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On the asymptotic and approximate distributions of the product of an inverse Wishart matrix and a Gaussian vector

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ON THE ASYMPTOTIC AND APPROXIMATE DISTRIBUTIONS OF THE PRODUCT OF AN INVERSE WISHART MATRIX AND A GAUSSIAN VECTOR

I. KOTSIUBA AND S. MAZUR

ABSTRACT. In this paper we study the distribution of the product of an inverse Wishart random matrix and a Gaussian random vector. We derive its asymptotic distribution as well as its approximate density function formula which is based on the Gaussian integral and the third order Taylor expansion. Furthermore, we compare obtained asymptotic and approximate density functions with the exact density which is obtained by Bodnar and Okhrin (2011). A good performance of obtained results is documented in the numerical study.

1. INTRODUCTION

The multivariate normal distribution is one of the most important and very useful distribution in multivariate statistical analysis. If we have sample of size n from the k-variate normal distribution then the sample unbiased estimators of the mean vector and the covariance matrix have k-variate normal and k-variate Wishart distributions, respectively, and they are independently distributed (see Section 3 of Mardia et al. (1980)).

Recently, several papers study the investigation of properties of the estimators for mean vector and covariance matrix. For instance, Stein (1956) and Jorion (1986) discuss improvement techniques for the mean estimator. Ledoit and Wolf (2004), Bodnar and Gupta (2010), Bai and Shi (2011), Cai et al. (2011), Cai and Yuan (2012), Cai and Zhou (2012), and Bodnar et al. (2014a, 2015a) improve the estimation techniques for the (inverse) covariance matrices. Papers by von Rosen (1988), Styan (1989), Díaz-García et al. (1997), Bodnar and Okhrin (2008), Drton et al. (2008), and Bodnar et al. (2015b) develop the distributional properties of the Wishart matrices, the inverse Wishart matrices and related quantities.

However, functions that depend on the product of the (inverse) Wishart random matrix and the Gaussian random vector are not comprehensively investigated in literature. Bodnar and Okhrin (2011), Bodnar et al. (2013, 2014b) derived the exact distributions of the product of the (inverse) Wishart random matrix and the Gaussian random vector, which have integral representations. We note that these products have direct applications in the portfolio theory and in the discriminant analysis. For example, the elements of the sample estimator of the discriminant function are expressed as products of the inverse Wishart random matrix and the Gaussian random vector, and the weights of the tangency portfolio are estimated by the same product. In this paper we extend the results obtained by Bodnar and Okhrin (2011) by providing the asymptotic and approximate density functions of this product.

The rest of the paper is structured as follows. The main results are presented in Section 2, where the asymptotic distribution of the product of an inverse Wishart random matrix and a Gaussian random vector is derived as Theorem 2.1. Its approximate density function which is based on the third order Taylor series approximation is obtained in Theorem 2.2. The numerical performance of the obtained results is outlined in details in Section 3, while Section 4 summarizes the paper.

2. Main Results

In this section we consider the product of the inverse Wishart random matrix and a normally distributed random vector. In particular, we derive its asymptotic and approximative density functions.

Let $\mathbf{A} \sim \mathcal{W}_k(n, \boldsymbol{\Sigma})$, i.e. the random matrix $\mathbf{A}$ has the k-dimensional Wishart distribution with n degrees of freedom and covariance matrix $\boldsymbol{\Sigma}$, which is positive definite. We consider the case when $k < n$, i.e. $\mathbf{A}$ is a non-singular random matrix. Also, let $\mathbf{z} \sim \mathcal{N}_k(\boldsymbol{\mu}, \lambda \boldsymbol{\Sigma})$, it means that the random vector $\mathbf{z}$ has the k-dimensional normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\lambda \boldsymbol{\Sigma}$, where $\lambda > 0$ is a constant. Furthermore, let $\mathbf{L}$ is the $p \times k$ non-zero matrix of constants of rank$(\mathbf{L}) = p < k$, the symbol $\mathbf{0}$ denotes the suitable vector of zeros, and $\mathbf{I}_k$ stands for the $k \times k$ identity matrix. Throughout the paper it is assumed that $\mathbf{A}$ and $\mathbf{z}$ are independently distributed. Then it holds that the distribution of $\mathbf{L}\mathbf{A}^{-1}\mathbf{z}|(\mathbf{z} = \mathbf{z}^*)$ equals to the distribution of $\mathbf{L}\mathbf{A}^{-1}\mathbf{z}^*$. It is noted that

$$\mathbf{L}\mathbf{A}^{-1}\mathbf{z}^* \stackrel{d}{=} \mathbf{z}^{*T}\boldsymbol{\Sigma}^{-1}\mathbf{z}^* \frac{\mathbf{L}\mathbf{A}^{-1}\mathbf{z}^*}{\mathbf{z}^{*T}\mathbf{A}^{-1}\mathbf{z}^*} \frac{\mathbf{z}^{*T}\mathbf{A}^{-1}\mathbf{z}^*}{\mathbf{z}^{*T}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*}, \tag{1}$$

where the symbol $\stackrel{d}{=}$ denotes the equality in distribution.

Using Theorem 3.2.12 of Muirhead (1982) we obtain that

$$\frac{\mathbf{z}^{*T}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*}{\mathbf{z}^{*T}\mathbf{A}^{-1}\mathbf{z}^*} \sim \chi^2_{n-k+1}$$

and is independently distributed of $\mathbf{z}^*$. Furthermore, it holds that $\mathbf{z}^{*T}\mathbf{A}^{-1}\mathbf{z}^*$ is independent of $\mathbf{L}\mathbf{A}^{-1}\mathbf{z}^*/\mathbf{z}^{*T}\mathbf{A}^{-1}\mathbf{z}^*$ for given $\mathbf{z}^*$ (see Theorem 3 of Bodnar and Okhrin (2008)). Consequently, it is independent of $\mathbf{z}^{*T}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*\mathbf{L}\mathbf{A}^{-1}\mathbf{z}^*/\mathbf{z}^{*T}\mathbf{A}^{-1}\mathbf{z}^*$ and respectively of $\mathbf{z}^T\boldsymbol{\Sigma}^{-1}\mathbf{z}\mathbf{L}\mathbf{A}^{-1}\mathbf{z}/\mathbf{z}^T\mathbf{A}^{-1}\mathbf{z}$. Using the proof of Theorem 1 of Bodnar and Schmid (2008) it follows that

