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A Perturbation Analysis of Non-Linear Diffusion from
a Permeable Solid into a Finite Volume Containing a Liquid

Gustav Lindberg, Per Ståhle, Ingrid Svensson

The equation

$$\dot{c}(x, t) = c''(x, t), \quad (1)$$

with the initial conditions

$$c(x, 0) = 1 \quad \text{in } 0 < x < 1, \quad (2)$$

and the nested boundary conditions

$$c(0, t) = c(1, t) = \chi(1 - \int_0^1 c(x, t) dx). \quad (3)$$

has a solution which is split into

$$c(x, t) = u(x, t) + v(x, t). \quad (4)$$

The part $u(x, t)$ fulfils the equation

$$\dot{u}(x, t) = u''(x, t), \quad (5)$$

with the initial conditions

$$u(x, 0) = 1 \quad \text{in } 0 < x < 1, \quad (6)$$

and the boundary conditions

$$u(0, t) = u(1, t) = 0. \quad (7)$$

The part $v(x, t)$ fulfils the equation

$$\dot{v}(x, t) = v''(x, t), \quad (8)$$

with the initial conditions

$$v(x, 0) = 0 \quad \text{in } 0 < x < 1, \quad (9)$$

and the boundary conditions

$$v(0, t) = v(1, t) = \chi(1 - \int_0^1 c(x, t) dx), \quad (10)$$

where χ is the ratio of the volume of the bone, V_B versus the volume of the container V_C , i.e., $\chi = V_B/V_C$. Obviously $v(x, t)$ may be expanded in a Taylor's series as follows:

$$v(x, t) = \chi\varphi_1(x, t) + \chi^2\varphi_2(x, t) + \chi^3\varphi_3(x, t) + \dots \quad (11)$$

Since χ is a free parameter the functions $\varphi_i(x, t)$ have to fulfil the equation

$$\dot{\varphi}_i(x, t) = \varphi_i''(x, t), \quad (12)$$

with the initial conditions

$$\dot{\varphi}_i(x, 0) = 0 \quad \text{in } 0 < x < 1. \quad (13)$$

The boundary conditions become recursive according to the following:

$$\varphi_1(0, t) = \varphi_1(1, t) = \phi_1(t) = (1 - \int_0^1 u(x, t) dx), \quad (14)$$

and

$$\varphi_i(0, t) = \varphi_i(1, t) = \phi_i(t) = - \int_0^1 \varphi_{i-1}(x, t) dx, \quad \text{for } i = 2, 3, 4, \dots \quad (15)$$

Duhamel's theorem allow us to solve the φ_i 's as an application of single step boundary conditions $d\phi_i(\tau) = \frac{d\phi_i(\tau)}{d\tau}d\tau$ representing a concentration $d\phi_i(t')$ applied at the time t' ,

$$d\varphi_i(x, t) = u(x, t - \tau) \frac{d\phi_i(\tau)}{d\tau} d\tau. \quad (16)$$

Integration from $\tau' = \tau$ to $\tau' = t$ gives

$$\varphi_i(x, t) = \int_0^t u(x, t - \tau) \frac{d\phi_i(\tau)}{d\tau} d\tau = [u(x, t - \tau)\phi_i(\tau)]_0^t - \int_0^t \frac{du(x, t - \tau)}{d\tau} \phi_i(\tau) d\tau. \quad (17)$$

Considering that $u(x, 0) = 0$ and $\varphi_i(0, \tau) = 0$. The general solution is given as

$$\varphi_i(x, t) = - \int_0^t \phi_i(\tau) \frac{du(x, t - \tau)}{d\tau} d\tau. \quad (18)$$

Insertion of the solution

$$u = \sum_{n=1,3,5,\dots} \frac{4e^{-n^2\pi^2 t} \sin(n\pi x)}{n\pi}, \quad (19)$$

gives

$$\varphi_i(x, t) = - \int_0^t \phi_i(\tau) \sum_{n=1,3,5,\dots} 4\pi n e^{-n^2\pi^2(t-\tau)} \sin(n\pi x) d\tau. \quad (20)$$

For large containers and, hence, small values of χ the solution for the concentration $u(x, t)$ according to (19), clearly gives the approximative solution as $\chi \rightarrow 0$.

