CONVERSION OF UPPER HYBRID WAVES INTO MAGNETOSPHERIC RADIATION

K. Rönnmark*

Abstract

A few years after the first observations of electromagnetic continuum (myriametric) radiation in the magnetosphere, the source of the radiation was identified as strong electrostatic upper hybrid waves frequently seen just outside the plasmapause. However, the process converting the electrostatic waves into electromagnetic radiation has still not been identified. Several mechanisms have been proposed, such as linear mode conversion in a density gradient, weak turbulence coalescence with low frequency electrostatic waves, coherent decay of the upper hybrid waves, and radiation from collapsing, strongly nonlinear cavitons. Introducing the concept of a phase space density of plasma waves, the efficiency of the linear mode conversion can be accurately estimated. Similar concepts also allow the efficiency of the nonlinear mechanisms to be evaluated in a form that can be quantitatively compared with observations.

1 Introduction

From observations [e.g., Kurth et al., 1979; Kurth, 1982; Etcheto et al., 1982] there is rather convincing evidence that the source of magnetospheric continuum radiation are strong electrostatic upper hybrid waves, which often are seen near the equatorial plasmapause. However, the mechanism converting these electrostatic waves into electromagnetic radiation has not been identified.

It was early suggested that TMR may be produced by linear mode conversion [Jones, 1976], and this possibility has since received considerable attention [Jones, 1982, 1987b; Jones et al., 1987; Budden and Jones, 1987a,b; Horne, 1988; 1990]. The efficiency of the linear mode conversion was questioned by Melrose [1981], Barbosa [1982] and Rönnmark [1989], and discrepancies between the beaming pattern predicted by this theory and observations were recently discussed by Morgan and Gurnett [1991]. Nonlinear conversion mechanisms have also been proposed to explain the radiation. Melrose [1981] and Rönnmark [1983] considered the coalescence of upper hybrid waves with low frequency

*Swedish Institute of Space Physics, University of Umeå, S–901 87 UMEÅ, Sweden

electrostatic waves, and concluded that this could produce the radiation if the low frequency waves were sufficiently common. However, there are no observations supporting that an electrostatic instability generating the required low frequency waves is commonly operating at the sources of myriametric radiation. The decay of upper hybrid waves into low frequency waves and electromagnetic radiation was studied by Murtaza and Shukla [1984] and Rönnmark [1985] under the assumption that the nonlinearities were weak. If, on the other hand, the upper hybrid waves are sufficiently strong, they may form cigar–shaped solitary waves, and the collapse of these solitary structures may also produce myriametric radiation [Christiansen et al., 1984].

In this study, we will apply phase space methods to the generation of magnetospheric continuum radiation. The phase space representation introduced by Rönnmark and Larsson [1988] provides new quantitative methods for the analysis of the generation and propagation of waves in inhomogeneous space plasmas, and it allows us to compare the linear mode conversion to the nonlinear decay mechanism.

2 Phase space description of plasma waves

Recently, there has been an increased interest in phase space descriptions of plasma waves [McDonald and Kaufman, 1985; McDonald, 1988; Rönnmark and Larsson, 1988]. In the review by Rönnmark [1990], these methods were discussed, and certain ambiguities concerning the starting point for the theory were pointed out. These ambiguities were resolved by Larsson [1989], who showed that in an inhomogeneous and nonstationary medium we should use the vector potential $\mathbf{A}$ and relate it to the current density $\mathbf{j}$ by admittance tensors $\mathbf{\Lambda}$ of the form shown below. In this section, I will briefly outline the phase space methods described in greater detail by Biro and Rönnmark [1992] and Rönnmark and Biro [1992].

In terms of the vector potential $\mathbf{A}$ describing the wave field, we define a phase space amplitude $\tilde{\mathbf{A}}(\mathbf{k}, \omega, \mathbf{r}, t)$ by the transformation

$$\tilde{\mathbf{A}}(\mathbf{k}, \omega, \mathbf{r}, t) = \int \frac{d\mathbf{r}'dt'}{\pi\sqrt{L^3T}} \mathbf{A}(\mathbf{r}', t') e^{-\frac{(\mathbf{r}'-\mathbf{r})^2}{2L^2} - \frac{(t'-t)^2}{2T^2} - i\mathbf{k}\cdot(\mathbf{r}'-\mathbf{r}) + i\omega(t'-t)} . \tag{2.1}$$

The wave field is here weighted by a Gaussian window, and the center of the window is taken as the origin of a Fourier transformation. The function $\tilde{\mathbf{A}}$ will clearly depend mainly on the behaviour of $\mathbf{A}$ in the vicinity of $\mathbf{r}$ and t, and it can thus be regarded as a local Fourier transform of the wave field.

