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Discrete-Time LQG with Cross-Terms in the Loss Function and the Noise Description

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1. Introduction

When presenting discrete-time linear quadratic Gaussian (LQG) control most textbooks [AM90, BH69, FPW90, Kai80, MG90, Oga87, PN84] do not pursue cross-terms in the loss function and/or the noise description. This may be well motivated for ease and clarity, especially since the cross-term in the loss function can be removed through an appropriate state-input transformation. It is still, however, interesting to explore the structure of the complete formulae, since cross-terms arise naturally when sampling continuous-time loss functions or when using an innovation's representation for the noise. The paper [Kwo91] claims to be the first complete treatment.

Åström and Wittenmark have considered some of the cross-term issues in [ÅW90], but the complete case is not treated. Our presentation aims at summarizing the formulae for LQG control with cross-terms, and it could be regarded as a generalization of the results in [ÅW90]. The presentation is terse and the reader is referred to [ÅW90, Ch. 11–12] for the basic material.

2. The Process

The discrete-time process is given by [ÅW90, p. 335]

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) + v(k) \\ y(k) &= Cx(k) + e(k) \end{aligned} \tag{1}$$

where $v(k)$ and $e(k)$ are discrete-time Gaussian white-noise processes with zero mean and [ÅW90, p. 336]

$$E \begin{pmatrix} v(k) \\ e(k) \end{pmatrix} \begin{pmatrix} v^T(k) & e^T(k) \end{pmatrix} = \begin{pmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{pmatrix}$$

The initial state $x(0)$ is Gaussian distributed with [ÅW90, p. 336]

$$E\, x(0) = m_0, \quad E\, x(0)x^T(0) = R_0$$

Underlying continuous-time process

Consider the continuous-time process [ÅW90, p. 182]

$$dx = Ax\, dt + Bu\, dt + dv \tag{2}$$

where v is a Wiener process with zero mean and uncorrelated increments. The incremental covariance of v is $R_{1c}\, dt$. By sampling (2) with a zero-order hold and sampling interval h we obtain [ÅW90, pp. 43–46]

$$x(k+1) = \Phi(h)x(k) + \Gamma(h)u(k) + v(k)$$

where

$$\begin{aligned} \Phi(h) &= e^{Ah} \\ \Gamma(h) &= \int_0^h e^{A\tau} B\, d\tau \end{aligned}$$

The process noise $v(k)$ is a discrete-time Gaussian white-noise process with zero mean and covariance [ÅW90, p. 182]

$$R_1(h) = \int_0^h e^{A\tau} R_{1c} e^{A^T\tau}\, d\tau$$

The actual measurements are often modeled as

$$y(k) = Cx(k) + e(k) \tag{3}$$

where $e(k)$ is a sequence of independent random variables modeling the measurement noise; $E\, e(k) = 0$ and $E\, e(k)e^T(k) = R_2$. The measurement noise $e(k)$ is often considered independent of the process noise $v(k)$, i.e. $R_{12} = 0$.

If the measurement noise is colored the state vector has to be extended with states modeling the noise characteristics. Then, R_{12} will have non-zero components. Similarly, R_{12} is also nonzero when using an innovation's representation for the noise.

Sometimes the measurements have been modeled using integrating sampling [Åst70]. This approach normally leads to $R_{12} \neq 0$.

3. Optimal State Feedback

The discrete-time loss function is given by [ÅW90, p. 337]

$$\begin{aligned} J &= E \sum_{k=0}^{N-1} \left\{ x^T(k)Q_1x(k) + 2x^T(k)Q_{12}u(k) + u^T(k)Q_2u(k) \right\} \\ &\quad + E\, x^T(N)Q_0x(N) \\ &= E \sum_{k=0}^{N-1} \begin{pmatrix} x^T(k) & u^T(k) \end{pmatrix} \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix} \begin{pmatrix} x(k) \\ u(k) \end{pmatrix} + E\, x^T(N)Q_0x(N) \end{aligned} \tag{4}$$

The discrete-time loss function can either be constructed directly or be obtained by sampling the continuous-time loss function [ÅW90, p. 336]

$$\begin{aligned} J = E &\int_0^{Nh} \left\{ x^T(t)Q_{1c}x(t) + 2x^T(t)Q_{12c}u(t) + u^T(t)Q_{2c}u(t) \right\} \\ &+ E\, x^T(Nh)Q_0x(Nh) \end{aligned} \tag{5}$$

If (4) were obtained by sampling then [ÅW90, p. 337]

$$\begin{aligned} Q_1 &= \int_0^h \Phi^T(\tau)Q_{1c}\Phi(\tau)\, d\tau \\ Q_{12} &= \int_0^h \Phi^T(\tau)\left[Q_{1c}\Gamma(\tau) + Q_{12c}\right]\, d\tau \\ Q_2 &= \int_0^h \left[\Gamma^T(\tau)Q_{1c}\Gamma(\tau) + 2\Gamma^T(\tau)Q_{12c} + Q_{2c}\right]\, d\tau \end{aligned}$$

When the stochastic case is considered, one additional term depending on the noise is obtained when sampling (5). This term reads

$$\sum_{k=0}^{N-1} \mathrm{tr}\left(Q_{1c} \int_0^h R_1(\tau)\, d\tau \right) = N\bar{J}$$

and should be added to (4). This noise term is the extra term mentioned in the text [ÅW90, p. 337]. It depends on h but is unaffected by the choice of control signal.