$$\mathbf{z}^{*T}\boldsymbol{\Sigma}^{-1}\mathbf{z}^* \frac{\mathbf{L}\mathbf{A}^{-1}\mathbf{z}^*}{\mathbf{z}^{*T}\mathbf{A}^{-1}\mathbf{z}^*} \sim t_p\left(n - k + 2; \mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*, \frac{1}{n-k+2}\mathbf{z}^{*T}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*\mathbf{L}\mathbf{R}_{\mathbf{z}^*}\mathbf{L}^T\right), \tag{2}$$

where $\mathbf{R}_{\mathbf{z}^*} = \boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\mathbf{z}^*\mathbf{z}^{*T}\boldsymbol{\Sigma}^{-1}/\mathbf{z}^{*T}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*$. The symbol $t_p(d; \mathbf{b}; \mathbf{B})$ stands for the p-dimensional multivariate t-distribution with d degrees of freedom, the location parameter $\mathbf{b}$, and the dispersion matrix $\mathbf{B}$.

Let $\xi = \mathbf{z}^{*T}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*/\mathbf{z}^{*T}\mathbf{A}^{-1}\mathbf{z}^*$. Then ξ and $\mathbf{z}$ are independent and $\xi \sim \chi^2_{n-k+1}$. As a result, we obtain the following stochastic representation of $\mathbf{L}\mathbf{A}^{-1}\mathbf{z}$ which is given by

$$\mathbf{L}\mathbf{A}^{-1}\mathbf{z} \stackrel{d}{=} \xi^{-1}\left(\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{z} + \sqrt{\frac{\mathbf{z}^T\boldsymbol{\Sigma}^{-1}\mathbf{z}}{n-k+2}}(\mathbf{L}\mathbf{R}_{\mathbf{z}}\mathbf{L}^T)^{1/2}\mathbf{t}_0\right), \tag{3}$$

where $\mathbf{R}_{\mathbf{z}} = \boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\mathbf{z}\mathbf{z}^T\boldsymbol{\Sigma}^{-1}/\mathbf{z}^T\boldsymbol{\Sigma}^{-1}\mathbf{z}$; $\xi \sim \chi^2_{n-k+1}$, $\mathbf{z} \sim \mathcal{N}_k(\boldsymbol{\mu}, \lambda\boldsymbol{\Sigma})$, and $\mathbf{t}_0 \sim t_p(n-k+2, \mathbf{0}_p, \mathbf{I}_p)$. Moreover, ξ, $\mathbf{z}$, and $\mathbf{t}_0$ are independently distributed.

Next, we consider the asymptotic distribution of $n\mathbf{L}\mathbf{A}^{-1}\mathbf{z}$. It is remarkable that usually $\lambda = 1/n$ (see Section 3 of Muirhead (1982)). Then

$$E(n\mathbf{L}\mathbf{A}^{-1}\mathbf{z}) = \frac{n}{n-k-1}\mathbf{L}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}. \tag{4}$$

Also, it is easy to see that

$$\lim_{n\to\infty} \frac{\xi}{n} = 1 \quad \text{and} \quad \lim_{n\to\infty} \mathbf{z} = \boldsymbol{\mu}. \tag{5}$$

Thus, from (3), (4), (5), and Theorem 6.15 of Arnold (1990), we get the asymptotic distribution of $n\mathbf{L}\mathbf{A}^{-1}\mathbf{z}$ which is presented in the following theorem.

Theorem 2.1. *Let matrix* $\mathbf{A} \sim W_k(n, \boldsymbol{\Sigma})$ *and vector* $\mathbf{z} \sim \mathcal{N}_k(\boldsymbol{\mu}, \lambda\boldsymbol{\Sigma})$ *with* $\boldsymbol{\Sigma}$ *positive definite and* $\lambda = 1/n$. *Furthermore, let* $\mathbf{A}$ *and* $\mathbf{z}$ *be independent and* $\mathbf{L}$ *be a* $p \times k$ *matrix of constants of* $\mathrm{rank}(\mathbf{L}) = p < k$. *Then the asymptotic distribution of* $n\mathbf{L}\mathbf{A}^{-1}\mathbf{z}$, *i.e.* $n \to \infty$, *is given by*

$$\sqrt{n}\left(n\mathbf{L}\mathbf{A}^{-1}\mathbf{z} - \mathbf{L}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\right) \xrightarrow{d} \mathcal{N}_p(\mathbf{0}, \widetilde{\boldsymbol{\Sigma}}), \tag{6}$$

where $\widetilde{\boldsymbol{\Sigma}} = (1 + \boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{L}^T - \mathbf{L}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\mathbf{L}^T$.

From Theorem 2.1 we get that the asymptotic distribution of the random vector $n\mathbf{L}\mathbf{A}^{-1}\mathbf{z}$ has a p-variate normal distribution.

In many applications the sample size n is not large. For this reason, we need an adjustment like a finite-sample approximation of the density. Bodnar and Okhrin (2011) derived an exact density function of $\mathbf{L}\mathbf{A}^{-1}\mathbf{z}$. However, density function is expressed as a four-dimensional integral of the generalized function. Hence, the aim of this paper is to derive a simpler approximative density function of the random vector $\mathbf{L}\mathbf{A}^{-1}\mathbf{z}$.

Let $\mathbf{Q} = \boldsymbol{\Sigma}^{-1/2}\mathbf{L}^T\left(\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{L}^T\right)^{-1}\mathbf{L}\boldsymbol{\Sigma}^{-1/2}$, $\mathbf{P} = \boldsymbol{\Sigma}^{-1/2}\mathbf{L}^T\left(\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{L}^T\right)^{-1/2}$, $\mathbf{S}_1 = \mathbf{r}_1\mathbf{r}_1^T$, $\mathbf{S}_2 = \mathbf{r}_3\mathbf{r}_3^T$, $\mathbf{r}_1 = \boldsymbol{\Sigma}^{-1/2}\boldsymbol{\mu}$, $\mathbf{r}_2 = \mathbf{P}^T\mathbf{r}_1$, $\mathbf{r}_3 = (\mathbf{I}_k - \mathbf{Q})\mathbf{r}_1$, $\mathbf{r}_4 = \mathbf{Q}\mathbf{r}_1$, $\mathbf{r}_5 = -(p-1)/a\mathbf{r}_1 + 1/b\mathbf{r}_3$, $a = \mathbf{r}_1^T\mathbf{r}_1$, $b = \mathbf{r}_1^T(\mathbf{I}_k - \mathbf{Q})\mathbf{r}_1$, $c = \mathbf{r}_1^T\mathbf{Q}\mathbf{r}_1$. It is noted that $\mathbf{Q}$ is a $k \times k$ singular and projection matrix with $\mathbf{Q} = \mathbf{P}\mathbf{P}^T$.