Choosing the gauge $\phi = 0$ and including nonlinear effects to lowest order, the evolution of the vector potential is governed by the wave equation

$$\varepsilon_0 \left[\partial_t^2 \mathbf{A}(\mathbf{r}, t) + c^2 \partial_{\mathbf{r}} \times \partial_{\mathbf{r}} \times \mathbf{A}(\mathbf{r}, t)\right] = \mathbf{j}_1(\mathbf{r}, t) + \mathbf{j}_2(\mathbf{r}, t) , \tag{2.2}$$

where the linear current $\mathbf{j}_1$ can be written as

$$\mathbf{j}_1(\mathbf{r}, t) = \int d\mathbf{r}'\, dt'\, \mathbf{\Lambda}_1(\mathbf{r}' - \mathbf{r}, t' - t, \frac{\mathbf{r}' + \mathbf{r}}{2}, \frac{t' + t}{2}) \cdot \mathbf{A}(\mathbf{r}', t') , \tag{2.3}$$

while the second order nonlinear current is

$$\mathbf{j}_2(\mathbf{r},t) = \int d\mathbf{r}'\, dt'\, d\mathbf{r}''\, dt'' \times \tag{2.4}$$
$$\times \Big[\mathbf{A}(\mathbf{r}',t') \cdot \mathbf{\Lambda}_2(\mathbf{r}'-\mathbf{r}, t'-t, \mathbf{r}''-\mathbf{r}, t''-t, \tfrac{(\mathbf{r}+\mathbf{r}'+\mathbf{r}'')}{3}, \tfrac{(t+t'+t'')}{3}) \cdot \mathbf{A}(\mathbf{r}'',t'')\Big]\,.$$

We now apply the transformation defined by (2.1) to Equation (2.3). This results in an exact equation for $\tilde{\mathbf{A}}$, which we according to Biro and Rönnmark [1992] and Rönnmark and Biro [1992] can write as

$$\mathbf{D}(\mathbf{k},\omega,\mathbf{r},t)\exp\Big[i\overleftarrow{\partial}_{\mathbf{r}}\cdot\overrightarrow{\partial}_{\mathbf{k}} - i\overleftarrow{\partial}_{\mathbf{k}}\cdot(\overrightarrow{\partial}_{\mathbf{r}} - i\mathbf{k}) - \tag{2.5}$$
$$i\overleftarrow{\partial}_t\overrightarrow{\partial}_\omega + i\overleftarrow{\partial}_\omega(\overrightarrow{\partial}_t + i\omega)\Big]\cdot\tilde{\mathbf{A}}(\mathbf{k},\omega,\mathbf{r},t) = \mathbf{J_2}(\mathbf{k},\omega,\mathbf{r},t) \quad .$$

The local dispersion tensor

$$\mathbf{D}(\mathbf{k},\omega,\mathbf{r},t) = \varepsilon_0[\omega^2 + (\mathbf{kk} - k^2)c^2] + \mathbf{\Lambda}_1(\mathbf{k},\omega,\mathbf{r},t) \tag{2.6}$$

is defined by standard Fourier transforms with respect to the first pair of arguments.