To get the optimal control law, solve the Riccati equation [ÅW90, p. 341]

$$\begin{aligned} S(k) &= \Phi^T S(k+1)\Phi + Q_1 \\ &\quad - \left[\Phi^T S(k+1)\Gamma + Q_{12}\right] \left[\Gamma^T S(k+1)\Gamma + Q_2\right]^{-1} \left[\Gamma^T S(k+1)\Phi + Q_{12}^T\right] \\ S(N) &= Q_0 \end{aligned}$$

and calculate (generalization of [ÅW90, p. 357] to $Q_{12} \neq 0$)

$$\begin{aligned} L(k) &= \left[\Gamma^T S(k+1)\Gamma + Q_2\right]^{-1} \left[\Gamma^T S(k+1)\Phi + Q_{12}^T\right] \\ L_v(k) &= \left[\Gamma^T S(k+1)\Gamma + Q_2\right]^{-1} \Gamma^T S(k+1) \end{aligned}$$

Using these expressions, the total loss J can be rewritten as (generalization of [ÅW90, p. 342] to $Q_{12} \neq 0$)

$$\begin{aligned} J &= N\bar{J} + E\left\{x^T(0)S(0)x(0) + \sum_{k=0}^{N-1} [u + Lx]^T \left[\Gamma^T S\Gamma + Q_2\right][u + Lx]\right. \\ &\qquad \left. + \sum_{k=0}^{N-1} v^T S v + \sum_{k=0}^{N-1} v^T S\left[\Phi x + \Gamma u\right] + \sum_{k=0}^{N-1} \left[\Phi x + \Gamma u\right]^T S v\right\} \qquad (6) \\ &= N\bar{J} + E\left\{x^T(0)S(0)x(0)\right. \\ &\qquad + \sum_{k=0}^{N-1} [u + Lx + L_v v]^T \left[\Gamma^T S\Gamma + Q_2\right][u + Lx + L_v v] \\ &\qquad + \sum_{k=0}^{N-1} v^T \left\{S - L_v^T \left[\Gamma^T S\Gamma + Q_2\right] L_v\right\} v \\ &\qquad \left. + \sum_{k=0}^{N-1} v^T S\left[\Phi - \Gamma L\right] x + \sum_{k=0}^{N-1} x^T \left[\Phi - \Gamma L\right]^T S v\right\} \qquad (7) \end{aligned}$$

where x, u, v, and L within the summations have time argument k, while S has argument $k+1$.

In addition, the recursion for S can be written

$$\begin{aligned} S(k) &= \left[\Phi - \Gamma L(k)\right]^T S(k+1)\left[\Phi - \Gamma L(k)\right] \\ &\quad + \begin{pmatrix} I & -L^T(k) \end{pmatrix} \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix} \begin{pmatrix} I \\ -L(k) \end{pmatrix} \end{aligned}$$

which gives the loss for any $u(k) = -L(k)x(k)$.

Different optimal control laws

Depending on if the disturbance $v(k)$ is regarded as unknown or known when forming $u(k)$, we get two different optimal control laws, i.e.

a) $u(k) = -Lx(k)$

b) $u(k) = -Lx(k) - L_v v(k)$

These two choices give rise to different J values. Using the expressions for $u(k)$ and performing the expectation in (6) and (7), respectively, results in

$$
\begin{aligned}
J_a &= N\bar{J} + \operatorname{tr} R_0 S(0) + \sum_{k=0}^{N-1} \operatorname{tr} R_1 S \\
J_b &= N\bar{J} + \operatorname{tr} R_0 S(0) + \sum_{k=0}^{N-1} \operatorname{tr} R_1 \left(S - L_v^T \left[\Gamma^T S \Gamma + Q_2\right] L_v\right) \\
&= J_a - \sum_{k=0}^{N-1} \operatorname{tr} L_v R_1 L_v^T \left[\Gamma^T S \Gamma + Q_2\right]
\end{aligned}
\tag{8}
$$

As can be seen the total loss decreases when $v(k)$ can be measured and used when forming $u(k)$.

