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by K-J Åström

Abstract

This paper presents recursive formulas which admits simultaneous test of stability and evaluation of quadratic lossfunctions for linear discrete time dynamical systems. The method admits a significant reduction of the number of computations in comparison with previously known methods.

1. Introduction

We will consider the evaluation of integrals of the type

$$I = \frac{1}{2\pi i} \oint \frac{B(z)\, B(z^{-1})}{A(z)\, A(z^{-1})} \cdot \frac{dz}{z} \tag{1}$$

where A and B are polynomials with real coefficients

$$A(z) = a_o z^n + a_1 z^{n-1} + \ldots + a_n \tag{2}$$

$$B(z) = b_o z^n + b_1 z^{n-1} + \ldots + b_n \tag{3}$$

and $\oint$ denotes the integral along the unit circle in the positive direction.

Integrals such as (1) occur in many control and communication problems. The sum of squares of the values of the impulse response of a dynamical system with the pulse transfer function B(z)/A(z) is e.g. given by (1). Evaluation of quadratic lossfunctions, generation of quadratic Lyapunov functions for linear systems and investigation of the accuracy of parameter estimation in linear systems also lead to integrals of type (1), see e.g. [1] and [3].
Closed form solutions of (1) for polynomials of low order are available in literature. See e.g. Jury [3, p. 298-299]. For large n, say $n \geq 4$, the closed form solutions are, however, very cumbersome to use. It is also wellknown that $a_o I$, where I is the integral defined by (1) can be obtained as the first component of the vector x which satisfies the following linear equation.

$$\begin{bmatrix} a_o & a_1 & a_2 & a_3 & & a_n \\ a_1 & a_o+a_2 & a_1+a_3 & a_2+a_4 & \cdots & a_{n-1} \\ a_2 & a_3 & a_o+a_4 & a_1+a_5 & \cdots & a_{n-2} \\ \vdots & & & & & \\ a_{n-1} & a_n & 0 & 0 & \cdots & a_1 \\ a_n & 0 & 0 & 0 & \cdots & a_o \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{n} b_i^2 \\ 2\sum_{i=0}^{n-1} b_i b_{i+1} \\ 2\sum b_i b_{i+2} \\ \vdots \\ 2\sum b_i b_{i+n-1} \\ 2b_o b_n \end{bmatrix} \qquad (3)$$

See e.g. [3, p. 168-172] .

In this paper we present recursive equations for the integral which leads to considerably fewer computations than a direct solution of (3). The recursive equations are derived using elementary results from the theory of analytic functions. The recursive equations can be conveniently used both for hand- and machine computation. Analogous results for continuous time systems have been obtained by Nekolný and Beneš [4].

2. Preliminaries and notations

We first observe that the integral (1) will always exist if the polynomial $A(z)$ has all its zeros inside the unit circle. In such a case we can always find a stable dynamical system with the pulse transfer function $B(z)/A(z)$ and the integral (1) is then simply the sum of squares of the ordinates of the impulse response of the system.

If $A(z)$ has zeros on the unit circle the integral diverges.

If $A(z)$ has zeros both inside and outside the unit circle but not on the unit circle, the integral (1) still exists. In such a case we can always find a polynomial $A'(z)$ with all its zeros inside the unit circle such that

$$A(z)\, A(z^{-1}) = A'(z)\, A'(z^{-1})$$

and the integral then represents the sum of squares of the impulse response of a stable dynamical system whose pulse transfer function is $B(z)/A'(z)$.

In many practical cases, however, we obtain the integral as a result of an analysis of a dynamical system whose pulse transfer function is $B(z)/A(z)$. In such a case it is naturally of great importance to test that $A(z)$ has all its zeros inside the unit circle because when this is not the case the dynamical system will be unstable although the integral (1) exists.