The exact formula for the transformed nonlinear current $\mathbf{J_2}$ will not be given here since it is rather complicated. However, it can be simplified if we assume that the medium is not simultaneously strongly dispersive and strongly inhomogeneous. By this we mean that we can find a length scale L such that the variation in the properties of the medium is small when $\mathbf{k}$ changes by $\Delta\mathbf{k} \sim L^{-1}$ and $\mathbf{r}$ changes by $\Delta\mathbf{r} \sim L$. For such a medium, we can write the equation for $\mathbf{J_2}$ as

$$\mathbf{J_2}(\mathbf{k},\omega,\mathbf{r},t) = 16\pi^3 L^{9/2} T^{3/2} \int \frac{d\mathbf{k}'d\omega' d\mathbf{k}'' d\omega''}{(2\pi)^8}\, e^{-L^2(\mathbf{k}-\mathbf{k}'-\mathbf{k}'')^2 - T^2(\omega-\omega'-\omega'')^2} \times$$
$$\times\, \tilde{\mathbf{A}}(\mathbf{k}',\omega',\mathbf{r},t)\cdot\mathbf{\Lambda}_2(\mathbf{k}',\omega',\mathbf{k}'',\omega'',\mathbf{r},t)\cdot\tilde{\mathbf{A}}(\mathbf{k}'',\omega'',\mathbf{r},t)\,. \tag{2.7}$$

The same assumptions also allow us to derive a kinetic equation for the wave density on phase space. In a weakly inhomogeneous medium we expect $\tilde{\mathbf{A}}(\mathbf{k},\omega,\mathbf{r},t)$ as a function of ω to be sharply peaked at the frequencies $\Omega_m = \Omega_m(\mathbf{k},\mathbf{r},t)$ that satisfy the lowest order local dispersion relation $\det\mathbf{D}'(\mathbf{k},\Omega_m,\mathbf{r},t) = 0$ where $\mathbf{D}'$ is the Hermitian part of $\mathbf{D}$. The field of each mode m also has a characteristic polarization in the direction of the unit polarization vector $\mathbf{a}_m(\mathbf{k},\mathbf{r},t)$, and the assumption that the total field is essentially a superposition of linear eigenmodes can thus be expressed as

$$\tilde{\mathbf{A}}(\mathbf{k},\omega,\mathbf{r},t) = \sum_m \tilde{A}_m(\mathbf{k},\omega,\mathbf{r},t)\mathbf{a}_m(\mathbf{k},\mathbf{r},t)\,. \tag{2.8}$$

By integrating over frequencies around the peak at Ω_m we define

$$\boldsymbol{\mathcal{A}}_m(\mathbf{k},\mathbf{r},t) = (\pi T^2)^{1/4}\int \tilde{\mathbf{A}}_m(\mathbf{k},\omega,\mathbf{r},t)\, d\omega/2\pi\,. \tag{2.9}$$

We also introduce the function $\mathcal{D}_m = \mathcal{D}'_m + i\mathcal{D}''_m$ by

$$\mathcal{D}_m(\mathbf{k},\Omega_m,\mathbf{r},t) = \mathbf{a}^*_m\cdot[\mathbf{D}'(\mathbf{k},\Omega_m,\mathbf{r},t) + i\mathbf{D}''(\mathbf{k},\Omega_m,\mathbf{r},t)]\cdot\mathbf{a}_m\,. \tag{2.10}$$

The wave density on (**k**, **r**) space in mode m can then be defined as

$$\begin{aligned}\mathcal{N}_m(\mathbf{k},\mathbf{r},t) &= \hbar^{-1}|\mathcal{A}_m|^2\partial_{\Omega_m}\mathcal{D}'_m \\ &= \hbar^{-1}\boldsymbol{\mathcal{A}}^*_m(\mathbf{k},\mathbf{r},t)\cdot\partial_{\Omega_m}\mathbf{D}'(\mathbf{k},\Omega_m,\mathbf{r},t')\cdot\boldsymbol{\mathcal{A}}_m(\mathbf{k},\mathbf{r},t)\,.\end{aligned} \tag{2.11}$$

A kinetic equation for the wave density can be derived by retaining only first derivatives in Equation (2.5), multiplying by $\tilde{\mathbf{A}}^*_m$, and integrating over ω. This procedure, which is described in detail by Biro and Rönnmark [1992] and Rönnmark and Biro [1992] leads to

$$\partial_t\mathcal{N}_m + (\partial_{\mathbf{k}}\Omega_m)\cdot\partial_{\mathbf{r}}\mathcal{N}_m - (\partial_{\mathbf{r}}\Omega_m)\cdot\partial_{\mathbf{k}}\mathcal{N}_m = 2\gamma_m\mathcal{N}_m + \Gamma\,. \tag{2.12}$$

