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Abstract

Propagation of transient electric and magnetic (TEM) pulses in nonstationary, linear, homogeneous, and isotropic dielectric and magnetic materials is investigated using an exact wave splitting. Key intrinsic properties are the index of refraction and the relative admittance, which are both temporal integral operators with kernels that depend on two time variables. In addition, the Sommerfeld forerunners in dispersive nonstationary materials are derived. A numerical example — a single-resonance Lorentz model with time-dependent plasma frequency — is presented.

1 Introduction

Propagation of transient electric and magnetic (TEM) pulses in nonstationary, linear, stratified dielectrics has been studied extensively by Åberg *et al.* using optical wave splitting [2–5]. Nonstationary materials are complex materials in the sense that the dielectric constant depends on time. Specifically, this description applies to nondispersive materials. However, real materials are known to be dispersive. In dispersive nonstationary materials, the electric susceptibility kernel depends on two times, the excitation time and the observation time.

Model problems, where time-dependent coefficients occur, can be found in telecommunication problems, for example, fading and modulation problems. Other interesting fields of applications arise in control theory [10]. In [10], Tzou shows how piezo-electric sensors can be used to control continua. Here the research fields of elastodynamics and electromagnetics are interconnected. Another interesting approach, that has proven successful, is to apply the nonstationary wave equations to wave propagation in weakly nonlinear electromagnetic media. Accordingly, two nonlinear applications — high-frequency switching and Kerr effect — are treated in [1]. Microwave propagation in ferrites also shows applications that can be modelled with time-dependent coefficients.

The analysis in the above references leads to the study of systems of coupled, hyperbolic integro-differential equations. In the present article, this propagation problem is revisited, and the theory is extended to include magnetic nonstationary materials as well.

Specifically, a so called dispersive wave splitting, leading to scalar hyperbolic integro-differential equations for the up-going and down-going fields, that is, one-way equations, is adopted. Dispersive wave splitting has been used before for time-invariant media [7]. The index of refraction, the relative admittance, and the relative impedance of the nonstationary medium are defined. These temporal integral operators are the key intrinsic properties of the medium. Using these concepts is believed to facilitate the direct and inverse scattering problems for the nonstationary slab.

Moreover, Sommerfeld forerunners or first precursors in these materials are defined and derived. The Sommerfeld forerunner is the highly oscillating early time-behavior of the signal in a dispersive material. These transients are generalizations of the classical results for Lorentz materials presented by Sommerfeld [8]. The adopted method is similar to the time-domain technique used by Karlsson and Rikte for time-invariant materials [7]. The early time-behavior of the signal can be obtained much easier by using the Sommerfeld approximation than by solving the one-way integro-differential equation for the propagating field. A numerical example illustrating the theory is given; specifically, a Lorentz model with a time-varying plasma frequency is discussed.

The problem is formulated in Section 2. In Section 3, the dispersive wave splitting is introduced and the intrinsic properties of the medium are defined. The properties of the one-way wave equations are presented in Section 4. Sommerfeld forerunners are discussed in Section 5.

2 Basic equations

Throughout the article, Cartesian coordinates $O(x, y, z)$ are used. The radius vector is written $\boldsymbol{r} = \boldsymbol{u}_x x + \boldsymbol{u}_y y + \boldsymbol{u}_z z$, where $\boldsymbol{u}_x$, $\boldsymbol{u}_y$, and $\boldsymbol{u}_z$ are the basis vectors in the $x-$direction, $y-$direction, and $z-$direction, respectively. Time is denoted by t.

The electric and magnetic field intensities at $(\boldsymbol{r}, t)$ are denoted by $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{H}(\boldsymbol{r}, t)$, respectively, and the corresponding flux densities are $\boldsymbol{D}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$. Each field vector is written in the form

$$\boldsymbol{E} = \boldsymbol{u}_x E_x(\boldsymbol{r}, t) + \boldsymbol{u}_y E_y(\boldsymbol{r}, t) + \boldsymbol{u}_z E_z(\boldsymbol{r}, t).$$

The speed of light in vacuum is c and the intrinsic impedance of vacuum η. The Dirac delta measure is denoted by $\delta(t)$ and the Heaviside step function by $H(t)$.

The Maxwell equations govern the dynamics of the fields in macroscopic media:

$$\nabla \times \boldsymbol{E}(\boldsymbol{r}, t) = -\partial_t \boldsymbol{B}(\boldsymbol{r}, t), \qquad \nabla \times \boldsymbol{H}(\boldsymbol{r}, t) = \boldsymbol{J}(\boldsymbol{r}, t) + \partial_t \boldsymbol{D}(\boldsymbol{r}, t),$$

where $\boldsymbol{J}(\boldsymbol{r}, t)$ denotes the imposed current density and the current density

$$\boldsymbol{J}_\sigma(\boldsymbol{r}, t) = \sigma(t)\boldsymbol{E}(\boldsymbol{r}, t),$$

due to a finite conductivity, $\sigma(t)$, has been included in the displacement current density, $\partial_t \boldsymbol{D}(\boldsymbol{r}, t)$. This conductivity may vary with time.