In order to present the result in a simple form we will first introduce some notations. Let A^{*} denote the polynomial defined by

$$A^{*}(z) = z^{n}A(z^{-1}) = a_{o} + a_{1}z + \ldots + a_{n}z^{n} \tag{4}$$

Further introduce the polynomials

$$A_{k}(z) = a_{o}^{k}z^{k} + a_{1}^{k}z^{k-1} + \ldots + a_{k}^{k} \tag{5}$$

$$B_{k}(z) = b_{o}^{k}z^{k} + b_{1}^{k}z^{k-1} + \ldots + b_{k}^{k} \tag{6}$$

which are defined recursively by

$$A_{k-1}(z) = z^{-1}\{A_k(z) - \alpha_k A^*_k(z)\} \tag{7}$$

$$B_{k-1}(z) = z^{-1}\{B_k(z) - \beta_k A^*_k(z)\} \tag{8}$$

where

$$\alpha_k = a_k^k / a_o^k \tag{9}$$

$$\beta_k = b_k^k / a_o^k \tag{10}$$

and

$$A_n(z) = A(z) \tag{11}$$

$$B_n(z) = B(z) \tag{12}$$

If these equations should have a meaning we must naturally require that all a_o^k are different from zero. To see the implications of this we will make use of the following theorem.

Theorem 1

The polynomial A(z) has all its zeros inside the unit circle if and only if $a_o^k > 0$ for all k.

This theorem is essentially the Schur-Cohn stability criterion for linear discrete time dynamical systems. See e.g. [2], [3, p 126], [6]. We also have the following result which will be used in the proof of our main result.

Theorem 2

Let the polynomial $A_k(z)$ have all its zeros inside the unit circle, then $A_{k-1}(z)$ also has all its zeros inside the unit circle.

This theorem is also given by Schur [6]. A simple proof is given by Růžička [5].

We thus find that the polynomials $A_k(z)$ can always be find if the original polynomial has all its zeros inside the unit circle. If this is not the case we will always get $a_o^k \leq 0$ at same step in the reduction. The equations (7) and (8) can thus be profitable exploited as a stability criterion.

3. The main result

We will now show that the integral (1) can be computed recursively. For this purpose we introduce the integrals I_k defined by

$$I_k = \frac{1}{2\pi i} \oint \frac{B_k(z)\ B_k(z^{-1})}{A_k(z)\ A_k(z^{-1})} \cdot \frac{dz}{z} \tag{13}$$

It follows from (1) that $I = I_n$. We now have

Theorem 3

Let the polynomial $A(z)$ have all its zeros inside the unit circle. The integrals I_k defined by (13) then satisfies the following recursive equations

$$\{1 - \alpha_k^2\}I_{k-1} = I_k - \beta_k^2 \tag{14}$$

$$I_o = \beta_o^2 \tag{15}$$

Proof

As $A(z)$ has all its zeros inside the unit circle it follows from Theorem 1 that all a_o^k are different from zero. It thus follows from (9) and (10) that all polynomials A_k and B_k can be defined. Furthermore it follows from Theorem 2 that all polynomials A_k have all zeros inside the unit circle. All integrals I_k thus exist.

To prove the theorem we will make use of the theory of analytic functions. The integral (13) equals the sum of residues at the poles of the function $B_k(z)\ B_k(z^{-1})/\{z\ A_k(z)\ A_k(z^{-1})\}$ inside the unit circle. As the integral is invariant under the change of variables $z \rightarrow 1/z$, we also find that the integral equals the sum of residues of the poles outside the unit circle.

Now consider

$$I_{k-1} = \frac{1}{2\pi i} \oint \frac{B_{k-1}(z)\ B_{k-1}(z^{-1})}{A_{k-1}(z)\ A_{k-1}(z^{-1})} \cdot \frac{dz}{z}$$

The poles of the integrand inside the unit circle are $z = 0$ and the zeros z_i of the polynomial $A_{k-1}(z)$. It follows from (7) and (4) that

$$A_k \quad (z_i) = \alpha_k A_k^*(z_i) = \alpha_k z_i^k A_k(z_i^{-1})$$

$$A_{k-1}(z_i^{-1}) = z_i\{A_k(z_i^{-1}) - \alpha_k A_k^*(z_i^{-1})\}$$

Hence

$$\begin{aligned} A_{k-1}(z_i^{-1}) &= z_i\{A_k(z_i^{-1}) - \alpha_k z_i^{-k} A_k(z_i)\} \\ &= (1 - \alpha_k^2) z_i \; A_k(z_i^{-1}) \end{aligned}$$