The linear growth rate γ_m is determined from the imaginary part $\mathcal{D}''_m$ of $\mathcal{D}_m$ as

$$\gamma_m(\mathbf{k},\mathbf{r},t) = -\frac{\mathcal{D}''_m}{\partial_{\Omega_m}\mathcal{D}'_m}\,, \tag{2.13}$$

and the nonlinear term is found to be

$$\Gamma = \frac{\pi}{2}\int\frac{d\mathbf{k}'\,d\mathbf{k}''}{(2\pi)^3}\delta(\mathbf{k}-\mathbf{k}'-\mathbf{k}'')\delta(\Omega-\Omega'-\Omega'')M(\mathbf{k},\mathbf{k}',\mathbf{k}'',\mathbf{r},t)\times$$

$$\times\Big[\mathcal{N}_{m'}(\mathbf{k}',\mathbf{r},t)\mathcal{N}_{m''}(\mathbf{k}'',\mathbf{r},t) - \mathcal{N}_m(\mathbf{k},\mathbf{r},t)\mathcal{N}_{m''}(\mathbf{k}'',\mathbf{r},t) - \mathcal{N}_m(\mathbf{k},\mathbf{r},t)\mathcal{N}_{m'}(\mathbf{k}',\mathbf{r},t)\Big]\,. \tag{2.14}$$

The coupling coefficient M can be written as

$$M(\mathbf{k},\mathbf{k}',\mathbf{k}'',\mathbf{r},t) = \frac{\hbar|V(\mathbf{k},\mathbf{k}',\mathbf{k}'',\mathbf{r},t)|^2}{\partial_\Omega\mathcal{D}'_m(\Omega)\,\partial_{\Omega'}\mathcal{D}'_{m'}(\Omega')\,\partial_{\Omega''}\mathcal{D}'_{m''}(\Omega'')}\,, \tag{2.15}$$

where V is related to the nonlinear admittance $\mathbf{\Lambda}_2$ as

$$V(\mathbf{k},\mathbf{k}',\mathbf{k}'',\mathbf{r},t) = \mathbf{a}^*_m(\mathbf{k},\mathbf{r},t)\cdot\Big[\mathbf{a}_{m'}(\mathbf{k}',\mathbf{r},t)\cdot\mathbf{\Lambda}_2(\mathbf{k}',\Omega',\mathbf{k}'',\Omega'',\mathbf{r},t)\cdot\mathbf{a}_{m''}(\mathbf{k}'',\mathbf{r},t)\Big]\,, \tag{2.16}$$

and $\Omega = \Omega_m(\mathbf{k})$, $\Omega' = \Omega_{m'}(\mathbf{k}')$, and $\Omega'' = \Omega_{m''}(\mathbf{k}'')$.

The observable electric field spectral density may be calculated from the wave density as

$$G(\omega,\mathbf{r},t) = \int\frac{d\mathbf{k}}{(2\pi)^3}Q_m(\mathbf{k},\mathbf{r},t)\delta(\omega-\Omega_m(\mathbf{k},\mathbf{r},t))\mathcal{N}_m(\mathbf{k},\mathbf{r},t)\,, \tag{2.17}$$

where the function Q_m may depend on properties of the antenna as well as on the wave mode. For an ideal omnidirectional dipole antenna, we have

$$Q_m(\mathbf{k},\mathbf{r},t) = \hbar\frac{(\omega^2+\partial_t^2)}{\partial_\Omega\mathcal{D}'_m} \approx \frac{\hbar\omega^2}{\partial_\Omega\mathcal{D}'_m}\,. \tag{2.18}$$

In practice, we can always neglect the term ∂_t^2 compared to ω^2 in this expression for Q_m, since we will assume $\omega T \gg 1$. Relations similar to (2.17) were first introduced by Storey and Lefeuvre [1974], and discussed in greater detail by Oscarsson and Rönnmark [1989].