The constitutive relations of a linear, homogeneous, and isotropic medium are

$$c\eta \boldsymbol{D}(\boldsymbol{r}, t) = [\mathcal{E}_r \boldsymbol{E}](\boldsymbol{r}, t), \qquad c\boldsymbol{B}(\boldsymbol{r}, t) = \eta[\mathcal{M}_r \boldsymbol{H}](\boldsymbol{r}, t), \tag{2.1}$$

where

$$\begin{aligned} \mathcal{E}_r &= \epsilon_r(t) + \chi^e(t, t')* = \big(\epsilon_r(t)\delta(t - t') + \chi^e(t, t')\big)* \\ \mathcal{M}_r &= \mu_r(t) + \chi^m(t, t')* = \big(\mu_r(t)\delta(t - t') + \chi^m(t, t')\big)* \end{aligned} \tag{2.2}$$

are the relative permittivity operator and the relative permeability operator, respectively. The electric and magnetic optical responses, $\epsilon_r(t) \geq 1$ and $\mu_r(t) \geq 1$, respectively, are assumed to be sufficiently regular (e.g., bounded and smooth) functions. Temporal dispersion and causality are modeled by integrals in time:

$$(\chi^e * \boldsymbol{E})(\boldsymbol{r}, t) = \int_{-\infty}^{+\infty} \chi^e(t, t')\boldsymbol{E}(\boldsymbol{r}, t')\, dt',$$

where the electric and magnetic susceptibility kernels, $\chi^e(t, t')$ and $\chi^m(t, t')$, respectively, are functions of two time-variables, namely, the observation time, t, and the excitation time, t'. The susceptibility kernels vanish for $t < t'$, and are assumed to be sufficiently regular (e.g., bounded and smooth) functions of both variables for $t > t'$. This implies causality: the wave-front of any nonpathological traveling plane wave propagates through the dispersive material with the speed

$$\frac{c}{\sqrt{\epsilon_r(t)\mu_r(t)}} \leq c.$$

For time-invariant materials, $\epsilon_r(t)$ and $\mu_r(t)$ are independent of time and the susceptibility kernels dependent of one time-variable only: $\chi^e(t, t') = \chi^e(t - t')$ and $\chi^m(t, t') = \chi^m(t - t')$. A non-dispersive medium is characterized by $\epsilon_r(t)$, $\mu_r(t)$, $\chi^e(t, t') = c\eta\sigma(t')H(t-t')$, and $\chi^m(t, t') = 0$. Nonmagnetic materials satisfy $\mathcal{M}_r = 1$.

The simplest radiation problem for the unbounded temporally dispersive dielectric medium with the constitutive relations (2.1) is to calculate the electromagnetic response to a transverse current source distribution:

$$\boldsymbol{J}(\boldsymbol{r}, t) = \boldsymbol{J}(z, t) = \boldsymbol{u}_x J_x(z, t) + \boldsymbol{u}_y J_y(z, t).$$

Such an initially quiescent and sufficiently regular source distribution supports transverse electric and magnetic (TEM) waves:

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{r}, t) &= \boldsymbol{E}(z, t) = \boldsymbol{u}_x E_x(z, t) + \boldsymbol{u}_y E_y(z, t), \\ \boldsymbol{H}(\boldsymbol{r}, t) &= \boldsymbol{H}(z, t) = \boldsymbol{u}_x H_x(z, t) + \boldsymbol{u}_y H_y(z, t). \end{aligned}$$

Concentrated sources terms on the form $\boldsymbol{J}(z, t) = \delta(z - z_0)\boldsymbol{j}(t)$, where $\boldsymbol{j}(t)$ is regular surface current density, arise at normal incidence at plane interfaces and such distributions are allowed as well. Subjected to TEM waves, the Maxwell equations reduce to

$$\partial_z \boldsymbol{E} = c^{-1}\partial_t \mathcal{M}_r \left(\boldsymbol{u}_z \times \eta \boldsymbol{H}\right), \quad \partial_z \left(\boldsymbol{u}_z \times \eta \boldsymbol{H}\right) = \eta \boldsymbol{J} + c^{-1}\partial_t \mathcal{E}_r \boldsymbol{E}. \tag{2.3}$$

3 Dispersive wave splitting

The Maxwell equations (2.3) couple the electric field and the magnetic field in the dispersive medium locally in space and nonlocally in time. In this section, the Maxwell equations are transformed into two uncoupled first-order integro-differential equations via a dispersive wave splitting. To this end, three temporal integral operators of the form (2.2) are introduced, namely, the index of refraction, the relative intrinsic admittance, and the relative intrinsic impedance. These operators are the intrinsic characteristics of the nonstationary medium as far as wave propagation is concerned. The obtained uncoupled first-order equations — the dynamical equations — are analyzed in the proceeding sections.

3.1 Index of refraction, relative intrinsic admittance, and relative intrinsic impedance

A wave splitting is a change of the dependent vector field variables, $\boldsymbol{E}(z,t)$ and $\boldsymbol{H}(z,t)$, such that the new field variables, $\boldsymbol{E}^{\pm}(z,t)$, represent the up-going waves and the down-going waves in the medium, respectively. The split vector field variables of the nonstationary medium are defined by

$$\boldsymbol{E} = \boldsymbol{E}^{+} + \boldsymbol{E}^{-}, \qquad \boldsymbol{u}_z \times \eta \boldsymbol{H} = -\mathcal{Y}_r \boldsymbol{E}^{+} + \mathcal{Y}_r \boldsymbol{E}^{-}, \tag{3.1}$$

where the temporal integral operator

$$\mathcal{Y}_r = y_r(t) + Y(t,t')* = \big(y_r(t)\delta(t-t') + Y(t,t')\big)*$$

is the relative intrinsic admittance. This operator, which is independent of the sources and depend on the permittivity and the permeability operators only, is defined precisely later in this section. The definition is such that $\mathcal{Y}_r$ inherits the properties of $\mathcal{E}_r$ and $\mathcal{M}_r$. In particular, the optical contribution, $y_r(t)$, depends on $\epsilon_r(t)$ and $\mu_r(t)$ only, and the admittance kernel, $Y(t,t')$, is regular for $t > t'$ and vanishes for $t < t'$. For time-invariant materials, $y_r(t)$ is independent of time and $Y(t,t') = Y(t-t')$. In general, $Y(t,t')$ is nonzero also in the non-dispersive case.