In the next theorem we suggest the approximate density function of $\mathbf{L}\mathbf{A}^{-1}\mathbf{z}$ which is based on the Gaussian integral and the third order Taylor series expansion.

Theorem 2.2. *Let matrix* $\mathbf{A} \sim W_k(n, \boldsymbol{\Sigma})$ *and vector* $\mathbf{z} \sim \mathcal{N}_k(\boldsymbol{\mu}, \lambda\boldsymbol{\Sigma})$ *with* $\boldsymbol{\Sigma}$ *positive definite. Furthermore, let* $\mathbf{A}$ *and* $\mathbf{z}$ *be independent and* $\mathbf{L}$ *be a* $p \times k$ *matrix of constants of* $\mathrm{rank}(\mathbf{L}) = p < k$. *Then the approximate density function of* $\mathbf{L}\mathbf{A}^{-1}\mathbf{z}$ *is given by*

$$\begin{aligned}
f_{\mathbf{L}\mathbf{A}^{-1}\mathbf{z}}(\mathbf{y}) &\approx \frac{|\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{L}^T|^{-1/2}}{a^{(p-1)/2}b^{1/2}\pi^{p/2}} \frac{\Gamma\left(\frac{p+n-k+2}{2}\right)}{\Gamma\left(\frac{n-k+2}{2}\right)} \\
&\quad \times \int_0^{+\infty} f_2^{-(p+n-k+2)/2}(s\mathbf{y}, \mathbf{0})\left[1 + \frac{\lambda}{2}\mathrm{tr}[g_2(s\mathbf{y})]\right] \\
&\quad \times s^p f_{\chi^2_{n-k+1}}(s)\mathbf{d}s,
\end{aligned} \tag{7}$$

where $\breve{\mathbf{y}} = s(\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{L}^T)^{-1/2}\mathbf{y}$, $f_2(s\mathbf{y}, \mathbf{0}) = 1 + \frac{1}{a}\left(\breve{\mathbf{y}}^T\breve{\mathbf{y}} - 2\breve{\mathbf{y}}^T\mathbf{r}_2 + c + \frac{(c - \breve{\mathbf{y}}^T\mathbf{r}_2)^2}{b}\right)$,

$$\boldsymbol{\omega}(s\mathbf{y}) = -(f_2(s\mathbf{y}, \mathbf{0}) - 1)\mathbf{r}_1 + \mathbf{r}_4 - \mathbf{P}\breve{\mathbf{y}} + \frac{c - \breve{\mathbf{y}}\mathbf{r}_2^T}{b}(2\mathbf{r}_4 - \mathbf{P}\breve{\mathbf{y}}) - \frac{(c - \breve{\mathbf{y}}\mathbf{r}_2)^2}{b^2}\mathbf{r}_3,$$

$$\begin{aligned}
g_2(s\mathbf{y}) = {}& \frac{p-1}{a}\left(\frac{2}{a}\mathbf{S}_1 - \mathbf{I}_k\right) + \frac{1}{b}\left(\frac{2}{b}\mathbf{S}_2 - (\mathbf{I}_k - \mathbf{Q})\right) + \mathbf{r}_5\left(\mathbf{r}_5^T + \frac{2(p+n-k+2)}{af_2(s\mathbf{y}, \mathbf{0})}\boldsymbol{\omega}^T(s\mathbf{y})\right) \\
& + \frac{p+n-k+2}{a^2 f_2^2(s\mathbf{y}, \mathbf{0})}\left\{\frac{p+n-k+4}{a^2}\boldsymbol{\omega}(s\mathbf{y})\boldsymbol{\omega}^T(s\mathbf{y})\right. \\
& + f_2(s\mathbf{y}, \mathbf{0})\left[a(f_2(s\mathbf{y}, \mathbf{0}) - 1)\left(\mathbf{I}_k - \frac{4}{a}\mathbf{S}_1\right) - 4\mathbf{r}_1\left[\boldsymbol{\omega}(s\mathbf{y}) + (f_2(s\mathbf{y}, \mathbf{0}) - 1)\mathbf{r}_1\right]^T\right] \\
& - a\mathbf{Q} + \frac{a(c - \breve{\mathbf{y}}^T\mathbf{r}_2)}{b^2}\left\{b(2\mathbf{r}_4 - \mathbf{P}\breve{\mathbf{y}})\left[\frac{2}{b}\mathbf{r}_3^T + \frac{2\mathbf{r}_4^T - \breve{\mathbf{y}}^T\mathbf{P}^T}{c - \breve{\mathbf{y}}^T\mathbf{r}_2}\right] + 2b\mathbf{Q}\right. \\
& \left.\left. - 2\mathbf{r}_3(2\mathbf{r}_4^T - \breve{\mathbf{y}}^T\mathbf{P}^T) - (c - \breve{\mathbf{y}}^T\mathbf{r}_2)(\mathbf{I}_k - \mathbf{Q}) - 4\left[2\mathbf{r}_4 - \mathbf{P}\breve{\mathbf{y}} - \frac{c - \breve{\mathbf{y}}^T\mathbf{r}_2}{b}\mathbf{r}_3\right]\mathbf{r}_3^T\right\}\right\}.
\end{aligned}$$

The symbol $f_{\chi^2_{n-k+1}}(\cdot)$ *denotes the density of the* χ^2-*distributed random variable with* $(n - k + 1)$ *degrees of freedom.*

