (1996): "Control of underactuated mechanical systems using switching and saturation." In *Proc. of the Block Island Workshop on Control Using Logic Based Switching*. Springer-Verlag.

Spong, M. and M. Vidyasagar (1989): *Robot Dynamics and Control*. Wiley, New York.

Sussmann, H. and P. Kokotović (1989): "Peaking and stabilization." In *Proceedings of the 28th IEEE Conference on Decision and Control*, pp. 1379–1384. Tampa, Florida.

Sussmann, H. and P. Kokotović (1991): "The peaking phenomenon and the global stabilization of nonlinear systems." *IEEE Transactions on Automatic Control*, **36**, pp. 424–439.

Takegaki, M. and S. Arimoto (1981): "A new feedback method for dynamic control of manipulators." *Trans. of ASME, Journal of Dynamic Systems, Measurements and Control*, **102**, pp. 119–125.

Teel, A. (1992): "Global stabilization and restricted tracking for multiple integrators with bounded controls." *System & Control Letters*, **18:3**, pp. 165–171.

Teel, A. and L. Praly (1994): "Global stabilizability and observability imply semi-global stabilizability by output feedback." *System & Control Letters*, **22**, pp. 313–325.

Teel, A. and L. Praly (1995): "Tools for semiglobal stabilization by partial state and output feedback." *SIAM Journal of Control Optimization*, **33:5**, pp. 1443–1488.

Thau, F. (1973): "Observing the state of nonlinear dynamic systems." *International Journal of Control*, **17**, pp. 471–479.

Tilbury, D., O. Sørdalen, L. Bushnell, and S. Sastry (1995): "A multisteering trailer system: conversion into chained form using dynamic feedback." *IEEE Transactions on Robotics and Automation*, **11:6**, pp. 807–818.

Tsinias, J. (1990): "Further results on the observer design problem." *System & Control Letters*, **14**, pp. 411–418.

Tsinias, J. (1991): "A generalization of Vidyasagar's theorem on stability using state detection." *System & Control Letters*, **17**, pp. 37–42.

Tsinias, J. (1993): "Sontag's 'input to state stability condition' and global stabilization using state detection." *System & Control Letters*, **20**, pp. 219–226.

Vidyasagar, M. (1980): "On the stabilization of nonlinear systems using state detection." *IEEE Transactions on Automatic Control*, **25**, pp. 504–509.

Wang, J. and W. J. Rugh (1989): "On the pseudo-linearization problem for nonlinear systems." *System & Control Letters*, **12:2**, pp. 161–167.

Wiener, N. (1949): *Extrapolation, Interpolation, and Smoothing of Stationary Time Series, with Engineeering Applications*. Wiley, New York.

Wiklund, M., A. Kristenson, and K. J. Åström (1993): "A new strategy for swinging up an inverted pendulum." In *Preprints of 12th IFAC World Congress*, vol. 9, pp. 151–154. Sydney.

Willems, J. C. (1972): "Dissipative dynamical systems, part I and II." *Arch. Rational Mechanics and Analysis*, **45**, pp. 321–393.

Willems, J. L. (1970): *Stability Theory of Dynamical Systems*. Academic Press, New York.

Yamamoto, S. and I. Hashimoto (1991): "Present status and future needs: The view from Japanese industry." In Arkun and Ray, Eds., *Chemical Process Control—CPCIV*. Proceedings of the Fourth Internation Conference on Chemical Process Control, Texas.

Zeitz, M. (1987): "The extended Luenberger observer for nonlinear systems." *System & Control Letters*, **9**, pp. 149–156.

Summary of publications

This thesis work has been conducted under the Lund Program on "Mobile Autonomous Systems". Below follows a list of the author's publications within that project:

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Johansson, R. and A. Robertsson (1999): "Extension of the Yakubovich-Kalman-Popov lemma for stability analysis of dynamic output feedback systems." In *Proceedings of IFAC'99*, vol. F, pp. 393–398. Beijing, China.

Johansson, R., A. Robertsson, and R. Lozano-Leal (1999): "Stability analysis of adaptive output feedback control." In *Proceedings of the 38th IEEE Conference on Decision and Control (CDC'99)*. Phoenix, Arizona.

Johansson, R., A. Robertsson, K. Nilsson, and M. Verhaegen (1997): "State-space system identification of robot manipulator dynamics." In *Proceedings of "SNART-97"*, pp. 152–158. Lund, Sweden.

Johansson, R., A. Robertsson, K. Nilsson, and M. Verhaegen (1999): "State-space system identification of robot manipulator dynamics." *Mechatronics*. In press.

Johansson, R., M. Verhaegen, C. T. Chou, and A. Robertsson (1998a): "Behavioral model identification." In *Proceedings of the 37th IEEE Conference on Decision and Control*, pp. 126–131. Tampa, Florida.

Johansson, R., M. Verhaegen, C. T. Chou, and A. Robertsson (1998b): "Residual models and stochastic realization in state-space system identification." In *Proceedings of the 37th IEEE Conference on Decision and Control*, pp. 3439–3444. Tampa, Florida.

Lefeber, E., A. Robertsson, and H. Nijmeijer (1999a): "Linear controllers for tracking chained-form systems." In Aeyels *et al*, Eds., *Stability and Stabilization of Nonlinear Systems*, vol. 246 of *Lecture Notes in Control and Information Sciences*, pp. 183–197. Springer-Verlag, Heidelberg. ISBN 1-85233-638-2.

Lefeber, E., A. Robertsson, and H. Nijmeijer (1999b): "Output feedback tracking of nonholonomic systems in chained form." In *Proceedings of the 5th European Control Conference (ECC'99)*. Karlsruhe, Germany. Paper 772 (AP-4).

Lefeber, E., A. Robertsson, and H. Nijmeijer (1999c): "Output feedback tracking of nonholonomic systems in chained form." In *18th Benelux meeting on System and Control*. Houthalen-Helchteren, Belgium.

Lefeber, E., A. Robertsson, and H. Nijmeijer (2000): "Linear controllers for exponential tracking of systems in chained form." *International Journal of Robust and Nonlinear Control: Special issue on Control of Underactuated Nonlinear Systems*, **10:4**. In press.

Nilsson, K., A. Robertsson, and R. Johansson (1994): "Robot control research in Lund." In *Preprints, Reglermöte'94*, pp. 1–6. Västerås, Sweden.

Robertsson, A., J. Eker, K. Nilsson, and R. Johansson (1998): "Application-specific control for industrial robots." In *Proceedings of the 3rd Portuguese Conference on Automatic Control — Controlo'98*, vol. 2, p. 877. Portuguese Association of Automatic Control, APCA, Coimbra, Portugal.

Robertsson, A. and R. Johansson (1997): "A note on 'Nonlinear control using only position feedback: An observer backstepping approach'." In *Proceedings of "Robotikdagarna 1997"*, pp. 1–6. Linköping, Sweden.

Robertsson, A. and R. Johansson (1998a): "Comments on 'Nonlinear output feedback control of dynamically positioned ships using vectorial observer backstepping'." *IEEE Transactions on Control Systems Technology*, **6:3**, pp. 439–441.

Robertsson, A. and R. Johansson (1998b): "Nonlinear observers and output feedback control with application to dynamically positioned ships." In *4th IFAC Nonlinear Control Systems Design Symposium (NOLCOS'98)*, vol. 3, pp. 817–822. Enschede, Netherlands.

Robertsson, A. and R. Johansson (1998c): "Observer backstepping and control design of linear systems." In *Proceedings of the 37th IEEE Conference Decision and Control*, pp. 4592–4593.

Robertsson, A. and R. Johansson (1999): "Observer backstepping for a class of nonminimum-phase systems." In *Proceedings of the 38th IEEE Conference on Decision and Control (CDC'99)*, pp. 4866–4871. Phoenix, Arizona.

Valera, A., A. Robertsson, K. Nilsson, and R. Johansson (1999): "Interactive on-line evalutaion of robot motion control." In *Proceedings of IEEE CCA-99*, pp. CCA–1039–1044. Hawaii.

A

Paper A

Linear Controllers for Tracking Chained-Form Systems

E. Lefeber, A. Robertsson, and H. Nijmeijer

Abstract

In this paper we study the tracking problem for the class of nonholonomic systems in chained-form. In particular, with the first and the last state component of the chained-form as measurable output signals, we suggest a solution for the tracking problem using output feedback by combining a time-varying state feedback controller with an observer for the chained-form system. For the stability analysis of the "certainty equivalence type" of controller, we use a cascaded systems approach. The resulting closed loop system is globally $\mathcal{K}$-exponentially stable.

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1. Introduction

In recent years a lot of interest has been devoted to (mainly) stabilization and tracking of nonholonomic dynamic systems, see e.g. [1; 5; 7; 15; 17]. One of the reasons for the attention is the lack of a continuous static state feedback control since Brockett's necessary condition for smooth stabilization is not met, see [3]. The proposed solutions to this problem follow mainly two routes, namely discontinuous and/or time-varying control. For a good overview, see the survey paper [11] and the references therein.

It is well known that the kinematic model of several nonholonomic systems can be transformed into a *chained-form system*. The global tracking problem for chained-form systems has recently been addressed in [14; 5; 6; 7; 17; 20]. In this paper we consider the tracking problem for chained form systems by means of output feedback, where we consider as output the first and last state component of the chained-form. To our knowledge, this problem has only been addressed in [8] where a backstepping approach is used. Our results are based on the construction of a linear time varying state feedback controller in combination with an observer. However, the stability analysis and design are based on results for (time-varying) cascaded systems [18]. In the design we divide the chained-form into a cascade of two sub-systems which we can stabilize independently of each other, and furthermore a similar partition into cascaded systems can be done for the controller-observer combination, where the same stability results apply. Regarding the latter part, similar ideas were recently presented for the combination of high-gain controllers and high-gain observer for a class of triangular nonlinear systems [2], see also [12].

The organization of the paper is as follows. Section 2 contains some definitions, preliminary results and the problem formulation. Section 3 addresses the tracking problem based on time-varying state feedback and in section 4 we design an exponentially convergent observer for the chained-form system. In section 5 we combine the control law from section 3 with the observer from section 4 in a "certainty equivalence" sense. This yields a globally $\mathcal{K}$-exponentially stable closed loop system under the condition of persistently exciting reference trajectories. Finally, section 6 concludes the paper.

2. Preliminaries and Problem Formulation

In this section we introduce the definitions and theorems used in the remainder of this paper and formulate the problem under consideration. We start with some basic stability concepts in 2, present a result for cascaded systems in 2 and recall some basic results from linear systems theory in 2. We conclude this section with the problem formulation in 2.

Stability

To start with, we recall some basic concepts (see e. g. [10; 23]).

DEFINITION A.1
A continuous function $\alpha : [0,a) \to [0,\infty)$ is said to belong to *class* $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. $\square$

DEFINITION A.2
A continuous function $\beta : [0,a) \times [0,\infty) \to [0,\infty)$ is said to belong to *class* $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r,s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r,s)$ is decreasing with respect to s and $\beta(r,s) \to 0$ as $s \to \infty$. $\square$

Consider the system

$$\dot{x} = f(t,x), \quad f(t,0) = 0 \quad \forall t \geq 0 \tag{1}$$

with $x \in I\!R^n$ and $f(t,x)$ piecewise continuous in t and locally Lipschitz in x.

DEFINITION A.3
The system (1) is *uniformly stable* if for each $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$, independent of t_0, such that

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall t \geq t_0 \geq 0. \tag{2}$$

$\square$

DEFINITION A.4
The system (1) is *globally uniformly asymptotically stable (GUAS)* if it is uniformly stable and globally attractive, that is, there exists a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ such that for every initial state $x(t_0)$:

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall t \geq t_0 \geq 0 \tag{3}$$

$\square$

DEFINITION A.5
The system (1) is *globally exponentially stable (GES)* if there exist $k > 0$ and $\gamma > 0$ such that for any initial state

$$\|x(t)\| \leq \|x(t_0)\| k \exp[-\gamma(t - t_0)]. \tag{4}$$

$\square$

A slightly weaker notion of exponential stability is the following

Definition A.6—cf. [22]
We call the system (1) *globally $\mathcal{K}$-exponentially stable* if there exist $\gamma > 0$ and a class $\mathcal{K}$ function $\kappa(\cdot)$ such that

$$\|x(t)\| \leq \kappa(\|x(t_0)\|) \exp[-\gamma(t - t_0)] \tag{5}$$

□

Definition A.7
We call the (locally integrable) vector-valued function

$$w(t) = [w_1(t), \ldots, w_n(t)]^T$$

persistently exciting if there exist $\delta, \varepsilon_1, \varepsilon_2 > 0$ such that for all $t > 0$:

$$\varepsilon_1 I \leq \int_t^{t+\delta} w(\tau) w(\tau)^T d\tau \leq \varepsilon_2 I \tag{6}$$

□

Cascaded systems

Consider the system

$$\left\{ \begin{array}{rcl} \dot{z}_1 & = & f_1(t, z_1) + g(t, z_1, z_2) z_2 \\ \dot{z}_2 & = & f_2(t, z_2) \end{array} \right. \tag{7}$$

where $z_1 \in \mathbb{R}^n$, $z_2 \in \mathbb{R}^m$, $f_1(t, z_1)$ is continuously differentiable in (t, z_1) and $f_2(t, z_2)$, $g(t, z_1, z_2)$ are continuous in their arguments, and locally Lipschitz in z_2 and (z_1, z_2) respectively.

We can view the system (7) as the system

$$\Sigma_1 : \dot{z}_1 = f_1(t, z_1) \tag{8}$$

that is perturbed by the state of the system

$$\Sigma_2 : \dot{z}_2 = f_2(t, z_2). \tag{9}$$

When Σ_2 is asymptotically stable, we have that z_2 tends to zero, which means that the z_1 dynamics in (7) asymptotically reduces to Σ_1. Therefore, we can hope that asymptotic stability of both Σ_1 and Σ_2 implies asymptotic stability of (7).

Unfortunately, this is not true in general. However, from the proof presented in [18] it can be concluded that:

THEOREM A.1—BASED ON [18]
The cascaded system (7) is GUAS if the following three assumptions hold:

- *assumption on Σ_1:* the system $\dot{z}_1 = f_1(t, z_1)$ is GUAS and there exists a continuously differentiable function $V(t, z_1) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ that satisfies

$$W(z_1) \leq V(t, z_1), \tag{10}$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial z_1} \cdot f_1(t, z_1) \leq 0, \qquad \forall \|z_1\| \geq \eta, \tag{11}$$
$$\left\| \frac{\partial V}{\partial z_1} \right\| \|z_1\| \leq c V(t, z_1), \qquad \forall \|z_1\| \geq \eta, \tag{12}$$

  where $W(z_1)$ is a positive definite proper function and $c > 0$ and $\eta > 0$ are constants,

- *assumption on the interconnection:* the function $g(t, z_1, z_2)$ satisfies for all $t \geq t_0$:

$$\|g(t, z_1, z_2)\| \leq \theta_1(\|z_2\|) + \theta_2(\|z_2\|)\|z_1\|, \tag{13}$$

  where $\theta_1, \theta_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions,

- *assumption on Σ_2:* the system $\dot{z}_2 = f_2(t, z_2)$ is GUAS and for all $t_0 \geq 0$:

$$\int_{t_0}^{\infty} \|z_2(t_0, t, z_2(t_0))\| dt \leq \kappa(\|z_2(t_0)\|), \tag{14}$$

  where the function $\kappa(\cdot)$ is a class $\mathcal{K}$ function,

□

REMARK A.1
Notice that the assumption on Σ_1 is slightly weaker than the one presented in [18]. However, under the assumption mentioned above the result can still be shown to be true by (almost) exactly copying the proof presented in [18]. □

LEMMA A.1—SEE [17]
If in addition to the assumptions in Theorem A.1 both $\dot{z}_1 = f_1(t, z_1)$ and $\dot{z}_2 = f_2(t, z_2)$ are globally $\mathcal{K}$-exponentially stable, then the cascaded system (7) is globally $\mathcal{K}$-exponentially stable. □

Linear time-varying systems

Consider the linear time-varying system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{15}$$

and let $\Phi(t, t_0)$ denote the state-transition matrix for the system

$$\dot{x} = A(t)x$$

We recall some results from linear control theory (cf. [9; 19]).

DEFINITION A.8
The pair $(A(t), B)$ is said to be *uniformly controllable* if there exist $\delta, \varepsilon_1, \varepsilon_2 > 0$ such that for all $t > 0$:

$$\varepsilon_1 I \leq \int_t^{t+\delta} \Phi(t,\tau)BB^T\Phi^T(t,\tau)d\tau \leq \varepsilon_2 I \tag{16}$$

□

DEFINITION A.9
The pair $(A(t), C)$ is said to be *uniformly observable* if there exist $\delta, \varepsilon_1, \varepsilon_2 > 0$ such that for all $t > 0$:

$$\varepsilon_1 I \leq \int_{t-\delta}^{t} \Phi^T(\tau, t-\delta)C^TC\Phi(\tau, t-\delta)d\tau \leq \varepsilon_2 I \tag{17}$$

□

From linear systems theory several methods are available to exponentially stabilize the linear time-varying system (15) via state or output feedback, in case the pairs $(A(t), B)$ and $(A(t), C)$ are uniformly controllable and observable respectively (cf. [19]):

THEOREM A.2
Suppose that the system (15) is uniformly controllable and define for $\alpha > 0$

$$W_\alpha(t, t+\delta) = \int_t^{t+\delta} 2e^{4\alpha(t-\tau)}\Phi(t,\tau)BB^T\Phi^T(t,\tau)d\tau \tag{18}$$

Then given any constant α the state feedback $u(t) = K_\alpha(t)x(t)$ where

$$K_\alpha(t) = -B^TW_\alpha^{-1}(t, t+\delta) \tag{19}$$

is such that the resulting closed-loop state equation is uniformly exponentially stable with rate α. □

THEOREM A.3
Suppose that the system (15) is uniformly controllable and uniformly observable and define for $\alpha > 0$

$$M_\alpha(t-\delta,t) = \int_{t-\delta}^{t} 2e^{4\alpha(\tau-t)}\Phi^T(\tau,t-\delta)C^T C\Phi(\tau,t-\delta)d\tau \tag{20}$$

Then given $\alpha > 0$, for any $\eta > 0$ the linear dynamic output feedback

$$\begin{aligned} u(t) &= K_{\alpha+\eta}(t)\hat{x}(t) && (21)\\ \dot{\hat{x}}(t) &= A(t)\hat{x}(t) + Bu(t) + H_{\alpha+\eta}(t)[y(t)-\hat{y}(t)], \quad \hat{x}(t_0) = \hat{x}_0 && (22)\\ \hat{y}(t) &= C\hat{x}(t) && (23) \end{aligned}$$

with feedback and observer gains

$$\begin{aligned} K_{\alpha+\eta}(t) &= -B^T W_{\alpha+\eta}^{-1}(t,t+\delta) && (24)\\ H_{\alpha+\eta}(t) &= \left[\Phi^T(t-\delta,t)M_{\alpha+\eta}(t-\delta,t)\Phi(t-\delta,t)\right]^{-1} C^T && (25) \end{aligned}$$

is such that the closed-loop state equation is uniformly exponentially stable with rate α. □

Problem formulation

The class of chained-form nonholonomic systems we study in this paper is given by the following equations

$$\begin{aligned} \dot{x}_1 &= u_1\\ \dot{x}_2 &= u_2\\ \dot{x}_3 &= x_2u_1\\ &\vdots\\ \dot{x}_n &= x_{n-1}u_1 \end{aligned} \tag{26}$$

where $x = (x_1,\ldots,x_n)$ is the state, u_1 and u_2 are control inputs.

Consider the problem of tracking a reference trajectory (x_r, u_r) generated by the chained-form system:

$$\begin{aligned} \dot{x}_{1,r} &= u_{1,r}\\ \dot{x}_{2,r} &= u_{2,r}\\ \dot{x}_{3,r} &= x_{2,r}u_{1,r}\\ &\vdots\\ \dot{x}_{n,r} &= x_{n-1,r}u_{1,r} \end{aligned} \tag{27}$$

where we assume $u_{1,r}(t)$ to $u_{2,r}(t)$ be continuous functions of time. This reference trajectory can be generated by any of the motion planning techniques available from the literature.

When we define the tracking error $x_e = x - x_r$ we obtain as tracking error dynamics

$$\begin{aligned}
\dot{x}_{1,e} &= u_1 - u_{1,r} &&= u_1 - u_{1,r} \\
\dot{x}_{2,e} &= u_2 - u_{2,r} &&= u_2 - u_{2,r} \\
\dot{x}_{3,e} &= x_2 u_1 - x_{2,r} u_{1,r} &&= x_{2,e} u_{1,r} + x_2(u_1 - u_{1,r}) \\
&\vdots &&\vdots \\
\dot{x}_{n,e} &= x_{n-1} u_1 - x_{n-1,r} u_{1,r} &&= x_{n-1,e} u_{1,r} + x_{n-1}(u_1 - u_{1,r})
\end{aligned} \tag{28}$$

The state feedback tracking control problem then can be formulated as

PROBLEM A.1—STATE FEEDBACK TRACKING CONTROL PROBLEM
Find appropriate state feedback laws u_1 and u_2 of the form

$$u_1 = u_1(t, x, x_r, u_r) \quad \text{and} \quad u_2 = u_2(t, x, x_r, u_r) \tag{29}$$

such that the closed-loop trajectories of (28,29) are globally uniformly asymptotically stable. □

Consider the system (26) with output

$$y = \begin{bmatrix} x_1 \\ x_n \end{bmatrix} \tag{30}$$

then it is easy to show (see e. g. [1]) that the system (26) with output (30) is locally observable at any $x \in I\!R^n$. Clearly, this is the minimal number of state components we need to know for solving the output-feedback tracking problem.

Now we can formulate the output feedback tracking problem as

PROBLEM A.2—OUTPUT FEEDBACK TRACKING CONTROL PROBLEM
Find appropriate control laws u_1 and u_2 of the form

$$u_1 = u_1(t, \hat{x}, y, x_r, u_r) \quad \text{and} \quad u_2 = u_2(t, \hat{x}, y, x_r, u_r) \tag{31}$$

where $\hat{x}$ is generated from an observer

$$\dot{\hat{x}} = f(t, \hat{x}, y, x_r, u_r) \tag{32}$$

such that the closed-loop trajectories of (28,31,32) are globally uniformly asymptotically stable. □

3. The State Feedback Problem

The approach we use to solve our problem is based on the recently developed studies on cascaded systems [4; 13; 16; 18; 21], and that of Theorem A.1 in particular, since it deals with time-varying systems.

We search for a subsystem which, with a stabilizing control law, can be written in the form $\dot{z}_2 = f_2(t, z_2)$ that is asymptotically stable. In the remaining dynamics we can then replace the appearance of z_2 by 0, leading to the system $\dot{z}_1 = f_1(t, z_1)$. If this system is asymptotically stable we might be able to conclude asymptotic stability of the overall system using Theorem A.1.

Consider the tracking error dynamics (28). We can stabilize the $x_{1,e}$ dynamics by using the linear controller

$$u_1 = u_{1,r} - c_1 x_{1,e} \tag{33}$$

which yields GES for $x_{1,e}$, provided $c_1 > 0$.

If we now set $x_{1,e}$ equal to 0 in (28) we obtain

$$\begin{aligned} \dot{x}_{2,e} &= u_2 - u_{2,r} \\ \dot{x}_{3,e} &= x_{2,e} u_{1,r} \\ &\vdots \\ \dot{x}_{n,e} &= x_{n-1,e} u_{1,r} \end{aligned} \tag{34}$$

where we used (33).

Notice that the system (34) is a linear time-varying system:

$$\begin{bmatrix} \dot{x}_{2,e} \\ \dot{x}_{3,e} \\ \dot{x}_{4,e} \\ \vdots \\ \dot{x}_{n,e} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \dots & \dots & \dots & 0 \\ u_{1,r}(t) & \ddots & & & \vdots \\ 0 & u_{1,r}(t) & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u_{1,r}(t) & 0 \end{bmatrix}}_{A(t)} \begin{bmatrix} x_{2,e} \\ x_{3,e} \\ x_{4,e} \\ \vdots \\ x_{n,e} \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{B} (u_2 - u_{2,r}) \tag{35}$$

that can be made exponentially stable by means of the controller $u(t) = K(t)x(t)$ provided the system (35) is uniformly controllable (cf. Theorem A.2).