The operator $\mathcal{Y}_r$ is defined such that the Maxwell equations reduce to the first-order integro-differential equations

$$\partial_z \boldsymbol{E}^{\pm} = \mp c^{-1}\partial_t \mathcal{N} \boldsymbol{E}^{\pm} \mp \mathcal{Z}_r \eta \boldsymbol{J}/2 \tag{3.2}$$

when the wave splitting is applied. The refractive index

$$\mathcal{N} = n(t) + N(t,t')* = \big(n(t)\delta(t-t') + N(t,t')\big)* \tag{3.3}$$

and the relative intrinsic impedance

$$\mathcal{Z}_r = z_r(t) + Z(t,t')* = \big(z_r(t)\delta(t-t') + Z(t,t')\big)*$$

are temporal integral operators to be determined. Since the split vector fields do not couple, $\boldsymbol{E}^{\pm}(z,t)$ can be interpreted as the up-going and the down-going electric fields in the medium, respectively. The dispersive wave equations (3.2) are analyzed in Section 4 using the fundamental solutions of the dispersive wave operators

$$\pm\partial_z + c^{-1}\partial_t \mathcal{N}. \tag{3.4}$$

The optical responses of the refractive index and the relative impedance, $n(t)$ and $z_r(t)$, respectively, are given explicitly below. The corresponding integral kernels, $N(t,t')$ and $Z(t,t')$, are given implicitly in terms of Volterra integral equations of the second kind. These kernels are regular for $t > t'$ and vanish for $t < t'$. For time-invariant materials, $n(t)$ and $z_r(t)$ are independent of time, $N(t,t') = N(t-t')$, and $Z(t,t') = Z(t-t')$.

Straightforward calculations show that the conditions on the operators $\mathcal{Y}_r$, $\mathcal{N}$, and $\mathcal{Z}_r$ for exact wave splitting leading to the dynamical equations (3.2) are

$$\begin{aligned} \partial_t \mathcal{E}_r &= \mathcal{Y}_r \partial_t \mathcal{N}, &(3.5)\\ \mathcal{M}_r \mathcal{Y}_r &= \mathcal{N}, &(3.6)\\ \mathcal{Y}_r \mathcal{Z}_r &= 1, &(3.7)\end{aligned}$$

where 1 is the identity operator. Combining these operator equalities gives

$$\begin{aligned} \mathcal{N}\partial_t \mathcal{N} &= \mathcal{M}_r \partial_t \mathcal{E}_r, &(3.8)\\ \mathcal{M}_r &= \mathcal{N}\mathcal{Z}_r. &(3.9)\end{aligned}$$

In general, these integral operators do not commute; however, by definition, the operators $\mathcal{Y}_r$ and $\mathcal{Z}_r$ always commute since they are inverses (resolvents) of one another, see, e.g., Tricomi [9]. For time-invariant media, all the integral operators reduce to convolution operators which are known to commute. Moreover, equations (3.8) and (3.5) can be integrated in this special case:

$$\mathcal{E}_r = \mathcal{Y}_r \mathcal{N}, \qquad \mathcal{N}^2 = \mathcal{M}_r \mathcal{E}_r.$$

Observe that $\mathcal{Y}_r = \mathcal{N}$ for nonmagnetic time-varying materials.

It is profitable to use the operators $\mathcal{Y}_r$, $\mathcal{N}$, and $\mathcal{Z}_r$ in direct and inverse scattering problems for the slab. In the direct scattering problem, the operators $\mathcal{E}_r$ and $\mathcal{M}_r$ are known. Equation (3.8) is used first to obtain $\mathcal{N}$ and then equation (3.6) to obtain $\mathcal{Y}_r$. Finally, equation (3.7) is used to obtain $\mathcal{Z}_r$. Knowing $\mathcal{Y}_r$, $\mathcal{N}$, and $\mathcal{Z}_r$, the direct scattering problem is relatively easy to analyze. In the inverse scattering problem, it is assumed that $\mathcal{N}$ and $\mathcal{Z}_r$ have been calculated from scattering data. The operators $\mathcal{E}_r$ and $\mathcal{M}_r$ are obtained by solving first equation (3.7) for $\mathcal{Y}_r$ and thereafter equation (3.9) for $\mathcal{M}_r$. Finally, equation (3.5) is employed to obtain $\mathcal{E}_r$.

3.2 Integral equations

For clarification and for numerical purposes, equations (3.5)–(3.9) are now written down explicitly. The kernel of the operator $\mathcal{M}_r \partial_t \mathcal{E}_r$ (with optical responses included) is found to be

$$\begin{aligned} &\epsilon_r(t')\mu_r(t')\delta'(t-t') - \epsilon_r(t')\mu_r'(t')\delta(t-t') + \\ &\quad + \mu_r(t)\partial_t \chi^e(t,t') - \epsilon_r(t')\partial_{t'}\chi^m(t,t') - (\partial_{t'}\chi^m * \chi^e)(t,t'). \end{aligned}$$

In this expression, the derivatives are to be interpreted in the sense of distributions:

$$\begin{aligned} \partial_t \chi^e(t,t') &= \chi^e(t,t-0)\,\delta(t-t') + \{\partial_t \chi^e(t,t')\}_{\text{classical}}, \\ \partial_{t'} \chi^m(t,t') &= -\chi^m(t,t-0)\,\delta(t-t') + \{\partial_{t'} \chi^m(t,t')\}_{\text{classical}}. \end{aligned}$$

Analogously, a similar expression can be obtained for the operator $\mathcal{N}\partial_t\mathcal{N}$. The most irregular terms in equation (3.8) contain derivatives of the delta distribution.