Proof. From (2) the unconditional density function of $\mathbf{z}^T\boldsymbol{\Sigma}^{-1}\mathbf{z}\mathbf{L}\mathbf{A}^{-1}\mathbf{z}/\mathbf{z}^T\mathbf{A}^{-1}\mathbf{z}$ is given by

$$\begin{aligned}
f_{\mathbf{z}^T\boldsymbol{\Sigma}^{-1}\mathbf{z}\frac{\mathbf{L}\mathbf{A}^{-1}\mathbf{z}}{\mathbf{z}^T\mathbf{A}^{-1}\mathbf{z}}}(\mathbf{y}) = {}& \int_{\mathbb{R}^k} f_{\mathbf{z}^{*T}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*\frac{\mathbf{L}\mathbf{A}^{-1}\mathbf{z}^*}{\mathbf{z}^*\mathbf{A}^{-1}\mathbf{z}^*}}(\mathbf{y}|\mathbf{z} = \mathbf{z}^*)f_{\mathbf{z}}(\mathbf{z}^*)\, d\mathbf{z}^* \\
= {}& \lambda^{-k/2}\frac{|\boldsymbol{\Sigma}|^{-1/2}}{(2\pi)^{k/2}}\frac{\Gamma\left(\frac{p+n-k+2}{2}\right)}{\pi^{p/2}\Gamma\left(\frac{n-k+2}{2}\right)} \\
& \times \int_{\mathbb{R}^k} \exp\left(-\frac{(\mathbf{z}^* - \boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{z}^* - \boldsymbol{\mu})}{2\lambda}\right)\frac{|\mathbf{L}\mathbf{R}_{\mathbf{z}^*}\mathbf{L}^T|^{-1/2}}{(\mathbf{z}^{*T}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*)^{p/2}} \\
& \times \left(1 + \frac{(\mathbf{y} - \mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*)^T(\mathbf{L}\mathbf{R}_{\mathbf{z}^*}\mathbf{L}^T)^{-1}(\mathbf{y} - \mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*)}{\mathbf{z}^{*T}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*}\right)^{-(p+n-k+2)/2} d\mathbf{z}^*.
\end{aligned}$$

From Theorem 18.2.8 of Harville(1997) it follows that

$$\begin{aligned}
(\mathbf{L}\mathbf{R}_{\mathbf{z}^*}\mathbf{L}^T)^{-1} = {}& (\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{L}^T)^{-1} + \frac{(\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{L}^T)^{-1}\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*\mathbf{z}^{*T}\boldsymbol{\Sigma}^{-1}\mathbf{L}^T(\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{L}^T)^{-1}}{\mathbf{z}^{*T}\boldsymbol{\Sigma}^{-1}\mathbf{z}^* - \mathbf{z}^{*T}\boldsymbol{\Sigma}^{-1}\mathbf{L}^T(\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{L}^T)^{-1}\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*}, \\
|\mathbf{L}\mathbf{R}_{\mathbf{z}^*}\mathbf{L}^T| = {}& |\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{L}^T|\frac{\mathbf{z}^{*T}\boldsymbol{\Sigma}^{-1}\mathbf{z}^* - \mathbf{z}^{*T}\boldsymbol{\Sigma}^{-1}\mathbf{L}^T(\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{L}^T)^{-1}\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*}{\mathbf{z}^{*T}\boldsymbol{\Sigma}^{-1}\mathbf{z}^*}.
\end{aligned}$$

Using the transformation $\mathbf{t} = \boldsymbol{\Sigma}^{-1/2}(\mathbf{z}^* - \boldsymbol{\mu})$ in the last integral with Jacobian $|\boldsymbol{\Sigma}|^{1/2}$ we get

$$\begin{aligned}
f_{\mathbf{z}^T\boldsymbol{\Sigma}^{-1}\mathbf{z}\frac{\mathbf{L}\mathbf{A}^{-1}\mathbf{z}}{\mathbf{z}^T\mathbf{A}^{-1}\mathbf{z}}}(\mathbf{y}) = {}& \lambda^{-k/2}\frac{|\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{L}^T|^{-1/2}}{(2\pi)^{k/2}}\frac{\Gamma\left(\frac{p+n-k+2}{2}\right)}{\pi^{p/2}\Gamma\left(\frac{n-k+2}{2}\right)} \\
& \times \int_{\mathbb{R}^k} \exp\left(-\frac{\mathbf{t}^T\mathbf{t}}{2\lambda}\right)\frac{\left[(\mathbf{t} + \mathbf{r}_1)^T(\mathbf{t} + \mathbf{r}_1)\right]^{-(p-1)/2}}{[(\mathbf{t} + \mathbf{r}_1)^T(\mathbf{I}_k - \mathbf{Q})(\mathbf{t} + \mathbf{r}_1)]^{1/2}} \\
& \times \left\{1 + \frac{1}{(\mathbf{t} + \mathbf{r}_1)^T(\mathbf{t} + \mathbf{r}_1)}\left[\widetilde{\mathbf{y}}^T\widetilde{\mathbf{y}} - 2\widetilde{\mathbf{y}}^T\mathbf{P}^T(\mathbf{t} + \mathbf{r}_1) + (\mathbf{t} + \mathbf{r}_1)^T\mathbf{Q}(\mathbf{t} + \mathbf{r}_1)\right.\right. \\
& \left.\left. + \frac{((\mathbf{t} + \mathbf{r}_1)^T\mathbf{Q}(\mathbf{t} + \mathbf{r}_1) - \widetilde{\mathbf{y}}^T\mathbf{P}^T(\mathbf{t} + \mathbf{r}_1))^2}{(\mathbf{t} + \mathbf{r}_1)^T(\mathbf{I}_k - \mathbf{Q})(\mathbf{t} + \mathbf{r}_1)}\right]\right\}^{-(p+n-k+2)/2} d\mathbf{t},
\end{aligned}$$

where $\mathbf{r}_1 = \boldsymbol{\Sigma}^{-1/2}\boldsymbol{\mu}$, $\widetilde{\mathbf{y}} = (\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{L}^T)^{-1/2}\mathbf{y}$, $\mathbf{P} = \boldsymbol{\Sigma}^{-1/2}\mathbf{L}^T(\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{L}^T)^{-1/2}$, and $\mathbf{Q} = \mathbf{P}\mathbf{P}^T$.