This observation leads to the following

PROPOSITION A.2
Assume that the reference trajectory (x_r, u_r) satisfying (27) to be tracked by our chained form system is given. Define

$$w_r(t,t_0) = \begin{bmatrix} 1 \\ \int_{t_0}^{t} u_{1,r}(\tau)d\tau \\ \left(\int_{t_0}^{t} u_{1,r}(\tau)d\tau\right)^2 \\ \vdots \\ \left(\int_{t_0}^{t} u_{1,r}(\tau)d\tau\right)^{n-2} \end{bmatrix} = \begin{bmatrix} 1 \\ x_{1,r}(t) - x_{1,r}(t_0) \\ (x_{1,r}(t) - x_{1,r}(t_0))^2 \\ \vdots \\ (x_{1,r}(t) - x_{1,r}(t_0))^{n-2} \end{bmatrix} \tag{36}$$

and assume that there exist $\delta, \varepsilon_1, \varepsilon_2 > 0$ such that for all $t > 0$:

$$\varepsilon_1 I \leq \int_{t}^{t+\delta} w_r(t,\tau) w_r(t,\tau)^T d\tau \leq \varepsilon_2 I. \tag{37}$$

Consider the system (28) in closed-loop with the linear controller

$$\begin{aligned} u_1 &= u_{1,r} - c_1 x_{1,e} \\ u_2 &= u_{2,r} + K(t) \begin{bmatrix} x_{2,e} \\ \vdots \\ x_{n,e} \end{bmatrix} \end{aligned} \tag{38}$$

where $c_1 > 0$ and $K(t)$ is given by

$$K(t) = -[1\ 0\ 0\ \ldots\ 0] \left[\int_{t}^{t+\delta} 2e^{4\alpha(t-\tau)} w_r(t,\tau) w_r(t,\tau)^T d\tau \right]^{-1} \tag{39}$$

with $\alpha > 0$. If $x_{2,r}(t), \ldots, x_{n-1,r}(t)$ are bounded then the closed-loop system (28,38) is globally $\mathcal{K}$-exponentially stable. □

Proof We can see the closed-loop system (28,38) as a system of the form (7) where

$$z_1 = [x_{2,e}, \ldots, x_{n,e}]^T \tag{40}$$
$$z_2 = x_{1,e} \tag{41}$$
$$f_1(t,z_1) = (A(t) - BK(t))z_1 \tag{42}$$
$$f_2(t,z_2) = -c_1 z_2 \tag{43}$$
$$g(t,z_1,z_2) = -c_1[0, x_2, x_3, \ldots, x_{n-1}]^T \tag{44}$$

with

$$A(t)=\begin{bmatrix} 0 & \dots & \dots & \dots & 0 \\ u_{1,r}(t) & \ddots & & & \vdots \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u_{1,r}(t) & 0 \end{bmatrix} \qquad B=\begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \tag{45}$$

To be able to apply Theorem A.1 we need to verify the three assumptions:

- assumption on Σ_1: Due to the assumption (37) on $u_{1,r}(t)$ we have that the system (35) is uniformly controllable (cf. Remark A.3). Therefore, from Theorem A.2 we know that $\dot{z}_1 = f_1(t, z_1)$ is GES and therefore GUAS. From converse Lyapunov theory (see e. g. [10]) the existence of a suitable V is guaranteed.

- assumption on connecting term: Since $x_{2,r}, \dots, x_{n-1,r}$ are bounded, we have

$$\begin{aligned} \|g(t, z_1, z_2)\| &\leq c_1 \left(\left\| \begin{bmatrix} 0 \\ x_{2,r} \\ \vdots \\ x_{n-1,r} \end{bmatrix} \right\| + \left\| \begin{bmatrix} 0 \\ x_{2,e} \\ \vdots \\ x_{n-1,e} \end{bmatrix} \right\| \right) && (46) \\ &\leq c_1 M + c_1 \|x\| && (47) \end{aligned}$$

- assumption on Σ_2: Follows from GES of $\dot{x}_2 = -c_1 x_2$.

Therefore, we can conclude GUAS from Theorem A.1. Since both Σ_1 and Σ_2 are GES, Lemma A.1 gives the desired result. ■

REMARK A.2
Notice that since

$$u_1(t) = u_{1,r}(t) - c_1 x_{1,e}(t_0) \exp(-c_1(t - t_0)) \tag{48}$$

the condition (37) on $u_{1,r}(t)$ is satisfied if and only if a similar condition on $u_1(t)$ is satisfied (i.e. in which the r is omitted).

Therefore, we can also see the closed-loop system (28,38) as a system of the form (7) where

$$\begin{aligned} z_1 &= [x_{2,e}, \ldots, x_{n,e}]^T && (49) \\ z_2 &= x_{1,e} && (50) \\ f_1(t,z_1) &= (A(t) - BK(t))z_1 && (51) \\ f_2(t,z_2) &= -c_1 z_2 && (52) \\ g(t,z_1,z_2) &= -c_1[0, x_{2,r}, x_{3,r}, \ldots, x_{n-1,r}]^T && (53) \end{aligned}$$

with

$$A(t) = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ u_1(t) & \ddots & & & \vdots \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & u_1(t) & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \tag{54}$$

Notice that we redefined $A(t)$ and that correspondingly the connecting term $g(t,z_1,z_2)$ changed. When we modify our controller accordingly, i.e. redefine $K(t)$ in (38) as

$$K(t) = -[1\ 0\ 0\ \ldots\ 0] \left[\int_t^{t+\delta} 2e^{4\alpha(t-\tau)} w(t,\tau)w(t,\tau)^T d\tau \right]^{-1} \tag{55}$$

with $\alpha > 0$, where

$$w(t,t_0) = \begin{bmatrix} 1 \\ \int_{t_0}^t u_1(\tau)d\tau \\ \left(\int_{t_0}^t u_1(\tau)d\tau\right)^2 \\ \vdots \\ \left(\int_{t_0}^t u_1(\tau)d\tau\right)^{n-2} \end{bmatrix} = \begin{bmatrix} 1 \\ x_1(t) - x_1(t_0) \\ (x_1(t) - x_1(t_0))^2 \\ \vdots \\ (x_1(t) - x_1(t_0))^{n-2} \end{bmatrix} \tag{56}$$

we can copy the proof.

Moreover, since the connecting term $g(t,z_1,z_2)$ now can be bounded by a constant, we can claim not only global $\mathcal{K}$-exponential stability, but even GES. However, the disadvantage of (55) in comparison to (39) is that it depends on the state and therefore can not be determined a priori for a known reference trajectory in contrast to (39). □

Remark A.3
Notice that in general it is not easy to compute $\Phi(t, t_0)$. However, for the system (35) this turns out not to be too difficult, due to the nice and simple structure of the matrix $A(t)$. We find:

$$\Phi(t, t_0) = \begin{bmatrix} f_0(t, t_0) & 0 & \dots & 0 \\ f_1(t, t_0) & f_0(t, t_0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ f_{n-2}(t, t_0) & \dots & f_1(t, t_0) & f_0(t, t_0) \end{bmatrix} \tag{57}$$

where

$$f_k(t, t_0) = \frac{1}{k!} \left[\int_{t_0}^{t} u_{1,r}(\sigma) d\sigma \right]^k = \frac{1}{k!} \left[x_{1,r}(t) - x_{1,r}(t_0) \right]^k \tag{58}$$

From this it is also straightforward to see that uniform controllability of the system (35) can also be rephrased as persistency of excitation of the vector

$$\begin{bmatrix} f_0(t, t_0) \\ f_1(t, t_0) \\ \vdots \\ f_{n-2}(t, t_0) \end{bmatrix} \tag{59}$$

□

Remark A.4
Notice that the persistency of excitation condition (37) is obviously met in case $\liminf_{t \to \infty} u_{1,r}(t) = \varepsilon > 0$, so that the results of [5; 6; 7; 17] are included in this result. □

4. An Observer

The observability property for chained-form systems was considered in [1], in which a (local) observer was proposed in case $u_1(t) = -c_1 x_1(t)$. In this section we propose a globally exponentially stable observer for the chained system under an observability condition which is related to the persistence of excitation with respect to the first component of the state.

PROPOSITION A.3
Consider the chained-form system (26) with output (30). Define

$$
w(t,t_0) = \begin{bmatrix} 1 \\ \int_{t_0}^{t} u_1(\tau)d\tau \\ \left(\int_{t_0}^{t} u_1(\tau)d\tau\right)^2 \\ \vdots \\ \left(\int_{t_0}^{t} u_1(\tau)d\tau\right)^{n-2} \end{bmatrix} = \begin{bmatrix} 1 \\ x_1(t) - x_1(t_0) \\ (x_1(t) - x_1(t_0))^2 \\ \vdots \\ (x_1(t) - x_1(t_0))^{n-2} \end{bmatrix} \tag{60}
$$

Assume that there exist $\delta, \varepsilon_1, \varepsilon_2 > 0$ such that for all $t > 0$:

$$
\varepsilon_1 I \leq \int_t^{t+\delta} w(t,\tau)w(t,\tau)^T d\tau \leq \varepsilon_2 I. \tag{61}
$$

Then the observer

$$
\begin{bmatrix} \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \\ \dot{\hat{x}}_4 \\ \vdots \\ \dot{\hat{x}}_n \end{bmatrix} = \begin{bmatrix} 0 & \dots & \dots & \dots & 0 \\ u_1 & \ddots & & & \vdots \\ 0 & u_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u_1 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \\ \vdots \\ \hat{x}_n \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u_2 + H(t)\tilde{x}_n \tag{62}
$$

where $\tilde{x}_n = x_n - \hat{x}_n$ and

$$
H(t) = \left[\Phi^T(t-\delta,t)M_\alpha(t-\delta,t)\Phi(t-\delta,t)\right]^{-1} C^T \quad (\alpha > 0) \tag{63}
$$

guarantees that the observation error $\tilde{x} = x - \hat{x}$ converges to zero exponentially. □

Proof Because of the assumption on $u_1(t)$, we have a uniformly observable linear time-varying system. The result follows readily from standard linear theory (see e. g. [19]). ■

5. The Output Feedback Problem

In section 3 we derived a state feedback controller for tracking a desired trajectory, whereas in section 4 we derived an observer for a system in chained-form. We can also combine these two results in a "certainty equivalence" sense:

PROPOSITION A.4
For the reference trajectory (x_r, u_r) satisfying (27) define

$$w_r(t,t_0) = \begin{bmatrix} 1 \\ \int_{t_0}^{t} u_{1,r}(\tau)d\tau \\ \left(\int_{t_0}^{t} u_{1,r}(\tau)d\tau\right)^2 \\ \vdots \\ \left(\int_{t_0}^{t} u_{1,r}(\tau)d\tau\right)^{n-2} \end{bmatrix} = \begin{bmatrix} 1 \\ x_{1,r}(t) - x_{1,r}(t_0) \\ (x_{1,r}(t) - x_{1,r}(t_0))^2 \\ \vdots \\ (x_{1,r}(t) - x_{1,r}(t_0))^{n-2} \end{bmatrix} \tag{64}$$

and assume that there exist $\delta, \varepsilon_1, \varepsilon_2 > 0$ such that for all $t > 0$:

$$\varepsilon_1 I \leq \int_{t}^{t+\delta} w_r(t,\tau)w_r(t,\tau)^T d\tau \leq \varepsilon_2 I. \tag{65}$$

Consider the system (28) in closed-loop with the linear controller-observer-combination

$$\begin{aligned} u_1 &= u_{1,r} - c_1 x_{1,e} \\ u_2 &= u_{2,r} + K(t)\begin{bmatrix} \hat{x}_{2,e} \\ \vdots \\ \hat{x}_{n,e} \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} \dot{\hat{x}}_{2,e} \\ \dot{\hat{x}}_{3,e} \\ \dot{\hat{x}}_{4,e} \\ \vdots \\ \dot{\hat{x}}_{n,e} \end{bmatrix} = \begin{bmatrix} 0 & \dots & \dots & \dots & 0 \\ u_{1,r} & \ddots & & & \vdots \\ 0 & u_{1,r} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u_{1,r} & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_{2,e} \\ \hat{x}_{3,e} \\ \hat{x}_{4,e} \\ \vdots \\ \hat{x}_{n,e} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u_2 + H(t)\tilde{x}_n \tag{66}$$

where $\tilde{x}_n = x_n - \hat{x}_n$, $c_1 > 0$ and $K(t)$ and $H(t)$ are given by

$$\begin{aligned} K(t) &= -[1\ 0\ 0\ \dots\ 0]\left[\int_{t}^{t+\delta} 2e^{4\alpha(t-\tau)} w_r(t,\tau)w_r(t,\tau)^T d\tau\right]^{-1} \\ H(t) &= \left[2e^{4\alpha(\tau-t)} w_r(\tau,t-\delta)w_r(\tau,t-\delta)^T d\tau\ \Phi(t-\delta,t)\right]^{-1} w_r(t,t-\delta) \end{aligned}$$

with $\alpha > 0$. If $x_{2,r}, \dots, x_{n-1,r}$ are bounded then the closed-loop system (28,66) is globally $\mathcal{K}$-exponentially stable. $\square$

Proof Similar to that of Proposition A.2. Note that due to the assumption on $u_{1,r}$ we have both uniform controllability and uniform controllability. From Theorem A.3 we then know that the system

$$\begin{bmatrix} \dot{z}_1 \\ \dot{\hat{z}}_1 \end{bmatrix} = \begin{bmatrix} A(t) & -BK(t) \\ A(t)+H(t)C & -BK(t)-H(t)C \end{bmatrix} \begin{bmatrix} z_1 \\ \hat{z}_1 \end{bmatrix} \tag{67}$$

is globally exponentially stable.

Since we can write the closed-loop system (28,66) as

$$\begin{aligned} \begin{bmatrix} \dot{z}_1 \\ \dot{\hat{z}}_1 \end{bmatrix} &= \begin{bmatrix} A(t) & -BK(t) \\ A(t)+H(t)C & -BK(t)-H(t)C \end{bmatrix} \begin{bmatrix} z_1 \\ \hat{z}_1 \end{bmatrix} \\ &\quad + \begin{bmatrix} g(t, \begin{bmatrix} z_1 \\ \hat{z}_1 \end{bmatrix}, z_2) \\ 0 \end{bmatrix} z_2 \\ \dot{z}_2 &= -c_1 z_2 \end{aligned}$$

where

$$\begin{aligned} z_1 &= [x_{2,e}, \ldots, x_{n,e}]^T \\ z_2 &= x_{1,e} \\ g(t, \begin{bmatrix} z_1 \\ \hat{z}_1 \end{bmatrix}, z_2) &= -c_1[0, x_2, x_3, \ldots, x_{n-1}]^T \end{aligned}$$

The proof can be completed similar to that of Proposition A.2. ∎

6. Conclusions

In this paper we considered the tracking problem for nonholonomic systems in chained-form by means of output feedback. We combined a time-varying state feedback controller with an observer for the chained-form in a "certainty equivalence" way. The stability of the closed loop system is shown using results from time-varying cascaded systems. Under a condition of persistence of excitation, we have shown globally $\mathcal{K}$-exponential stability of the closed loop system.

Bibliography

[1] Astolfi, A. (1995): "Discontinuous output feedback control of nonholonomic chained systems." In *Proceedings of the 3rd European Control Conference*, pp. 2626–2629. Rome, Italy.

[2] Besançon, G. (1998): "State-affine systems and observer based control." In *Proceedings of the Fourth IFAC Symposium on Nonlinear Control Systems Design (NOLCOS'98)*, vol. 2, pp. 399–404. Enschede, The Netherlands.

[3] Brockett, R. (1983): "Asymptotic stability and feedback stabilization." In Brockett *et al*, Eds., *Differential Geometric Control Theory*, pp. 181–191. Birkhauser, Boston, MA.

[4] Jankovic, M., R. Sepulchre, and P. Kokotovic (1996): "Constructive Lyapunov design of nonlinear cascades." *IEEE Transactions on Automatic Control*, **41:12**, pp. 1723–1735.

[5] Jiang, Z.-P., E. Lefeber, and H. Nijmeijer (1998): "Stabilization and tracking of a nonholonomic mobile robot with saturating actuators." In *Proceedings of CONTROLO'98, Third Portuguese Conference on Aut omatic Control*, vol. 1, pp. 315–320. Coimbra, Portugal.

[6] Jiang, Z.-P. and H. Nijmeijer (1997a): "Backstepping-based tracking control of nonholonomic chained systems." In *Proceedings of the 4th European Control Conference*. Brussels, Belgium. Paper 672 (TH-M A2).

[7] Jiang, Z.-P. and H. Nijmeijer (1997b): "Tracking control of mobile robots: a case study in backstepping." *Automatica*, **33:7**, pp. 1393–1399.

[8] Jiang, Z.-P. and H. Nijmeijer (1999): "Observer-controller design for nonholonomic systems." In Nijmeijer and Fossen, Eds., *New Trends in Nonlinear Observer Design*, Lecture Notes in Control and Information Sciences. Springer Verlag, London.

[9] Kailath, T. (1980): *Linear Systems*. Prentice-Hall.

[10] Khalil, H. (1996): *Nonlinear Systems*, second edition. Prentice-Hall, Upper Saddle River, NJ USA.

[11] Kolmanovsky, I. and N. McClamroch (1995): "Developments in nonholonomic control problems." *IEEE Control Systems Magazine*, **16:6**, pp. 20–36.

[12] Loría, A., E. Panteley, H. Nijmeijer, and T. Fossen (1998): "Robust adaptive control of passive systems with unknown disturbances." In *Proceedings of the Fourth IFAC Symposium on Nonlinear Control Systems Design (NOLCOS'98)*, vol. 3, pp. 866–871. Enschede, The Netherlands.

[13] Mazenc, F. and L. Praly (1996): "Adding integrators, saturated controls and global asymptotic stibilization of feedforward systems." *IEEE Transactions on Automatic Control*, **41**, pp. 1559–1579.

[14] M.Egerstedt, X.Hu, and A.Stotsky (1998): "Control of a car-like robot using a virtual vehicle approach." In *Proceedings of the 5th International Workshop on Advanced Motion Control*. Coimbra, Portugal.

[15] Morin, P. and C. Samson (1997): "Application of backstepping techniques to the time-varying exponential stabilisation of chained form systems." *European Journal of Control*, **3**, pp. 15–36.

[16] Ortega, R. (1991): "Passivity properties for stabilization of nonlinear cascaded systems." *Automatica*, **29**, pp. 423–424.

[17] Panteley, E., E. Lefeber, A. Loría, and H. Nijmeijer (1998): "Exponential tracking control of a mobile car using a cascaded approach." In *Proceedings of the IFAC Workshop on Motion Control*. Grenoble.

[18] Panteley, E. and A. Loría (1998): "On global uniform asymptotic stability of nonlinear time-varying systems in cascade." *Systems and Control Letters*, **33:2**, pp. 131–138.

[19] Rugh, W. (1996): *Linear Systems Theory*. Prentice-Hall.

[20] Samson, C. and K. Ait-Abderrahim (1991): "Feedback control of a nonholonomic wheeled cart in cartesian space." In *Proceedings of the 1991 IEEE International Conference on Robotics and Automation*, pp. 1136–1141. Sacramento, USA.

[21] Sontag, E. (1989): "Smooth stabilization implies coprime factorization." *IEEE Transactions on Automatic Control*, **34**, pp. 435–443.

[22] Sørdalen, O. and O. Egeland (1995): "Exponential stabilization of nonholonomic chained systems." *IEEE Transactions on Automatic Control*, **40:1**, pp. 35–49.

[23] Vidyasagar, M. (1993): *Nonlinear Systems Analysis*, second edition. Prentice-Hall, Englewood Cliffs, New Jersey USA.

B

Paper B

Linear Controllers for Exponential Tracking of Systems in Chained Form

E. Lefeber, A. Robertsson, and H. Nijmeijer

Abstract

In this paper we address the tracking problem for a class of nonholonomic chained form control systems. We present a simple solution for both the state feedback and the dynamic output feedback problem. The proposed controllers are linear and render the tracking error dynamics globally $\mathcal{K}$-exponentially stable. We also deal with both control problems under input saturation.

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1. Introduction

In recent years the control, and in particular the stabilization, of nonholonomic dynamic systems has received considerable attention. One of the reasons for this is that no smooth stabilizing static state-feedback control law exists for these systems, since Brockett's necessary condition for smooth stabilization is not met [3]. For an overview we refer to the survey paper [21] and references cited therein.

Although the stabilization problem for nonholonomic control systems is now well understood, the tracking control problem has received less attention. In fact, it is unclear how the stabilization techniques available can be extended directly to tracking problems for nonholonomic systems.

In [7; 8; 17; 27; 28; 29] tracking control schemes have been proposed based on the linearization of the corresponding error model. All these papers solve the local tracking problem for some classes of nonholonomic systems. To our knowledge, the first global tracking control law was proposed in [36] for a two-wheel driven mobile car. Other global results can be found in [6; 12; 13; 15; 25; 31].

In this paper we study the tracking problem for the class of nonholonomic systems in chained form [27]. It is well known that many mechanical systems with nonholonomic constraints can be locally, or globally, converted to the chained form under coordinate change and state feedback.

A disadvantage of most of the aforementioned tracking controllers is their lack of a clear interpretation. Complicated changes of coordinates and difficult Lyapunov analysis are needed to show that the proposed control laws yield asymptotic stability of the tracking error dynamics.

The purpose of this paper is to develop *simple* tracking controllers for the class of nonholonomic systems in chained form. Based on a result for (time-varying) cascaded systems [32] we divide the tracking error dynamics into a cascade of two linear sub-systems which we can stabilize independently of each other with simple (i. e., linear) controllers.

Using the same approach we also consider the tracking problem for chained form systems by means of dynamic output-feedback. To our knowledge, the only papers that addressed the dynamic output-feedback problem are [1; 2] that concern the stabilization problem and [12; 24] dealing with the tracking problem.

Last, we partially deal with the tracking control problem under input constraints. The only results on saturated tracking control of nonholonomic systems that we are aware of, are [12] which deals with this problem for a mobile robot with two degrees of freedom, and [14] that deals with general chained form systems.

The organization of the paper is as follows: In Section 2 we present the class of systems and state the problem formulation. Based on the theory from Section 2, section 3 deals with the design of simple tracking-controllers, for both the state-feedback case and for the output-feedback case. Also both control problems under input saturation are studies in this section. Section 4 illustrates the presented design methods with simulations of an articulated vehicle and comparisons with other recent design methods are made. Finally, Section 5 concludes the paper.

2. Preliminaries and Problem Formulation

In this section we introduce definitions and theorems used in the remainder of this paper and formulate the problem under consideration. We start with some basic stability concepts in 2, present a result for cascaded systems in 2 and recall present in 2 some results from linear systems theory we use. We conclude this section with the problem formulation in 2.

Stability

To start with, we recall some basic concepts (see e.g. [19; 42]).

DEFINITION B.1
A continuous function $\alpha : [0,a) \to [0,\infty)$ is said to belong to *class* $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. □

DEFINITION B.2
A continuous function $\beta : [0,a) \times [0,\infty) \to [0,\infty)$ is said to belong to *class* $\mathcal{K}L$ if, for each fixed s, the mapping $\beta(r,s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r,s)$ is decreasing with respect to s and $\beta(r,s) \to 0$ as $s \to \infty$. □

Consider the system

$$\dot{x} = f(t,x), \quad f(t,0) = 0, \quad \forall t \geq 0 \tag{1}$$

with $x \in \mathbb{R}^n$ and $f(t,x)$ piecewise continuous in t and locally Lipschitz in x.