We notice from Equation (2.12) that Hamilton's equations of geometric optics, $\dot{\mathbf{r}} = \partial_{\mathbf{k}}\Omega$ and $\dot{\mathbf{k}} = -\partial_{\mathbf{r}}\Omega$, define rays in phase space along which the wave density is conserved in the absence of dissipation. The flow of $\mathcal{N}$ in phase space is always incompressible, and many problems involving wave convection will thus appear much simpler when viewed in phase space. To illustrate this, we will show that Equation (2.12) conveniently can be taken as the starting point for a quantitative analysis of linear as well as nonlinear conversion of upper hybrid waves into electromagnetic magnetospheric radiation.

3 The model

The sources of magnetospheric continuum radiation are mainly located at the equatorial plasmapause, and sometimes at the magnetopause. These source regions are characterized by a density gradient approximately perpendicular to the constant magnetic field $\mathbf{B}_0$. We will henceforth use a coordinate system with the z axis along $\mathbf{B}_0$ and assume that the plasma density gradient is in the x direction and is characterized by a scale length $\mathcal{L} = n_e(x)|\partial_x n_e(x)|^{-1}$. In the model we consider, the wave density as well as the plasma density is independent of y and z.

Electrostatic upper hybrid waves are driven unstable by hot electrons (thermal velocity $V_h \sim 10^7$ m/s, corresponding to 300 eV) with a loss cone distribution in the presence of cool plasmaspheric particles ($V_c \sim 10^6$ m/s, or 3 eV). These instabilities have been studied by several authors such as Young et al. [1973], Rönnmark and Christiansen [1981], and Kennel and Ashour–Abdalla [1982]. Rönnmark [1989] used the known properties of the instability mechanism to model the wave density in $\mathbf{k}$ space at a point x_I as

$$\mathcal{N}_U(\mathbf{k}, x_I) = \begin{cases} \widehat{\mathcal{N}}_U, & \text{if } |k_z| \leq k_{Iz},\ k_\perp \sim k_{I\perp} \\ 0, & \text{otherwise} \end{cases} \tag{3.1}$$

The initial wave numbers characterizing the instability are taken to as $k_{Iz} \sim \omega_{ce}/(2V_h)$ and $k_{I\perp} \sim \omega/V_c$. Since we assume all functions to be independent of y, z, and t, we will for notational convenience henceforth omit these coordinates from the formulas. The constant $\widehat{\mathcal{N}}_U$ can be related to the observed spectral densities through Equation (2.17) as

$$\begin{aligned} G_U(\omega, x_I) &= Q_U \int \mathcal{N}_U(\mathbf{k}, x_I)\, \delta(\omega - \Omega_U(\mathbf{k}, x_I)) \frac{d\mathbf{k}}{(2\pi)^3} \\ &= \frac{Q_U}{2\pi^2} \frac{k_{Iz} k_{I\perp}}{|\partial_{k_\perp} \Omega_U|} \widehat{\mathcal{N}}_U \end{aligned} \tag{3.2}$$

Notice that since the delta function can be transformed as $\delta(\omega - \Omega_U(\mathbf{k}, x_I)) \sim \delta(k_\perp - k_{I\perp})|\partial_{k_\perp}\Omega_U|^{-1}$, we need not specify the $k_\perp$ dependence of $\mathcal{N}$ in detail. From equation (3.2) we have

$$\widehat{\mathcal{N}}_U = \frac{2\pi^2 |\partial_{k_\perp} \Omega_U|}{Q_U k_{Iz} k_{I\perp}} G_U(\omega, x_I) \tag{3.3}$$

and with this normalization we can use Equation (3.1) as input to a calculation of the radiation emitted due to linear mode conversion.

4 Linear mode conversion

The efficiency of the linear mode conversion that can be expected from the model described above was considered by Rönnmark [1989]. By integrating Hamilton's equations of geometric optics, $\dot{x} = \partial_{k_x}\Omega_U(k_x, x)$ and $\dot{k}_x = -\partial_x \Omega_U(k_x, x)$ we obtain the ray in (k_x, x) phase space shown in Figure 1. The density is assumed to decrease in the x direction, and along the ray the values of ω, $k_z = k_w$, and $k_y = 0$ are constant. The detailed shape