4. Optimal Kalman Filter

To obtain the Kalman filter, solve the Riccati equation [ÅW90, p. 352]

$$
\begin{aligned}
P(k+1) &= \Phi P(k) \Phi^T + R_1 \\
&\quad - \left[\Phi P(k) C^T + R_{12}\right] \left[C P(k) C^T + R_2\right]^{-1} \left[C P(k) \Phi^T + R_{12}^T\right] \\
P(0) &= R_0
\end{aligned}
$$

and calculate

$$
\begin{aligned}
K_f(k) &= P(k) C^T \left[C P(k) C^T + R_2\right]^{-1} \\
K_v(k) &= R_{12} \left[C P(k) C^T + R_2\right]^{-1} \\
K(k) &= \Phi K_f(k) + K_v(k)
\end{aligned}
$$

The Kalman filter is then given by [ÅW90, p. 353]

$$
\begin{aligned}
\hat{x}(k+1|k) &= \Phi \hat{x}(k|k) + \Gamma u(k) + \hat{v}(k|k) \\
&= \left[\Phi - K(k) C\right] \hat{x}(k|k-1) + \Gamma u(k) + K(k) y(k) \\
\hat{v}(k+1|k) &= 0 \\
\hat{x}(0|-1) &= m_0 \\
\hat{x}(k|k) &= \hat{x}(k|k-1) + K_f(k) \left[y(k) - C\hat{x}(k|k-1)\right] \\
\hat{v}(k|k) &= K_v(k) \left[y(k) - C\hat{x}(k|k-1)\right]
\end{aligned}
\tag{9}
$$

with

$$
E \begin{pmatrix} \tilde{x}(k|k-1) \\ \tilde{v}(k|k-1) \end{pmatrix} \begin{pmatrix} \tilde{x}^T(k|k-1) & \tilde{v}^T(k|k-1) \end{pmatrix} = \begin{pmatrix} P(k) & 0 \\ 0 & R_1 \end{pmatrix}
$$

$$
\begin{aligned}
&E \begin{pmatrix} \tilde{x}(k|k) \\ \tilde{v}(k|k) \end{pmatrix} \begin{pmatrix} \tilde{x}^T(k|k) & \tilde{v}^T(k|k) \end{pmatrix} \\
&= \begin{pmatrix} P(k) & 0 \\ 0 & R_1 \end{pmatrix} - \begin{pmatrix} P(k) C^T \\ R_{12} \end{pmatrix} \begin{pmatrix} C P(k) C^T + R_2 \end{pmatrix}^{-1} \begin{pmatrix} C P(k) & R_{12}^T \end{pmatrix}
\end{aligned}
$$

where $P = P(k)$.

In addition, the recursion

$$P(k+1) = [\Phi - K(k)C]\,P(k)\,[\Phi - K(k)C]^T + \begin{pmatrix} I & -K(k) \end{pmatrix} \begin{pmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{pmatrix} \begin{pmatrix} I & -K(k) \end{pmatrix}^T$$

gives the error variance in case of any observer gain $K(k)$, and not just the optimal one.

5. The Complete Controller

The separation property [ÅW90, pp. 356–357] follows directly from (6) and (7). It implies that the estimated state and disturbance should be used in the optimal control law. Due to orthogonality between $u + L\hat{x}$ and $\tilde{x}$ ($u + L\hat{x} + L_v\hat{v}$ and $[\tilde{x},\ \tilde{v}]$) the cross-terms vanish when substituting $u + Lx = u + L\hat{x} + L\tilde{x}$ into (6) and $u + Lx + L_v v = u + L\hat{x} + L\tilde{x} + L_v\hat{v} + L_v\tilde{v}$ into (7), see e.g. [Åst70, p. 282].

There are two main cases: $u(k|Y_{k-1})$, where the control signal is based on measurements up to time $k-1$, and $u(k|Y_k)$, where the control signal is based on measurements up to time k. We get (generalization of [ÅW90, p. 357] to $Q_{12} \neq 0$)

$$\begin{aligned} u(k|Y_{k-1}) &= -L\hat{x}(k|k-1) - L_v\hat{v}(k|k-1) = -L\hat{x}(k|k-1) \\ u(k|Y_k) &= -L\hat{x}(k|k) - L_v\hat{v}(k|k) \\ &= -L\hat{x}(k|k-1) - [LK_f + L_vK_v]\,[y(k) - C\hat{x}(k|k-1)] \\ &= -\,[L - LK_fC - L_vK_vC]\,\hat{x}(k|k-1) - [LK_f + L_vK_v]\,y(k) \\ &= -\,[L - MC]\,\hat{x}(k|k-1) - My(k) \end{aligned}$$

where L, L_v, K_v, and K_f all have time argument k, and

$$M = LK_f + L_vK_v.$$

Substituting the expressions for $u(k)$ into (6) and (7), respectively, and performing the expectation results in