DEFINITION B.3
The system (1) is *uniformly stable* if for each $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$, independent of t_0, such that

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall t \geq t_0 \geq 0. \tag{2}$$

□

DEFINITION B.4
The system (1) is *globally uniformly asymptotically stable (GUAS)* if it is uniformly stable and globally attractive, that is, there exists a class $\mathcal{K}L$ function $\beta(\cdot,\cdot)$ such that for every initial state $x(t_0)$:

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t-t_0), \quad \forall t \geq t_0 \geq 0 \tag{3}$$

□

DEFINITION B.5
The system (1) is *globally exponentially stable (GES)* if there exist $k > 0$ and $\gamma > 0$ such that for any initial state $x(t_0)$:

$$\|x(t)\| \leq \|x(t_0)\| k \exp[-\gamma(t-t_0)]. \tag{4}$$

□

A slightly weaker notion of exponential stability is the following

DEFINITION B.6—CF. [37]
We call the system (1) *globally $\mathcal{K}$-exponentially stable* if there exist $\gamma > 0$ and a class $\mathcal{K}$ function $\kappa(\cdot)$ such that

$$\|x(t)\| \leq \kappa(\|x(t_0)\|) \exp[-\gamma(t-t_0)] \tag{5}$$

□

Cascaded systems

Consider the system

$$\begin{cases} \dot{z}_1 &= f_1(t,z_1) + g(t,z_1,z_2)z_2 \\ \dot{z}_2 &= f_2(t,z_2) \end{cases} \tag{6}$$

where $z_1 \in \mathbb{R}^n$, $z_2 \in \mathbb{R}^m$, $f_1(t,z_1)$ is continuously differentiable in (t,z_1) and $f_2(t,z_2)$, $g(t,z_1,z_2)$ are continuous in their arguments, and locally Lipschitz in z_2 and (z_1,z_2) respectively.

We can view the system (6) as the system

$$\Sigma_1 : \dot{z}_1 = f_1(t,z_1) \tag{7}$$

that is perturbed by the state of the system

$$\Sigma_2 : \dot{z}_2 = f_2(t,z_2). \tag{8}$$

When Σ_2 is asymptotically stable, we have that z_2 tends to zero, which means that the z_1 dynamics in (6) asymptotically reduces to Σ_1. Therefore, we can hope that asymptotic stability of both Σ_1 and Σ_2 implies asymptotic stability of (6).

Unfortunately, this is not true in general. However, from the proof presented in [32] it can be concluded that:

THEOREM B.1—BASED ON [32]
The cascaded system (6) is GUAS if the following three assumptions hold:

- *assumption on Σ_1:* the system $\dot{z}_1 = f_1(t, z_1)$ is GUAS and there exists a continuously differentiable function $V(t, z_1) : I\!R_+ \times I\!R^n \to I\!R$ that satisfies

$$W_1(z_1) \leq V(t, z_1) \leq W_2(z_1), \qquad \forall t \geq 0, \quad \forall z_1 \in I\!R^n, \tag{9}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial z_1} \cdot f_1(t, z_1) \leq 0, \qquad \forall \|z_1\| \geq \eta, \tag{10}$$

$$\left\| \frac{\partial V}{\partial z_1} \right\| \|z_1\| \leq c V(t, z_1), \qquad \forall \|z_1\| \geq \eta, \tag{11}$$

  where $W_1(z_1)$ and $W_2(z_1)$ are positive definite proper function and $c > 0$ and $\eta > 0$ are constants,

- *assumption on the interconnection:* the function $g(t, z_1, z_2)$ satisfies for all $t \geq t_0$:

$$\|g(t, z_1, z_2)\| \leq \theta_1(\|z_2\|) + \theta_2(\|z_2\|)\|z_1\|, \tag{12}$$

  where $\theta_1, \theta_2 : I\!R_+ \to I\!R_+$ are continuous functions,

- *assumption on Σ_2:* the system $\dot{z}_2 = f_2(t, z_2)$ is GUAS and for all $t_0 \geq 0$:

$$\int_{t_0}^{\infty} \|z_2(t_0, t, z_2(t_0))\| dt \leq \kappa(\|z_2(t_0)\|), \tag{13}$$

  where the function $\kappa(\cdot)$ is a class $\mathcal{K}$ function.

$\square$

REMARK B.1
Notice the assumption on Σ_1 is slightly weaker than the one presented in [32]. However, under the assumption mentioned above the result can still be shown to be true by (almost) exactly copying the proof presented in [32]. $\square$

LEMMA B.5—SEE [31]
If in addition to the assumptions in Theorem B.1 both $\dot{z}_1 = f_1(t, z_1)$ and $\dot{z}_2 = f_2(t, z_2)$ are globally $\mathcal{K}$-exponentially stable, then the cascaded system (6) is globally $\mathcal{K}$-exponentially stable. □

Linear time-varying systems

Consider the linear time-varying system

$$\begin{aligned} \dot{z} &= \underbrace{\begin{bmatrix} 0 & \dots & \dots & \dots & 0 \\ \psi(t) & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \psi(t) & 0 \end{bmatrix}}_{A(t)} z + \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}}_{B(t)} u \\ y &= \underbrace{\begin{bmatrix} 0 & \dots & \dots & 0 & 1 \end{bmatrix}}_{C(t)} z \end{aligned} \tag{14}$$

where $z \in I\!R^m$ and let $\Phi(t, t_0)$ denote the state-transition matrix for the system $\dot{z} = A(t)z$. We recall two definitions from linear control theory (cf. [16; 34]).

DEFINITION B.7
The pair $(A(t), B(t))$ is *uniformly completely controllable (UCC)* if there exist $\delta, \varepsilon_1, \varepsilon_2 > 0$ such that for all $t > 0$:

$$\varepsilon_1 I \leq \int_t^{t+\delta} \Phi(t,\tau)B(\tau)B(\tau)^T\Phi^T(t,\tau)d\tau \leq \varepsilon_2 I \tag{15}$$

□

DEFINITION B.8
The pair $(A(t), C(t))$ is *uniformly completely observable (UCO)* if there exist $\delta, \varepsilon_1, \varepsilon_2 > 0$ such that for all $t > 0$:

$$\varepsilon_1 I \leq \int_{t-\delta}^{t} \Phi^T(\tau, t-\delta)C(\tau)^T C(\tau)\Phi(\tau, t-\delta)d\tau \leq \varepsilon_2 I \tag{16}$$

□

From linear systems theory several methods are available to exponentially stabilize the linear time-varying system (14) via state or dynamic output-feedback, in case the pairs $(A(t), B(t))$ and $(A(t), C(t))$ are uniformly completely controllable and observable respectively.

ASSUMPTION B.6
We assume that $\psi(t) : [0, \infty) \to I\!R$ is a bounded continuously differentiable Lipschitz function that does not converge to zero. More precise, we assume that

- there exists a constant M such that for all t: $|\psi(t)| \leq M$,
- $\psi(t)$ is a continuously differentiable function with respect to t,
- there exists a constant L such that for all $t_1, t_2 \in [0, \infty)$: $|\psi(t_1) - \psi(t_2)| \leq L|t_1 - t_2|$,
- there exist $\delta > 0$ and $\varepsilon > 0$ such that for all $t \geq 0$ there exists an $s \in [t, t + \delta]$ such that $|\psi(s)| \geq \varepsilon$.

□

PROPOSITION B.7
Assume $\psi(t)$ satisfies the conditions of Assumption B.6. Then the system (14) is uniformly completely controllable and uniformly completely observable. □

Proof This is a direct consequence of Theorem 2 in [18]. ■

THEOREM B.2
Consider the system (14) in closed-loop with the controller

$$u = -k_1 z_1 - k_2 \psi(t) z_2 - k_3 z_3 - k_4 \psi(t) z_4 - \dots \tag{17}$$

where k_i $(i = 1, \dots, m)$ are such that the polynomial

$$\lambda^m + k_1 \lambda^{m-1} + \dots + k_{m-1} \lambda + k_m \tag{18}$$

is Hurwitz (i. e., has its roots in the left half of the open complex plane). If $\psi(t)$ meets Assumption B.6, then the closed-loop system (14,17) is GES.

□

Proof See Appendix. ■

REMARK B.2
Notice we use a linear controller of the form $u = K(t)x$ with a special choice of the gain $K(t)$. Clearly, several other choices can be made. One possibility is to use the gain as known from 'standard linear control theory' [34] as we used in [24], or a gain as proposed in [5] (c.f. [25]), based on pole-placement [41; 40] or based on any robust design method for LTV systems. □

THEOREM B.3
Consider the system (14) in closed-loop with the controller

$$u = -k_1\hat{z}_1 - k_2\psi(t)\hat{z}_2 - k_3\hat{z}_3 - k_4\psi(t)\hat{z}_4 - \dots \tag{19}$$

where $\hat{z}$ is generated from the observer

$$\begin{aligned} \dot{\hat{z}} &= \begin{bmatrix} 0 & \dots & \dots & \dots & 0 \\ \psi(t) & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \psi(t) & 0 \end{bmatrix} \hat{z} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} u + \begin{bmatrix} \vdots \\ l_4\psi(t) \\ l_3 \\ l_2\psi(t) \\ l_1 \end{bmatrix} (y - \hat{y}) \\ \hat{y} &= \begin{bmatrix} 0 & \dots & \dots & 0 & 1 \end{bmatrix} \hat{z} \end{aligned} \tag{20}$$

and k_i, l_i $(i = 1, \dots, m)$ are such that the polynomials

$$\begin{gathered} \lambda^m + k_1\lambda^{m-1} + \dots + k_{m-1}\lambda + k_m \\ \lambda^m + l_1\lambda^{m-1} + \dots + l_{m-1}\lambda + l_m \end{gathered} \tag{21}$$

are Hurwitz (i. e., have their roots in the left half of the open complex plane). If $\psi(t)$ meets Assumption B.6, then the closed-loop system (14,19,20) is GES. □

Proof See Appendix. ■

Problem formulation

The class of chained-form nonholonomic systems we study in this paper is given by the following equations

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_2 u_1 \\ &\vdots \\ \dot{x}_n &= x_{n-1} u_1 \end{aligned} \tag{22}$$

where $x = (x_1, \dots, x_n)$ is the state, u_1 and u_2 are control inputs.
Consider the problem of tracking a reference trajectory (x_r, u_r) generated

by the chained-form system:

$$\begin{aligned} \dot{x}_{1,r} &= u_{1,r} \\ \dot{x}_{2,r} &= u_{2,r} \\ \dot{x}_{3,r} &= x_{2,r}u_{1,r} \\ &\vdots \\ \dot{x}_{n,r} &= x_{n-1,r}u_{1,r} \end{aligned} \tag{23}$$

where we assume $u_{1,r}(t)$ and $u_{2,r}(t)$ to be continuous functions of time. This reference trajectory can be generated by any of the motion planning techniques available from the literature.

When we define the tracking error $x_e = x - x_r$ we obtain as tracking error dynamics

$$\begin{aligned} \dot{x}_{1,e} &= u_1 - u_{1,r} & &= u_1 - u_{1,r} \\ \dot{x}_{2,e} &= u_2 - u_{2,r} & &= u_2 - u_{2,r} \\ \dot{x}_{3,e} &= x_2u_1 - x_{2,r}u_{1,r} & &= x_{2,e}u_{1,r} + x_2(u_1 - u_{1,r}) \\ &\vdots & &\vdots \\ \dot{x}_{n,e} &= x_{n-1}u_1 - x_{n-1,r}u_{1,r} & &= x_{n-1,e}u_{1,r} + x_{n-1}(u_1 - u_{1,r}) \end{aligned} \tag{24}$$

The state-feedback tracking control problem then can be formulated as

Problem B.3—State-feedback tracking control problem
Find appropriate state feedback laws u_1 and u_2 of the form

$$u_1 = u_1(t, x, x_r, u_r) \quad \text{and} \quad u_2 = u_2(t, x, x_r, u_r) \tag{25}$$

such that the closed-loop trajectories of (24,25) are globally uniformly asymptotically stable. □

Consider the system (22) with output

$$y = \begin{bmatrix} x_1 \\ x_n \end{bmatrix} \tag{26}$$

then it is easy to show (see e.g. [1]) that the system (22) with output (26) is locally observable at any $x \in \mathbb{R}^n$.

Now we can formulate the dynamic output-feedback tracking problem as

PROBLEM B.4—DYNAMIC OUTPUT-FEEDBACK TRACKING CONTROL PROBLEM
Find appropriate control laws u_1 and u_2 of the form

$$u_1 = u_1(t, \hat{x}, y, x_r, u_r) \quad \text{and} \quad u_2 = u_2(t, \hat{x}, y, x_r, u_r) \tag{27}$$

where $\hat{x}$ is generated from an observer

$$\dot{\hat{x}} = f(t, \hat{x}, y, x_r, u_r) \tag{28}$$

such that the closed-loop trajectories of (24,27,28) are globally uniformly asymptotically stable. □

3. Controller Design

As mentioned in the introduction, our goal is to find simple controllers that globally stabilize the tracking error dynamics (24). The approach used in [15] is based on the integrator backstepping idea [4; 20; 22; 39] which consists of searching a stabilizing function for a subsystem of (24), assuming the remaining variables to be controls. Then, new variables are defined, describing the difference between the desired dynamics and the true dynamics. Subsequently a stabilizing controller for this 'new system' is looked for.

This approach has the advantage that it can lead to globally stabilizing controllers for systems in chained form. A disadvantage, however, is that the controller is also expressed in these 'new coordinates'. When written in the 'original' chained form coordinates, usually complex expressions are obtained. Especially since a change of coordinates is required to bring the dynamics (24) in a triangular form suitable for applying the integrator backstepping technique.

To arrive at simple controllers, our approach is different. We use the ideas of cascaded systems [11; 26; 30] and in particular the result for time-varying systems as presented in [32]. With the result of Theorem B.1 in mind, we try to look for a subsystem which, with a stabilizing control law, can be written in the form $\dot{z}_2 = f_2(t, z_2)$ and is asymptotically stable. In the remaining dynamics we can then replace the appearance of z_2 by 0, leading to the system $\dot{z}_1 = f_1(t, z_1)$. As a result we can write the system in the form (6). If both the subsystems $\dot{z}_1 = f_1(t, z_1)$ and $\dot{z}_2 = f_2(t, z_2)$ are asymptotically stable we might be able to conclude asymptotic stability of the overall system by means of Theorem B.1.

One could remark that for arriving at the chained form, usually complex changes of coordinates and state feedback are needed. Therefore, a simple controller in chained form coordinates is no guarantee for a simple controller in the coordinates of the original model. However, using the same idea simple controllers in the original coordinates can also be found, as was shown in [31] for a two-wheel driven mobile car.

Consider the tracking error dynamics

$$\begin{aligned}
\dot{x}_{1,e} &= u_1 - u_{1,r} \\
\dot{x}_{2,e} &= u_2 - u_{2,r} \\
\dot{x}_{3,e} &= x_{2,e}u_{1,r} + x_2(u_1 - u_{1,r}) \\
&\vdots \\
\dot{x}_{n,e} &= x_{n-1,e}u_{1,r} + x_{n-1}(u_1 - u_{1,r})
\end{aligned} \tag{29}$$

It is very easy to stabilize only the $x_{1,e}$ dynamics, for example by using

$$u_1 = u_{1,r} - kx_{1,e} \quad k > 0 \tag{30}$$

Clearly, other choices can be made as well.

Once the $x_{1,e}$ dynamics are asymptotically stable, we have determined a subsystem of the form $\dot{z}_2 = f_2(t, z_2)$. In order to arrive at the $\dot{z}_1 = f_1(t, z_1)$ dynamics, we can assume we already have stabilized the $\dot{x}_{1,e}$ dynamics, i.e., we assume $x_{1,e}(t) \equiv 0$. As a result also $u_1(t) - u_{1,r}(t) \equiv 0$. Then the remaining dynamics become

$$\begin{aligned}
\dot{x}_{2,e} &= u_2 - u_{2,r} \\
\dot{x}_{3,e} &= x_{2,e}u_{1,r} \\
&\vdots \\
\dot{x}_{n,e} &= x_{n-1,e}u_{1,r}
\end{aligned} \tag{31}$$

which is equivalent to

$$\underbrace{\begin{bmatrix} \dot{x}_{2,e} \\ \dot{x}_{3,e} \\ \vdots \\ \vdots \\ \dot{x}_{n,e} \end{bmatrix}}_{\dot{z}_1} = \underbrace{\begin{bmatrix} 0 & \dots & \dots & \dots & 0 \\ u_{1,r}(t) & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u_{1,r}(t) & 0 \end{bmatrix}}_{A(t)} \underbrace{\begin{bmatrix} x_{2,e} \\ x_{3,e} \\ \vdots \\ \vdots \\ x_{n,e} \end{bmatrix}}_{z_1} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}}_{B} (u_2 - u_{2,r}) \tag{32}$$

Now we only have to make sure that the system (32) in closed-loop with a suitably chosen feedback controller for u_2 is asymptotically stable, and hope that Theorem B.1 enables us to conclude asymptotic stability of the tracking error dynamics (29).

As a result, we have reduced the tracking control problem to the problem of finding a control law for u_1 that stabilizes the linear system

$$\dot{x}_{1,e} = u_1 - u_{1,r} \tag{33}$$

and finding a control law for u_2 that stabilizes the LTV system (32).

State-feedback

In order to solve the state-feedback tracking control problem (Problem B.3) we stabilize the systems (32) and (33). For stabilizing (32) we use the result of Theorem B.2 and for stabilizing (33) we use (30). As a result we get

THEOREM B.4
Consider the tracking error dynamics (29). Assume that $u_{1,r}(t)$ satisfies Assumption B.6 and that $x_{2,r}, \ldots, x_{n-1,r}$ are bounded.

Then the control law

$$\begin{aligned} u_1 &= u_{1,r} - k_1 x_{1,e} \\ u_2 &= u_{2,r} - k_2 x_{2,e} - k_3 u_{1,r}(t) x_{3,e} - k_4 x_{4,e} - k_5 u_{1,r}(t) x_{5,e} \ldots \end{aligned} \tag{34}$$

results in closed-loop dynamics that are globally $\mathcal{K}$-exponentially stable, provided $k_1 > 0$ and k_i $(i = 2, \ldots, n)$ are such that the polynomial

$$\lambda^{n-1} + k_2 \lambda^{n-2} + \cdots + k_{n-1} \lambda + k_n \tag{35}$$

is Hurwitz (i. e., has its roots in the left half of the open complex plane). □

Proof We can see the closed-loop system (29,34) as a system of the form (6) where

$$\begin{aligned} z_1 &= [x_{2,e}, \ldots, x_{n,e}]^T && (36) \\ z_2 &= x_{1,e} && (37) \\ f_1(t, z_1) &= (A(t) - BK(t)) z_1 && (38) \\ f_2(t, z_2) &= -k_1 z_2 && (39) \\ g(t, z_1, z_2) &= -k_1 [0, x_2, x_3, \ldots, x_{n-1}]^T && (40) \end{aligned}$$

with

$$A(t) = \begin{bmatrix} 0 & \dots & \dots & \dots & 0 \\ u_{1,r}(t) & \ddots & & & \vdots \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u_{1,r}(t) & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \tag{41}$$

$$K(t) = [\, k_2 \quad k_3 \cdot u_{1,r}(t) \quad k_4 \quad k_5 \cdot u_{1,r}(t) \quad \dots \quad]$$

To be able to apply Theorem B.1 we need to verify the three assumptions:

- assumption on Σ_1: Due to the assumption on $u_{1,r}(t)$ we have from Theorem B.2 that $\dot{z}_1 = f_1(t, z_1)$ is GES (and therefore GUAS). From converse Lyapunov theory (see e. g. [19]) the existence of a suitable V is guaranteed.

- assumption on connecting term: Since $x_{2,r}, \dots, x_{n-1,r}$ are bounded, we have

$$\begin{aligned} \|g(t, z_1, z_2)\| &\leq k_1 \left(\| \begin{bmatrix} 0 \\ x_{2,r} \\ \vdots \\ x_{n-1,r} \end{bmatrix} \| + \| \begin{bmatrix} 0 \\ x_{2,e} \\ \vdots \\ x_{n-1,e} \end{bmatrix} \| \right) && (42) \\ &\leq k_1 M + k_1 \|z_1\| && (43) \end{aligned}$$

- assumption on Σ_2: Follows from GES of $\dot{z}_2 = -k_1 z_2$.

Therefore, we conclude GUAS from Theorem B.1. Since both Σ_1 and Σ_2 are GES, Lemma B.5 gives the desired result. ∎

Dynamic output-feedback

In order to solve the dynamic output-feedback tracking control problem

(Problem B.4) we stabilize the systems

$$
\begin{aligned}
\begin{bmatrix} \dot{x}_{2,e} \\ \dot{x}_{3,e} \\ \vdots \\ \vdots \\ \dot{x}_{n,e} \end{bmatrix} &= \begin{bmatrix} 0 & \dots & \dots & \dots & 0 \\ u_{1,r}(t) & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u_{1,r}(t) & 0 \end{bmatrix} \begin{bmatrix} x_{2,e} \\ x_{3,e} \\ \vdots \\ \vdots \\ x_{n,e} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} (u_2 - u_{2,r}) \\
y_1 &= x_{n,e}
\end{aligned} \tag{44}
$$

and

$$
\begin{aligned}
\dot{x}_{1,e} &= u_1 - u_{1,r} \\
y_2 &= x_{1,e}
\end{aligned} \tag{45}
$$

For stabilizing (44) we use the result of Theorem B.3 and for stabilizing (33) we use again (30). As a result we obtain

THEOREM B.5
Consider the tracking error dynamics (29). Assume that $u_{1,r}(t)$ satisfies Assumption B.6 and that $x_{2,r}, \dots, x_{n-1,r}$ are bounded.
Then the control law

$$
\begin{aligned}
u_1 &= u_{1,r} - k_1 x_{1,e} \\
u_2 &= u_{2,r} - k_2 \hat{x}_{2,e} - k_3 u_{1,r}(t) \hat{x}_{3,e} - k_4 \hat{x}_{4,e} - k_5 u_{1,r}(t) \hat{x}_{5,e} \dots
\end{aligned} \tag{46}
$$

where $[\hat{x}_{2,e}, \dots, \hat{x}_{n,e}]^T$ is generated by the observer

$$
\begin{aligned}
\begin{bmatrix} \dot{\hat{x}}_{2,e} \\ \dot{\hat{x}}_{3,e} \\ \vdots \\ \vdots \\ \dot{\hat{x}}_{n,e} \end{bmatrix} &= \begin{bmatrix} -k_2 & -k_3 u_{1,r}(t) & \dots & \dots & \dots \\ u_{1,r}(t) & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u_{1,r}(t) & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_{2,e} \\ \hat{x}_{3,e} \\ \vdots \\ \vdots \\ \hat{x}_{n,e} \end{bmatrix} \\
&+ \begin{bmatrix} \vdots \\ l_5 u_{1,r}(t) \\ l_4 \\ l_3 u_{1,r}(t) \\ l_2 \end{bmatrix} (x_{n,e} - \hat{x}_{n,e})
\end{aligned} \tag{47}
$$

results in closed-loop dynamics that are globally $\mathcal{K}$-exponentially stable, provided that $k_1 > 0$ and k_i, l_i $(i = 2, \ldots, n)$ are such that the polynomials

$$\lambda^{n-1} + k_2\lambda^{n-2} + \cdots + k_{n-1}\lambda + k_n \tag{48}$$
$$\lambda^{n-1} + l_2\lambda^{n-2} + \cdots + l_{n-1}\lambda + l_n \tag{49}$$

are Hurwitz (i. e., have their roots in the left half of the open complex plane). □

Proof We can see the closed-loop system (29,34) as a system of the form (6) where

$$z_1 = [x_{2,e}, \ldots, x_{n,e}, \tilde{x}_{2,e}, \ldots, \tilde{x}_{n,e}]^T \tag{50}$$
$$z_2 = x_{1,e} \tag{51}$$
$$f_1(t, z_1) = \begin{bmatrix} A(t) - BK(t) & -BK(t) \\ 0 & A(t) - L(t)C \end{bmatrix} z_1 \tag{52}$$
$$f_2(t, z_2) = -k_1 z_2 \tag{53}$$
$$g(t, z_1, z_2) = -k_1[0, x_2, x_3, \ldots, x_{n-1}, \underbrace{0, \ldots, 0}_{n-1}]^T \tag{54}$$

and $\tilde{x}_{i,e} = x_{i,e} - \hat{x}_{i,e}$ $(i = 2, \ldots, n)$. To be able to apply Theorem B.1 we need to verify the three assumptions:

- assumption on Σ_1: Due to the assumption on $u_{1,r}(t)$, we have from Theorem B.3 that $\dot{z}_1 = f_1(t, z_1)$ is GES (and therefore GUAS). From converse Lyapunov theory (see e. g. [19]) the existence of a suitable V is guaranteed.
- assumption on connecting term: Since $x_{2,r}, \ldots, x_{n-1,r}$ are bounded, we have again

$$\|g(t, z_1, z_2)\| \leq k_1 M + k_1\|z_1\| \tag{55}$$

- assumption on Σ_2: Follows from GES of $\dot{z}_2 = -k_1 z_2$.