Let

$$f_1(\mathbf{t}) = \frac{\left[(\mathbf{t}+\mathbf{r}_1)^T(\mathbf{t}+\mathbf{r}_1)\right]^{-(p-1)/2}}{[(\mathbf{t}+\mathbf{r}_1)^T(\mathbf{I}_k - \mathbf{Q})(\mathbf{t}+\mathbf{r}_1)]^{1/2}} \quad \text{and} \quad f_2(\mathbf{y},\mathbf{t}) = 1 + \frac{1}{(\mathbf{t}+\mathbf{r}_1)^T(\mathbf{t}+\mathbf{r}_1)}\widetilde{f}_2(\mathbf{y},\mathbf{t})$$

with

$$\begin{aligned}
\widetilde{f}_2(\mathbf{y},\mathbf{t}) &= \widetilde{\mathbf{y}}^T\widetilde{\mathbf{y}} - 2\widetilde{\mathbf{y}}^T\mathbf{P}^T(\mathbf{t}+\mathbf{r}_1) + (\mathbf{t}+\mathbf{r}_1)^T\mathbf{Q}(\mathbf{t}+\mathbf{r}_1) \\
&\quad + \frac{\left[(\mathbf{t}+\mathbf{r}_1)^T\mathbf{Q}(\mathbf{t}+\mathbf{r}_1) - \widetilde{\mathbf{y}}^T\mathbf{P}^T(\mathbf{t}+\mathbf{r}_1)\right]^2}{(\mathbf{t}+\mathbf{r}_1)^T(\mathbf{I}_k - \mathbf{Q})(\mathbf{t}+\mathbf{r}_1)}.
\end{aligned}$$

Next, we approximate the function $f_1(\mathbf{t})f_2^{-(p+n-k+2)/2}(\mathbf{y},\mathbf{t})$ using a Taylor series expansion of the third order at $\mathbf{t} = \mathbf{0}$. Notice, that the integrals of summands in Taylor series expansion, which correspond to the odd derivatives at point $\mathbf{t} = \mathbf{0}$ are all zero as the moments of the odd order from the multivariate normal distribution with zero mean vector. As a result, we get the next approximation

$$f_1(\mathbf{t})f_2^{-(p+n-k+2)/2}(\mathbf{y},\mathbf{t}) = g_0(\mathbf{y}) + \frac{1}{2}\mathbf{t}^T g_2(\mathbf{y})\mathbf{t} + o(\|\mathbf{t}\|^4), \tag{8}$$

where

$$g_0(\mathbf{y}) = f_1(\mathbf{0})f_2^{-(p+n-k+2)/2}(\mathbf{y},\mathbf{0}),$$

$$g_2(\mathbf{y}) = \left.\frac{\partial^2\left[f_1(\mathbf{t})f_2^{-(p+n-k+2)/2}(\mathbf{y},\mathbf{t})\right]}{\partial\mathbf{t}\partial\mathbf{t}^T}\right|_{\mathbf{t}=\mathbf{0}} = f_2^{-(p+n-k+2)/2}(\mathbf{y},\mathbf{0})\left.\frac{\partial^2 f_1(\mathbf{t})}{\partial\mathbf{t}\partial\mathbf{t}^T}\right|_{\mathbf{t}=\mathbf{0}}$$

$$+ 2\left.\frac{\partial f_1(\mathbf{t})}{\partial\mathbf{t}}\right|_{\mathbf{t}=\mathbf{0}}\left.\frac{\partial f_2^{-(p+n-k+2)/2}(\mathbf{y},\mathbf{t})}{\partial\mathbf{t}^T}\right|_{\mathbf{t}=\mathbf{0}} + f_1(\mathbf{0})\left.\frac{\partial^2 f_2^{-(p+n-k+2)/2}(\mathbf{y},\mathbf{t})}{\partial\mathbf{t}\partial\mathbf{t}^T}\right|_{\mathbf{t}=\mathbf{0}}.$$

It is easy to see that

$$\frac{\partial f_1(\mathbf{t})}{\partial\mathbf{t}} = \frac{\varphi_1(\mathbf{t})}{\varphi_2(\mathbf{t})}$$

with

$$\begin{aligned}
\varphi_1(\mathbf{t}) &= -(p-1)\left[(\mathbf{t}+\mathbf{r}_1)^T(\mathbf{t}+\mathbf{r}_1)\right]^{-1}(\mathbf{t}+\mathbf{r}_1) \\
&\quad -[(\mathbf{t}+\mathbf{r}_1)^T(\mathbf{I}_k - \mathbf{Q})(\mathbf{t}+\mathbf{r}_1)]^{-1}(\mathbf{I}_k - \mathbf{Q})(\mathbf{t}+\mathbf{r}_1), \\
\varphi_2(\mathbf{t}) &= \left[(\mathbf{t}+\mathbf{r}_1)^T(\mathbf{t}+\mathbf{r}_1)\right]^{(p-1)/2}[(\mathbf{t}+\mathbf{r}_1)^T(\mathbf{I}_k - \mathbf{Q})(\mathbf{t}+\mathbf{r}_1)]^{1/2}.
\end{aligned}$$

Then the second order derivative $\frac{\partial^2 f_1(\mathbf{t})}{\partial\mathbf{t}\partial\mathbf{t}^T}$ can be computed as

$$\frac{\partial^2 f_1(\mathbf{t})}{\partial\mathbf{t}\partial\mathbf{t}^T} = \frac{\varphi_2(\mathbf{t})\frac{\partial\varphi_1(\mathbf{t})}{\partial\mathbf{t}^T} - \varphi_1(\mathbf{t})\frac{\partial\varphi_2(\mathbf{t})}{\partial\mathbf{t}^T}}{\varphi_2^2(\mathbf{t})},$$

where

$$\begin{aligned}
\frac{\partial\varphi_1(\mathbf{t})}{\partial\mathbf{t}^T} &= \frac{p-1}{(\mathbf{t}+\mathbf{r}_1)^T(\mathbf{t}+\mathbf{r}_1)}\left[2\frac{(\mathbf{t}+\mathbf{r}_1)(\mathbf{t}+\mathbf{r}_1)^T}{(\mathbf{t}+\mathbf{r}_1)^T(\mathbf{t}+\mathbf{r}_1)} - \mathbf{I}_k\right] + \frac{1}{(\mathbf{t}+\mathbf{r}_1)^T(\mathbf{I_k} - \mathbf{Q})(\mathbf{t}+\mathbf{r}_1)} \\
&\quad \times \left[2\frac{(\mathbf{I}_k - \mathbf{Q})(\mathbf{t}+\mathbf{r}_1)(\mathbf{t}+\mathbf{r}_1)^T(\mathbf{I}_k - \mathbf{Q})}{(\mathbf{t}+\mathbf{r}_1)^T(\mathbf{I}_k - \mathbf{Q})(\mathbf{t}+\mathbf{r}_1)} - (\mathbf{I}_k - \mathbf{Q})\right], \\
\frac{\partial\varphi_2(\mathbf{t})}{\partial\mathbf{t}^T} &= \left[(\mathbf{t}+\mathbf{r}_1)^T(\mathbf{t}+\mathbf{r}_1)\right]^{(p-1)/2}[(\mathbf{t}+\mathbf{r}_1)^T(\mathbf{I}_k - \mathbf{Q})(\mathbf{t}+\mathbf{r}_1)]^{1/2} \\
&\quad \times \left\{\frac{(\mathbf{t}+\mathbf{r}_1)^T(\mathbf{I}_k - \mathbf{Q})}{(\mathbf{t}+\mathbf{r}_1)^T(\mathbf{I}_k - \mathbf{Q})(\mathbf{t}+\mathbf{r}_1)} + \frac{p-1}{(\mathbf{t}+\mathbf{r}_1)^T(\mathbf{t}+\mathbf{r}_1)}(\mathbf{t}+\mathbf{r}_1)^T\right\}.
\end{aligned}$$