$$\begin{aligned} J(Y_{k-1}) &= N\bar{J} + \operatorname{tr} R_0S(0) + \sum_{k=0}^{N-1} \operatorname{tr} R_1S + \sum_{k=0}^{N-1} \operatorname{tr} PL^T \left[\Gamma^TS\Gamma + Q_2\right] L \\ &= J_a + \sum_{k=0}^{N-1} \operatorname{tr} LPL^T \left[\Gamma^TS\Gamma + Q_2\right] \\ J(Y_k) &= N\bar{J} + \operatorname{tr} R_0S(0) + \sum_{k=0}^{N-1} \operatorname{tr} R_1S + \sum_{k=0}^{N-1} \operatorname{tr} PL^T \left[\Gamma^TS\Gamma + Q_2\right] L \\ &\quad - \sum_{k=0}^{N-1} \operatorname{tr} \left[CPC^T + R_2\right]^{-1} \left[LPC^T + L_vR_{12}\right]^T \\ &\qquad\qquad \left[\Gamma^TS\Gamma + Q_2\right] \left[LPC^T + L_vR_{12}\right] \\ &= J(Y_{k-1}) - \sum_{k=0}^{N-1} \operatorname{tr} M \left[CPC^T + R_2\right] M^T \left[\Gamma^TS\Gamma + Q_2\right] \end{aligned} \tag{10}$$

It is quite clear that the total loss decreases when the current measurement is used when forming $u(k)$.

Reference value

The reference value can be introduced in many different ways [ÅW90, pp. 264–271]. The most simple way is to add the term $L_r y_r(k)$ to the expression for $u(k)$, i.e.

$$
\begin{aligned}
u(k|Y_{k-1}) &= L_r y_r(k) - L\hat{x}(k|k-1) \\
u(k|Y_k) &= L_r y_r(k) - [L - MC]\,\hat{x}(k|k-1) - My(k)
\end{aligned}
$$

By choosing the parameter L_r such that

$$
L_r C\,(I - \Phi + \Gamma L)^{-1}\,\Gamma = I
$$

the static gain from y_r to y equals I.

Transfer function of the controller

The complete controller can be summarized as

$$
\begin{aligned}
\varepsilon(k) &= y(k) - C\hat{x}(k|k-1) \\
u(k) &= L_r y_r(k) - L\hat{x}(k|k-1) - M\varepsilon(k) \\
\hat{x}(k+1|k) &= \Phi\hat{x}(k|k-1) + \Gamma u(k) + K(k)\varepsilon(k)
\end{aligned}
\tag{11}
$$

where

$$
M = \begin{cases} 0, & u(k|Y_{k-1}) \\ LK_f + L_v K_v, & u(k|Y_k) \end{cases}
$$

The form (11) is well suited for implementation. It is easy to include extra features such as saturation models for the control signal and validation of the measurements through comparisons between $\varepsilon(k)$ and $P(k)$.

From (11) it is not too difficult to calculate the transfer function of the controller. Some calculations give

$$
u(k) = H_{ff}(q)y_r(k) - H_{fb}(q)y(k)
$$

where

$$
\begin{aligned}
H_{ff}(q) &= [L - MC]\,[qI - \Phi + \Gamma L + KC - \Gamma MC]^{-1}\,[-\Gamma L_r] + L_r \\
H_{fb}(q) &= [L - MC]\,[qI - \Phi + \Gamma L + KC - \Gamma MC]^{-1}\,[K - \Gamma M] + M
\end{aligned}
\tag{12}
$$

Closed loop system

Introduce $\tilde{x}(k) = x(k) - \hat{x}(k|k-1)$. Then, both for the controller with $u(k|Y_{k-1})$ and $u(k|Y_k)$, (generalization of [ÅW90, p. 358])

$$
\begin{aligned}
\begin{pmatrix} x(k+1) \\ \tilde{x}(k+1) \end{pmatrix} &= \begin{pmatrix} \Phi - \Gamma L & \Gamma L - \Gamma MC \\ 0 & \Phi - KC \end{pmatrix} \begin{pmatrix} x(k) \\ \tilde{x}(k) \end{pmatrix} \\
&\quad + \begin{pmatrix} \Gamma L_r \\ 0 \end{pmatrix} y_r(k) + \begin{pmatrix} I \\ I \end{pmatrix} v(k) + \begin{pmatrix} -\Gamma M \\ -K \end{pmatrix} e(k)
\end{aligned}
\tag{13}
$$

It is interesting to note that the closed loop poles are the same for both controllers.