Therefore, we conclude GUAS from Theorem B.1. Since both Σ_1 and Σ_2 are GES, Lemma B.5 gives the desired result. ■

Saturated control

As in [14] we can consider the Problems B.3 and B.4 under the additional design constraint that

$$|u_1(t)| \leq u_{1,\max} \quad \forall t \geq 0 \tag{56}$$

where $u_{1,\max}$ is a constant such that $\sup_t |u_{1,r}(t)| < u_{1,\max}$.

It is obvious that if we replace the expression $u_1 = u_{1,r} - k_1 x_{1,e}$ with

$$u_1 = u_{1,r} - \sigma(x_{1,e}) \tag{57}$$

where $\sigma(\cdot)$ is any differentiable function that satisfies

- $x\sigma(x) > 0$ for all $x \neq 0$,
- $\sup_s |\sigma(s)| \leq u_{1,\max} - \sup_t |u_{1,r}(t)|$,
- $\dfrac{d\sigma}{dx}(0) > 0$,

the results of Theorems B.4 and B.5 still hold.

More interesting is the case where we not only assume the design constraint (56) on u_1, but also a design constraint on u_2:

$$|u_2(t)| \leq u_{2,\max} \quad \forall t \geq 0 \tag{58}$$

where $u_{2,\max}$ is a constant such that $\sup_t |u_{2,r}(t)| < u_{2,\max}$. To our knowledge, no saturated controller for stabilizing the general LTV system (14) has been derived in the literature yet. However, for the case that $u_{1,r}$ is constant for all t, the system (14) reduces to a time-invariant linear system. In that case the results of [38] can be used to solve the problem for both the state and dynamic output-feedback problem.

4. Simulations: Car with Trailer

In this section we apply the proposed state- and output-feedback designs for the tracking control of a well-known benchmark problem; a towing car with a single trailer, see e. g. [15; 27; 35].

The state configuration of the articulated vehicle consists of the position of the car, $(x_c,\ y_c)$, the steering angle ϕ, and the angles, $(\theta_0,\ \theta_1)$, of the car and the trailer with respect to the x-axis, see Fig. 1. The rear wheels of the car and the trailer are aligned with the chassis and are not allowed to slip sideways. The two input signals are the driving velocity of the front wheels, v, and the steering velocity, ω. The kinematic equations

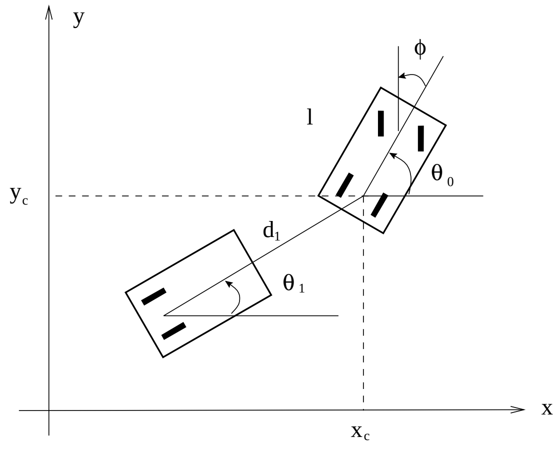

Figure 1. Car with a trailer, see [27].

of motion for the vehicle can be described by (c.f. [27]):

$$\begin{aligned}
\dot{x}_c &= v\cos\theta_0 \\
\dot{y}_c &= v\sin\theta_0 \\
\dot{\phi} &= \omega \\
\dot{\theta}_0 &= \frac{1}{l}\tan\phi \\
\dot{\theta}_1 &= \frac{1}{d_1}v\sin(\theta_0-\theta_1)
\end{aligned} \tag{59}$$

Via a (local) change of coordinates the system can be transformed to the following system in chained form.

$$\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= u_1x_2 \\
\dot{x}_4 &= u_1x_3 \\
\dot{x}_5 &= u_1x_4
\end{aligned} \tag{60}$$

(61)

We refer to [15] for explicit expressions of the transformation.

For the simulations, we have considered tracking of a reference model (23) moving along a straight line,

$$u_{1,r}=1,\ u_{2,r}=0,$$

with the initial conditions

$$\begin{aligned} x_{ir}(0) &= 0.0, \quad i = 1, \ldots, 5 \\ x_1(0) &= 1.0, \quad x_2(0) = x_3(0) = x_4(0) = x_5(0) = 0.5. \end{aligned}$$

The state-feedback *(SF)* and the output feedback controller *(OF)* used in the simulations are

$$u_{1,SF} = u_{1,r} - k_1 x_{1,e} \tag{62}$$
$$u_{2,SF} = u_{2,r} - k_2 x_{2,e} - k_3 u_{1,r} x_{3,e} - k_4 x_{4,e} - k_5 u_{1,r} x_{5,e} \tag{63}$$
$$u_{1,OF} = u_{1,r} - k_1 x_{1,e} \tag{64}$$
$$u_{2,OF} = u_{2,r} - k_2 \hat{x}_{2,e} - k_3 u_{1,r} \hat{x}_{3,e} - k_4 \hat{x}_{4,e} - k_5 u_{1,r} \hat{x}_{5,e} \tag{65}$$

where the ‘controller polynomial’ (48) has all the roots in $\lambda = -2$ and the ‘observer polynomial’ (49) has its roots in $\lambda = -3$.

In Figure 2 the behavior of the closed-loop system for the state-feedback controller *(SF)* and the output-feedback controller *(OF)* are compared to a recently presented state-feedback controller, *J&N(106-7)*, based on a backstepping design [15].

$$u_{2,JN} = -k_4 z_4 - 2k_4 z_2 - u_{1r}(3z_3 + z_1) \tag{66}$$
$$u_{1,JN} = u_{1r} - k_5 z_5 - [k_4 z_4 + 2k_4 z_2 + u_{1r}(3z_3 + z_1)] \cdot \tag{67}$$
$$\left[z_1 + z_3 - \frac{5}{2} z_2 z_5 - z_4 z_5 + \frac{(2z_1 + z_3) z_5^2}{6} \right] \tag{68}$$

where

$$\begin{aligned} z_1 &= x_5 - x_4 x_{1,e} + \frac{1}{2} x_3 x_{1,e}^2 - \frac{1}{6} x_2 x_{1,e}^3 \\ z_2 &= x_4 - x_3 x_{1,e} + \frac{1}{2} x_2 x_{1,e}^2 \\ z_3 &= x_3 - x_2 x_{1,e} \\ z_4 &= x_2 \\ z_5 &= x_{1,e} \end{aligned} \tag{69}$$

For the case of constant $u_{1,r}$ we can apply the ideas from [38] for bounded control also on u_2. Figure 3 and 4 show the tracking error in the y-direction using bounded control of u_2 for the state-feedback and the output-feedback case. The saturated state-feedback controller [38] has the structure

$$u_{1,sat} = u_{1,r} - \sigma_1(x_{1,e}) \tag{70}$$
$$u_{2,sat} = u_{2,r} - \sum_{j=1}^{4} \varepsilon^i \sigma_2(y_i) \tag{71}$$

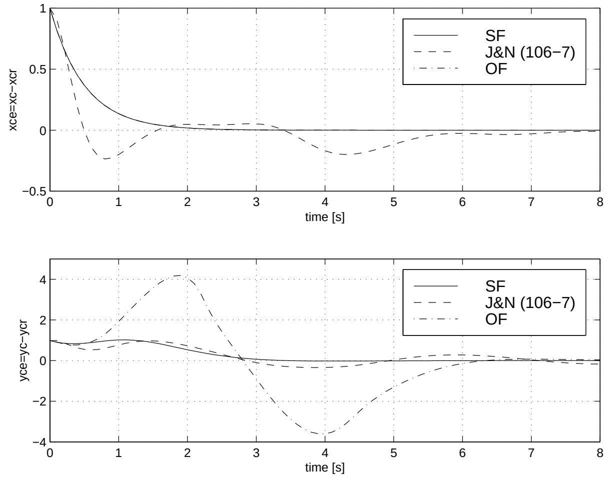

Figure 2. The tracking errors xce and yce for the state-feedback controller (SF), the output-feedback controller (OF) and the state-feedback controller in [15]. Note that xce is identical for SF and OF.

where

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \varepsilon & 0 & 0 \\ 1 & \varepsilon^2+\varepsilon & \varepsilon^3 & 0 \\ 1 & \varepsilon^3+\varepsilon^2+\varepsilon & \varepsilon^5+\varepsilon^4+\varepsilon^3 & \varepsilon^6 \end{bmatrix} \begin{bmatrix} x_{2,e} \\ x_{3,e} \\ x_{4,e} \\ x_{5,e} \end{bmatrix} \tag{72}$$

and the saturated output-feedback controller uses the state estimations from the observer (47) in a certainty equivalence sense.

5. Concluding Remarks

In this paper we addressed the problem of designing simple global tracking controllers for nonholonomic systems in chained form under both state and dynamic output feedback.

We divided the (nonlinear) tracking control problem into two simpler and ‘independent’ linear control problems. We showed by means of cascaded system theory that the two linear controllers that solve the two linear control problems also solve the tracking problem.

The state and dynamic output feedback tracking problem under input saturation were globally solved in case we have input saturation only on

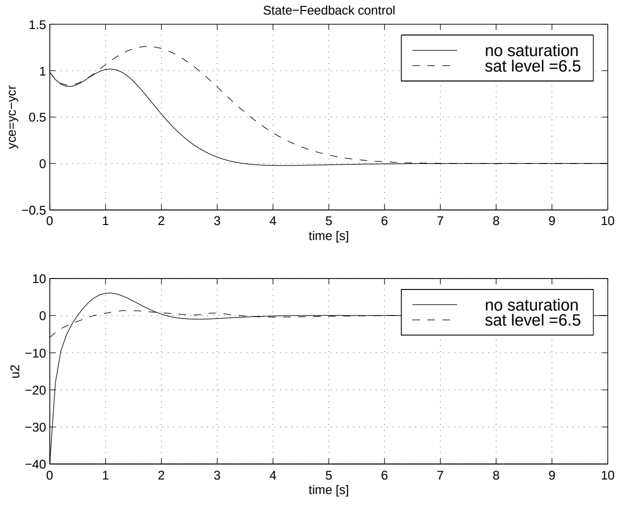

Figure 3. State feedback control with and without saturated u_2.

u_1. In case of input saturation on u_1 and u_2 both problems were solved for constant $u_{1,r}$.

We illustrated our results by means of a simulation of a car with a trailer.

Challenging questions that remain open are the tracking problem under input saturation on u_1 and u_2 for arbitrary $u_{1,r}$ and the study for robustness of the proposed schemes.

Appendix: Proofs of Theorems B.2 and B.3

To start with, we consider the stability of the differential equation

$$\frac{d^m}{dt^m}y(t) + a_1\frac{d^{m-1}}{dt^{m-1}}y(t) + \cdots + a_{m-1}\frac{d}{dt}y(t) + a_m y(t) = 0 \tag{73}$$

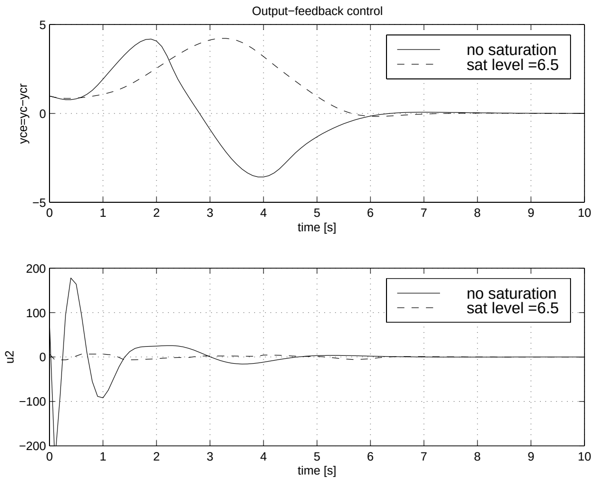

Figure 4. Output feedback control with and without saturated u_2.

For this system we can define the Hurwitz-determinants

$$\Delta_i = \begin{vmatrix} a_1 & a_3 & a_5 & \dots & a_{2i-1} \\ 1 & a_2 & a_4 & \dots & a_{2i-2} \\ 0 & a_1 & a_3 & \dots & a_{2i-3} \\ 0 & 1 & a_2 & \dots & a_{2i-4} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_i \end{vmatrix} \quad (i = 1, \dots, m) \tag{74}$$

where if an element a_j appears in Δ_i with $j > i$ it is assumed to be zero. It is well known [9] that the system (73) is asymptotically stable, if and only if the determinants Δ_i are all positive. Less known is a proof of this result by means of the second method of Lyapunov. If we define

$$b_1 = \Delta_1, \quad b_2 = \frac{\Delta_2}{\Delta_1}, \quad b_3 = \frac{\Delta_3}{\Delta_1 \Delta_2}, \quad \dots, \quad b_i = \frac{\Delta_{i-3}\Delta_i}{\Delta_{i-2}\Delta_{i-1}} \quad (i = 4, \dots, m) \tag{75}$$

it was shown in [33] that the system (73) can also be represented as

$$\dot{w} = \begin{bmatrix} -b_1 & -b_2 & 0 & \dots & 0 \\ 1 & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -b_m \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix} w \tag{76}$$

Differentiating the Lyapunov function candidate

$$V = b_1 w_1^2 + b_1 b_2 w_2^2 + \cdots + b_1 b_2 \cdots b_{m-1} w_{m-1}^2 + b_1 b_2 \cdots b_m w_m^2 \tag{77}$$

(which is positive definite if and only if the determinants Δ_i are all positive) along solutions of (76) results in

$$\dot{V} = -b_1^2 w_1^2 \tag{78}$$

Asymptotic stability then can be shown by invoking LaSalle's theorem [23].

Inspired by the result of [33] we look for a state-transformation $z = Sw$, that transforms the system (76) into

$$\dot{z} = \begin{bmatrix} -a_1 & -a_2 & \dots & \dots & -a_m \\ 1 & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix} z \tag{79}$$

To start with, we define

$$z_m = w_m \tag{80}$$

Since $\dot{w}_m = w_{m-1}$ and we would like $\dot{z}_m = z_{m-1}$ we define

$$z_{m-1} = w_{m-1} \tag{81}$$

Since $\dot{w}_{m-1} = w_{m-2} - b_m w_m$ and we would like $\dot{z}_{m-1} = z_{m-2}$ we define

$$z_{m-2} = w_{m-2} - b_m w_m \tag{82}$$

Proceeding similarly we define all z_k and obtain an expression that look like

$$z_k = w_k + s_{k,k+2} \cdot w_{k+2} + s_{k,k+4} \cdot w_{k+4} + \dots \tag{83}$$

By this construction of the state-transformation, we are guaranteed to meet the $m-1$ final equations of (79). The only thing that remains to be verified is if the equation for $\dot{z}_1$ holds. From the structure displayed in (83) we know the matrix S is nonsingular, so therefore we can write

$$\dot{z}_1 = -\alpha_1 z_1 - \alpha_2 z_2 - \cdots - \alpha_n z_n, \quad \alpha_i \in \mathbb{R}, \ \ (i = 1, \dots, m). \tag{84}$$

The characteristic polynomial of the transformed system then becomes

$$\lambda^m + \alpha_1 \lambda^{m-1} + \cdots + \alpha_{m-1}\lambda + \alpha_m \tag{85}$$

Since a state-transformation does not change the characteristic polynomial and we know from [33] that the characteristic polynomial of (76) equals

$$\lambda^m + a_1 \lambda^{m-1} + \cdots + a_{m-1}\lambda + a_m \tag{86}$$

clearly $\alpha_i = a_i$ $(i = 1, \dots, m)$.

Before we can prove Theorems B.2 and B.3 we need to remark one thing about this transformation. When we define $T = S^{-1}$, we know that

$$\begin{aligned} w_1 &= z_1 + t_{1,3} z_3 + t_{1,5} z_5 + \dots && (87) \\ w_2 &= z_2 + t_{2,4} z_4 + t_{2,6} z_6 + \dots && (88) \end{aligned}$$

But also $\dot{w}_1 = -a_1 w_1 - b_2 w_2$ (notice that $b_1 = a_1$). Therefore,

$$\begin{aligned} \dot{w}_1 &= \dot{z}_1 + t_{1,3}\dot{z}_3 + t_{1,5}\dot{z}_5 + \dots && (89) \\ &= (-a_1 z_1 - a_2 z_2 - \cdots - a_n z_n) + t_{1,3} z_2 + t_{1,5} z_4 + \dots && (90) \\ &= [-a_1 z_1 - a_3 z_3 - \dots] + [(t_{1,3} - a_2) z_2 + (t_{1,5} - a_4) z_4 + \dots] && (91) \end{aligned}$$

So obviously

$$w_1 = z_1 + \frac{a_3}{a_1} z_3 + \frac{a_5}{a_1} z_5 + \dots \tag{92}$$

Knowing this state-transformation and (92) we can start proving Theorems B.2 and B.3.

Proof of Theorem B.2 The closed-loop system (14,17) is given by

$$\dot{z} = \begin{bmatrix} -a_1 & -a_2 u_{1,r}(t) & -a_3 & -a_4 u_{1,r}(t) & \dots \\ u_{1,r}(t) & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u_{1,r}(t) & 0 \end{bmatrix} z \tag{93}$$

This can we rewritten as

$$\begin{aligned} \dot{z} = u_{1,r}(t) & \begin{bmatrix} -a_1 & -a_2 & \dots & \dots & -a_m \\ 1 & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix} z \\ + (u_{1,r}(t) - 1) & \begin{bmatrix} a_1 z_1 + a_3 z_3 + \dots \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \end{aligned} \tag{94}$$

When we apply the change of coordinates $z = Sw$ as defined before, we obtain

$$\begin{aligned} \dot{w} = u_{1,r}(t) & \begin{bmatrix} -b_1 & -b_2 & 0 & \dots & 0 \\ 1 & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -b_m \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix} w \\ + (u_{1,r}(t) - 1) & \begin{bmatrix} 1 & * & \dots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 w_1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \end{aligned} \tag{95}$$

which (using $a_1 = b_1$) can we rewritten as

$$\dot{w} = \begin{bmatrix} -b_1 & -b_2 u_{1,r}(t) & 0 & \dots & 0 \\ u_{1,r}(t) & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -b_m u_{1,r}(t) \\ 0 & \dots & 0 & u_{1,r}(t) & 0 \end{bmatrix} w \tag{96}$$

Consider the Lyapunov function candidate

$$V = b_1 w_1^2 + b_1 b_2 w_2^2 + \cdots + b_1 b_2 \cdots b_{m-1} w_{m-1}^2 + b_1 b_2 \cdots b_m w_m^2 \tag{97}$$

which is positive definite if and only if

$$\lambda^m + a_1 \lambda^{m-1} + \cdots + a_{m-1} \lambda + a_m \tag{98}$$

is a Hurwitz-polynomial. Differentiating (97) along solutions of (96) results in

$$\dot{V} = -b_1^2 w_1^2 \tag{99}$$

It is well known [19] that the origin of the system (96) is GES if the pair

$$\left(\begin{bmatrix} -b_1 & -b_2 u_{1,r}(t) & 0 & \dots & 0 \\ u_{1,r}(t) & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -b_m u_{1,r}(t) \\ 0 & \dots & 0 & u_{1,r}(t) & 0 \end{bmatrix}, [b_1, 0, \dots, 0] \right) \tag{100}$$

is uniformly completely observable (UCO).

If $u_{1,r}(t)$ satisfies Assumption B.6 it follows immediately from Theorem 2 in [18] that the pair (100) is UCO, which completes the proof. ∎

Proof of Theorem B.3 We can write the closed-loop system (14,19,20) as

$$\begin{bmatrix} \dot{z} \\ \dot{\tilde{z}} \end{bmatrix} = \begin{bmatrix} A(t) - BK(t) & -BK(t) \\ 0 & A(t) - L(t)C \end{bmatrix} \begin{bmatrix} z \\ \tilde{z} \end{bmatrix} \tag{101}$$

where $\tilde{z} = z - \hat{z}$.

Since $u_{1,r}(t)$ satisfies Assumption B.6 and k_i, l_i are such that the polynomials (21) are Hurwitz, we know from Theorem B.2 that the systems $\dot{z} = [A(t) - BK(t)]z$ and $\dot{\tilde{z}} = [A(t) - L(t)C]\tilde{z}$ are GES.

Then the result follows immediately from Theorem B.1, (since $K(t)$ is bounded), and the fact that a LTV system which is GUAS also is GES [10; 19]. ∎

Bibliography

[1] Astolfi, A. (1995): "Discontinuous output feedback control of nonholonomic chained systems." In *Proceedings of the 3rd European Control Conference*, pp. 2626–2629. Rome, Italy.

[2] Astolfi, A. and W. Schaufelberger (1997): "State and output feedback stabilization of multiple chained systems with discontinuous control." *Systems and Control Letters*, **32**, pp. 49–56.

[3] Brockett, R. (1983): "Asymptotic stability and feedback stabilization." In Brockett *et al*, Eds., *Differential Geometric Control Theory*, pp. 181–191. Birkhauser, Boston, MA.

[4] Byrnes, C. and A. Isidori (1989): "New results and examples in nonlinear feedback stabilization." *Systems and Control Letters*, **24:12**, pp. 437–442.

[5] Chen, M.-S. (1997): "Control of linear time-varying systems by the gradient algorithm." In *Proceedings of the 36th Conference on Decision and Control*, pp. 4549–4553. San Diego, California USA.

[6] Escobar, G., R. Ortega, and M. Reyhanoglu (1998): "Regulation and tracking of the nonholonomic double integrator: A field-oriented control approach." *Automatica*, **34:1**, pp. 125–132.

[7] Fierro, R. and F. Lewis (1995): "Control of a nonholonomic mobile robot: backstepping kinematics into dynamics." In *Proceedings 34th IEEE Conference on Decision and Control*, pp. 3805–3810. New Orleans, LA.

[8] Fliess, M., J. Levine, P. Martin, and P. Rouchon (1995): "Design of trajectory stabilizing feedback for driftless flat systems." In *Proceedings of the 3rd European Control Conference*, pp. 1882–1887. Rome, Italy.

[9] Hurwitz, A. (1895): "Über die Bediengungen unter welchen eine Gleichung nur Wurzeln mit negativen reellen Teilen besitzt." *Mathematische Annalen*, **46**, pp. 273–284.

[10] Ioannou, P. and J. Sun (1996): *Robust Adaptive Cotrnol*. Prentice-Hall, Upper Saddle River, NJ USA.

[11] Jankovic, M., R. Sepulchre, and P. Kokotovic (1996): "Constructive Lyapunov design of nonlinear cascades." *IEEE Transactions on Automatic Control*, **41:12**, pp. 1723–1735.

[12] Jiang, Z.-P., E. Lefeber, and H. Nijmeijer (1998): "Stabilization and tracking of a nonholonomic mobile robot with saturating actuators." In *Proceedings of CONTROLO'98, Third Portuguese Conference on Aut omatic Control*, vol. 1, pp. 315–320. Coimbra, Portugal.

[13] Jiang, Z.-P. and H. Nijmeijer (1997): "Tracking control of mobile robots: a case study in backstepping." *Automatica*, **33:7**, pp. 1393–1399.

[14] Jiang, Z.-P. and H. Nijmeijer (1999a): "Observer-controller design for nonholonomic systems." In Nijmeijer and Fossen, Eds., *New Trends in Nonlinear Observer Design*, Lecture Notes in Control and Information Sciences. Springer Verlag, London.