Comparing these terms gives

$$
\begin{aligned}
&-n(t')n'(t')\delta(t-t') + n(t)\partial_t N(t,t') - n(t')\partial_{t'} N(t,t') - (\partial_{t'} N * N)(t,t') = \\
&\quad = -\epsilon_r(t')\mu_r'(t')\delta(t-t') + \mu_r(t)\partial_t \chi^e(t,t') - \epsilon_r(t')\partial_{t'}\chi^m(t,t') \\
&\qquad -(\partial_{t'}\chi^m * \chi^e)(t,t'),
\end{aligned} \tag{3.10}
$$

where

$$n(t) = \sqrt{\epsilon_r(t)\mu_r(t)}.$$

Integrating this equation with respect to t gives a Volterra integral equation of the second kind in the refractive kernel:

$$
\begin{aligned}
&n(t)N(t,t') + n(t')N(t',t'-0) + \int_{t'}^{t} \big(N(t'',t''-0) - n'(t'')\big) N(t'',t')\, dt'' \\
&\quad - n(t') \int_{t'}^{t} D(t'',t')\, dt'' - \int_{t'}^{t} \left(\int_{t'}^{t''} D(t'',t''')N(t''',t')\, dt''' \right) dt'' - n(t')n'(t') \\
&\quad = \mu_r(t)\chi^e(t,t') + \epsilon_r(t')\chi^m(t',t'-0) + \int_{t'}^{t} \big(\chi^m(t'',t''-0) - \mu_r'(t'')\big)\chi^e(t'',t')\, dt'' \\
&\quad - \epsilon_r(t') \int_{t'}^{t} \{\partial_{t'}\chi^m(t'',t')\}\, dt'' - \int_{t'}^{t} \left(\int_{t'}^{t''} \{\partial_{t'''}\chi^m(t'',t''')\}\, \chi^e(t''',t')\, dt''' \right) dt'' \\
&\quad - \epsilon_r(t')\mu_r'(t'), \qquad t' \le t,
\end{aligned}
$$

where

$$D(t,t') = \{\partial_{t'} N(t,t')\}$$

denotes the classical derivative of $N(t,t')$ with respect to t'. Letting $t' \nearrow t$ gives the initial condition

$$N(t,t-0) = \frac{n(t)n'(t) - \epsilon_r(t)\mu_r'(t) + \epsilon_r(t)\chi^m(t,t-0) + \mu_r(t)\chi^e(t,t-0)}{2n(t)}.$$

This shows that $N(t,t') \neq 0$ also in the absence of memory terms in the constitutive relations ($\chi^e(t,t') = \chi^m(t,t') = 0$). In other words, a pure optical response in the refractive operator is not possible unless $\epsilon_r(t)$ and $\mu_r(t)$ are constant.

Equation (3.6) can be written in the form

$$\mu_r(t)Y(t,t') + y_r(t')\chi^m(t,t') + \int_{t'}^{t} \chi^m(t,t'')Y(t'',t')\, dt'' = N(t,t'),$$

where

$$y_r(t) = \frac{n(t)}{\mu_r(t)} = \sqrt{\frac{\epsilon_r(t)}{\mu_r(t)}}.$$

This is a linear Volterra integral equation of the second kind. Letting $t' \nearrow t$ gives the initial condition

$$Y(t,t-0) = \frac{N(t,t-0) - y_r(t)\chi^m(t,t-0)}{\mu_r(t)}.$$

Similarly, equation (3.7) can be written as

$$y_r(t)Z(t,t') + z_r(t')Y(t,t') + \int_{t'}^{t} Y(t,t'')Z(t'',t')\,dt'' = 0,$$

where

$$z_r(t) = \frac{1}{y_r(t)} = \sqrt{\frac{\mu_r(t)}{\epsilon_r(t)}}.$$

This is also a linear Volterra integral equation of the second kind. In particular,

$$Z(t,t-0) = -z_r^2(t)Y(t,t-0).$$

The susceptibility kernels can be expressed in terms of the refractive kernel and the relative admittance and impedance kernels. Integrating equation (3.5) with respect to t gives

$$\begin{aligned}\chi^e(t,t') &= y_r(t)N(t,t') + n(t')Y(t',t'-0) + \int_{t'}^{t} \big(Y(t'',t''-0) - y_r'(t'')\big)N(t'',t')\,dt'' \\ &- n(t')\int_{t'}^{t} \{\partial_{t'}Y(t'',t')\}\,dt'' - \int_{t'}^{t}\left(\int_{t'}^{t''} \{\partial_{t'''}Y(t'',t''')\}\,N(t''',t')\,dt'''\right)dt'' \\ &- n(t')y_r'(t'), \qquad t' \le t.\end{aligned}$$

Equation (3.9) reduces to

$$\chi^m(t,t') = n(t)Z(t,t') + z_r(t')N(t,t') + \int_{t'}^{t} N(t,t'')Z(t'',t')\,dt''.$$

4 One-way wave equations

The first-order dispersive wave equations (3.2) for the up-going and down-going fields read

$$\begin{aligned}&\big(\pm\partial_z + c^{-1}n(t)\partial_t\big)\,\boldsymbol{E}^{\pm}(z,t) + c^{-1}\big(n'(t) + N(t,t-0)\big)\boldsymbol{E}^{\pm}(z,t) + \\ &+ \int K(t,t')\boldsymbol{E}^{\pm}(z,t')\,dt' = -z_r(t)\eta\boldsymbol{J}(z,t)/2 - \int Z(t,t')\eta\boldsymbol{J}(z,t')\,dt'/2,\end{aligned} \tag{4.1}$$

where the wave-number kernel, $K(t,t')$, is the classical time-derivative of the refractive kernel, $N(t,t')$, divided by c:

$$K(t,t') = c^{-1}\left\{\partial_t N(t,t')\right\}.$$

The kernel $K(t,t')$ vanishes for $t < t'$ since $N(t,t')$ has this property. In this section, these scalar dispersive wave equations are analyzed using the method of characteristics.

4.1 Characteristics

Temporal dispersion, being a lower-order, however, important effect, do not significantly affect the propagation of jump discontinuities; therefore, the characteristics of a dispersive equation are defined in the same way as for the non-dispersive equation. This is the approach of Åberg *et al.* [4].