Further it follows from (4) and (7) that

$$A_{k-1}^*(z) = A_k^*(z) - \alpha_k A_k(z)$$

Hence

$$A_{k-1}^*(0) = A_k^*(0) - \alpha_k A_k(0) = a_o^k - \alpha_k a_k^k = a_o^k(1 - \alpha_k^2)$$

The functions

$$\frac{B_{k-1}(z)\ B_{k-1}(z^{-1})}{A_{k-1}(z)\ A_{k-1}(z^{-1})} \cdot \frac{1}{z} = \frac{B_{k-1}(z)\ B_{k-1}^*(z)}{A_{k-1}(z)\ A_{k-1}^*(z)} \cdot \frac{1}{z}$$

and

$$\frac{B_{k-1}(z)\ B_{k-1}(z^{-1})}{A_{k-1}(z)\{z(1-\alpha_k^2)A_k(z^{-1})\}} \cdot \frac{1}{z} = \frac{B_{k-1}(z)\ B_{k-1}^*(z)}{A_{k-1}(z)\{(1-\alpha_k^2)A_k^*(z)\}} \cdot \frac{1}{z}$$

have the same poles inside the unit circle and the same residues at these poles. Hence

$$\begin{aligned} I_{k-1} &= \frac{1}{1-\alpha_k^2} \cdot \frac{1}{2\pi i} \oint \frac{B_{k-1}(z)\ B_{k-1}(z^{-1})}{A_{k-1}(z)\ A_k(z^{-1})} \frac{dz}{z^2} \\ &= \frac{1}{1-\alpha_k^2} \cdot \frac{1}{2\pi i} \oint \frac{B_{k-1}(z)\ B_{k-1}(z^{-1})}{A_k(z)\ A_{k-1}(z^{-1})} dz \end{aligned} \qquad (16)$$

where the second equality is obtained by making the variable substitution $z \to z^{-1}$. The integrand has poles at the zeros of

$A_k(z)$. It follows, however, from (7) that

$$A_{k-1}(z^{-1}) = z\{A_k(z^{-1}) - \alpha_k A_k^*(z^{-1})\}$$
$$= z\{A_k(z^{-1}) - \alpha_k z^{-k} A_k(z)\}$$

Hence for z_i such that $A_k(z_i) = 0$ we get

$$A_{k-1}(z_i^{-1}) = z_i A_k(z_i^{-1})$$

The functions

$$\frac{B_{k-1}(z)\ B_{k-1}(z^{-1})}{A_k(z)\ A_{k-1}(z^{-1})}$$

and

$$\frac{B_{k-1}(z)\ B_{k-1}(z^{-1})}{A_k(z)\ A_k(z^{-1})} \cdot \frac{1}{z} = \frac{B_{k-1}(z)\ B_{k-1}^*(z)}{A_k(z)\ A_k^*(z)}$$

thus have the same poles inside the unit circle and the same residues at these poles. The integral of these functions around the unit circle are thus the same. The equation (16) now gives

$$I_{k-1} = \frac{1}{1-\alpha_k^2} \frac{1}{2\pi i} \oint \frac{B_{k-1}(z)\ B_{k-1}(z^{-1})}{A_k(z)\ A_k(z^{-1})} \frac{dz}{z}$$

Now introduce (8) and we find

$$(1-\alpha_k^2)I_{k-1} = \frac{1}{2\pi i} \oint \frac{\{B_k(z) - \beta_k A_k^*(z)\}\{B_k(z^{-1}) - \beta_k A_k^*(z^{-1})\}}{A_k(z)\ A_k(z^{-1})} \frac{dz}{z}$$

$$= \frac{1}{2\pi i} \oint \frac{B_k(z)\ B_k(z^{-1})}{A_k(z)\ A_k(z^{-1})} \cdot \frac{dz}{z} - \frac{\beta_k}{2\pi i} \oint \frac{B_k(z)\ A_k^*(z^{-1})}{A_k(z)\ A_k(z^{-1})} \cdot \frac{dz}{z}$$

$$- \frac{\beta_k}{2\pi i} \oint \frac{A_k^*(z)\ B_k(z^{-1})}{A_k(z)\ A_k(z^{-1})} \cdot \frac{dz}{z} + \frac{\beta_k^2}{2\pi i} \oint \frac{A_k^*(z)\ A_k^*(z^{-1})}{A_k(z)\ A_k(z^{-1})} \cdot \frac{dz}{z} \qquad (17)$$