[15] Jiang, Z.-P. and H. Nijmeijer (1999b): "A recursive technique for tracking control of nonholonomic systems in chained form." *IEEE Transactions on Automatic Control*, **44:2**, pp. 265–279.

[16] Kailath, T. (1980): *Linear Systems*. Prentice-Hall.

[17] Kanayama, Y., Y. Kimura, F. Miyazaki, and T. Noguchi (1990): "A stable tracking control scheme for an autonomous mobile robot." In *Proceedings IEEE International Conference on Robotics and Automation*, pp. 384–389.

[18] Kern, G. (1982): "Uniform controllability of a class of linear time-varying systems." *IEEE Transactions on Automatic Control*, **27:1**, pp. 208–210.

[19] Khalil, H. (1996): *Nonlinear Systems*, second edition. Prentice-Hall, Upper Saddle River, NJ USA.

[20] Koditschek, D. (1987): "Adaptive techniques for mechanical systems." In *Proceedings 5th Yale Workshop on Adaptive Systems*, pp. 259–265. New Haven, CT.

[21] Kolmanovsky, I. and N. McClamroch (1995): "Developments in nonholonomic control problems." *IEEE Control Systems Magazine*, **16:6**, pp. 20–36.

[22] Krstić, M., I. Kanellakopoulos, and P. Kokotović (1995): *Nonlinear and Adaptive Control Design*. Series on Adaptive and Learning Systems for Signal Processing, Communications, and Control. John Wiley and Sons.

[23] LaSalle, J. (1960): "Some extensions of Lyapunov's second method." *IRE Transactions on Circuit Theory*, **7:4**, pp. 520–527.

[24] Lefeber, E., A. Robertsson, and H. Nijmeijer (1999a): "Linear controllers for tracking chained-form systems." In Aeyels *et al*, Eds., *Stability and Stabilization*, Lecture Notes in Control and Information Sciences. Springer Verlag.

[25] Lefeber, E., A. Robertsson, and H. Nijmeijer (1999b): "Output feedback tracking of nonholonomic systems in chained form." In *Proceedings of the 5th European Control Conference*. Karlsruhe, Germany. Paper 772 (AP-4).

[26] Mazenc, F. and L. Praly (1996): "Adding integrators, saturated controls and global asymptotic stibilization of feedforward systems." *IEEE Transactions on Automatic Control*, **41**, pp. 1559–1579.

[27] Micaelli, A. and C. Samson (1993): "Trajectory tracking for unicycle-type and two-steering-wheels mobile robots." Technical Report 2097. INRIA.

[28] Murray, R., G. Walsh, and S. Sastry (1992): "Stabilization and tracking for nonholonomic control systems using time-varying state feedback." In Fliess, Ed., *Proceedings of the Second IFAC Symposium on Nonlinear Control Systems Design (NOLCOS'92)*, pp. 109–114. Bordeaux.

[29] Oelen, W. and J. van Amerongen (1994): "Robust tracking control of two-degrees-of-freedom mobile robots." *Control Engineering Practice*, **2**, pp. 333–340.

[30] Ortega, R. (1991): "Passivity properties for stabilization of nonlinear cascaded systems." *Automatica*, **29**, pp. 423–424.

[31] Panteley, E., E. Lefeber, A. Loría, and H. Nijmeijer (1998): "Exponential tracking control of a mobile car using a cascaded approach." In *Proceedings of the IFAC Workshop on Motion Control*. Grenoble.

[32] Panteley, E. and A. Loría (1998): "On global uniform asymptotic stability of nonlinear time-varying systems in cascade." *Systems and Control Letters*, **33:2**, pp. 131–138.

[33] Parks, P. (1962): "A new proof of the Routh-Hurwitz stability criterion using the second method of Liapunov." *Proceedings of the Cambridge Philosophical Society*, **58:4**, pp. 694–702.

[34] Rugh, W. (1996): *Linear Systems Theory*. Prentice-Hall.

[35] Samson, C. (1995): "Control of chained systems application to path following and time-varying point-stabilization of mobile robots." *IEEE Transactions on Automatic Control*, **40:1**, pp. 64–77.

[36] Samson, C. and K. Ait-Abderrahim (1991): "Feedback control of a nonholonomic wheeled cart in cartesian space." In *Proceedings of the 1991 IEEE International Conference on Robotics and Automation*, pp. 1136–1141. Sacramento, USA.

[37] Sørdalen, O. and O. Egeland (1995): "Exponential stabilization of nonholonomic chained systems." *IEEE Transactions on Automatic Control*, **40:1**, pp. 35–49.

[38] Sussmann, H., E. Sontag, and Y. Yang (1994): "A general result on the stabilization of linear systems using bounded controls." *IEEE Transactions on Automatic Control*, **39:12**, pp. 2411–2425.

[39] Tsinias, J. (1989): "Sufficient Lyapunov-like conditions for stabilization." *Math. Control Signals Systems*, **2**, pp. 343–357.

[40] Valášek, M. and N. Olgac (1995a): "Efficient eigenvalue assignment for general linear MIMO systems." *Automatica*, **31:11**, pp. 1605–1617.

[41] Valášek, M. and N. Olgac (1995b): "Efficient pole placement technique for linear time-variant SISO systems." *IEE Proceedings — Control Theory and Applications*, **142:5**, pp. 451–458.

[42] Vidyasagar, M. (1993): *Nonlinear Systems Analysis*, second edition. Prentice-Hall, Englewood Cliffs, New Jersey USA.

Paper C

Extension of the Yakubovich-Kalman-Popov Lemma and Stability Analysis of Dynamic Output Feedback Systems

R. Johansson and A. Robertsson

Abstract

This paper presents theory for extension of the Yakubovich-Kalman-Popov (YKP) lemma for stability analysis relevant for observer-based feedback control systems. We show that minimality is not necessary for existence of Lur'e-Lyapunov functions. Implications for output feedback stabilization, positivity, factorization and passivity are given.

Keywords: Stability, Nonlinear Systems, Output Feedback, Factorization

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1. Introduction

There are at least three contexts in which the Yakubovich-Kalman-Popov (YKP) lemma turns out to be instrumental for a solution: the absolute stability problem of nonlinear feedback control; *cf.* [8], [13], [23], [5]; [2]; and adaptive control based on output error feedback; [18]; and the covariance-matrix factorization of realization theory; [1];. We quote the lemma deriving from Yakubovich (1962), Kalman (1963), and Popov (1961)

YKP Lemma: Let $G_0(s) = C(sI-A)^{-1}B+D$ be a $p\times p$ transfer function where A is Hurwitzian, (A,B) is controllable and (A,C) is observable. Then, $G_0(s)$ is strictly positive real if and only if there exist a positive symmetric matrix P, matrices L, R, and a positive constant ε such that

$$\begin{aligned} PA + A^TP &= -LL^T - \varepsilon P \\ PB - C^T &= -LR^T \\ D + D^T &= RR^T \end{aligned} \tag{1}$$

□

This important lemma supports results of stability theory of nonlinear feedback (circle criterion; Popov criterion) and adaptive system theory and for extensions of Lyapunov theory such as passivity theory; [5], [17]. Although the lemma is very powerful in its predictions, the strictly positive real (SPR) conditions imposed on the transfer function are, unfortunately, rather restrictive for application. As for the absolute stability problem of nonlinear feedback systems, the starting point is the Lur'e problem of [10] with a linear system and nonlinear feedback of cone-bounded nonlinear variation described by the function $\psi(\cdot)$

$$\dot{x} = Ax + Bu = Ax - B\psi(z) \tag{2}$$

$$z = Cx \tag{3}$$

$$u = -\psi(z), \quad \psi^T(z)(\psi(z) - Kz) \le 0 \tag{4}$$

A Lyapunov function candidate is the Lur'e-Lyapunov function

$$V(x,z) = x^TPx + \alpha \int_0^z \psi^T(\zeta)Kd\zeta, \tag{5}$$

$$P = P^T > 0, \quad \alpha > 0 \tag{6}$$

which satisfies requirements on 'positivity', 'radial growth', 'continuity' and 'differentiation'. As guaranteed by the YKP lemma, the stability condition $\dot{V} \le 0$ holds under fairly restrictive SPR conditions. The circle theorem for SPR systems deals with the time-varying case under the assumption that $\psi(\cdot,t)$ belongs to the cone $[0,\infty]$ and that $\inf_\omega \operatorname{Re} G(j\omega) > 0$ for which the Lur'e system is $\mathcal{L}_2$−stable; [5]. As the Lyapunov functions generated by backstepping techniques apparently cannot be designed using the YKP lemma, there might exist other relevant classes of non-SPR systems for which extended regions of stability might exist; [7]. In our results, we extend stability theory to include non-SPR transfer functions and dynamic output feedback or observer-based design violating the controllability condition of the YKP lemma. A constructive method to provide quadratic Lyapunov functions and Lur'e-Lyapunov function for nonlinear output feedback of non-SPR transfer functions is given.

2. Problem Formulation

We will consider a class of problems which include observer-based feedback control and other classes of dynamic output feedback described by the state-space system

$$\begin{aligned}
\frac{dx_0}{dt} &= A_0 x_0 + B_0 u && (7)\\
\frac{d\xi}{dt} &= A_\xi \xi + B_\xi u + K_\xi C_0 x_0 && (8)\\
z &= \begin{pmatrix} C_0 & C_\xi \end{pmatrix} \begin{pmatrix} x_0 \\ \xi \end{pmatrix} + (D_0 + D_\xi)\, u && (9)\\
u &= -\psi(z) && (10)
\end{aligned}$$

where $\{A_0, B_0, C_0, D_0\}$ and the state x_0 represent the original system—*i.e.*, the control object—whereas $\{A_\xi, B_\xi, C_\xi, D_\xi, K_\xi\}$ and ξ represent the observer dynamics added to the system description. As all observer states without restriction may be assumed available to measurement, the matrix C_ξ may be chosen freely, say, as motivated by calculated output feedback parameters. This class of systems includes systems which violate the controllability assumption of the YKP lemma. In shorter notation, we have

$$\begin{aligned}
\frac{dx}{dt} &= Ax + Bu, \quad x = \begin{pmatrix} x_0 \\ \xi \end{pmatrix} \in \mathbb{R}^n, && (11)\\
z &= Cx + Du, \quad z \in \mathbb{R}^m, \quad u \in \mathbb{R}^m && (12)\\
u &= -\psi(z) && (13)
\end{aligned}$$

and for the Lur'e-Lyapunov function

$$V(x) = \frac{1}{2} x^T P x + \alpha \int_0^z \psi^T(\zeta) K d\zeta \qquad (14)$$

A Lyapunov Equation

Let $Q = Q^T \geq 0$ be given. The standard YKP matrix equation may then be formulated as

$$\begin{aligned} -Q &= \begin{pmatrix} PA + A^T P & PB - C^T \\ B^T P - C & -(D + D^T) \end{pmatrix} \\ &= \begin{pmatrix} P & 0 \\ 0 & I_m \end{pmatrix} \begin{pmatrix} A & B \\ -C & -D \end{pmatrix} \\ &+ \begin{pmatrix} A & B \\ -C & -D \end{pmatrix}^T \begin{pmatrix} P & 0 \\ 0 & I_m \end{pmatrix} \end{aligned} \tag{15}$$

Let

$$\mathcal{A} = \begin{pmatrix} A & B \\ -C & -D \end{pmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}, \tag{16}$$

$$\mathcal{P}_o = \begin{pmatrix} P & 0 \\ 0 & I_m \end{pmatrix} \in \mathbb{R}^{(n+m)\times(n+m)} \tag{17}$$

Then, the YKP matrix equation (15) may be reformulated as the Lyapunov equation

$$\mathcal{P}_o \mathcal{A} + \mathcal{A}^T \mathcal{P}_o = -Q \tag{18}$$

Conversely, a matrix triple $\{\mathcal{A}, Q, \mathcal{P}\}$ resulting from such a Lyapunov equation may provide a Lyapunov function or a Lur'e-Lyapunov function relevant for stability analysis. This observation is pursued throughout the paper. We will show that the Lyapunov equation (18) may contribute to stability theory for non-SPR systems.

3. Results and Extensions

LEMMA C.1—YKP LYAPUNOV EQUATION
Given a linear system $\Sigma(A, B, C, D)$ with the system matrix

$$\mathcal{A} = \begin{pmatrix} A & B \\ -C & -D \end{pmatrix} \in \mathbb{R}^{(n+m)\times(n+m)} \tag{19}$$

and a matrix $Q \in \mathbb{R}^{(n+m)\times(n+m)}$, $Q = Q^T > 0$, there is solution $\mathcal{P} = \mathcal{P}^T > 0$ to the Lyapunov equation

$$-Q = \mathcal{P}\mathcal{A} + \mathcal{A}^T \mathcal{P} \tag{20}$$

$$\mathcal{P} = \begin{pmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{12}^T & \mathcal{P}_{22} \end{pmatrix} \in \mathbb{R}^{(n+m)\times(n+m)} \tag{21}$$

if and only if all eigenvalues $\{\lambda_k\}_{k=1}^{n+m}$ of $\mathcal{A}$ have negative real part. Moreover, every positive definite solution $\mathcal{P} = \mathcal{P}^T > 0$ to the Lyapunov equation (20) provides a solution $P \in \mathbb{R}^{n\times n}$ to the YKP matrix equation

$$-Q \quad = \quad \begin{pmatrix} PA + A^T P & PB - C^T \\ B^T P - C & -(D + D^T) \end{pmatrix} \tag{22}$$

as

$$P = \mathcal{P}_{11} - \mathcal{P}_{12}\mathcal{P}_{22}^{-1}\mathcal{P}_{12}^T, \quad P = P^T > 0 \tag{23}$$

under the feedback transformation

$$\mathcal{T} \quad = \quad \begin{pmatrix} I_n & 0 \\ -\mathcal{P}_{22}^{-1}\mathcal{P}_{12}^T & \mathcal{P}_{22}^{-1/2} \end{pmatrix} \tag{24}$$

The transfer function of the system matrix obtained from the feedback transformation $\mathcal{T}^{-1}\mathcal{A}\mathcal{T}$ is stable and positive real. □

Proof By the correspondence of Eq. (18) and Eq. (15), it is clear that the YKP matrix equation can be reformulated as Lyapunov equation. The converse is not obvious as the Lyapunov equation (18) resulting from Eq. (15) requires the solution $\mathcal{P}$ to be block diagonal. To that end, consider any matrix solution $\mathcal{P} = \mathcal{P}^T > 0$ with decomposition

$$\mathcal{P} = \begin{pmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{12}^T & \mathcal{P}_{22} \end{pmatrix} \tag{25}$$

If $\mathcal{P}$ is of full rank, then $\mathcal{P}_{22}$ is invertible and there always exists the similarity transformation matrix of Eq. (24) which renders

$$\mathcal{P}_o = \mathcal{T}^T\mathcal{P}\mathcal{T} = \begin{pmatrix} \mathcal{P}_{11} - \mathcal{P}_{12}\mathcal{P}_{22}^{-1}\mathcal{P}_{12}^T & 0 \\ 0 & I_m \end{pmatrix} > 0$$

Left multiplication of the Lyapunov equation by $\mathcal{T}^T$ and right multiplication by $\mathcal{T}$ preserves the Lyapunov equation structure but renders the solution $\mathcal{T}^T\mathcal{P}\mathcal{T}$ on the block-diagonal form of Eq. (17) with a resultant similarity transformation $\mathcal{T}^{-1}\mathcal{A}\mathcal{T}$. ■

Remark C.1

The appropriate block-diagonal form can be obtained via feedback design—*e.g.*, as suggested by $\mathcal{T}$. In order to accomplish positive realness, the right multiplication implicitly suggests a state feedback control law to stabilize the system whereas the left multiplication suggests an observer structure. Also, by choice of Q such that

$$Q = \mathcal{T}^{-T} Q_0 \mathcal{T}^{-1} \tag{26}$$

the Lyapunov equation may have a solution on the block-diagonal form for Q_0 given. □

Example Consider the state-space system

$$\begin{aligned} \dot{x} &= x(t) - 2u(t) && (27) \\ y(t) &= -2x(t) + 2u(t) && (28) \end{aligned}$$

and $G_0(s) = 2(s+1)/(s-1)$. Although the system is unstable, the parameter matrix

$$\mathcal{A} = \begin{pmatrix} A & B \\ -C & -D \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & -2 \end{pmatrix}, \tag{29}$$

has eigenvalues with negative real part. The feedback transformation matrix

$$\mathcal{T} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \tag{30}$$

gives the transformed system matrix. For

$$\begin{aligned} \mathcal{A}_{\mathcal{T}} &= \mathcal{T}^{-1}\mathcal{A}\mathcal{T} = \begin{pmatrix} -1 & -2 \\ 1 & 0 \end{pmatrix}, \quad G_{\mathcal{T}}(s) = \frac{2}{s+1} \\ Q &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{P}_o = \mathcal{P}_o^T = \begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix} > 0, \end{aligned} \tag{31}$$

there is a solution to the Lyapunov equation $\mathcal{P}_o\mathcal{A}_{\mathcal{T}} + \mathcal{A}_{\mathcal{T}}^T\mathcal{P}_o = -Q$. Note that the $\mathcal{A}_{\mathcal{T}}$ resulting from the feedback transformation of the unstable system provides a block-diagonal solution $\mathcal{P} = \mathcal{P}^T > 0$ and thus $P = P^T = 0.5 > 0$.

LEMMA C.2—POSITIVITY AND FACTORIZATION
Let $\{A, B, C, D\}$ be a LTI state-space system and

$$\mathcal{A} = \begin{pmatrix} A & B \\ -C & -D \end{pmatrix} \tag{32}$$

If $(\mathcal{A}, Q^{1/2})$ is observable and if all eigenvalues of $\mathcal{A}$ are in the open left-half plane, then the Lyapunov equation

$$\mathcal{P}\mathcal{A} + \mathcal{A}^T\mathcal{P} = -Q^{T/2}Q^{1/2} = -Q \tag{33}$$

provide a unique positive definite solution $P = P^T > 0$ to the YKP matrix equation. Moreover, there exist rational functions $G(s) = C(sI - A)^{-1}B + D$ and

$$\begin{aligned} \Gamma(s) &= \begin{pmatrix} \Gamma_1(s) & \Gamma_2(s) \end{pmatrix} && (34) \\ &= \begin{pmatrix} Q_1 & Q_1(sI - A)^{-1}B + Q_2 \end{pmatrix} && (35) \end{aligned}$$

with Q_1, Q_2 matrices satisfying

$$\begin{aligned} \Omega &= \Gamma^T(-s)\Gamma(s) \\ &= \begin{pmatrix} -(PA + A^TP) & G_{12}(s) \\ G_{12}^T(-s) & G(s) + G^T(-s) \end{pmatrix} && (36) \\ 0 &\leq \Gamma_1^T(-i\omega)\Gamma_1(i\omega) = G(i\omega) + G^T(-i\omega) && (37) \end{aligned}$$

□

Proof See Appendix. ■

4. Observers and Nonlinear Feedback

Consider the nonlinear feedback stabilization problem of Eqs. (7)–(10) and assume that $\{A_0, B_0, C_0, D_0\}$ be given and that the observer $\{A_\xi, B_\xi, C_\xi, D_\xi, K_\xi\}$ be chosen appropriately. Nonlinear feedback stabilization by means of observer design and SPR design of loop transfer function can be approached by:

Algorithm: Lur'e-Lyapunov function design

1. Find state-space matrices A_0, B_0, C_0, D_0;
2. Introduce augmented system dynamics and state-space system matrices A_ξ, B_ξ, C_ξ, D_ξ, K_ξ and arrange the aggregate system matrix

$$\mathcal{A} = \begin{pmatrix} A & B \\ -C & -D \end{pmatrix} \tag{38}$$

If necessary, modify the augmented system dynamics $\{A_\xi, B_\xi, C_\xi, D_\xi, K_\xi\}$ so that the eigenvalues of $\mathcal{A}$ have negative real part.

3. Choose $Q = Q^{T/2}Q^{1/2} \geq 0$ such that $(\mathcal{A}, Q^{1/2})$ is observable and solve the Lyapunov equation

$$\mathcal{P}\mathcal{A} + \mathcal{A}^T\mathcal{P} = -Q, \quad Q, \mathcal{P} \in \mathbb{R}^{(n+m)\times(n+m)} \tag{39}$$

4. Find the positive definite solution $\mathcal{P}$ with block decomposition

$$\mathcal{P} = \begin{pmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{12}^T & \mathcal{P}_{22} \end{pmatrix}, \quad \mathcal{P}_{11} \in \mathbb{R}^{n\times n} \tag{40}$$

5. Find Schur complement matrix P and the feedback transformation matrix

$$\mathcal{T} = \begin{pmatrix} I_n & 0 \\ -\mathcal{P}_{22}^{-1}\mathcal{P}_{12}^T & \mathcal{P}_{22}^{-1/2} \end{pmatrix} \tag{41}$$

$$P = \mathcal{P}_{11} - \mathcal{P}_{12}\mathcal{P}_{22}^{-1}\mathcal{P}_{12}^T, \tag{42}$$

6. Assign Lyapunov function candidate or Lur'e-Lyapunov function

$$V(x) = x^T P x + \alpha \int_0^z \psi^T(s) K ds \tag{43}$$

Example ([21])

Consider the state-space system

$$\begin{aligned} \frac{dx_0}{dt} &= \begin{pmatrix} -31 & -259 & -229 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} x_0 + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u \\ z_0 &= \begin{pmatrix} 0 & 0 & 458 \end{pmatrix} x_0 \end{aligned} \tag{44}$$

with the transfer function

$$G_0(s) = \frac{2}{s+1} \cdot \frac{229}{s^2 + 30s + 229} \tag{45}$$

By the relative degree 3, the conditions of the YKP lemma are violated and the Popov criterion predicts a finite gain margin for output feedback. For

$$A = \begin{pmatrix} A_0 & 0_{3\times3} \\ K_\xi C_0 & A_\xi \end{pmatrix}, \quad K_\xi = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad Q = I_6$$

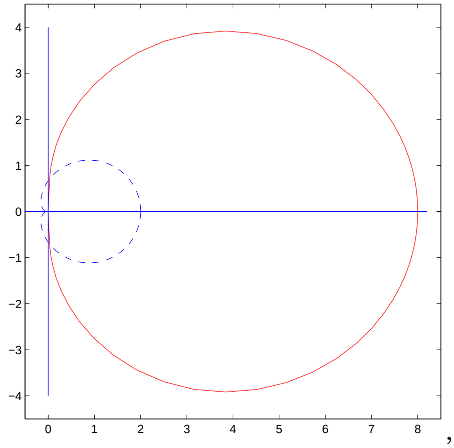

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Figure 1. Nyquist diagram of original system (*dashed line*) and transformed system (*solid line*).

we have a solution $P = P^T > 0$ to the Lyapunov equation $PA + A^T P = -Q$ found from $\mathcal{P}$ by the Schur-complement formula and with all eigenvalues of A possessing negative real part. An observer-transformed positive real loop-transfer function can be obtained as

$$G(s) = \frac{(12s + 14)(s + 2)^2(s^2 + 30s + 263)}{(s + 1)(s + 2)^3(s^2 + 30s + 229)}$$

with Nyquist diagram with the positive real property (Fig. 1); Popov diagram of the transformed system suggesting high gain-margin; stable system transients and Lyapunov function transients of system under feedback control (Fig. 2). Simulation of simple negative feedback connection of the loop-transfer function $40G_0(s)$ *via* a nonlinearity $\psi(\cdot)$ within the cone $[0, \infty)$ provides a limit cycle (Fig. 2). As predicted by the theory presented, the observer-compensated system eliminates the limit cycle and provides asymptotically stable behavior with Lyapunov function $V(x)$—see Fig. 2.

5. Discussion

We have provided an interpretation of the YKP matrix equation as a Lyapunov equation and have shown that this equation might provide Lyapunov functions for stability analysis (asymptotic stability and absolute stability) also for a class of systems without SPR properties. In particular, we have shown that Lyapunov functions may be provided also for observer-based feedback systems with possible pole-zeros cancellations introduced purposely. Such systems are not covered by regular formulations of the YKP lemma due the violation of minimality, controllability and observability. Actually, the minimality condition appears to be crucial only for the necessity part of the proof for the YKP lemma; *cf.* [5]. As pointed out by several authors—*e.g.*, [13], [14], [20]—there may exist $P = P^T \geq 0$ without any controllability assumption satisfied.