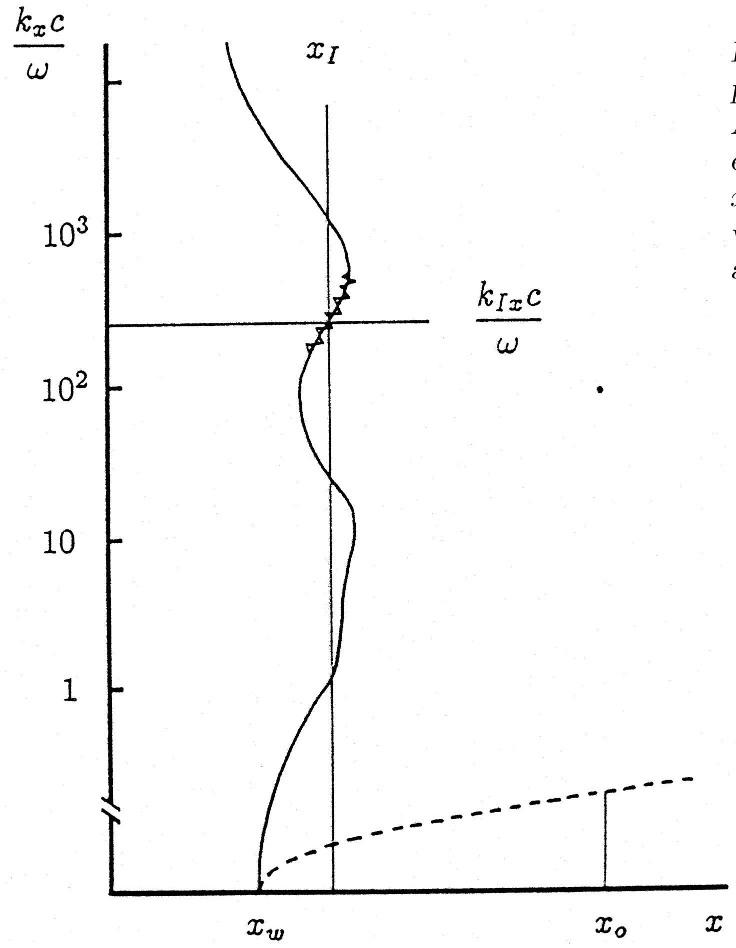

Figure 1: The full line shows a sample upper hybrid ray in (x, k_x) space. Along the hatched part of the ray, the electrostatic waves will grow, and at $x = x_w$, $k_x = 0$ they may be converted to electromagnetic radiation and escape along the dashed ray.

of the ray will of course depend on the density profile, and also on the mixture of hot and cold particles. Along the hatched section of the ray, around $k_{I\perp}$, the waves will grow. Neglecting possible damping and nonlinear effects along the rest of the ray, we then know from Equation (2.12) that $\mathcal{N}_U(\mathbf{k}, x)$ will be constant along the ray. Following the ray in the direction of decreasing k_x, we eventually reach $k_x = 0$ at the point $x = x_w$ where the local plasma frequency equals the wave frequency, and here the upper hybrid waves may be converted to radiation in the ordinary mode.

In the derivation of Equation (2.12) it was assumed that different wave modes could be separated. However, there are degenerate points $(\mathbf{k}_w, \mathbf{r}_w)$ where two modes have the same frequency. In the case of interest for this study this means $\Omega_U(\mathbf{k}_w, \mathbf{r}_w) = \Omega_O(\mathbf{k}_w, \mathbf{r}_w)$, and this can happen when the frequency matches the local plasma frequency $\omega_p(\mathbf{r}_w)$. The coupling point is in $\mathbf{k}$ space located at $k_\perp = 0$ with the component parallel to $\mathbf{B}_0$ given by $k_w c = \omega\sqrt{\omega_{ce}/(\omega + \omega_{ce})}$. The linear mode conversion has been studied by, among others, Mjølhus [1984] who calculated the coefficient for conversion from upper hybrid waves to O mode radiation

$$T(\mathbf{k}) = \exp\left\{-\Delta_z^2(k_z - k_w)^2 - \Delta_y^2 k_y^2\right\} , \tag{4.1}$$

where

$$\Delta_y^2 = \pi\left(\frac{\omega_{ce}}{2\omega}\right)^{\frac{1}{2}} \frac{c\mathcal{L}}{\omega} ,$$

$$\Delta_z^2 = 2\,\frac{\omega+\omega_{ce}}{\omega}\pi\left(\frac{\omega_{ce}}{2\omega}\right)^{\frac{1}{2}}\frac{c\mathcal{L}}{\omega}\,. \tag{4.2}$$

Full wave calculations by Hansen et al. [1988] have confirmed that this expression is accurate for $\mathcal{L} \geq 4c/\omega$.