The first integral equals I_k. The second integral can be reduced as follows

$$\frac{\beta_k}{2\pi i} \oint \frac{B_k(z)\, A_k^*(z^{-1})}{A_k(z)\, A_k(z^{-1})} \cdot \frac{dz}{z} = \frac{\beta_k}{2\pi i} \oint \frac{B_k(z)\, A_k(z)}{A_k(z)\, A_k^*(z)} \cdot \frac{dz}{z} =$$

$$= \frac{\beta_k}{2\pi i} \oint \frac{B_k(z)}{A_k^*(z)} \frac{dz}{z} = \beta_k \frac{B_k(0)}{A_k^*(0)} = \beta_k \frac{b_k^k}{a_o^k} = \beta_k^2$$

Where the first equality follows from (4), the third from the residues theorem and the fifth from (10). Similarly we find that the third integral of the right number also equals β_k^2.

Using (4) the fourth term of the right member of (17) can be reduced as follows

$$\frac{\beta_k^2}{2\pi i} \oint \frac{A_k^*(z)\, A^*(z^{-1})}{A_k(z)\, A_k(z^{-1})} \frac{dz}{z} = \frac{\beta_k^2}{2\pi i} \oint \frac{dz}{z} = \beta_k^2$$

Summarizing we find (14). When k = 0 we get from (13)

$$I_o = \frac{1}{2\pi i} \oint \left(\frac{b_o^o}{a_o^o} \right)^2 \frac{dz}{z} = \beta_o^2$$

which completes the proof os the theorem.

4. The algorithm

Having obtained the recursive formula (14) we will now develop a computer algorithm for the evaluation of (1). We first observe that if we carry out the reduction of the polynomials A_k and B_k given by the equations (7) and (8), we obtain α_k and β_k in descending order whereas a direct evaluation of (14) requires the coefficients in ascending order. A direct application of (7), (8) and (14) will thus require temporary storage of α_k and β_k. To avoid this we observe that (14) is a linear difference equation. The solution can thus be written as

$$I_k = C_k I_n + D_k \tag{18}$$

where

$$C_{k-1} = \frac{C_k}{1 - \alpha_k^2} \tag{19}$$

$$D_{k-1} = \frac{D_k - \beta_k^2}{1 - \alpha_k^2} \tag{20}$$

and

$$C_n = 1 \tag{21}$$

$$D_n = 0 \tag{22}$$

For k = 0 we get

$$I_o = C_o I_n + D_o = \beta_o^2$$

Hence

$$I = I_n = \frac{\beta_o^2 - D_o}{C_o} \tag{23}$$

The integral (1) can now be computed recursively as follows

1. Set $k = n$, $C_n = 1$ and $D_n = 0$
2. Test if $A(z)$ is stable by checking $a_o^k > 0$. Stop computation if unstable otherwise proceed.
3. Compute α_k and β_k from (9) and (10) and a_i^{k-1} and b_i^{k-1} from (7) and (8).
4. Compute C_{k-1} and D_{k-1} from (19) and (20).
5. Repeat steps 2; 3 and 4 until $k = 1$
6. Compute $I = I_n$ from (23)

The number of algebraic operations required to compute I by this procedure is shown in table 1a.As seen by this we obtain a considerable saving in comparison with a straight forward solution of (3) for x_1 using gaussian reduction.

For comparison we show in table 1b the number of operations required for a straight forward solution of (3) for x_1 using gaussian reduction.

Table 1a - Number of arithmetic operations required to compute I using the recursive formulas

Order of system	Add/Subtract	Multiply	Divide
2	13	11	10
5	46	41	22
10	141	131	42
n	n^2+4n+1	n^2+3n+1	$4n+2$

Table 1b - Number of arithmetic operations required to compute the integral form (3) using gaussian reduction

Order of system	Add/Subtract	Multiply	Divide
2	9	13	4
5	82	81	16
10	476	461	66
n	$\frac{4n^3+15n^2+20n+12(9)}{12}$	$\frac{2n^3+6n^2+16n+6}{6}$	$\frac{n^2+n+2}{2}$

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