The nonstandard evaluation of the eigenvalues of $\mathcal{A}$ makes sense in that the eigenvalues are invariant under the similarity transformation of state-space transformation. As for eigenvalue assignment for $\mathcal{A}$, it is relevant to evaluate other matrix operations on $\mathcal{A}$ such as left and right multiplication by

$$\mathcal{L} \;=\; \begin{pmatrix} I_n & K \\ 0 & I_m \end{pmatrix}, \text{ and } \mathcal{R} = \begin{pmatrix} I_n & 0 \\ -L & I_m \end{pmatrix} \tag{46}$$

with the resultant matrix $\mathcal{A}' = \mathcal{L}\mathcal{A}\mathcal{R}$. Such matrix multiplications relate to feedback transformations of observer design and state-feedback control stability, respectively, as found by [6]. In fact, the similarity transformation provided by the algorithm implicitly suggest both a stabilizing state feedback control law (the right multiplication) and a linear combination of states and output to provide an SPR transfer function. By adding dynamics (integrators) up to the complexity of a full-order observer, it is possible to accomplish the SPR property needed for subsequent application of control algorithms of nonlinear control, adaptive control or realization theory; *cf.* [12], [3].

6. Conclusions

We have shown that there are solutions to the Yakubovich-Kalman-Popov equation in the form of nonminimal positive real systems. Thus, the very restrictive conditions of strict positive realness relevant to observer-based feedback control systems can be significantly relaxed. Important applications of such systems can be found in nonlinear stability theory, observer design and design of feedback stabilization—*e.g.*, by means of the Popov criterion or circle criterion. A method for construction of Lur'e-Lyapunov functions for systems with observer-based feedback control is given.

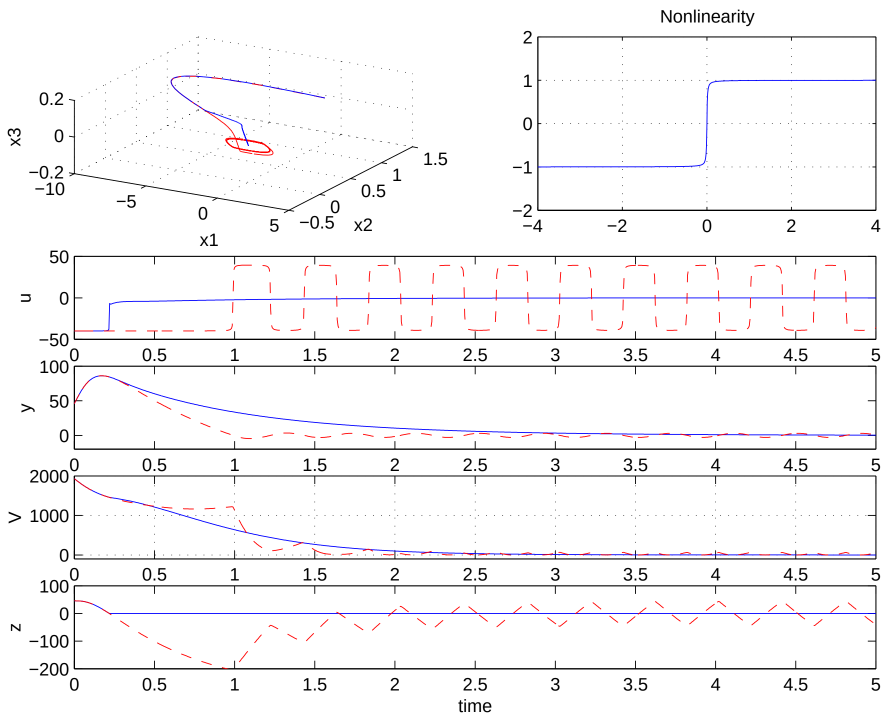

Figure 2. State-space diagram (*upper left*) with state trajectory of stabilized system and original limit-cycle; Feedback nonlinearity $\psi(\cdot)$ (*upper right*); Input u *vs.* time; output y *vs.* time; Lyapunov function $V(x)$ *vs.* time and behavior of $V(x)$ for non-compensated system; feedback signal z—trajectories of original feedback system (*dashed*) and stabilized system (*solid*), respectively.

Bibliography

[1] Anderson, B. D. O. and J. B. Moore (1979): *Optimal Filtering*. Prentice Hall, Englewood Cliffs, NJ.

[2] Boyd, S., L. El Ghaoui, E. Feron, and V. Balakrishnan (1994): *Linear Matrix Inequalities in System and Control Theory*, vol. 15. SIAM, Philadelphia, PA.

[3] Coron, J. M. and L. Praly (1991): "Adding an integrator for the stabilization problem." *Systems and Control Letters*, **17**, pp. 89–104.

[4] Kalman, R. E. (1963): "Lyapunov functions for the problem of Lur'e in automatic control." *Proc. Nat. Acad. Sci*, **49:2**.

[5] Khalil, H. K. (1996): *Nonlinear Systems*. Prentice Hall, Upper Saddle River, NJ.

[6] Kokotović, P. V. and H. J. Sussman (1989): "A positive real condition for global stabilization of nonlinear systems." *Systems and Control Letters*, **13**, pp. 125–133.

[7] Krstić, M., I. Kanellakopoulos, and P. Kokotović (1995): *Nonlinear and Adaptive Control Design*. John Wiley & Sons, New York.

[8] Lefschetz, S. (1965): *Stability of Nonlinear Control Systems*. Acad. Press, New York.

[9] Lozano Leal, R. and S. R. Joshi (1990): "Strictly positive real transfer functions revisited." *IEEE Transactions on Automatic Control*, **35:11**, pp. 1243–1245.

[10] Lur'e, A. I. and V. N. Postnikov (1944): "К Теории Устойчивости Регулируемых Систем (On the theory of stability of regulation system)." *Prikladnaya Matematika i Mekhanika*, **VIII**, pp. 246–248.

[11] Lyapunov, A. M. (1892): Общая Задача об Устойчивости Движения *(The general problem of the stability of motion)*. Kharkov Mathematical Society, Kharkov, Russia.

[12] Mazenc, F. and L. Praly (1996): "Adding integrations, saturated control and global asymptotic stabilization for feedforward systems." *IEEE Trans. Autom. Control*, **41**.

[13] Meyer, K. R. (1966): "On the existence of Lyapunov functions for the problem of Lur'e." *SIAM J. Control*, **3:3**.

[14] Narendra, K. S. and A. M. Annaswamy (1989): *Stable Adaptive Systems*. Prentice Hall, Englewood Cliffs, NJ.

[15] Narendra, K. S. and J. H. Taylor (1973): *Frequency Domain Criteria for Absolute Stability*. Academic Press, New York.

[16] Nijmeijer, H. and A. J. van der Schaft (1990): *Nonlinear Dynamical Control Systems*. Springer-Verlag.

[17] Ortega, R., A. Loria, R. Kelly, and L. Praly (1995): "On output feedback global stabilization of Euler-Lagrange systems." *Int. J. Robust and Nonlinear Cont.*, **5:4**, pp. 313–324.

[18] Parks, P. C. (1966): "Liapunov redesigns of model reference adaptive control systems." *IEEE Trans. Automatic Control*, **AC-11**, pp. 362–367.

[19] Popov, V. M. (1961): "Absolute stability of nonlinear systems of automatic control." *Avtomatika i Telemekhanika*, **22:8**.

[20] Rantzer, A. (1996): "On the Kalman-Yakubovich-Popov lemma." *Systems and Control Letters*, **28:1**, pp. 7–10.

[21] Rohrs, C. E., L. Valavani, M. Athans, and G. Stein (1982): "Robustness of adaptive control algorithms in the presence of unmodeled dynamics." In *Proc. Conf. Decision and Control*. Orlando, FL.

[22] Yakubovich, V. A. (1962): "Решение Некоторых Матричных Неравенств Встречающихся в Теории Автоматического Регулирования (Solution of certain matrix inequalities occuring in the theory of automatic control)." *Dokl. Akad. Nauk SSSR*, **143:6**, pp. 1304–1307.

[23] Zames, G. (1966): "On the input-output stability of time-varying nonlinear feedback systems." *IEEE Trans. Automatic Control*, **AC-11:2 (Part I), 3 (Part II)**.

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Appendix—Proof of Lemma 2

Proof We make the following constructive proof of positive realness: Let

$$E = \begin{pmatrix} I_n & 0_{n\times m} \\ 0_{m\times n} & 0_m \end{pmatrix} \tag{47}$$

$$L(s) = \begin{pmatrix} I_n & 0 \\ C(sI_n - A)^{-1} & I_m \end{pmatrix} \tag{48}$$

$$R(s) = \begin{pmatrix} I_n & -(sI_n - A)^{-1}B \\ 0 & I_m \end{pmatrix} \tag{49}$$

First, the transmission zeros of the system $\{A, B, C, D\}$ are found from

$$sE - \mathcal{A} = \begin{pmatrix} sI_n - A & -B \\ C & D \end{pmatrix} \tag{50}$$

$$= L(s)\begin{pmatrix} sI_n - A & 0 \\ 0 & G(s) \end{pmatrix} R(s) \tag{51}$$

Let $Q^{1/2}$ denote a matrix factor so that

$$Q = Q^{T/2}Q^{1/2} = \begin{pmatrix} Q_1^T \\ Q_2^T \end{pmatrix} \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} \tag{52}$$

Assume that for $(\mathcal{A}, Q^{1/2})$ observable and eigenvalues of $\mathcal{A}$ with negative real part, the Lyapunov equations

$$\begin{aligned} \mathcal{P}\mathcal{A} + \mathcal{A}^T\mathcal{P} &= -Q && (53)\\ PA + A^T P &= -Q_1^T Q_1 \overset{\text{def}}{=} -Q_{11} && (54) \end{aligned}$$

have provided the positive definite solutions $\mathcal{P}$ and P and that the solution obtained has been brought to block-diagonal form of Eq. (17). Then, expand the Lyapunov equation

$$Q = \mathcal{P}(sE - \mathcal{A}) + (-sE - \mathcal{A}^T)\mathcal{P} \tag{55}$$

By Eq. (50), it follows that

$$\mathcal{P}(sE - \mathcal{A}) = \begin{pmatrix} P(sI_n - A) & 0 \\ C & G(s) \end{pmatrix} R(s)$$

Thus, for $\Gamma(s) = Q^{1/2} R^{-1}(s)$ and

$$\begin{aligned} \Gamma(s) &= \begin{pmatrix} Q_1 & Q_1(sI - A)^{-1}B + Q_2 \end{pmatrix} \\ G_{12}(s) &= C^T + (-sI_n - A^T)P(sI_n - A)^{-1}B \\ &= Q_{11}(sI_n - A)^{-1}B + C^T - PB && (56) \end{aligned}$$

one finds that

$$\begin{aligned} \Omega &= \Gamma^T(-s)\Gamma(s) = R^{-T}(-s)Q^T Q R^{-1}(s) \\ &= R^{-T}(\mathcal{P}(sE - \mathcal{A}) + (-sE - \mathcal{A}^T)\mathcal{P})R^{-1} \\ &= \begin{pmatrix} -(PA + A^T P) & G_{12}(s) \\ G_{12}^T(-s) & G(s) + G^T(-s) \end{pmatrix} && (57) \end{aligned}$$

with rank deficit only at the transmission zeros of $\{A, B, C, D\}$. By the matrix equations (15, 57), the simultaneous transfer-function positivity and the positive definite Lyapunov equation properties follow from the diagonal matrix equation blocks of $\Gamma^T(-s)\Gamma(s)$ that

$$\begin{aligned} -(PA + A^T P) &= \Gamma_1^T(-s)\Gamma_1(s) = Q_1^T Q_1 \\ G(s) + G(-s) &= \Gamma_2^T(-s)\Gamma_2(s) && (58) \end{aligned}$$

As for $s = i\omega$, it follows that the 'positive-real' condition be fulfilled. ■

D

Paper D

Observer Backstepping for a Class of Nonminimum-Phase Systems

A. Robertsson and R. Johansson

Abstract

In this paper the stabilization of nonlinear nonminimum-phase systems is considered. In particular the observer backstepping design of Krstić *et al.* [14] is extended to stabilization of a class of nonlinear systems in output-feedback form with linear unstable zero-dynamics.

Keywords: Backstepping, Output feedback, Observers, Non-minimum-phase systems

Extended version of

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1. Introduction

The *backstepping design* [11; 14; 18] is a systematic design method for systems in strict feedback form assuming full state information. In [14] the output-feedback tracking problem is considered and the *observer backstepping procedure* is proposed for systems which are (weakly) minimum phase, *i.e.*, systems with stable zero-dynamics. The observer-backstepping procedure is roughly described as a backstepping design with respect to estimated states rather than to measured ones and it takes care of the disturbances the estimation errors may cause. Fossen and Grøvlen generalized the method to vectorial observer backstepping in [4], see also [19]. In [12; 20] a different approach is taken in that the backstepping procedure is used for nonlinear observer design.

In this paper, we consider the output-feedback problem and suggest a method how to use the observer backstepping design for a class of nonlinear systems with linear, unstable zero-dynamics. For other results on output feedback control of nonminimum-phase systems, see [8; 7; 3; 16; 5] and the references therein.

The outline of the paper is as follows: First, we provide an example that demonstrates the problem of stabilization of nonminimum-phase system by means of ordinary observer-backstepping. Secondly, we introduce a state-space transformation and an algorithm that extends the application of observer-backstepping and stabilization to a class of systems with nonminimum-phase properties.

2. Preliminaries and Problem Formulation

The observer backstepping methodology described in [14] provides a systematic design procedure for nonlinear systems with relative degree $r = n - m$ in the *output-feedback form* of

$$\begin{aligned}
\dot{x}_1 &= x_2 + \varphi_1(y) \\
\dot{x}_2 &= x_3 + \varphi_2(y) \\
&\vdots \\
\dot{x}_r &= x_{r+1} + \beta(y)u + \varphi_r(y) \\
\dot{x}_{r+1} &= x_{r+2} + b_1\beta(y)u + \varphi_{r+1}(y) \\
&\vdots \\
\dot{x}_{n-1} &= x_n + b_{m-1}\beta(y)u + \varphi_{n-1}(y) \\
\dot{x}_n &= b_m\beta(y)u + \varphi_n(y)
\end{aligned} \tag{1}$$

where the nonlinearities $\{\varphi_i\}_1^n$ only depend on the measured output

$$y = x_1$$

The system can be stabilized and output tracking can be achieved under the following two assumptions:

Assumption 1 (A1): $\beta(y) \neq 0$ for all $y \in I\!R$. □

Assumption 2 (A2): The m-dimensional linear zero-dynamics are asymptotically stable—*i.e.*, the roots of the polynomial

$$s^m + b_1 s^{m-1} + \ldots + b_m$$

lie strictly in the left half plane. □

To illustrate the problem of direct application to non-minimum phase systems, violating the assumption **(A2)**, we will (naively) use backstepping in the following simple example.

EXAMPLE D.1
Consider the linear system

$$Y(s) = \frac{s-1}{s^4} U(s) \tag{2}$$

which has a zero in the right half plane. The state-space realization in output-feedback form is for linear systems also known as *the observer canonical form* [10].

$$\begin{aligned} \dot{x} &= A_1 x + B_1 u = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} u \\ y &= C_1 x = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x \end{aligned} \tag{3}$$

The system is in strict feedback form, and applying the backstepping design, we will reach the control input u after three steps. Any stabilizing linear controller for the first three states will have the form

$$u_3 = -l_1 x_1 - l_2 x_2 - l_3 x_3, \quad l_i > 0, \; i = 1 \ldots 3$$

However, the state x_4, which represents the zero-dynamics, will be unstable and we can not neglect it in the design as we could have done if the zero-dynamics were stable. Even worse, it is not even possible to re-use our "stabilizing" control law u_3 and extend it with additional feedback from the state in the zero-dynamics to stabilize the whole system, as shown below.

Using

$$u(x) = u_3(x_1, x_2, x_3) - l_4 x_4$$

the closed loop system has the characteristic polynomial

$$\lambda(s) = s^4 + (-l_4 + l_3)\, s^3 + (l_2 - l_3)\, s^2 + (-l_2 + l_1)\, s - l_1$$

which is clearly unstable. Thus, the example shows that the observer-backstepping method will fail to stabilize such a nonminimum-phase system.

An alternative realization to system (2) is in *output-feedback canonical form*

$$\begin{aligned} \dot{\overline{x}} &= A_2\, \overline{x} + B_2\, u = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 \end{bmatrix} \overline{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \\ y &= C_2\, \overline{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \overline{x} \end{aligned} \tag{4}$$

which also is in strict-feedback form. The integrator chain between the first state and the input has the same length as the order of the system, and therefore (state feedback) backstepping will stabilize the system.

The change of variables

$$\overline{x} = \begin{bmatrix} \overline{x}_4 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{5}$$

relates the states for the different realizations and consists of a permutation of the states $\{x_1 \ldots x_r\}$ and a transformation for the state representing the zero-dynamics. □

3. Output-Feedback Stabilization

In this section we will use a state-transformation as indicated in the example above to stabilize nonlinear systems in output-feedback form with unstable zero-dynamics.

One-dimensional zero-dynamics

Here we consider systems with one-dimensional zero-dynamics of the form

$$\begin{aligned} \dot{x}_1 &= x_2 + \varphi_1(y) \\ \dot{x}_2 &= x_3 + \varphi_2(y) \\ &\vdots \\ \dot{x}_r &= x_{r+1} + \beta(y)u + \varphi_r(y) \\ \dot{x}_{r+1} &= b_1\beta(y)u + \varphi_{r+1}(y) \\ y &= x_1 \end{aligned} \tag{6}$$

where $n = r + 1$.

Assumption 3 (A3): The linear decomposition of Eq. (6) is stabilizable—e. g. there is no unstable pole-zero cancellation due to linear terms in $\{\varphi_i\}_1^n$. □

THEOREM D.1
Given a nonlinear system described by

$$\begin{aligned} \dot{x}_1 &= x_2 + \varphi_1(y) \\ \dot{x}_2 &= x_3 + \varphi_2(y) \\ &\vdots \\ \dot{x}_r &= x_n + \beta(y)u + \varphi_r(y) \\ \dot{x}_n &= b_1\beta(y)u + \varphi_n(y) \\ y &= x_1 \end{aligned} \tag{7}$$

Under the assumptions **(A1)** and **(A3)**, the nonlinear system of Eq. (7) with linear, one-dimensional unstable zero-dynamics, can be stabilized by output-feedback control using observer backstepping. □

Proof The system on output-feedback form (7) is via a state-transformation equivalent to

$$\begin{aligned} \dot{x}_1 &= x_2 + \varphi_1(y) \\ \dot{x}_2 &= x_3 + \varphi_2(y) \\ &\vdots \\ \dot{x}_r &= L^T[x_1, \ldots, \overline{x}_n]^T + \beta(y)u + \varphi_r(y) \\ \dot{\overline{x}}_n &= x_1 - b_1\overline{x}_n + \overline{\varphi}_n(y) \\ y &= x_1 \end{aligned} \tag{8}$$

in *output-feedback canonical form* [6]. The first $r = n-1$ states of the systems in Eq. (7) and Eq. (8) are identical and $L \in I\!R^n$, is given by the state space transformation. The vector $\overline{\varphi}_n(y)$ is a linear combination of the nonlinearities $\{\varphi_i(y)\}_1^n$.
For clarity, we introduce the variables, χ, and renumber the output nonlinearities accordingly

$$\chi = \begin{bmatrix} \chi_1 \\ \chi_2 \\ \vdots \\ \chi_n \end{bmatrix} = \begin{bmatrix} \overline{x}_n \\ x_1 \\ \vdots \\ x_r \end{bmatrix}, \qquad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{bmatrix} = \begin{bmatrix} \overline{\varphi}_n \\ \varphi_1 \\ \vdots \\ \varphi_r \end{bmatrix} \tag{9}$$

This is merely a permutation of the states. The system of Eq. (7) then transforms to

$$\begin{aligned} \dot{\chi}_1 &= -b_1\chi_1 + \chi_2 + \phi_1(y) \\ \dot{\chi}_2 &= \chi_3 + \phi_2(y) \\ &\vdots \\ \dot{\chi}_n &= L^T\chi + \beta(y)u + \phi_n(y) \\ y &= \chi_2 \end{aligned} \tag{10}$$

which is in *strict feedback form*. An exponentially stable observer for system (10) is readily constructed as

$$\begin{aligned} \dot{\hat{\chi}} &= A\hat{\chi} + K(y-\hat{y}) + \phi(y) + B\beta(y)u \\ \hat{y} &= C\hat{\chi} \end{aligned} \tag{11}$$

where K is chosen such that $A - KC$ is Hurwitz. The matrices A, B, C, and ϕ refer to the matrices appearing in Eq. (10) written in vector form. A Lyapunov function for the estimation error will be

$$\begin{aligned} V_{obs} &= \tilde{\chi}^T P_o \tilde{\chi} \\ \dot{V}_{obs} &= -\tilde{\chi}^T Q_o \tilde{\chi} \end{aligned} \tag{12}$$

where $P_o = P_o^T > 0$ satisfies the Lyapunov equation

$$\begin{aligned} &(A-KC)^T P_o + P_o(A-KC) = -Q_o \\ &Q_o = Q_o^T > 0 \end{aligned} \tag{13}$$

After the transformation of Eq. (9) we have the system in strict feedback form, but direct application of the *observer backstepping design procedure* as proposed in [14] is not feasible. A problem is the fact that the first state in the integrator chain of Eq. (10) is not measured but estimated only. Similarly to what is done in regular observer backstepping, extra damping, usually referred to as *nonlinear damping* [11], is introduced with respect to the observer dynamics in the first state, corresponding to the zero-dynamics.

Backstepping design, step 1

Consider the first row in the system of Eq. (10)

$$\begin{aligned}\dot{\chi}_1 &= -b_1\chi_1 + \chi_2 + \phi_1(y)\\ &= -b_1\hat{\chi}_1 + \hat{\chi}_2 + \phi_1(y) - b_1\tilde{\chi}_1 + \tilde{\chi}_2\end{aligned}$$

where $\hat{\chi}_2$ is the exponentially converging estimate for the state χ_2 and

$$\tilde{\chi}_2(t) \leq \tilde{\chi}_2(0) \cdot e^{-k_2 t},\ t \geq 0,\ \ k_2 > 0$$

is the estimation error $\chi_2 - \hat{\chi}_2$. Using the state estimates as virtual controls, we also need to modify the first step in the standard observer backstepping design to handle the estimation error for χ_1. Introducing the error variable

$$z_2 = \hat{\chi}_2 - \alpha_1(\hat{\chi}_1, y) \tag{14}$$

rather than $z_2 = \chi_2 - \alpha_1(\chi_1)$ as χ_1 is not measurable, we choose the stabilizing function

$$\begin{aligned}\alpha_1(\hat{\chi}_1, y) = (b_1 - c_1)\hat{\chi}_1 - \phi_1(y) - (d_{11} + d_{12})\hat{\chi}_1\\ c_1 > 0,\ d_{11} > 0,\ d_{12} > 0\end{aligned} \tag{15}$$

The damping term $-(d_{11} + d_{12})\hat{\chi}_1$ is introduced to compensate for the estimation errors in χ_1 and χ_2.