Phase space rays from the point $(\mathbf{k}, x_I)$, where the waves are generated, will at some point x_w in space pass through the plane $k_x = 0$. From Equation (4.1) we see that the radio window is a transparent region in this plane and has the shape of an ellipse centered on $k_y = 0$ and $k_z = k_w$. Depending on the initial value of $\mathbf{k}$, the ray may cut the (k_y, k_z) plane more or less close to the center of the window, and the rate of conversion is determined by this distance. Hence, the distribution of O waves generated by the conversion process is

$$\mathcal{N}_O(\mathbf{k}, x_w) = T(\mathbf{k})\,\mathcal{N}_U(\mathbf{k}, x_w)\,. \tag{4.3}$$

This radiation will propagate to larger x, as indicated by the dashed line in Figure 1, to the point x_o where the spectral density measured by a satellite is

$$G_O(\omega, x_o) = Q_O\int \mathcal{N}_O(\mathbf{k}, x_o)\,\delta(\omega - \Omega_O(\mathbf{k}, x_o))\,\frac{d\mathbf{k}}{(2\pi)^3}\,. \tag{4.4}$$

Since $\mathcal{N}$ according to Equation (2.12) is conserved along rays, we can use that $\mathcal{N}_U(k_{Ix}, x_I) = \mathcal{N}_U(0, x_w)$ and $\mathcal{N}_O(0, x_w) = \mathcal{N}_O(k_{ox}, x_o)$, where we have suppressed the constant arguments k_y and k_z. Using these relations we find

$$\begin{aligned} G_O(\omega, x_o) &= Q_O\int T(\mathbf{k})\mathcal{N}_U(\mathbf{k}, x_I)\,\delta(\omega - \Omega_O(\mathbf{k}, x_o))\,\frac{d\mathbf{k}}{(2\pi)^3} \\ &= \frac{Q_O}{|\partial_{k_x}\Omega_O|}\int \exp\left[-\Delta_z^2(k_z - k_w)^2 - \Delta_y^2 k_y^2\right]\widehat{\mathcal{N}}_U\,\frac{dk_y\,dk_z}{(2\pi)^3} \\ &\approx \left|\frac{\partial_{k_\perp}\Omega_U}{\partial_{k_x}\Omega_O}\right|\frac{G_U(\omega, x_I)}{4\Delta_y\Delta_z k_{Iz}k_{I\perp}}\,. \end{aligned} \tag{4.5}$$

Inserting reasonable parameter values as described by Rönnmark [1989] the result is that

$$G_O \leq 10^{-8} G_U\,. \tag{4.6}$$

Since the typical observed spectral density of the continuum radiation is $\sim 10^{-13}$ $\mathrm{V^2m^{-2}Hz^{-1}}$ and the very strongest upper hybrid waves reach 10^{-8} $\mathrm{V^2m^{-2}Hz^{-1}}$, this conversion rate is at least three orders of magnitude too small.

5 Nonlinear decay

The derivation presented by Rönnmark and Biro [1992] shows that the decay of upper hybrid (U) waves into lower hybrid (L) waves and electromagnetic radiation with left–handed or ordinary (O) polarization is described by the set of equations

$$d_t\mathcal{N}_U(\mathbf{k}, \mathbf{r}, t) = 2\gamma_U(\mathbf{k})\,\mathcal{N}_U(\mathbf{k}, \mathbf{r}, t) - 2\pi\int \frac{d\mathbf{k}'\,d\mathbf{k}''}{(2\pi)^3}\,\mathcal{S}(\mathbf{k}, \mathbf{k}', \mathbf{k}'', \mathbf{r}, t)\,,$$