6. Polynomial Formulation of SISO LQG

From (13) it is clear that the closed loop poles are defined by the roots of the two polynomials

$$P(z) = \det(zI - \Phi + \Gamma L), \qquad C(z) = \det(zI - \Phi + KC)$$

The process (1) can also be expressed in polynomial form (we now assume that the process is SISO)

$$\frac{B(z)}{A(z)} = C\,(zI - \Phi)^{-1}\,\Gamma$$

and similarly for the controller (12)

$$H_{fb}(z) = \frac{S(z)}{R(z)}$$

These polynomials are related through the Diophantine-Aryabhatta-Bezout identity

$$A(z)R(z) + B(z)S(z) = P(z)C(z). \tag{14}$$

It is now not necessary to solve the state feedback and Kalman filter problem in state space to obtain $P(z)$ and $C(z)$. Introduce

$$\begin{aligned} C_x^T C_x &= Q_1 - Q_{12}Q_2^{-1}Q_{12}^T, & \Gamma_v\Gamma_v^T &= R_1 - R_{12}R_2^{-1}R_{12}^T, \\ \rho^2 &= Q_2, & \sigma^2 &= R_2 \\ q^2 &= \Gamma^T S\Gamma + Q_2, & r^2 &= CPC^T + R_2 \end{aligned}$$

The monic stable polynomials $P(z)$ and $C(z)$ are then obtained by solving the spectral factorizations (generalization of [ÅW90, p. 385]

$$\begin{aligned} q^2 P(z)P(z^{-1}) &= B_1^T(z)B_1(z^{-1}) + [B_2(z) + \rho A(z)]\,[B_2(z^{-1}) + \rho A(z^{-1})] \\ r^2 C(z)C(z^{-1}) &= B_3(z)B_3^T(z^{-1}) + [B_4(z) + \sigma A(z)]\,[B_4(z^{-1}) + \sigma A(z^{-1})] \end{aligned}$$

where

$$\begin{aligned} \frac{B_1(z)}{A(z)} &= C_x\,(zI - \Phi)^{-1}\,\Gamma, & \frac{B_2(z)}{A(z)} &= \rho^{-1}Q_{12}^T\,(zI - \Phi)^{-1}\,\Gamma, \\ \frac{B_3(z)}{A(z)} &= C\,(zI - \Phi)^{-1}\,\Gamma_v, & \frac{B_4(z)}{A(z)} &= C\,(zI - \Phi)^{-1}\,R_{12}\sigma^{-1} \end{aligned}$$

Note that $B_1(z)$ and $B_3(z)$ are column/row vectors with number of elements equal to $\operatorname{rank}(Q_1)$ and $\operatorname{rank}(R_1)$, respectively.

In the case of minimum variance control, $Q_2 = 0$ (and thus $Q_{12} = 0$), the spectral factorization for $P(z)$ changes to

$$q^2 P(z)P(z^{-1}) = B_1^T(z)B_1(z^{-1})$$

Similarly, when perfect measurements are available, $R_2 = 0$ (and thus $R_{12} = 0$), the spectral factorization for $C(z)$ changes to

$$r^2 C(z)C(z^{-1}) = B_3^T(z)B_3(z^{-1})$$

Provided A and B coprime, the solution R, S to (14) is unique for controllers with $u(k|Y_{k-1})$, i.e. $\deg S \leq \deg R - 1$. The case $u(k|Y_k)$, i.e. $\deg S \leq \deg R$, is more complicated. The identity (14) gives $2n - 1$ equations while $S(z)$ and $R(z)$ contain $2n$ free variables. For $Q_{12} = 0$ or $R_{12} = 0$ it can be shown that $S(0) = 0$. In the special case where A and B are coprime, and $A(0) \neq 0$, this reduces the number of unknown variables by one, making the problem directly solvable. The general case is, however, more involved and [ÅW90, p. 392] gives another approach.

7. Loss Function Dependence on the Sampling Interval

As a numerical example we will investigate how the total loss J depends on the sampling interval h for a simple first order process (the example is due to Bo Bernhardsson [Ber91]). The LQG control problem is solved with a set of Matlab routines [TFRT-7454] developed at the Department of Automatic Control in Lund. The routines are able to handle both cross-terms in the loss function as well as in the noise description.

Consider the continuous-time process

$$dx = ax\,dt + u\,dt + dv,$$

with $m_0 = 0$, $R_0 = 0$, and $R_{1c} = 1$. The process is controlled using a discrete-time controller minimizing

$$\min_u E(y^2 + 10^{-4}u^2)$$

i.e. $Q_{1c} = 1$, $Q_{12c} = 0$, and $Q_{2c} = 10^{-4}$. The measurement noise is modeled as discrete-time Gaussian white noise with $R_2 = 10^{-2}$ and $R_{12} = 0$.