Equation (4.1) shows that the characteristic projections depend on the optical contribution $n(t)$ only. The intensities of jump discontinuities in the split vector fields depend on the initial values of the memory kernel $N(t,t')$ as well.

Consider the first-order dispersive wave equation for the up-going field, and let

$$\zeta \to \tau(z-\zeta,t)$$

be a parametrization of the characteristic projection passing through the point (z,t). This notation is a specialization of the parametrization $\zeta \to \tau^+(\zeta,z,t)$ employed by Åberg *et al.* [4] for materials stratified in the z-direction. By definition, $\tau(0,t)=t$. Moreover, the characteristic projection through the space-time point (z_1+z_2,t) passes through $(z_1,\tau(z_2,t))$; therefore,

$$\tau(z_1,\tau(z_2,t)) = \tau(z_1+z_2,t) \text{ for all } z_1,\ z_2, \text{ and } t.$$

In particular,

$$\tau(z,\tau(-z,t)) = t = \tau(-z,\tau(z,t)) \text{ for all } z,\ t.$$

By definition, the characteristic projection through (z,t) satisfies the ordinary differential equation

$$\frac{d\tau(z-\zeta,t)}{d\zeta} = \frac{n(\tau(z-\zeta,t))}{c}.$$

Integration gives

$$\tau(z,t) = t - c^{-1}\int_0^z n\left(\tau(z-\zeta,t)\right)\, d\zeta.$$

For time-invariant materials, $\tau(z,t)=t-zn/c$. It is now straightforward to show that

$$\frac{\partial\tau}{\partial z}(z,t) + c^{-1}n(t)\frac{\partial\tau}{\partial t}(z,t) = 0. \tag{4.2}$$

and

$$\frac{d}{d\zeta}\ln n\left(\tau(z-\zeta,t)\right) = \frac{1}{c}n'\left(\tau(z-\zeta,t)\right). \tag{4.3}$$

These results may be used to derive fundamental solutions of the dispersive wave operators.

Now consider the first-order wave equation for the down-going field. Due to symmetry

$$\zeta \to \tau(-z+\zeta,t)$$

is a parametrization of the characteristic projection passing through the point (z,t). This is a specialization of the parametrization $\zeta \to \tau^-(\zeta,z,t)$ employed by Åberg *et al.* [4] for stratified media.

4.2 Propagation of discontinuities

Discontinuities in the split vector fields, $\boldsymbol{E}^{\pm}(z,t)$, propagate along the characteristic projections. If the current density, $\boldsymbol{J}(z,t)$, is assumed to be regular, the intensity of a propagating jump discontinuity, $\left[\boldsymbol{E}^{\pm}(z,t)\right]$, satisfies the ordinary differential equation

$$\pm\frac{d}{d\zeta}\left[\boldsymbol{E}^{\pm}(\zeta,\tau(\pm z\mp\zeta,t))\right]+c^{-1}n'\left(\tau(\pm z\mp\zeta,t)\right)\left[\boldsymbol{E}^{\pm}(\zeta,\tau(\pm z\mp\zeta,t))\right]+$$
$$+c^{-1}N\left(\tau(\pm z\mp\zeta,t),\tau(\pm z\mp\zeta,t)-0\right)\left[\boldsymbol{E}^{\pm}(\zeta,\tau(\pm z\mp\zeta,t))\right]=0.$$

The solution of this equation can be written in the form

$$\left[\boldsymbol{E}^{\pm}(z,t)\right]=Q(\pm z\mp\zeta,\tau(\pm z\mp\zeta,t))\left[\boldsymbol{E}^{\pm}(\zeta,\tau(\pm z\mp\zeta,t))\right]$$

or

$$\left[\boldsymbol{E}^{\pm}(z,\tau(\mp z\pm\zeta,t))\right]=Q(\pm z\mp\zeta,t)\left[\boldsymbol{E}^{\pm}(\zeta,t)\right],$$

where wave-front factor, $Q(z,t)$, is regular and satisfies the ordinary differential equation

$$\partial_z Q(z,t)+c^{-1}\Big(n'\left(\tau(-z,t)\right)+N\left(\tau(-z,t),\tau(-z,t)-0\right)\Big)Q(z,t)=0,\ Q(0,t)=1.$$

Straightforward integration using equation (4.3) gives

$$Q(z,t)=\frac{n\left(t\right)}{n\left(\tau(-z,t)\right)}\exp\left(-\frac{1}{c}\int_0^z N\left(\tau(-\zeta,t),\tau(-\zeta,t)-0\right)\,d\zeta\right).$$

For time-invariant media, this expression reduces to $Q(z,t)=\exp\left(-\frac{z}{c}N\left(+0\right)\right)$.

4.3 Fundamental solutions

In order to solve the dynamical equations (3.2), or, equivalently, equations (4.1), the fundamental solutions of the dispersive wave operators (3.4) are needed. These fundamental solutions are denoted by $\mathcal{E}^{\pm}(z;t,t')$, respectively, and satisfy the first-order dispersive wave equations

$$\left(\pm\partial_z+c^{-1}\partial_t\mathcal{N}\right)\mathcal{E}^{\pm}=\delta_0\otimes\delta_{t'},$$

where the refractive index is of the form (3.3) and the excitation time, t', is regarded as a parameter. Explicitly, $\mathcal{E}^{\pm}(z;t,t')$ satisfy the fundamental equations

$$\left(\pm\partial_z+c^{-1}n(t)\partial_t\right)\mathcal{E}^{\pm}(z;t,t')+c^{-1}\big(n'(t)+N(t,t-0)\big)\mathcal{E}^{\pm}(z;t,t')+$$
$$+\int K(t,t'')\mathcal{E}^{\pm}(z;t'',t')\,dt''=\delta(z)\delta(t-t').$$

$\mathcal{E}^{\pm}(z;t,t')$ can be considered as the material response at the point (z,t) due to a concentrated source at $(0,t')$ (although this is not entirely correct). Since $\mathcal{E}^{+}(z;t,t')$ is up-going and $\mathcal{E}^{-}(z;t,t')$ down-going, the conditions

$$\mathcal{E}^{\pm}(\pm z;t,t')=0,\qquad z<0$$

are imposed on the fundamental solutions. These fields are required to be time-retarded (causal) as well:

$$\mathcal{E}^{\pm}(z;t,t') = 0, \qquad t' > \tau(|z|,t).$$

For time-invariant media, $\mathcal{E}^{\pm}(z;t,t') = \mathcal{E}^{\pm}(z;t-t')$.