The positive definite function $(k_1 > 0,\ k_2 > 0)$

$$V_1(\chi_1, \tilde{\chi}_1, \tilde{\chi}_2) = \frac{1}{2}\chi_1^2 + \frac{1}{2d_{12}k_2}\tilde{\chi}_2^2 + \frac{f_1}{2k_1}\tilde{\chi}_1^2$$

has the derivative

$$\begin{aligned}
\dot{V}_1 &= \chi_1\left(-b_1\chi_1 + z_2 + \phi_1 + \alpha_1 + \tilde{\chi}_2\right) - \frac{1}{d_{12}}\tilde{\chi}_2^2 - f_1\tilde{\chi}_1^2 \\
&\leq -c_1\chi_1^2 + \chi_1 z_2 + \left(-d_{11}\chi_1^2 + (-b_1 + c_1 + d_{11} + d_{12})\chi_1\tilde{\chi}_1 - f_1\tilde{\chi}_1^2\right) \\
&+ (-d_{12}\chi_1^2 + \chi_1\tilde{\chi}_2 - \frac{1}{d_{12}}\tilde{\chi}_2^2) \\
&\leq -c_1\chi_1^2 + \chi_1 z_2 - d_{11}\left[(\chi_1 - \kappa_1\tilde{\chi}_1)^2 + (\frac{f_1}{d_{11}} - \kappa_1^2)\tilde{\chi}_1^2\right] \\
&- d_{12}\left[(\chi_1 - \frac{1}{2d_{12}}\tilde{\chi}_2)^2 + \frac{3}{4d_{12}^2}\tilde{\chi}_2^2\right] \\
&\leq -c_1\chi_1^2 + \chi_1 z_2 - d_{11}(\frac{f_1}{d_{11}} - \kappa_1^2)\tilde{\chi}_1^2 - \frac{3}{4d_{12}}\tilde{\chi}_2^2
\end{aligned}$$

where

$$\begin{aligned}
&\kappa_1 = (\frac{-b_1 + c_1 + d_{11} + d_{12}}{2d_{11}}) > 0, \\
&f_1 > \kappa_1^2 d_{11}.
\end{aligned}$$

Step 2

For the next step in the design, we choose the Lyapunov function candidate ($k_3 > 0$)

$$\begin{aligned}
V_2 &= V_1 + \frac{1}{2}z_2^2 + \frac{f_{21}}{2k_1}\tilde{\chi}_1^2 + \frac{f_{23}}{2k_3}\tilde{\chi}_3^2 \\
\dot{V}_2 &= \dot{V}_1 + z_2(\dot{\hat{\chi}}_2 - \dot{\alpha}_1) - f_{21}\tilde{\chi}_1^2 - f_{23}\tilde{\chi}_3^2
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
\dot{\hat{\chi}}_2 &= \hat{\chi}_3 + \phi_2(y) + K_2(y - \hat{y}) \\
&= z_3 + \alpha_2 + \phi_2(y) + K_2(y - \hat{y}).
\end{aligned} \tag{17}$$

Now

$$\begin{aligned}
\dot{V}_2 \leq &-c_1\chi_1^2 - d_{11}(\frac{f_1}{d_{11}} - \kappa_1^2)\tilde{\chi}_1^2 - \frac{3}{4d_{12}}\tilde{\chi}_2^2 - f_{21}\tilde{\chi}_1^2 - f_{23}\tilde{\chi}_3^2 \\
&+ z_2[\hat{\chi}_1 + \tilde{\chi}_1 + z_3 + \alpha_2(\hat{\chi}_1, z_2) + \phi_2(y) \\
&\quad - \frac{\partial \alpha_1}{\partial y}\cdot(\hat{\chi}_3 + \tilde{\chi}_3 + \phi_2(y)) - \frac{\partial \alpha_1}{\partial \hat{\chi}_1}\cdot\dot{\hat{\chi}}_1]
\end{aligned} \tag{18}$$

and by choosing

$$
\begin{aligned}
\alpha_2 = \alpha_2(\hat{\chi}_1, z_2, y) &= -\hat{\chi}_1 - c_2 z_2 - \phi_2(y) - K_2(y - \hat{y}) \\
&+ \frac{\partial \alpha_1}{\partial y} \cdot (\hat{\chi}_3 + \phi_2(y)) + \frac{\partial \alpha_1}{\partial \hat{\chi}_1} \dot{\hat{\chi}}_1 - d_{21} z_2 - d_{23} (\frac{\partial \alpha_1}{\partial y})^2 z_2
\end{aligned}
\tag{19}
$$

we get

$$
\begin{aligned}
\dot{V}_2 \leq & -c_1 \chi_1^2 - c_2 z_2^2 - d_{11}(\frac{f_1}{d_{11}} - \kappa_1^2)\tilde{\chi}_1^2 - \frac{3}{4d_{12}}\tilde{\chi}_2^2 \\
& - d_{21}(z_2 - \frac{1}{2d_{21}}\tilde{\chi}_1)^2 - (f_{21} - \frac{1}{4d_{21}^2})\tilde{\chi}_1^2 \\
& - d_{23}(\frac{\partial \alpha_1}{\partial y})^2 (z_2 - \frac{1}{2d_{23}\frac{\partial \alpha_1}{\partial y}}\tilde{\chi}_3)^2 \\
& - \left(f_{23} - \frac{1}{(2d_{23}\frac{\partial \alpha_1}{\partial y})^2} \right) \tilde{\chi}_3^2 + z_2 z_3
\end{aligned}
\tag{20}
$$

where f_{21} and f_{23} are chosen such that

$$
\begin{aligned}
f_{21} &> 1/(2\,d_{21})^2 > 0 \\
f_{23} &> 1/(2\,d_{23}\frac{\partial \alpha_1}{\partial y})^2 > 0
\end{aligned}
\tag{21}
$$

Similarly to the ordinary observer backstepping design, the estimation errors will show up and have to be compensated for with additional damping in each step of the design. For each step in the recursion the derivative of the Lyapunov candidate contains a negative definite part plus the indefinite cross-term $z_i z_{i+1}$, which will be compensated for in the last step.

Step n

Introducing the error vector

$$Z_{n-1} = \begin{bmatrix} z_1 \\ \vdots \\ z_{n-1} \\ \tilde{\chi}_1 \\ \vdots \\ \tilde{\chi}_{n-1} \end{bmatrix} \in I\!R^{2(n-1)} \tag{22}$$

we can write the derivative of V_{n-1} as

$$\dot{V}_{n-1} = -Z_{n-1}^T Q_{n-1} Z_{n-1} + z_{n-1} z_n, \quad Q_{n-1} > 0 \tag{23}$$

Now define the last Lyapunov function candidate

$$V_n = V_{n-1} + \frac{1}{2} z_n^2 + \frac{f_{n3}}{2k_3} \tilde{\chi}_3^2 \tag{24}$$

where the dynamics for the nth observer state is

$$\begin{aligned} \dot{\hat{\chi}}_n &= L^T \hat{\chi} + K_n(y - \hat{y}) + \beta(y) u + \phi_n(y) \\ z_n &= \hat{\chi}_n - \alpha_{n-1} \end{aligned} \tag{25}$$

The time derivative of V_n will be

$$\begin{aligned} \dot{V}_n &= \dot{V}_{n-1} + z_n \dot{z}_n + \frac{f_{n3}}{k_3} \tilde{\chi}_3 \dot{\tilde{\chi}}_3 \\ &= -Z_{n-1}^T Q_{n-1} Z_{n-1} + z_n \left[z_{n-1} + \dot{\hat{\chi}}_n - \dot{\alpha}_{n-1} \right] - f_{n3} \tilde{\chi}_3^2 \\ &\leq -Z_{n-1}^T Q_{n-1} Z_{n-1} + z_n [z_{n-1} + L^T \hat{\chi} + K_n(y - \hat{y}) \\ &\quad + \beta(y) u + \phi_n(y) - \dot{\alpha}_{n-1}] - f_{n3} \tilde{\chi}_3^2 \end{aligned} \tag{26}$$

$$\begin{aligned} \dot{\alpha}_{n-1} &= \frac{\partial \alpha_{n-1}}{\partial y} [\hat{\chi}_3 + \tilde{\chi}_3 + \phi_2(y)] \\ &\quad + \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{\chi}_j} [\hat{\chi}_{j+1} + K_j(y - \hat{y}) + \phi_j(y)] \end{aligned} \tag{27}$$

Finally, by choosing the control law

$$u = \frac{1}{\beta(y)}\alpha_n \tag{28}$$

with

$$\begin{aligned}\alpha_n = &-c_n z_n - L^T\hat{\chi} - \phi_n(y) - K_n(y-\hat{y}) + \frac{\partial\alpha_{n-1}}{\partial y}\cdot(\hat{\chi}_3 + \phi_2(y))\\ &+\sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{\chi}_j}\dot{\hat{\chi}}_j - d_{n3}(\frac{\partial\alpha_{n-1}}{\partial y})^2 z_n\end{aligned} \tag{29}$$

we have that the derivative of V_n is negative definite:

$$\begin{aligned}\dot{V}_n \le &-Z_{n-1}^T Q_{n-1} Z_{n-1} - c_n z_n^2\\ &- d_{n3}(\frac{\partial\alpha_{n-1}}{\partial y})^2(z_n - \frac{1}{2d_{n3}\frac{\partial\alpha_{n-1}}{\partial y}}\tilde{\chi}_3)^2\\ &-\left(f_{n3} - \frac{1}{(2d_{n3}\frac{\partial\alpha_{n-1}}{\partial y})^2}\right)\tilde{\chi}_3^2 < 0\end{aligned} \tag{30}$$

where f_{n3} is chosen such that

$$f_{n3} > 1/(2\,d_{n3}\frac{\partial\alpha_{n-1}}{\partial y})^2 > 0 \tag{31}$$

To show stability of the closed loop system, we take as a Lyapunov function the combination of "error states" and estimation errors

$$V = V_n(Z) + V_{obs}(\tilde{\chi}) > 0,\ Z \ne 0,\ \tilde{\chi} \ne 0 \tag{32}$$

where V_{obs} is defined in Eq. (12). Since V is positive definite, decrescent and radially unbounded with time derivative

$$\dot{V} \le -Z_n^T Q_n Z_n - \tilde{\chi}^T Q_o \tilde{\chi} < 0,\ \ Z \ne 0,\ \tilde{\chi} \ne 0 \tag{33}$$

we conclude global asymptotic stability for the closed loop system. ■

The complete algorithm for deriving the control law will be summarized in the next Section, see Eq. (52). For the stabilization problem $\chi_{1\mathrm{d}}(t) \equiv 0$ should be used.

Higher-order zero-dynamics

We introduce the following notation:

$$
\begin{aligned}
J_i &= \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix} \in \mathbb{R}^{i\times i} \\
\mathbf{b}_i &= \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_i \end{bmatrix}, \quad \mathtt{f}_i = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathtt{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^i
\end{aligned}
\tag{34}
$$

$$
\mathcal{A}_i(\mathbf{b_i}) = J_i - \mathbf{b}_i \mathtt{f}_i^T = \begin{bmatrix} -b_1 & 1 & & \\ \vdots & 0 & \ddots & \\ \vdots & \vdots & \ddots & 1 \\ -b_i & 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{i\times i}
$$

$$
x_{(i:j)} = \begin{bmatrix} x_i \\ x_{i+1} \\ \vdots \\ x_j \end{bmatrix} \in \mathbb{R}^{j-i+1}, \quad L_{(i:j)} = \begin{bmatrix} l_i \\ l_{i+1} \\ \vdots \\ l_j \end{bmatrix} \in \mathbb{R}^{j-i+1}
$$

Systems in output-feedback form may then be written as

$$
\begin{aligned}
\begin{bmatrix} \dot{x}_{(1:r)} \\ \\ \dot{x}_{(r+1:n)} \end{bmatrix} &= \begin{bmatrix} J_r & 0 \\ \\ 0 & J_{n-r} \end{bmatrix} \begin{bmatrix} x_{(1:r)} \\ \\ x_{(r+1:n)} \end{bmatrix} + \begin{bmatrix} \mathtt{e}_r \\ \\ \mathbf{b}_m \end{bmatrix} \beta(y) u \\
&\quad + \begin{bmatrix} \varphi_{(1:r)}(y) \\ \\ \varphi_{(m:n)}(y) \end{bmatrix} \\
y &= x_1
\end{aligned}
\tag{35}
$$

Even when $\{\varphi_i(y)\}_1^n$ are merely linear functions of the output y, we have natural restrictions on stabilizability, namely that there should not be any unstable pole-zero cancellations, which motivates the following assumption.

Assumption 4 (A4):
The linear decomposition of Eq. (35) is stabilizable—i. e., there is no unstable pole-zero cancellation due to linear terms in $\{\varphi_i\}_1^n$. □

The state space transformation from the original x-coordinates to the χ-coordinates has the structure of

$$\begin{bmatrix} \chi_{(1;m)} \\ \chi_{(m+1:n)} \end{bmatrix} = \begin{bmatrix} T_2 & T_1 \\ I_r & 0_{r\times m} \end{bmatrix} \begin{bmatrix} \overline{x}_{(r+1:n)} \\ x_{(1:r)} \end{bmatrix} \tag{36}$$

where $T_1 \in I\!R^{m\times m}$ is invertible and $T_2 \in I\!R^{m\times r}$.

Furthermore, the condition for preserving the strict feedback form in the transformed coordinates χ will be fulfilled under the following assumption.

Assumption 5 (A5):

$$\sigma_m(k)^T \left(T_1\varphi_{(m:n)}(y) + T_2\varphi_{(1:r)}(y)\right) = 0, \quad k = 1, \ldots, m-1 \tag{A5}$$

where

$$\sigma_m(k) = [0 \quad \ldots \quad 0 \quad 1 \quad 0 \quad \ldots \quad 0]^T \in I\!R^m$$

has a one in the kth position. □

Remark D.2
Assumption **(A5)** contains $m-1$ constraints and allows for an output nonlinearity $\overline{\varphi}_n(y)$ to enter in the equation of $\dot{\chi}_m$. This assumption needs thus not to be considered in the case of one-dimensional zero-dynamics. **(A5)** may also be relaxed to allow for linear terms, $\bar{b}\chi_{m+1}$ in the first m states as long as the subsystem $[\mathcal{A}_m(\mathbf{b_m}),\ \mathsf{e}_m + \bar{b}]$ is stabilizable. □

Theorem D.2
Under assumptions **(A1)**, **(A4)**, and **(A5)** the nonlinear system in output-feedback form of

$$\begin{aligned} \begin{bmatrix} \dot{x}_{(1:r)} \\ \\ \dot{x}_{(r+1:n)} \end{bmatrix} &= \begin{bmatrix} J_r & 0 \\ \\ 0 & J_{n-r} \end{bmatrix} \begin{bmatrix} x_{(1:r)} \\ \\ x_{(r+1:n)} \end{bmatrix} + \begin{bmatrix} \mathsf{e}_r \\ \\ \mathbf{b}_m \end{bmatrix} \beta(y)u \\ &\quad + \begin{bmatrix} \varphi_{(1:r)}(y) \\ \\ \varphi_{(m:n)}(y) \end{bmatrix} \\ y &= x_1 \end{aligned} \tag{37}$$

with linear, m-dimensional unstable zero-dynamics, is output feedback stabilizable using observer backstepping. □

Proof The system in Eq. (37) can be transformed into the output feedback canonical form of

$$\begin{aligned}\begin{bmatrix} \dot{x}_{(1:r)} \\ \\ \dot{\overline{x}}_{(r+1:n)} \end{bmatrix} &= \begin{bmatrix} J_r + \mathrm{e}_r \overline{L}^T_{(1:r)} & \mathrm{e}_r \overline{L}^T_{(r+1:n)} \\ \\ \mathrm{e}_m \mathrm{f}^T_r & \mathcal{A}_m(\mathbf{b_m}) \end{bmatrix} \begin{bmatrix} x_{(1:r)} \\ \\ \overline{x}_{(r+1:n)} \end{bmatrix} \\ &\quad + \begin{bmatrix} \mathrm{e}_r \\ \\ 0_m \end{bmatrix} \beta(y)u + \begin{bmatrix} \varphi_{(1:r)}(y) \\ \\ \mathrm{e}_m \overline{\varphi}_n(y) \end{bmatrix} \\ y &= x_1 \end{aligned} \tag{38}$$

and via the permutation

$$\chi = \begin{bmatrix} \overline{x}_{(r+1:n)} \\ \\ x_{(1:r)} \end{bmatrix} = \begin{bmatrix} \chi_{(1;m)} \\ \\ \chi_{(m+1:n)} \end{bmatrix} \tag{39}$$

written in the strict feedback-form

$$\begin{aligned}\dot{\chi} &= \begin{bmatrix} \mathcal{A}_m(\mathbf{b_m}) & \mathrm{e}_m \mathrm{f}^T_r \\ \\ \mathrm{e}_r L^T_{(1:m)} & J_r + \mathrm{e}_r L^T_{(m+1:n)} \end{bmatrix} \chi \\ &\quad + \begin{bmatrix} 0 \\ \\ \mathrm{e}_r \end{bmatrix} \beta(y)\,u + \begin{bmatrix} \mathrm{e}_m \overline{\varphi}_n(y) \\ \\ \varphi_{(1:r)}(y) \end{bmatrix} \\ y &= \chi_{m+1} \end{aligned} \tag{40}$$

As the (unstable) linear subsystem $[\mathcal{A}_m(\mathbf{b_m}),\ \mathrm{e}_m]$ is completely controllable, we can use χ_{m+1} as a virtual control to stabilize the subsystem using block backstepping [14], see also the notion of vectorial observer backstepping in [4].

Consider the subsystem

$$\begin{aligned}\dot{\chi}_{(1:\mathrm{m})} &= \mathcal{A}_m(\mathbf{b_m})\chi_{(1:\mathrm{m})} + \mathrm{e}_m(\chi_{m+1} + \overline{\varphi}_n) \\ &= \mathcal{A}_m(\mathbf{b_m})\chi_{(1:\mathrm{m})} + \mathrm{e}_m(\hat{\chi}_{m+1} + \tilde{\chi}_{m+1} + \overline{\varphi}_n) \end{aligned} \tag{41}$$

where $\tilde{\chi}_{m+1}$ is the exponentially converging estimate for the state χ_{m+1}. Introduce the error variable

$$z_{m+1} = \hat{\chi}_{m+1} - \alpha_{(1:\mathrm{m})}(y, \hat{\chi}_1, \ldots, \hat{\chi}_m)$$

For stabilization, we can use ordinary pole-placement to find

$$\begin{aligned} \alpha_{(1:\mathrm{m})} &= -[c(1:m) + d_{(1:\mathrm{m})}]\hat{\chi}_{(1:\mathrm{m})} - \overline{\varphi}_n \\ &= -C_m\hat{\chi}_{(1:\mathrm{m})} - \overline{\varphi}_n,\ C_m^T \in I\!R^m \end{aligned} \tag{42}$$

such that the matrix

$$\mathcal{A}_C = \mathcal{A}(\mathbf{b_m}) - \mathsf{e}_m C_m \tag{43}$$

is Hurwitz. This implies the existence of symmetric, positive definite matrices P_C and Q_C such that the Lyapunov equation

$$\mathcal{A}_C^T P_C + P_C \mathcal{A}_C = -Q_C - 2I_m \tag{44}$$

is fulfilled. Here $-Q_C$ represents the effect of the "ordinary" stabilization and the two identity matrices, $-2I_m$, represent the effect of the extra damping to take care of the estimation errors $\tilde{\chi}_{(1:\mathrm{m})}$ and $\tilde{\chi}_{m+1}$. This could be compared with the role of the terms c_1 and (d_{11}, d_{12}) in the one-dimensional case in the previous section.

By Eq. (11) and the deterministic Kalman filter properties the estimation errors $\tilde{\chi}_{(1:\mathrm{m})}$ and $\tilde{\chi}_{m+1}$ converge exponentially,

$$\begin{aligned} \tilde{\chi}_{(1:\mathrm{m})}(t) &\leq \tilde{\chi}_{(1:\mathrm{m})}(0) \cdot e^{-\Lambda_k t} \\ \tilde{\chi}_{m+1}(t) &\leq \tilde{\chi}_{m+1}(0) \cdot e^{-k_{m+1} t} \end{aligned} \tag{45}$$

for some diagonal $\Lambda_k > 0$ and some $k_{m+1} > 0$.
The positive definite function

$$V_{(1:\mathrm{m})} = \chi_{(1:\mathrm{m})}^T P_C \chi_{(1:\mathrm{m})} + \frac{f_{m+1}\,\tilde{\chi}_{m+1}^2}{2k_{m+1}} + \tilde{\chi}_{(1:\mathrm{m})}^T \Lambda_f \Lambda_k^{-1} \tilde{\chi}_{(1:\mathrm{m})} \tag{46}$$

where

$$\begin{aligned} &\Lambda_f > \frac{1}{2}(P_C \mathsf{e}_m C_m)^T (P_C \mathsf{e}_m C_m), \quad \Lambda_f \ \text{ diagonal} \\ &f_{m+1} > \mathsf{e}_m^T P_C^T P_C \mathsf{e}_m \end{aligned} \tag{47}$$

has the derivative

$$
\begin{aligned}
\dot{V}_{(1:\mathrm{m})} &= 2\chi_{(1:\mathrm{m})}^{T}[P_C\mathcal{A}(\mathbf{b_m})\chi_{(1:\mathrm{m})} \\
&\qquad + \mathsf{e}_m(z_{m+1} + \alpha_{(1:\mathrm{m})}(y, \hat{\chi}_{(1:\mathrm{m})}) + \tilde{\chi}_{m+1} + \overline{\varphi}_n)] \\
&\qquad - f_{m+1}\tilde{\chi}_{m+1}^2 - 2\tilde{\chi}_{(1:\mathrm{m})}^{T}\Lambda_f\tilde{\chi}_{(1:\mathrm{m})} \\
&= \chi_{(1:\mathrm{m})}^{T}\underbrace{\left[\mathcal{A}_C^TP_C + P_C\mathcal{A}_C\right]}_{-Q_C - I_m - I_m}\chi_{(1:\mathrm{m})} \\
&\quad + 2\chi_{(1:\mathrm{m})}^{T}P_C\mathsf{e}_mC_m\tilde{\chi}_{(1:\mathrm{m})} + 2\chi_{(1:\mathrm{m})}^{T}P_C\mathsf{e}_m\tilde{\chi}_{m+1} \\
&\quad - f_{m+1}\tilde{\chi}_{m+1}^2 - 2\tilde{\chi}_{(1:\mathrm{m})}^{T}\Lambda_f\tilde{\chi}_{(1:\mathrm{m})} + 2\chi_{(1:\mathrm{m})}^{T}P_C\mathsf{e}_mz_{m+1}
\end{aligned}
\tag{48}
$$

Completing the squares gives

$$
\begin{aligned}
\dot{V}_{(1:\mathrm{m})} &= -\chi_{(1:\mathrm{m})}^{T}Q_C\,\chi_{(1:\mathrm{m})} \\
&\quad - \left(\chi_{(1:\mathrm{m})} - P_C\mathsf{e}_mC_m\tilde{\chi}_{(1:\mathrm{m})}\right)^T\left(\chi_{(1:\mathrm{m})} - P_C\mathsf{e}_mC_m\tilde{\chi}_{(1:\mathrm{m})}\right) \\
&\quad - \tilde{\chi}_{(1:\mathrm{m})}^{T}\left[2\Lambda_f - (P_C\mathsf{e}_mC_m)^T(P_C\mathsf{e}_mC_m)\right]\tilde{\chi}_{(1:\mathrm{m})} \\
&\quad - \left(\chi_{(1:\mathrm{m})} - P_C\mathsf{e}_m\tilde{\chi}_{m+1}\right)^T\left(\chi_{(1:\mathrm{m})} - P_C\mathsf{e}_m\tilde{\chi}_{m+1}\right) \\
&\quad - \left[f_{m+1} - \mathsf{e}_m^TP_C^TP_C\mathsf{e}_m\right]\tilde{\chi}_{(1:\mathrm{m})}^2 \\
&\qquad + 2\chi_{(1:\mathrm{m})}^{T}P_C\mathsf{e}_mz_{m+1} \\
&\leq -\chi_{(1:\mathrm{m})}^{T}Q_C\,\chi_{(1:\mathrm{m})} - \tilde{\chi}_{(1:\mathrm{m})}^{T}\Lambda_F\tilde{\chi}_{(1:\mathrm{m})} \\
&\quad - F_{m+1}\tilde{\chi}_{m+1}^2 + 2\chi_{(1:\mathrm{m})}^{T}P_C\mathsf{e}_mz_{m+1}
\end{aligned}
\tag{49}
$$

where

$$
\begin{aligned}
\Lambda_F &= 2\Lambda_f - (P_C\mathsf{e}_mC_m)^T(P_C\mathsf{e}_mC_m) > 0 \\
F_{m+1} &= f_{m+1} - \mathsf{e}_m^TP_C^TP_C\mathsf{e}_m > 0
\end{aligned}
\tag{50}
$$