The problem was solved for a set of different sampling intervals using the Matlab routine in Listing 1. The total loss J can be written as (cf. (8) and (10))

$$J = J_{\text{samp}} + J_{\text{init}} + J_{\text{load}} + J_{\text{meas}}$$

The first term J_{samp} is the continuous-time process noise contribution that is independent of the control law. It is a consequence of using a discrete-time controller to control a continuous-time process. The second term depends on the initial state $x(0)$. In our example this term is zero since $m_0 = 0$ and $R_0 = 0$. The third term J_{load} comes from the contribution of the process noise that can be influenced with the control signal. The last term J_{meas} is the contribution due to imperfect measurements. It comes in two versions depending on if $u(k)$ is based on Y_{k-1} or Y_k.

Case 1, $a = -1$ Figure 1 depicts the different contributions to J normalized with h for the case $a = -1$. The component J_{samp} decreases when $h \to 0$, as the difference between the discrete-time controller and a corresponding continuous-time controller gets smaller. As $h \to \infty$ the correlation between consecutive output measurements tends to zero and J_{samp} reaches a stationary value ($\|1/(s+1)\|_{H_2} = 1/2$). The process is stable and for sampling intervals that are larger than the dominating process time constants, it is not possible to find a control signal that reduces the variance of the output. The stationary value of J_{samp} corresponds to no control.

```
function loss = lqgloss(A,B,C,Q1c,Q12c,Q2c,R1c,R12,R2,h)
%LQGLOSS Calculates loss per timeunit for LQG control
%
%        loss = lqgloss(A,B,C,Q1c,Q12c,Q2c,R1c,R12,R2,h)
%
%        The different contributions to the total loss is calculated for
%        the sampling intervals in h. loss contains the components
%        [Jsamp Jload Jmeas1 Jmeas2]. They are all normalized with h.

loss = [];
for hcur=h
  [Phi,Gam,Q1,Q2,Q12,R1,Je] = lqgsamp(A,B,hcur,Q1c,Q2c,Q12c,R1c);
  [L,Lv,lr,S] = lqrd(Phi,Gam,C,Q1,Q2,Q12);
  [K,Kf,Kv,P,Pf] = lqed(Phi,C,R1,R2,R12);
  Jsamp = Je/hcur;
  Jload = trace(R1*S)/hcur;
  Jmeas1 = trace(P*L'*(Gam'*S*Gam+Q2)*L)/hcur;
  Jmeas2 = Jmeas1-..
    trace((L*Kf+Lv*Kv)*(C*P*C'+R2)*(L*Kf+Lv*Kv)'*(Gam'*S*Gam+Q2))/hcur;
  loss = [loss; ..
    Jsamp Jload Jmeas1 Jmeas2 Jsamp+Jload+Jmeas1 Jsamp+Jload+Jmeas2];
end
```

Listing 1. A listing of the Matlab routine used to solve the example. The routine is based on the LQG-routines described in [TFRT-7454].

For h of medium size the term J_{load} decreases as h. As h gets smaller it is possible to eliminate more and more of the effect from v, and J_{load} decreases. For very small h values J_{load} reaches a steady state value due to Q_{2c}. The penalty on the control signal makes it impossible to further reduce J_{load}. If $Q_{2c} = 0$, J_{load} continues to decrease as h when $h \to 0$. J_{load} also decreases when $h \to \infty$. This should not be interpreted as if the controller manages to eliminate the effect from v; it merely demonstrates that for sampling intervals large enough the control signal can not influence the effects of v (the effect of v is captured in J_{samp}).

The term J_{meas} behaves similarly to J_{load}. The number of measurements per time unit increase when h decreases, and J_{meas} also decreases. In contrast to J_{load}, the asymptotic behavior for J_{meas} is $\sqrt{h}$ instead of h. As expected J_{meas} is larger for $u(k|Y_{k-1})$ than for $u(k|Y_k)$. The difference is more pronounced for large sampling intervals.

Case 2, $a = 1$ Figure 2 depicts the different contributions to J normalized with h for the case $a = 1$. The behavior is quite similar to the case $a = -1$ except for large h. The process is unstable and it has to be controlled actively also when h gets large. Due to the instability the different components of J grow exponentially a $h \to \infty$. This phenomenon gets pronounced as h exceeds the dominating time constants of the process.

The total loss for both Case 1 and Case 2 is depicted in Figure 3. The loss is smallest when complete state information is available, e.g. no measurement noise, and largest for the control law $u(k|Y_{k-1})$. The case $u(k|Y_k)$ is somewhere in between.