Under suitable assumptions, Schwartz' kernel theorem [6, pp. 128-129] is applicable, and the solutions of the propagation problems (3.2) can be written as

$$\boldsymbol{E}^{\pm}(z,t) = \iint \mathcal{E}^{\pm}(z-z';t,t')\boldsymbol{S}(z',t')\,dz'\,dt', \tag{4.4}$$

where the source term is

$$\boldsymbol{S}(z,t) = -z_r(t)\eta\boldsymbol{J}(z,t)/2 - \int Z(t,t')\eta\boldsymbol{J}(z,t')\,dt'/2.$$

The electric and magnetic fields are obtained from equation (3.1) and electric and magnetic flux densities from the constitutive relations (2.1).

The retarded fundamental solutions are defined in the following theorem:

Theorem 4.1. *The distribution*

$$\mathcal{E}^{\pm}(z;t,t') = H(\pm z)Q(|z|,t')\Big(\delta\left(\tau(|z|,t)-t'\right) + P\left(|z|;\tau(|z|,t),t'\right)\Big),$$

where $Q(z,t)$ is the wave-front factor and the regular propagator kernel, $P(z;t,t')$, satisfies the integro-differential equation

$$\begin{aligned}
&\partial_z P(z;t,t') + K\left(\tau(-z,t),\tau(-z,t')\right)\partial_{t'}\tau(-z,t') + \\
&\quad + \int_{t'}^{t} K\left(\tau(-z,t),\tau(-z,t'')\right)\partial_{t''}\tau(-z,t'')P(z;t'',t')\,dt'' + \\
&\quad + c^{-1}\big(N\left(\tau(-z,t),\tau(-z,t)-0\right) - N\left(\tau(-z,t'),\tau(-z,t')-0\right)\big)P(z;t,t') + \\
&\quad + c^{-1}\big(n'\left(\tau(-z,t)\right) - n'\left(\tau(-z,t')\right)\big)P(z;t,t') = 0, \\
&P(0;t,t') = 0,
\end{aligned}$$

are retarded fundamental solutions (causal Green's functions) of the dispersive first-order wave operators (3.4), respectively. Thus, $P(z;t,t') = 0$ for $t < t'$. Explicitly, the initial condition for the propagator kernel is

$$P(z;t,t-0) = -\int_0^z K\left(\tau(-z',t),\tau(-z',t)-0\right)\partial_t\tau(-z',t)dz'.$$

Proof. Straightforward differentiation using equation (4.2) gives

$$\begin{aligned}
\left(\pm\partial_z + c^{-1}n(t)\partial_t\right)\mathcal{E}^{\pm}(z;t,t') &= \delta(z)Q(0,t)\left(\delta(t-t') + P(0;t,t')\right) + \\
&\quad + H(\pm z)Q(|z|,t')\partial_{|z|}P(|z|;t,t')|_{t=\tau(|z|,t)} + \\
&\quad + H(\pm z)\partial_{|z|}Q(|z|,t')\left(\delta(\tau(|z|,t)-t') + P(|z|;\tau(|z|,t),t')\right).
\end{aligned}$$

The change of variables $t'' = \tau(-|z|, t''')$ shows that

$$\int K(t,t'')\mathcal{E}^{\pm}(z;t'',t')\,dt'' = H(\pm z)Q(|z|,t')K\left(t,\tau(-|z|,t')\right)\partial_{t'}\tau(-|z|,t') + \\ +H(\pm z)Q(|z|,t')\int K\left(t,\tau(-|z|,t'')\right)\partial_{t''}\tau(-|z|,t'')P(|z|;t'',t')\,dt''.$$

Substituting these equations into the fundamental equation, matching contributions of the same regularity, and introducing wave-front time yield the desired result. Assuming unique solubility for the integro-differential equation shows that $P(z;t,t')$ vanishes for $t < t'$ since $K(t,t')$ vanishes for $t < t'$. Therefore, $\mathcal{E}^{\pm}(z;t,t')$ are causal, which finishes the proof.

Now consider a concentrated source distributed over the plane $z = 0$:

$$\boldsymbol{J}(z,t) = \boldsymbol{j}(t)\delta(z).$$

The wave splitting (3.1) shows that this current density generates the electric field

$$\boldsymbol{E}(0,t) = \boldsymbol{E}^{+}(+0,t) = \boldsymbol{E}^{-}(-0,t) = -\left(z_r(t)\boldsymbol{j}(t) + \int Z(t,t')\boldsymbol{j}(t')\,dt'\right)\eta/2$$

in the plane $z = 0$. Using this and equation (4.4), the up-going and down-going electric fields can be written as

$$\boldsymbol{E}^{\pm}(z,t) = \int \mathcal{E}^{\pm}(z;t,t')\boldsymbol{E}(0,t')\,dt'$$

or

$$\boldsymbol{E}^{\pm}(z,t) = H(\pm z)Q(|z|,\tau(|z|,t))\boldsymbol{E}(0,\tau(|z|,t)) + \\ + H(\pm z)\int_{-\infty}^{\tau(|z|,t)} Q(|z|,t')P(|z|;\tau(|z|,t),t')\boldsymbol{E}(0,t')\,dt'.$$

This is the canonical problem in the study of forerunners.