The boundary conditions for φ_1 are obtained after integration of (19),

$$\phi_1(t) = 1 - \sum_{m=1,3,5,\dots} \frac{8}{m^2\pi^2} e^{-m^2\pi^2 t}. \quad (21)$$

The solution of $\varphi_1(x, t)$ with the boundary conditions $\phi_1(t)$ $v(0, t)$ is the concentration in the container considering the escape of matter from the bone sample and is

$$\varphi_1(x, t) = - \int_0^t \left(1 - \sum_{m=1,3,5,\dots} \frac{8}{m^2\pi^2} e^{-m^2\pi^2 \tau} \right) \times \sum_{n=1,3,5,\dots} 4\pi n e^{-n^2\pi^2(t-\tau)} \sin(n\pi x) d\tau. \quad (22)$$

$$\begin{aligned} \varphi_1(x, t) = & - \left[\sum_{n=1,3,5,\dots} \frac{4}{n\pi} (1 - e^{-n^2\pi^2 t}) - \right. \\ & \left. - \sum_{\substack{m=1,3,5,\dots \\ m \neq n}} \frac{32n}{m^2(n^2 - m^2)\pi^3} (e^{-m^2\pi^2 t} - e^{-n^2\pi^2 t}) + f(n, t) \right] \sin(n\pi x), \end{aligned} \quad (23)$$

where

$$f(n, t) = \frac{64}{m^2(n^2 - m^2)\pi^4} (e^{-m^2\pi^2 t} - e^{-n^2\pi^2 t}) \quad \text{as } m^2 \rightarrow n^2, \quad (24)$$

which is written

$$f(n, t) = \lim_{n^2 - m^2 \rightarrow \epsilon} \frac{64}{n^2 \epsilon \pi^4} e^{-n^2\pi^2 t} (e^{-\epsilon\pi^2 t} - 1) = - \frac{64t}{n^2\pi^2} e^{-n^2\pi^2 t}. \quad (25)$$

The contribution to the concentration in the container from φ_1 becomes

$$\begin{aligned}
\phi_2 = \int_0^1 \varphi_1 dx = & - \sum_{n=1,3,5,\dots} \frac{8}{n^2 \pi^2} (1 - e^{-n^2 \pi^2 t}) \\
& + \left[\sum_{\substack{m=1,3,5,\dots \\ m \neq n}} \frac{64}{m^2(n^2-m^2)\pi^4} (e^{-m^2 \pi^2 t} - e^{-n^2 \pi^2 t}) \right] + f(n, t),
\end{aligned} \tag{26}$$

This now reads

$$\begin{aligned}
\phi_2 = & -1 - \frac{8}{\pi^2} + \sum_{n=1,3,5,\dots} \frac{8}{n^2 \pi^2} (1 - 8t) e^{-n^2 \pi^2 t} - \\
& = \\
& -1 - \frac{8}{\pi^2} + \sum_{n=1,3,5,\dots} \left\{ \frac{8}{n^2 \pi^2} (1 - 8t) e^{-n^2 \pi^2 t} + \right. \\
& \left. \sum_{\substack{m=1,3,5,\dots \\ m \neq n}} \frac{128 e^{-n^2 \pi^2 t}}{n^2(n^2-m^2)\pi^4} \right\} = -1 - \frac{8}{\pi^2} + \\
& \sum_{n=1,3,5,\dots} \frac{8}{n^2 \pi^2} e^{-n^2 \pi^2 t} \left\{ 1 - 8t + \sum_{\substack{m=1,3,5,\dots \\ m \neq n}} \frac{16}{(n^2-m^2)\pi^2} \right\},
\end{aligned} \tag{27}$$

that may serve as boundary conditions for the function φ_2 . Considering a fairly small ratio $\chi \approx 0.01$ we anticipate the third order term $\chi^2 \varphi_2$ to be of the order of 10^{-4} . The work to do this does not seem meaningful, with the accuracy of the present measurements in mind. Figure 1 shows the difference between the solution without considering the increasing concentration in the container and same with a first order correction.