The way J depends on h gives information about reasonable sampling intervals. For our example we note that it does not pay off to sample faster than $h \approx 0.005$. The value on J will not get smaller even if h is decreased. The reason is the penalty on the control signal. In addition, it is no use having

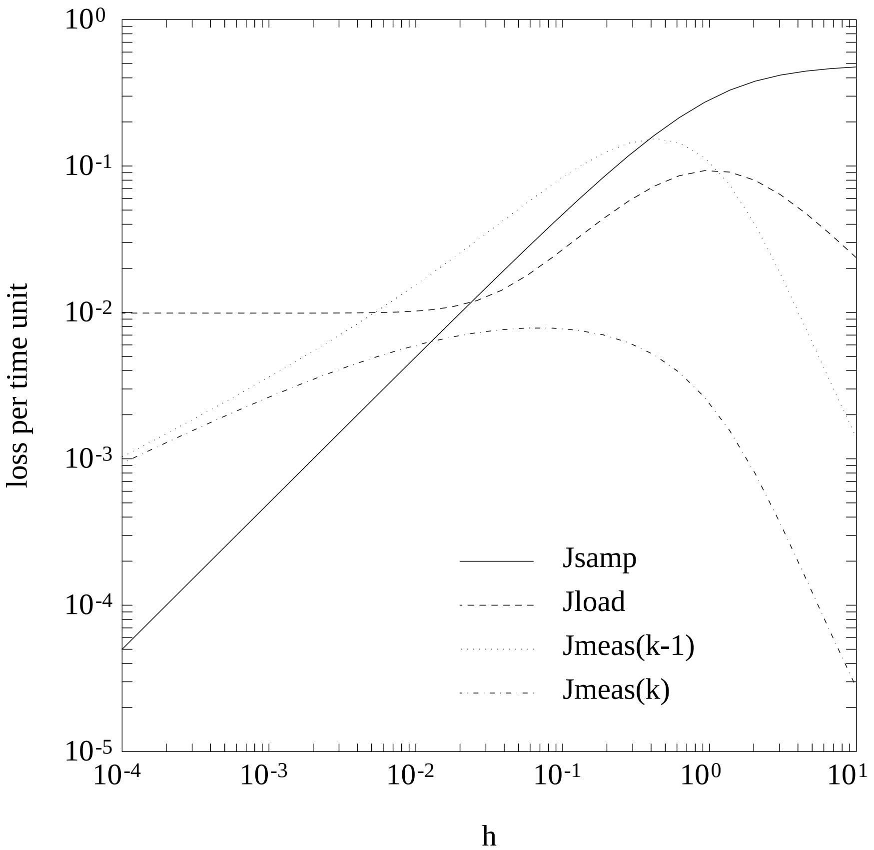

Figure 1. Different contributions to the total loss J as function of the sampling interval h for the case $a = -1$.

$h > 0.5$. For the stable process such slow sampling makes it impossible to affect J constructively, and in the unstable case the sampling is already to slow to handle the instability reasonably well.

It is worth noting the overall behavior for J as $h \to 0$. Bo Bernhardsson conjectures [Ber91] that for small h

$$J = c_1 + c_2\sqrt{h} + c_3 h \tag{15}$$

where $c_1 = 0$ if $Q_{2c} = 0$, and $c_2 = 0$ if $R_2 = 0$. This behavior is demonstrated in our example, but differs from what is claimed in [ÅW90, p. 360]. It is, however, important to realize that our result hinges strongly on the assumptions made. We have assumed that the measurement noise is independent between consecutive samples. In the case of very fast sampling this assumption seems unrealistic, and the noise model should be changed, e.g. [Åst70] uses integrating sampling and assumes independent increments in the measurement noise. Depending on how this is done the result (15) may or may not change.

8. Concluding Remarks

We have presented complete formulae for discrete-time LQG with cross-terms in both the loss function and the noise description. The complete case leads to a moderate increase of complexity compared to the standard case, and it is surprising that most textbooks do not present it.