Finally, using wave-front time, the fundamental solutions can be written as

$$\mathcal{E}^{\pm}(z;\tau(-|z|,t),t') = H(\pm z)Q(|z|,t')\Big(\delta\left(t-t'\right) + P\left(|z|;t,t'\right)\Big).$$

The integral operator

$$\mathcal{P}(|z|) = Q(|z|,t')\Big(\delta\left(t-t'\right) + P\left(|z|;t,t'\right)\Big)*$$

is referred to as the wave propagator.

5 Sommerfeld forerunners

In this section, expressions for Sommerfeld forerunners are derived in the heuristic sense of [7]. The Sommerfeld forerunner or the first precursor in a dispersive medium is the early time-behavior of the propagating field at a distant field-point shortly after the arrival of the wave-front. In the one-dimensional case, finding this field is tantamount to obtaining an approximation to the propagator kernel $P(z;t+t',t')$ for large $z>0$ and for small $t>0$. In this section, the leading edge behavior of this field in nonstationary materials is derived. The response is referred to as the Sommerfeld forerunner since it corresponds to Sommerfeld's result for the time-invariant single-resonance Lorentz medium [8]. The early time-behavior of the field can be obtained much quicker by using the Sommerfeld approximation instead of solving the integro-differential equation in (4.1).

For simplicity, assume that

$$\epsilon_r(t)=\mu_r(t)=1, \qquad \chi^e(t,t-0)=\chi^m(t,t-0)=0. \tag{5.1}$$

This special case is, perhaps, the most physical one, and applies, for instance, to nonstationary Lorentz materials, see Åberg *et al.* [4]:

$$\chi^e(t,t')=\omega_p^2(t)\frac{\sin\left(\nu_0(t-t')\right)}{\nu_0}\exp\left(-\frac{\nu}{2}(t-t')\right)H(t-t'), \qquad \nu_0=\sqrt{\omega_0^2-\nu^2/4},$$

where the natural frequency, ω_0, and the collision frequency, ν, are constant and the plasma frequency, $\omega_p(t)$, depends on time. Generally, Lorentz materials are assumed nonmagnetic for the case the anomalous dispersion occurs in the optical regime.

From equation (5.1), it is a straightforward matter to show that

$$n(t)=1, \qquad N(t,t-0)=0$$

and

$$\partial_t\chi^i(t,t-0)=-\partial_{t'}\chi^i(t,t-0), \quad i=e,m, \qquad cK(t,t-0)=-D(t,t-0).$$

Using these results and equation (3.10) gives the initial values

$$2cK(t,t-0)=\partial_t\chi^e(t,t-0)+\partial_t\chi^m(t,t-0)=-\partial_{t'}\chi^e(t,t-0)-\partial_{t'}\chi^m(t,t-0).$$

The wave-front factor $Q(z,t)=1$ and the integro-differential equation for the propagator kernel reduces to

$$\partial_z P(z;t,t')+K\left(t+\frac{z}{c},t'+\frac{z}{c}\right)+\int_{t'}^{t}K\left(t+\frac{z}{c},t''+\frac{z}{c}\right)P(z;t'',t')\,dt''=0,$$

where $P(0;t,t')=0$.

In order to obtain an approximation to $P(z;t+t',t')$ for small $t\geq 0$, it makes sense apply the approximation

$$K\left(t+\frac{z}{c},t'+\frac{z}{c}\right)=K\left(t'+\frac{z}{c},t'+\frac{z}{c}-0\right)H(t-t')$$

to this integro-differential equation. The excitation time, t', can now be treated as a parameter. Denoting the Sommerfeld forerunner kernel by $P_S(z;t+t',t')$, one arrives at an integro-differential equation in the variables (z,t) only:

$$\partial_z P_S(z;t+t',t')+K\left(t'+\frac{z}{c},t'+\frac{z}{c}-0\right)\Big(H(t)+\big(H(\cdot)*P_S(z;\cdot+t',t')\big)(t)\Big)=0,$$

where $H(t)$ is the Heaviside step function. Then, formally, for the corresponding temporal integral operator, one can write

$$1 + P_S(z; \cdot + t', t')* = \exp\left(-f(z,t')H(\cdot)*\right),$$

where frequency $f(z,t')$ is

$$f(z,t') = \int_0^z K\left(t' + \frac{z'}{c}, t' + \frac{z'}{c} - 0\right) dz',$$

the asterisk $(*)$ represents temporal convolution, and the exponential convolution operator is interpreted in terms of its Taylor series, see Karlsson and Rikte [7]. Exploiting that

$$H(t) \sum_{i=1}^{\infty} \big(- f(z,t')\big)^i \frac{t^{i-1}}{(i-1)!i!}$$

is a Bessel-function expansion, the Sommerfeld forerunner kernel becomes

$$P_S(z; t+t'; t') = -f(z,t') \left(J_0\left(2\sqrt{f(z,t')t}\right) + J_2\left(2\sqrt{f(z,t')t}\right)\right) H(t).$$

This agrees with the general result in the time-invariant case [7]. Other ways to obtain this result is to use the method of successive approximations or to employ Laplace transform techniques.