After the initial step where the linear $m \times m$-subsystem is stabilized, we proceed with the ordinary observer backstepping, where each step adds one additional error state, z_i. Should some of the nonlinearities $\varphi_i(y)$, $i \geq m$, consecutively be absent, we can use block-stabilization again. The rest of the proof follows the same outline as the proof of Theorem D.1. ■

Output tracking

Via ordinary observer backstepping the tracking error

$$y(t) - y_{\mathrm{d}}(t) \to 0$$

as $t \to \infty$, where $y(t) = x_1$ and $y_d(t)$ is the desired output trajectory. For our system this would correspond to

$$\lim_{t\to\infty}[\chi_1(t) - \chi_{1d}(t)] = 0$$

and not to the output $y = x_1 = \chi_{m+1}$. The control law for regulation of the tracking error is

$$u = \frac{1}{\beta(y)}[\alpha_n - \chi_{1d}^{(n)}] \tag{51}$$

where α_n will be derived through the recursive design below. Except for the extra added nonlinear damping, the following algorithm aligns with the one described in Theorem 7.1 of [14] for nonminimum-phase systems:

$$\begin{aligned}
z_1 &= \chi_1 - \chi_{1d} \\
z_i &= \hat{\chi}_i - \alpha_{i-1}(\hat{\chi}_1, \ldots, \hat{\chi}_{i-1}, \chi_{1d}, \ldots, \chi_{1d}^{(i-2)}) \\
&\quad - \chi_{1d}^{(i-1)}, \quad i = 2 \ldots n \\
\alpha_1 &= (b_1 - c_1 - d_{11} - d_{12})z_1 \\
\alpha_i &= -c_i z_i - d_{i1}\hat{\chi}_1 - d_{i2}(\frac{\partial \alpha_{i-1}}{\partial \hat{\chi}_1})^2 z_i \\
&\quad - K_i(y - \hat{y}) - \varphi_i(y) + \frac{\partial \alpha_{i-1}}{\partial \hat{\chi}_1}\chi_2 \\
&\quad + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \hat{\chi}_j}[\hat{\chi}_{j+1} + K_j(y - \hat{y}) + \varphi_j(y)] \\
&\quad + \sum_{j=1}^{i-2} \frac{\partial \alpha_{i-1}}{\partial \chi_{1d}^{(j)}}\chi_{1d}^{(j+1)}, \quad i = 2 \ldots n-1
\end{aligned} \tag{52}$$

$$\begin{aligned}
\alpha_n &= -c_n z_n - L_{(1:n)}^T \hat{\chi}_{(1:n)} - d_{n1}\hat{\chi}_1 - d_{n2}(\frac{\partial \alpha_{n-1}}{\partial \hat{\chi}_1})^2 z_n \\
&\quad - K_n(y - \hat{y}) - \varphi_n(y) + \frac{\partial \alpha_{n-1}}{\partial \hat{\chi}_1}\chi_2 \\
&\quad + \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{\chi}_j}[\hat{\chi}_{j+1} + K_j(y - \hat{y}) + \varphi_j(y)] \\
&\quad + \sum_{j=1}^{n-2} \frac{\partial \alpha_{n-1}}{\partial \chi_{1d}^{(j)}}\chi_{1d}^{(j+1)}
\end{aligned}$$

where

$$\begin{aligned} c_i > 0,\ d_{ij} > 0, \quad & i = 1 \dots n \\ b_i = 0, \quad & i > m \\ \varphi_i(y) = 0, \quad & i \leq m \end{aligned} \tag{53}$$

$\chi_{1\mathrm{d}}$ is a consistent state reference trajectory with respect to the desired reference output y_d. For a discussion on the reference trajectory generation problem for nonminimum-phase systems, see for instance [1; 2; 16; 3].

4. Discussion

Recently Tan *et al.* [22] presented an observer–based controller design for nonlinear systems in the "extended" output-feedback form

$$\begin{aligned} \dot{x}_1 &= x_2 + \varphi_{0,1}(x_1) \\ \dot{x}_2 &= x_3 + \varphi_{0,2}(x_1) + \varphi_{1,2}(x_1)x_2 \\ &\vdots \\ \dot{x}_r &= x_{r+1} + \beta(y)u + \varphi_{0,r}(x_1) + \varphi_{1,r}(x_1)x_2 \\ \dot{x}_{r+1} &= x_{r+2} + b_1\beta(y)u + \varphi_{0,r+1}(x_1) \\ &\qquad + \varphi_{1,r+1}(x_1)x_2 \\ &\vdots \\ \dot{x}_n &= b_m\beta(y)u + \varphi_{0,n}(x_1) + \varphi_{1,n}(x_1)x_2 \\ y &= x_1 \end{aligned} \tag{54}$$

where additional terms linear in the output derivative are allowed. Using a nonlinear transformation depending on the output y, the dynamics of the unmeasured states can be linearized and an exponentially convergent observer for the states may be designed.

Using the proposed observer in [22], the assumption that the numerator polynomial

$$B(s) = s^m + b_1 s^{m-1} + \ldots + b_m$$

is Hurwitz, can be relaxed under assumptions (A1), (A4), and under an additional restriction on (A5), namely

$$\begin{aligned} & T_1(\varphi_{(0,m:n)} + \varphi_{(1,m:n)}x_2) \\ & \quad + T_2(\varphi_{(0,1:r)} + \varphi_{(1,1:r)}x_2) = \mathsf{e}_m \overline{\varphi}_n(y) \end{aligned} \tag{55}$$

where $\overline{\varphi}_n(y)$ is an arbitrary nonlinear function of y.

Note that in contrast to the use of vectorial backstepping in [4], where it was an elegant way of reducing the number of steps in the backstepping procedure, it is here an important part of the proof technique, as the ordinary observer backstepping procedure would not be applicable to systems with unstable higher-order zero-dynamics without completing the proof with results from cascaded designs such as [9; 17; 15]. The reason for this is that although the system in Eq. (40) is in strict feedback form, the change of variables $\hat{\chi}_i \to z_i$ would introduce the (linear) correction terms, $k_i(y - \hat{y}) = k_i(y - \hat{\chi}_{m+1})$, from the observer dynamics into the expression for $\dot{z}_i$. This would violate the strict feedback form for the first $m - 1$ error variables.

By first adding an output nonlinearity to the control input u, affecting the last $m + 1$ states, we introduce an additional degree of freedom for assumption **A5**.

5. Conclusions

In this paper the output feedback problem for a class of nonlinear nonminimum-phase systems is considered. Our main contribution is extending the existing *observer backstepping method* [14] to cover a class of nonlinear systems in *output-feedback form* with linear unstable zero-dynamics and to provide a recursive algorithm for an observer-based controller. The design achieves global stabilization and allows reference tracking. In the case of higher-order zero-dynamics, the main restriction is assumption **A5**, which is a condition for preserving strict-feedback form in the state transformation used for the backstepping design.

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Bibliography

[1] Degang, C. and B. Paden (1996): “Stable inversion of nonlinear nonminimum phase systems.” *International Journal of Control*, **64:1**, pp. 81–97.

[2] Devasia, S., C. Degang, and B. Paden (1996): "Nonlinear inversion-based output tracking." *IEEE Trans. on Automatic Control*, **41:7**, pp. 930–942.

[3] Fliess, M., H. Sira-Ramirez, and R. Marquez (1998): "Regulation of non-minimum phase outputs: A flatness based approach." In Normand-Cyrot, Ed., *Perspectives in Control*. Springer-Verlag, Berlin.

[4] Fossen, T. I. and Å. Grøvlen (1998): "Nonlinear output feedback control of dynamically positioned ships using vectorial observer backstepping." *IEEE Trans. Control System and Technology*, **6**, pp. 121–128.

[5] Hauser, J., S. Sastry, and G. Meyer (1992): "Nonlinear control design for slightly nonminimum-phase systems: Applications to V/STOL aircraft." *Automatica*, **28:4**, pp. 665–679.

[6] Isidori, A. (1995): *Nonlinear Control Systems*, 3rd edition. Springer-Verlag.

[7] Isidori, A. (1998a): "Semiglobal practical stabilization of uncertain non-minimum-phase nonlinear systems via output feedback." In *Proceedings of the Fourth IFAC Symposium on Nonlinear Control Systems Design (NOLCOS'98)*, pp. 643–648. Enschede, The Netherlands.

[8] Isidori, A. (1998b): "Stabilization of output feedback revisited." In Normand-Cyrot, Ed., *Perspectives in Control*. Springer-Verlag, Berlin.

[9] Jiang, Z., I. Mareels, and Y. Wang (1996): "A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems." *Automatica*, **32**, pp. 1211–1215.

[10] Kailath, T. (1980): *Linear Systems*. Prentice-Hall, Inc, Englewood Cliffs, New Jersey.

[11] Kanellakopoulos, I., P. Kokotović, and S. Morse (1992): "A toolkit for nonlinear feedback control." *System & Control Letters*, **18**, June, pp. 83–92.

[12] Kang, W. and A. Krener (1998): "Nonlinear observer design, a backstepping approach." *Submitted to IEEE Trans on Automatic Control*.

[13] Kokotović, P. (1992): "The joy of feedback." *IEEE Control System Magazine*, **12**, June, pp. 7–17.

[14] Krstić, M., I. Kanellakopoulos, and P. Kokotović (1995): *Nonlinear and Adaptive Control Design*. John Wiley & Sons, New York.

[15] Loría, A., E. Panteley, H. Nijmeijer, and T. Fossen (1998): "Robust adaptive control of passive systems with unknown disturbances." In *Proceedings of the Fourth IFAC Symposium on Nonlinear Control Systems Design (NOLCOS'98)*, vol. 3, pp. 866–871. Enschede, The Netherlands.

[16] Martin, P., S. Devasia, and B. Paden (1996): "A different look at output tracking: Control of a VTOL aircraft." *Automatica*, **32:1**, pp. 101–107.

[17] Panteley, E. and A. Loría (1998): "On global uniform asymptotic stability of nonlinear time-varying systems in cascade." *Systems & Control Letters*, **33:2**, pp. 131–138.

[18] Praly, L. (1992): "Lyapunov design of a dynamic output feedback for system linear in their unmeasured state components." In *Proceedings of IFAC Nonlinear Control Systems Design Symposium (NOLCOS'92)*, pp. 31–36. Bordeaux, France.

[19] Robertsson, A. and R. Johansson (1998): "Comments on 'Nonlinear output feedback control of dynamically positioned ships using vectorial observer backstepping'." *IEEE Transactions on Control Systems Technology*, **6:3**, pp. 439–441.

[20] Song, R.-Y., S. Ishijima, and A. Kojima (1997): "Design of nonlinear observer by a backstepping approach." *Electrical Engineering in Japan*, **121:3**, pp. 53–59.

[21] Sussmann, H. and P. Kokotović (1989): "Peaking and stabilization." In *Proceedings of the 28th IEEE Conference on Decision and Control*, pp. 1379–1384. Tampa, Florida.

[22] Tan, Y., I. Kanellakopoulos, and Z.-P. Jiang (1998): "Nonlinear observer/controller design for a class of nonlinear systems." In *Proceedings of the 37th IEEE Conference on Decision and Control*, pp. 2503–2508. Tampa, Florida.

Paper E

Comments on “Nonlinear Output Feedback Control of Dynamically Positioned Ships Using Vectorial Observer Backstepping”

A. Robertsson and R. Johansson

Abstract

The decomposition of nonlinear output feedback control into an observer and a state feedback control is an open problem. A solution for dynamic positioning of ships has been proposed in the papers by Fossen and Grøvlen, and by Grøvlen and Fossen, where an observer-based backstepping method is used.

This note points out that the observer design in the papers mentioned above does not cover unstable ship dynamics and suggests a remedy for an extended class of ships. The proof for the nonlinear observer used in the design in the above-mentioned papers only applies to ships with stable sway-yaw dynamics. In the above-mentioned papers an example concerning thruster assisted mooring of a tanker is given, which does not fulfill the needed stability properties, so an extension to this case is highly motivated. We propose a method to extend the results, under a detectability condition. This condition implies stable surge dynamics, which is a natural assumption for ships.

Keywords: Marine systems, nonlinear systems, observer design, stability, state estimation

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1. Introduction

We are using the problem formulation and notation from [1] motivated by the application to ship model dynamics. We propose an extension of the results in [1], which covers the case of unstable sway-yaw dynamics, which are prevalent among e. g. large tankers.

Kinematics & Dynamics

An earth-fixed frame is used for the ship position (x, y) and the yaw angle ψ while the surge, sway, and yaw velocities (u, v, r) refer to a body-fixed frame. With $\eta = [x, y, \psi]^T$ and $\nu = [u, v, r]^T$ we have

$$\dot{\eta} = J(\eta)\nu, \quad J(\eta) = \begin{bmatrix} cos(\psi) & -sin(\psi) & 0 \\ sin(\psi) & cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1}$$

with the resulting system model

$$\begin{aligned} \dot{\eta} &= J(\eta)\nu \\ \dot{\nu} &= A_1\eta + A_2\nu + B\tau \end{aligned} \tag{2}$$

where

$$\begin{aligned} M &= \begin{bmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & m_{23} \\ 0 & m_{32} & m_{33} \end{bmatrix} > 0, \quad D = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & d_{23} \\ 0 & d_{32} & d_{33} \end{bmatrix} > 0 \\ A_1 &= -M^{-1}K, \quad A_2 = -M^{-1}D, \quad B = M^{-1}, \end{aligned} \tag{3}$$

are all constant matrices.[1]

The matrix K represents the mooring forces and τ is the control vector of forces from the thruster system. For a more complete model description, see [1], [2], and the references therein.

2. Observer Design and Analysis

In this section we first look at the observer structure proposed in [1], [2] and then extend the design procedure for the observer gains when the system matrix A_2 is non-Hurwitz. A detectability condition is stated.

[1] Note: The positive definiteness of M and D does not imply that A_2 is Hurwitz.

Nonlinear Observer

The design proposed in [1] is based on the assumption that only η is available for measurements and includes the observer structure

$$\begin{aligned}\dot{\hat{\eta}} &= J(\eta)\hat{\nu} + K_1\tilde{\eta}\\ \dot{\hat{\nu}} &= A_1\hat{\eta} + A_2\hat{\nu} + B\tau + K_2\tilde{\eta}\end{aligned} \tag{4}$$

with the resulting error dynamics

$$\begin{aligned}\dot{\tilde{\eta}} &= -K_1\tilde{\eta} + J(\eta)\tilde{\nu}\\ \dot{\tilde{\nu}} &= (A_1 - K_2)\tilde{\eta} + A_2\tilde{\nu} = -\overline{K}_2\tilde{\eta} + A_2\tilde{\nu}\end{aligned} \tag{5}$$

where $\tilde{\eta} = \eta - \hat{\eta}$ and $\tilde{\nu} = \nu - \hat{\nu}$ denote the error estimates for the position and velocity respectively. For notational simplicity $\overline{K}_2 = K_2 - A_1$ is introduced.

A Problem of Stability

Whereas the Lyapunov-based proof in [1] uses equation (18)

$$\frac{1}{2}(P_2A_2 + A_2^TP_2) = -Q_2 \qquad \text{[1], Eq. (18)}$$

with the claim that both Q_2 and P_2 should be positive definite symmetric matrices, this stability argument does not hold if A_2 is not Hurwitz.

Observer Design

We will now relax the Hurwitz condition on A_2, and show how a globally exponentially stable observer still can be found.

Consider the Lyapunov function candidate $V_{obs} = \tilde{x}^TP\tilde{x}$ where

$$P = \begin{bmatrix} P_1 & P_{12}\\ P_{12}^T & P_2\end{bmatrix} = P^T > 0, \quad \tilde{x} = \begin{bmatrix}\tilde{\eta}\\ \tilde{\nu}\end{bmatrix}. \tag{6}$$

For a constant P, we have

$$\dot{V}_{obs} = \dot{\tilde{x}}^TP\tilde{x} + \tilde{x}^TP\dot{\tilde{x}} = \tilde{x}^T\begin{bmatrix} Q_1 & Q_{12}\\ Q_{12}^T & Q_2\end{bmatrix}\tilde{x} \tag{7}$$

where

$$\begin{aligned}Q_1 &= -P_1K_1 - K_1^TP_1 - P_{12}\overline{K}_2 - \overline{K}_2^TP_{12}^T\\ Q_{12} &= -K_1^TP_{12} - \overline{K}_2^TP_2 + P_1J + P_{12}A_2\\ Q_2 &= P_{12}^TJ + J^TP_{12} + P_2A_2 + A_2^TP_2\end{aligned} \tag{8}$$

To show that Q is negative definite, we will show that we can find K_1 and $\overline{K}_2$ such that $Q_2 < 0$, $Q_{12} = 0$ and finally $Q_1 < 0$.[2]
By introducing the cross-term $P_{12} = kC_2^T C_2$, where $C_2 = [0 \quad 0 \quad 1]$, the condition $Q_2 < 0$ is equivalent to the linear matrix inequality

$$P_{12}^T J + J^T P_{12} + P_2 A_2 + A_2^T P_2 = 2kC_2^T C_2 + A_2^T P_2 + P_2 A_2 < 0 \tag{9}$$

A subproblem is to find, if there exist, a scalar k and
a matrix $P_2 = P_2^T > 0$, such that

$$2kC_2^T C_2 + A_2^T P_2 + P_2 A_2 < 0.$$

This inequality can be written as

$$(A_2 + kP_2^{-1} C_2^T C_2)^T P_2 + P_2(A_2 + kP_2^{-1} C_2^T C_2) < 0,$$

which has a solution if

$$A_2 + kP_2^{-1} C_2^T C_2 = A_2 + k(P_2^{-1})_{(:,3)} C_2 \tag{10}$$

is Hurwitz, where $(P_2^{-1})_{(:,3)}$ denotes the last column in P_2^{-1},
i.e., **the pair** (C_2, A_2) **should be detectable.**

Note 1: We can always choose $(P_2^{-1})_{(3,3)} > 0$ since we have freedom in k to change sign.

Positive definiteness of P_2 can be achieved as we only have constraints on its last column through (10). Furthermore, we choose the same block structure for P_2 as in A_2 (see Eq. (3)), which simplifies the analysis below.

Note 2: As there is no coupling between the surge and the sway-yaw system in the ship model dynamics, the detectability condition constrains us to the model class with stable surge dynamics, but this is not any restriction for ships due to the dissipative forces of the water.

Now consider

$$Q_{12} = -kK_1^T C_2^T C_2 - \overline{K}_2^T P_2 + P_1 J(\eta) + kC_2^T C_2 A_2. \tag{11}$$

We can always choose $\overline{K}_2 = \overline{K}_2(\eta)$ such that Q_{12} vanishes. We are left with the condition $Q_1 < 0$. Since

$$Q_1 = -P_1 K_1 - K_1^T P_1 - k(C_2^T C_2 \overline{K}_2 + \overline{K}_2^T C_2^T C_2), \tag{12}$$

[2] It is sufficient that $Q_2 < 0$ and (the Schur-complement) $Q_1 - Q_{12} Q_2^{-1} Q_{12}^T < 0$, but we choose $Q_{12} = 0$ for simplicity.

we have a coupling between (11) and (12) as our choice of $\overline{K}_2$ depends on P_1 and K_1.

Let $P_1 = \text{diag}(p_{ii})$, $K_1 = \text{diag}(k_{ii}) \Rightarrow$

$$kK_2^T C_2^T C_2 = k(-K_1^T P_{12} + P_1 J(\eta) + P_{12}A_2)P_2^{-1}C_2^T C_2 = \\ = \begin{bmatrix} 0 & 0 & f_1(p_{11},\eta) \\ 0 & 0 & f_2(p_{22},\eta) \\ 0 & 0 & f_3(p_{33},k_{33}) \end{bmatrix}$$

where $f_i(\cdot,\cdot)$, $i = 1..3$, are linear functions in the elements of P_1 and K_1 indicated above.

$$Q_1 = \left[\begin{array}{c|c|c} -2p_{11}k_{11} & 0 & -f_1(p_{11},\eta) \\ \hline 0 & -2p_{22}k_{22} & -f_2(p_{22},\eta) \\ \hline -f_1(p_{11},\eta) & -f_2(p_{22},\eta) & -2(p_{33}k_{33} + f_3(p_{33},k_{33})) \end{array}\right] \begin{array}{l} \\ Q_1^{(2)} \\ Q_1^{(3)} \end{array}$$

with submatrices as indicated. The structure of Q_1 lets us conclude, by an iterative procedure,

$$\begin{aligned} &\exists k_{33} \text{ such that } Q_1^{(3)} < 0\,, \forall p_{33} > 0, \\ &\exists k_{22} \text{ such that } Q_1^{(2)} < 0\,, \forall p_{22} > 0, \\ &\exists k_{11} \text{ such that } Q_1 \; < 0\,, \forall p_{11} > 0. \end{aligned} \tag{13}$$

Finally, as we have freedom to choose the coefficients in P_1, we can always choose them so that the Schur-complement $P_1 - P_{12}P_2^{-1}P_{12}^T > 0$, which together with $P_2 > 0$ implies $P > 0$.

We have shown that V_{obs} is a global Lyapunov function for the observer error dynamics (5). In contrast to the results in [1], our $Q_1 = Q_1(\eta)$ is not constant. However, when solving for the elements of $K_1(\eta)$, we can design with respect to a constant matrix which constitutes an upper bound for Q_1. By subtracting any constant, negative definite, diagonal matrix, say Q_{1C}, there still always exists $k_{33}, k_{22}(\eta)$, and $k_{11}(\eta)$ such that the conditions in Eq. (13) (now for $Q_1 - Q_{1C}$) can be fulfilled. Finally, as $Q_1(\eta) - Q_{1C} < 0 \iff Q_1(\eta) < Q_{1C} (< 0)$, we get Q_{1C} as an upper bound. By choosing $K_1(\eta)$ and $\overline{K}_2(\eta)$ as indicated above, we have a constant $P > 0$ and $Q(\eta) < 0$, upper bounded by a constant, negative definite matrix, which implies global exponential stability for the observer.

Note : If A_2 is Hurwitz, we can choose $k = 0$ which will give the Lyapunov function and the observer gains proposed in [1], [2].

3. A Comment on the Observer Backstepping Procedure

In the observer backstepping procedure of [1] a constant observer gain matrix K_1 was used. It shows up in the term $K_1\tilde{\eta}$ and is handled as a disturbance and compensated for by adding damping in the backstepping procedure, see e. g. [3], [4]. For a non-Hurwitz A_2, the proposed observer design above will give a nonlinear observer-gain matrix $K_1(\eta)$, but as the nonlinear elements in $K_1(\eta)$ (sinusoidal functions) are upper bounded, this does not cause any problem in the subsequent design or analysis.

4. Conclusions

The nonlinear output feedback problem stated in [1], [2] was not completely solved. A remedy to the case of a non-Hurwitz system matrix A_2 has been presented under the condition of $([\,0 \quad 0 \quad 1\,], A_2)$ being detectable. Still, also with the remedy proposed, it is not possible to handle models with unstable surge dynamics, due to the structure of decoupling in the system. The observer design presented will coincide with that of [1], [2] for the Hurwitz case. The observer backstepping procedure proposed in [1] still holds for our choice of gain matrices, which renders a globally exponentially stable nonlinear control law.

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Bibliography

[1] Fossen, T. I. and Å. Grøvlen (1998): "Nonlinear output feedback control of dynamically positioned ships using vectorial observer backstepping." *IEEE Trans. Control System and Technology*, **6**, pp. 121–128.

[2] Grøvlen, Å. and T. I. Fossen (1996): "Nonlinear control of dynamic positioned ships using only position feedback: An observer backstepping approach." In *35th IEEE CDC Proceedings*, pp. 3388–3393. Kobe, Japan.

[3] Kanellakopoulos, I., P. Kokotović, and S. Morse (1992): "A toolkit for nonlinear feedback control." *System & Control Letters*, **18**, June, pp. 83–92.

[4] Krstić, M., I. Kanellakopoulos, and P. Kokotović (1995): *Nonlinear and Adaptive Control Design*. John Wiley & Sons, New York.

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