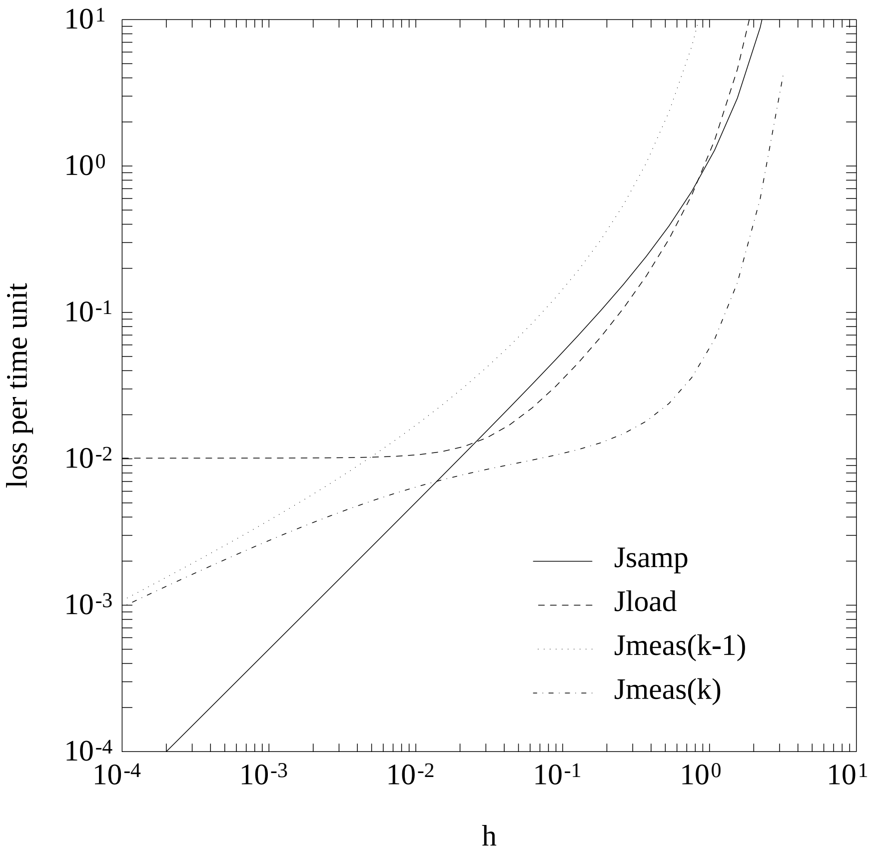

Figure 2. Different contributions to the total loss J as function of the sampling interval h for the case $a = 1$.

When designing discrete-time LQG controllers it is good practice, if possible, to use an underlying continuous-time loss function. This makes the controller minimize a criterion based on the whole sampling interval and not just what happens at the sampling instances. In addition, a continuous-time loss function provides a means to compare controllers with different sampling intervals. Even if the continuous-time loss function does not include cross-terms, its sampled version most certainly will, and then it is practical to have access to the complete formulae.

When allowing the Kalman filter to use the most recent measurements for its state estimates, you get a direct path from the measurement to the control signal. This path is given by the matrix M (cf. Section 5), which consists of two terms: LK_f and $L_v K_v$. The most recent measurement provides for a correction of the state estimate and the first term corresponds to a feedback from this correction. The second term, which is omitted in many presentations, like for an innovation realization, is more subtle. If the measurement noise and the process noise are correlated, then the most recent measurement provides indirect information about the process noise. The second term in M includes this information into the control signal.

The different contributions to J have different stepsize dependence, and plotting them us function of h is quite informative. Such plots reveal quite clearly in what region h should be chosen.

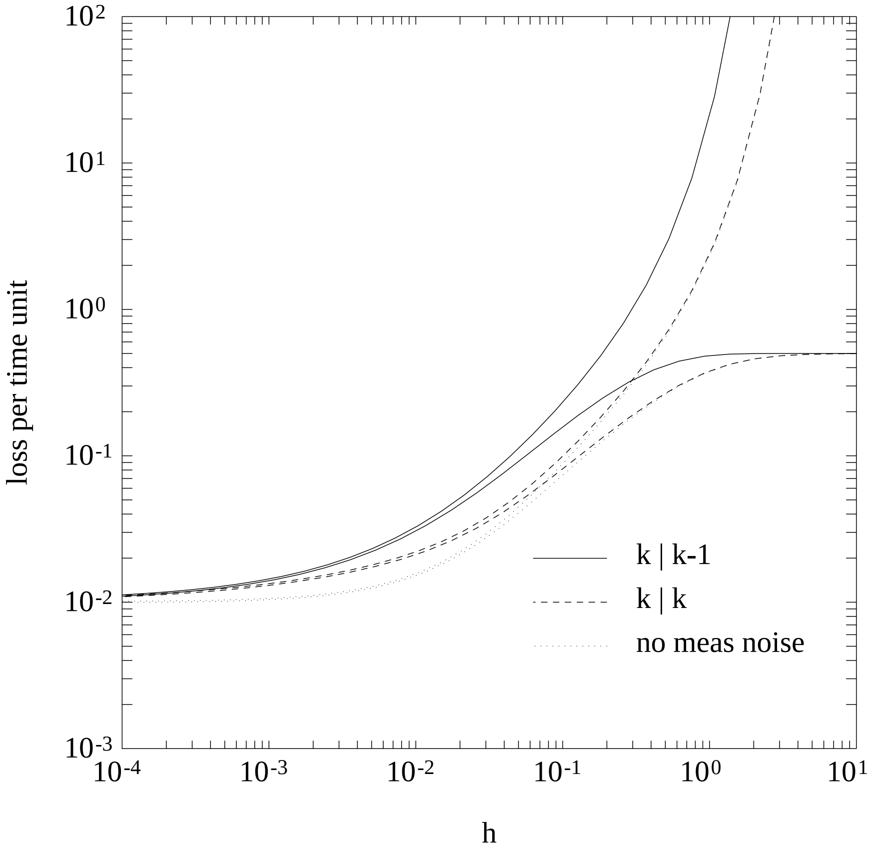

Figure 3. Total loss as function of h when $a = -1$ and $a = 1$.

9. References

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