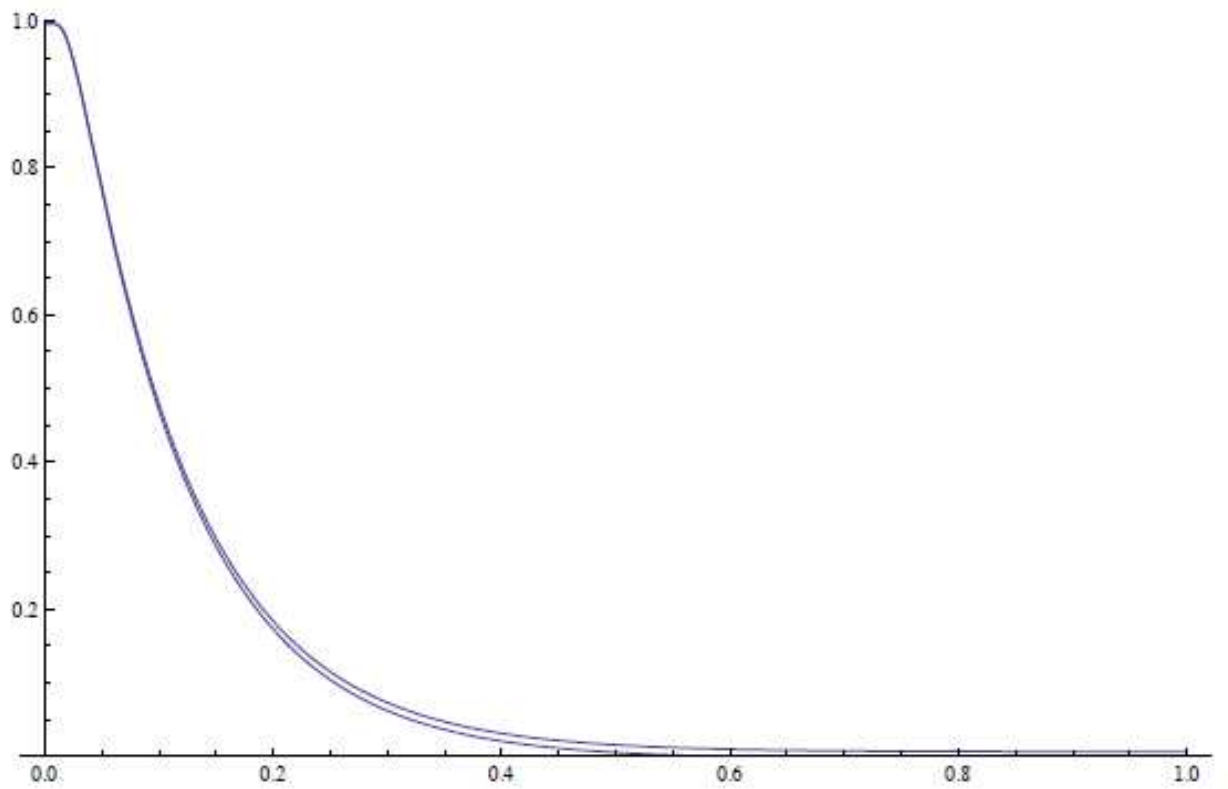


Figure 1: The solution for escaped ions for a sample in an infinite container $1 - \int_0^1 u dx$ and the same with first order correction $1 - \int_0^1 (u + \chi\varphi_1) dx$.