Similarly, the first order partial derivative of $f_2^{-(p+n-k+2)/2}(\mathbf{y}, \mathbf{t})$ is determined by the following formula

$$
\begin{aligned}
\frac{\partial f_2^{-(p+n-k+2)/2}(\mathbf{y}, \mathbf{t})}{\partial \mathbf{t}} &= -(p+n-k+2)\frac{f_2^{-(p+n-k+4)/2}(\mathbf{y}, \mathbf{t})}{(\mathbf{t}+\mathbf{r}_1)^T(\mathbf{t}+\mathbf{r}_1)} \\
&\times \left[-\frac{\widetilde{f}_2(\mathbf{y}, \mathbf{t})}{(\mathbf{t}+\mathbf{r}_1)^T(\mathbf{t}+\mathbf{r}_1)}(\mathbf{t}+\mathbf{r}_1) + \frac{1}{2}\frac{\partial \widetilde{f}_2(\mathbf{y}, \mathbf{t})}{\partial \mathbf{t}} \right],
\end{aligned} \quad (9)
$$

where

$$
\begin{aligned}
\frac{\partial \widetilde{f}_2(\mathbf{y}, \mathbf{t})}{\partial \mathbf{t}} &= -2\mathbf{P}\widetilde{\mathbf{y}} + 2\mathbf{Q}(\mathbf{t}+\mathbf{r}_1) + 2\frac{\psi_1(\mathbf{y}, \mathbf{t}) - \psi_2(\mathbf{y}, \mathbf{t})}{\psi_3(\mathbf{t})}, \\
\psi_1(\mathbf{y}, \mathbf{t}) &= \left[(\mathbf{t}+\mathbf{r}_1)^T(\mathbf{I}_k - \mathbf{Q})(\mathbf{t}+\mathbf{r}_1)\right]\left[(\mathbf{t}+\mathbf{r}_1)^T\mathbf{Q}(\mathbf{t}+\mathbf{r}_1) - \widetilde{\mathbf{y}}^T\mathbf{P}^T(\mathbf{t}+\mathbf{r}_1)\right] \\
&\times \left[2\mathbf{Q}(\mathbf{t}+\mathbf{r}_1) - \mathbf{P}\widetilde{\mathbf{y}}\right], \\
\psi_2(\mathbf{y}, \mathbf{t}) &= \left[(\mathbf{t}+\mathbf{r}_1)^T\mathbf{Q}(\mathbf{t}+\mathbf{r}_1) - \widetilde{\mathbf{y}}^T\mathbf{P}^T(\mathbf{t}+\mathbf{r}_1)\right]^2 (\mathbf{I}_k - \mathbf{Q})(\mathbf{t}+\mathbf{r}_1), \\
\psi_3(\mathbf{t}) &= \left[(\mathbf{t}+\mathbf{r}_1)^T(\mathbf{I}_k - \mathbf{Q})(\mathbf{t}+\mathbf{r}_1)\right]^2.
\end{aligned}
$$

Using (9) we can calculate the second partial derivative of $f_2^{-(p+n-k+2)/2}(\mathbf{y}, \mathbf{t})$ which has the following representation

$$
\begin{aligned}
\frac{\partial^2 f_2^{-(p+n-k+2)/2}(\mathbf{y}, \mathbf{t})}{\partial \mathbf{t} \partial \mathbf{t}^T} &= \frac{(p+n-k+2)}{\left[(\mathbf{t}+\mathbf{r}_1)^T(\mathbf{t}+\mathbf{r}_1)\right]^2} f_2^{-(p+n-k+6)/2}(\mathbf{y}, \mathbf{t})\left\{(p+n-k+4)\right. \\
&\times \left[-\frac{\widetilde{f}_2(\mathbf{y}, \mathbf{t})}{(\mathbf{t}+\mathbf{r}_1)^T(\mathbf{t}+\mathbf{r}_1)}(\mathbf{t}+\mathbf{r}_1) + \frac{1}{2}\frac{\partial \widetilde{f}_2(\mathbf{y}, \mathbf{t})}{\partial \mathbf{t}} \right] \\
&\times \left[-\frac{\widetilde{f}_2(\mathbf{y}, \mathbf{t})}{(\mathbf{t}+\mathbf{r}_1)^T(\mathbf{t}+\mathbf{r}_1)}(\mathbf{t}+\mathbf{r}_1) + \frac{1}{2}\frac{\partial \widetilde{f}_2(\mathbf{y}, \mathbf{t})}{\partial \mathbf{t}} \right]^T \\
&+ f_2(\mathbf{y}, \mathbf{t})\left[\widetilde{f}_2(\mathbf{y}, \mathbf{t})\left(\mathbf{I}_k - \frac{4(\mathbf{t}+\mathbf{r}_1)(\mathbf{t}+\mathbf{r}_1)^T}{(\mathbf{t}+\mathbf{r}_1)^T(\mathbf{t}+\mathbf{r}_1)}\right) \right. \\
&\left.\left. + 2(\mathbf{t}+\mathbf{r}_1)\frac{\partial \widetilde{f}_2(\mathbf{y}, \mathbf{t})}{\partial \mathbf{t}^T} - \frac{(\mathbf{t}+\mathbf{r}_1)^T(\mathbf{t}+\mathbf{r}_1)}{2}\frac{\partial^2 \widetilde{f}_2(\mathbf{y}, \mathbf{t})}{\partial \mathbf{t} \partial \mathbf{t}^T} \right]\right\},
\end{aligned}
$$