For the Lorentz model, $K(t, t-0) = \omega_p^2(t)/(2c)$ and

$$f(z,t') = \frac{1}{2c} \int_0^z \omega_p^2 \left(t' + \frac{z'}{c}\right) dz' = \frac{1}{2} \int_0^{z/c} \omega_p^2\left(t' + \zeta\right) d\zeta.$$

The choice of parameters

$$\omega_p(t) = \omega\left(1 + \alpha \sin\left(\beta t\right)\right)$$

employed by Åberg *et al.* [4] gives

$$\begin{aligned} \frac{f(z,t')}{\omega^2} &= \frac{1}{2}\left(1 + \frac{\alpha^2}{2}\right)\frac{z}{c} + \frac{\alpha^2}{8\beta}\left(\sin\left(2\beta t'\right) - \sin\left(2\beta\left(t' + \frac{z}{c}\right)\right)\right) \\ &+ \frac{\alpha}{\beta}\left(\cos\left(\beta t'\right) - \cos\left(\beta\left(t' + \frac{z}{c}\right)\right)\right). \end{aligned}$$

For a fixed propagation depth z, the Sommerfeld propagator kernel, $P_S(z; t+t'; t')$, for this special Lorentz model is depicted in Figure 1. The choice of parameters is, as in [4], $\omega = \sqrt{2} \cdot c/z$, $\alpha = .4$, $\beta = 10 \cdot c/z$, (and $\nu = 1 \cdot c/z$, $\nu_0 = 2 \cdot c/z$).

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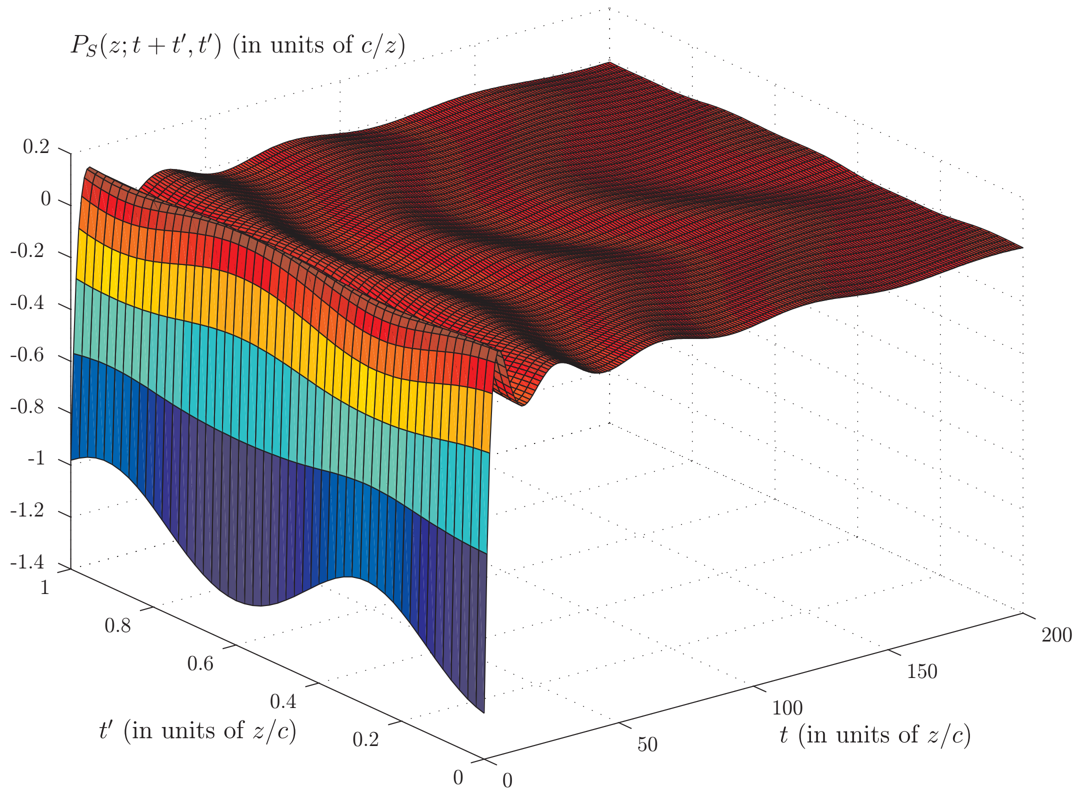

Figure 1: The Sommerfeld propagator kernel, $P_S(z;t+t';t')$, at a fixed propagation depth z in a Lorentz material with a periodic plasma frequency in the time interval $0 < t < 200 \cdot z/c$ and in the excitation time interval $0 < t' < z/c$.

References

[1] I. Åberg. High-frequency switching and Kerr effect — nonlinear problems solved with nonstationary time domain techniques. *Progress in Electromagnetics Research*, **17**, 185–235, 1997.

[2] I. Åberg. Transient waves from internal sources in non-stationary media - Numerical implementation. *Wave Motion*, **27**(1), 1–21, 1998.

[3] I. Åberg and A. Karlsson. The source problem—Transient waves propagating from internal sources in non-stationary media. *Wave Motion*, **26**, 43–68, 1997.

[4] I. Åberg, G. Kristensson, and D. J. N. Wall. Propagation of transient electromagnetic waves in time-varying media—direct and inverse scattering problems. *Inverse Problems*, **11**(1), 29–49, 1995.

[5] I. Åberg, G. Kristensson, and D. J. N. Wall. Transient waves in non-stationary media. *J. Math. Phys.*, **37**(5), 2229–2252, 1996.

[6] L. Hörmander. *The Analysis of Linear Partial Differential Operators I.*

Grundlehren der mathematischen Wissenschaften 256. Springer-Verlag, Berlin Heidelberg, 1983.

[7] A. Karlsson and S. Rikte. The time-domain theory of forerunners. *J. Opt. Soc. Am. A*, **15**(2), 487–502, 1998.

[8] A. Sommerfeld. Über die Fortpflanzung des Lichtes in dispergierenden Medien. *Ann. Phys.*, **44**, 177–202, 1914.

[9] F. G. Tricomi. *Integral Equations.* Dover Publications, New York, 1957.

[10] H. S. Tzou. *Piezoelectric Shells.* Kluwer Academic Publishers Group, Dordrecht, 1993.