where

$$\frac{\partial^2 \widetilde{f}_2(\mathbf{y},\mathbf{t})}{\partial\mathbf{t}\partial\mathbf{t}^T} = 2\mathbf{Q} + 2\frac{\psi_3(\mathbf{t})\left(\frac{\partial\psi_1(\mathbf{y},\mathbf{t})}{\partial\mathbf{t}^T} - \frac{\partial\psi_2(\mathbf{y},\mathbf{t})}{\partial\mathbf{t}^T}\right) - (\psi_1(\mathbf{y},\mathbf{t}) - \psi_2(\mathbf{y},\mathbf{t}))\frac{\partial\psi_3(\mathbf{t})}{\partial\mathbf{t}^T}}{\psi_3^2(\mathbf{t})},$$

$$\frac{\partial\psi_1(\mathbf{y},\mathbf{t})}{\partial\mathbf{t}^T} = \left[(\mathbf{t}+\mathbf{r}_1)^T(\mathbf{I}_k - \mathbf{Q})(\mathbf{t}+\mathbf{r}_1)\right]\left[(\mathbf{t}+\mathbf{r}_1)^T\mathbf{Q}(\mathbf{t}+\mathbf{r}_1) - \widetilde{\mathbf{y}}^T\mathbf{P}^T(\mathbf{t}+\mathbf{r}_1)\right]$$

$$\times\left\{2\mathbf{Q} + [2\mathbf{Q}(\mathbf{t}+\mathbf{r}_1) - \mathbf{P}\widetilde{\mathbf{y}}]\left[\frac{2(\mathbf{t}+\mathbf{r}_1)^T(\mathbf{I}_k-\mathbf{Q})}{(\mathbf{t}+\mathbf{r}_1)^T(\mathbf{I}_k-\mathbf{Q})(\mathbf{t}+\mathbf{r}_1)}\right.\right.$$

$$\left.\left.+\frac{2(\mathbf{t}+\mathbf{r}_1)^T\mathbf{Q} - \widetilde{\mathbf{y}}^T\mathbf{P}^T}{(\mathbf{t}+\mathbf{r}_1)^T\mathbf{Q}(\mathbf{t}+\mathbf{r}_1) - \widetilde{\mathbf{y}}^T\mathbf{P}^T(\mathbf{t}+\mathbf{r}_1)}\right]\right\},$$

$$\frac{\partial\psi_2(\mathbf{y},\mathbf{t})}{\partial\mathbf{t}^T} = \left[(\mathbf{t}+\mathbf{r}_1)^T\mathbf{Q}(\mathbf{t}+\mathbf{r}_1) - \widetilde{\mathbf{y}}^T\mathbf{P}^T(\mathbf{t}+\mathbf{r}_1)\right]^2$$

$$\times\left[\frac{2(\mathbf{I}_k-\mathbf{Q})(\mathbf{t}+\mathbf{r}_1)\left[2(\mathbf{t}+\mathbf{r}_1)^T\mathbf{Q} - \widetilde{\mathbf{y}}^T\mathbf{P}^T\right]}{(\mathbf{t}+\mathbf{r}_1)^T\mathbf{Q}(\mathbf{t}+\mathbf{r}_1) - \widetilde{\mathbf{y}}^T\mathbf{P}^T(\mathbf{t}+\mathbf{r}_1)} + (\mathbf{I}_k-\mathbf{Q})\right],$$

$$\frac{\partial\psi_3(\mathbf{t})}{\partial\mathbf{t}^T} = 4\left[(\mathbf{t}+\mathbf{r}_1)^T(\mathbf{I}_k-\mathbf{Q})(\mathbf{t}+\mathbf{r}_1)\right](\mathbf{t}+\mathbf{r}_1)^T(\mathbf{I}_k-\mathbf{Q}).$$

By taking all partial derivatives at $\mathbf{t}=\mathbf{0}$ and putting them in (8), we obtain the Taylor series expansion of the function $f_1(\mathbf{t})f_2^{-(p+n-k+2)/2}(\mathbf{y},\mathbf{t})$. Then, the approximate density function for $\mathbf{z}^T\mathbf{\Sigma}^{-1}\mathbf{z}\mathbf{L}\mathbf{A}^{-1}\mathbf{z}/\mathbf{z}^T\mathbf{A}^{-1}\mathbf{z}$ is given by

$$f_{\mathbf{z}^T\mathbf{\Sigma}^{-1}\mathbf{z}\frac{\mathbf{L}\mathbf{A}^{-1}\mathbf{z}}{\mathbf{z}^T\mathbf{A}^{-1}\mathbf{z}}}(\mathbf{y}) \approx \lambda^{-k/2}\frac{|\mathbf{L}\mathbf{\Sigma}^{-1}\mathbf{L}^T|^{-1/2}}{(2\pi)^{k/2}}\frac{\Gamma\left(\frac{p+n-k+2}{2}\right)}{\pi^{p/2}\Gamma\left(\frac{n-k+2}{2}\right)}$$

$$\times \int_{\mathbb{R}^k}\left[g_0(\mathbf{y}) + \frac{1}{2}\mathbf{t}^T g_2(\mathbf{y})\mathbf{t}\right]\exp\left(-\frac{\mathbf{t}^T\mathbf{t}}{2\lambda}\right)d\mathbf{t}.$$

Using Harville (1997, p. 321), the last integral leads to

$$f_{\mathbf{z}^T\mathbf{\Sigma}^{-1}\mathbf{z}\frac{\mathbf{L}\mathbf{A}^{-1}\mathbf{z}}{\mathbf{z}^T\mathbf{A}^{-1}\mathbf{z}}}(\mathbf{y}) \approx \frac{|\mathbf{L}\mathbf{\Sigma}^{-1}\mathbf{L}^T|^{-1/2}}{\pi^{p/2}}\frac{\Gamma\left(\frac{p+n-k+2}{2}\right)}{\Gamma\left(\frac{n-k+2}{2}\right)}\left[g_0(\mathbf{y}) + \frac{\lambda}{2}\mathrm{tr}[g_2(\mathbf{y})]\right].$$

Finally, applying the facts that $\mathbf{z}^T\mathbf{\Sigma}^{-1}\mathbf{z}\mathbf{L}\mathbf{A}^{-1}\mathbf{z}/\mathbf{z}^T\mathbf{A}^{-1}\mathbf{z}$ and $\mathbf{z}^T\mathbf{A}^{-1}\mathbf{z}/\mathbf{z}^T\mathbf{\Sigma}^{-1}\mathbf{z}$ are independently distributed, and $\mathbf{z}^T\mathbf{\Sigma}^{-1}\mathbf{z}/\mathbf{z}^T\mathbf{A}^{-1}\mathbf{z} \sim \chi^2_{n-k+1}$ we obtain that

$$f_{\mathbf{L}\mathbf{A}^{-1}\mathbf{z}}(\mathbf{y}) = \int_0^{+\infty} f_{\mathbf{z}^T\mathbf{\Sigma}^{-1}\mathbf{z}\frac{\mathbf{L}\mathbf{A}^{-1}\mathbf{z}}{\mathbf{z}^T\mathbf{A}^{-1}\mathbf{z}}}(\mathbf{y}s)f_{\chi^2_{n-k+1}}(s)s^p d s.$$

Summarizing the above, we get the statement of the theorem. $\qquad\square$

3. Numerical study

In this section we compare the area under the approximative density which is derived in Theorem 2.2, with the target value of unity. Moreover, the asymptotic and the approximate densities obtained in Section 2 are compared with the exact one which is derived by Bodnar and Okhrin (2011, Theorem 1). The comparison is done for $\lambda = 1/n$, $\boldsymbol{\mu} = (1,...,1)^T$, and $\mathbf{\Sigma} = \mathbf{I}_k$.

In the first numerical study, we investigate the performance of the approximate density function which is obtained in Theorem 2.2. For this reason the area under the approximative density function is compared with one. This comparison allows us to conclude how many information is ignored by deleting the moments of order higher than two which are positive. If the deviation from 1 is very big, then we should include the moments of higher order in the Taylor series expansion.

k n	2	4	6	8	10
30	0.999974	1.000000	1.000000	1.000000	1.000000
60	1.000000	0.999917	0.999943	0.999972	0.999989
90	1.000000	0.999999	0.999993	0.999946	0.999475
120	1.000000	0.999999	0.999902	0.999898	0.999494

TABLE 1. Area under the approximate density function which is given in Theorem 2.2 for $n \in \{30, 60, 90, 120\}$, $k \in \{2, 4, 6, 8, 10\}$, $p = 1$, $\mathbf{l} = (1, 0 \ldots, 0)^T$.

k n	11	12	13	14	15
30	0.999766	0.999678	0.999753	0.999958	0.999968
60	0.999662	0.999568	0.999893	0.9997898	0.999734
90	0.999659	0.999779	0.999753	0.999582	0.9987399
120	0.999677	0.999738	0.999431	0.999578	0.998789

TABLE 2. Area under the approximate density function which is given in Theorem 2.2 for $n \in \{30, 60, 90, 120\}$, $k \in \{11, 12, 13, 14, 15\}$, $p = 10$, $\mathbf{L} = (\mathbf{I}_p, \ \mathbf{0}_{p,k-p})$.

The results of the first simulation study are presented in Tables 1 and 2 for a several values of k with $p = \{1, 10\}$, and $\mathbf{L} = (\mathbf{I}_p, \ \mathbf{0}_{p,k-p})$. From Tables 1 and 2 we observe that our approximate density works quite good for different values p.

In the second numerical study, we compare the asymptotic and the approximate density functions with the exact one. The results are given for $k = \{2, 4\}$, $p = 1$, and $\mathbf{l}^T = (1, 0, ..., 0)$. We note that in this case the exact density is given as the two-dimensional integral (see Theorem 1 b) of Bodnar and Okhrin (2011)). Also, it is remarkable that all integrals can be easily evaluated using any mathematical software, e.g., Mathematica. Here, in Figure 1, the Taylor series approximation is shown by the short-dashed line, the asymptotic density by the long-dashed line, and the exact one by the solid line. We observe that the approximative density function coincides with the exact one. Furthermore, we observe that they are slightly skewed to the right. Also, we see that the asymptotic density leads very closed to the exact one and the approximate densities.

4. SUMMARY

In this paper, the product of a random matrix which includes an inverse Wishart distributed and a normally distributed vector is studied. We derived its asymptotic distribution. Moreover, we obtained the formula of an approximative density of linear functions of the product which is based on the Gaussian integral and the third order Taylor expansion. In the numerical study we documented the good performance of the approximate and asymptotic densities.
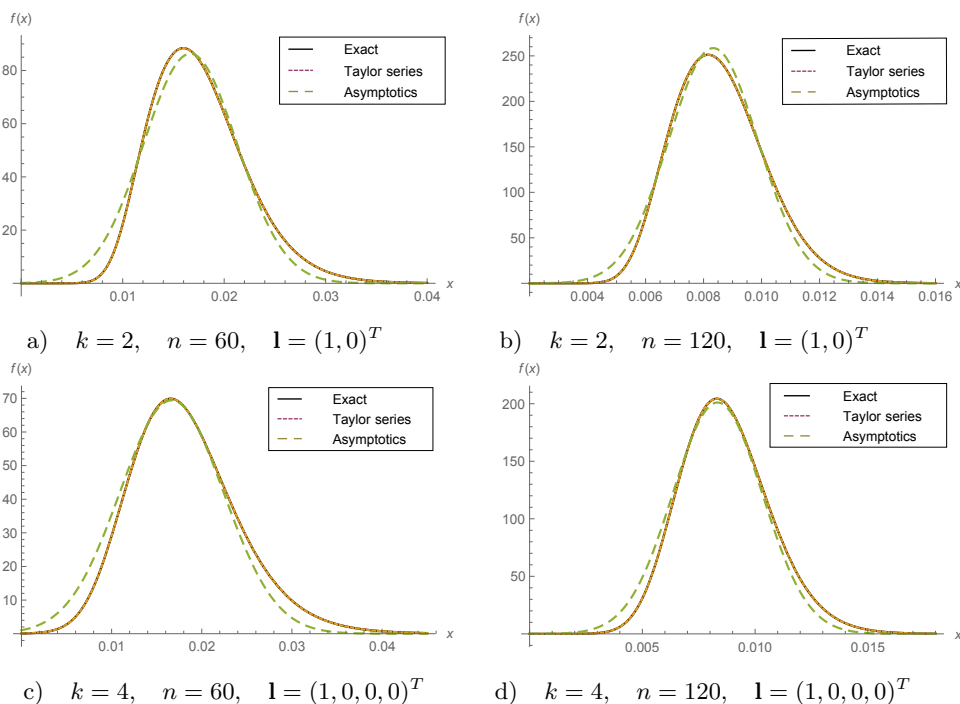
FIGURE 1. The exact density and its two approximations as given in Section 2 with $k \in \{2, 4\}$, $n \in \{60, 120\}$, and $l \in \{(1,0)^T, (1,0,0,0)^T\}$.

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