$$d_t\mathcal{N}_L(\mathbf{k}',\mathbf{r},t) \;=\; 2\gamma_L(\mathbf{k}')\,\mathcal{N}_L(\mathbf{k}',\mathbf{r},t)\;+2\pi\int\frac{d\mathbf{k}\,d\mathbf{k}''}{(2\pi)^3}\,\mathcal{S}(\mathbf{k},\mathbf{k}',\mathbf{k}'',\mathbf{r},t)\,, \tag{5.1}$$

$$d_t\mathcal{N}_O(\mathbf{k}'',\mathbf{r},t) \;=\; 2\gamma_O(\mathbf{k}'')\,\mathcal{N}_O(\mathbf{k}'',\mathbf{r},t)\;+2\pi\int\frac{d\mathbf{k}\,d\mathbf{k}'}{(2\pi)^3}\,\mathcal{S}(\mathbf{k},\mathbf{k}',\mathbf{k}'',\mathbf{r},t)\,,$$

where

$$\begin{aligned}\mathcal{S}(\mathbf{k},\mathbf{k}',\mathbf{k}'',\mathbf{r},t) \;=\; & \delta(\mathbf{k}-\mathbf{k}'-\mathbf{k}'')\,\delta(\Omega_U(\mathbf{k})-\Omega_L(\mathbf{k}')-\Omega_O(\mathbf{k}''))\\ & M\,[\mathcal{N}_U(\mathbf{k})\mathcal{N}_O(\mathbf{k}'')+\mathcal{N}_U(\mathbf{k})\mathcal{N}_L(\mathbf{k}')-\mathcal{N}_L(\mathbf{k}')\mathcal{N}_O(\mathbf{k}'')]\;.\end{aligned} \tag{5.2}$$

In a stationary state ($\partial_t\mathcal{N}=0$) we neglect linear damping of the electromagnetic radiation ($\gamma_O=0$) and convective loss of the electrostatic waves ($d_t\mathcal{N}_U=d_t\mathcal{N}_L=0$). We assume that the upper hybrid waves are linearly unstable ($\gamma_U>0$) and the lower hybrid waves are damped ($\gamma_L<0$). Electromagnetic radiation is lost by convection ($d_t\mathcal{N}_O\approx\partial_{\mathbf{k}}\Omega_O\cdot\partial_{\mathbf{r}}\mathcal{N}_O$). With these assumptions, the set of equations describing the evolution of the wave densities can be written

$$\gamma_U\mathcal{N}_U(\mathbf{k},\mathbf{r}) \;=\; \pi\int\frac{d\mathbf{k}'\,d\mathbf{k}''}{(2\pi)^3}\,\mathcal{S}\,, \tag{5.3a}$$

$$\gamma_L\mathcal{N}_L(\mathbf{k}',\mathbf{r}) \;=\; -\pi\int\frac{d\mathbf{k}\,d\mathbf{k}''}{(2\pi)^3}\,\mathcal{S}\,, \tag{5.3b}$$

$$\partial_{\mathbf{k}''}\Omega_O\cdot\partial_{\mathbf{r}}\mathcal{N}_O(\mathbf{k}'',\mathbf{r}) \;=\; 2\pi\int\frac{d\mathbf{k},\,d\mathbf{k}'}{(2\pi)^3}\,\mathcal{S}\,. \tag{5.3c}$$

As shown in Figure 2, rays of electromagnetic O mode waves will reach a satellite at x_o outside the plasmapause after reflection at a level x_r where $\omega^2=\omega_p^2(x_r)+{k''_{yz}}^2c^2$ with ${k''_{yz}}^2={k''_y}^2+{k''_z}^2$. Within the shaded region in Figure 2, where the electrostatic turbulence is in the right frequency range, the radiation will be amplified due to the nonlinear interaction. The decay thus takes place within a layer of thickness $x_m-x_r\approx\mathcal{L}(\omega_{ce}^2-{k''_{yz}}^2c^2)\omega_{pe}^{-2}$. Here, x_m is the level where the upper hybrid frequency drops below the wave frequency ($\omega_p^2(x_m)+\omega_{ce}^2=\omega^2$). Each ray passes this layer twice, and the magnitude of the group velocity can be estimated as

$$\partial_{k''_x}\Omega''_O\approx c\sqrt{1-(\omega_{pe}(x)/\omega)^2}\approx c\frac{\omega_p(x_r)}{\omega}[(x-x_r)/\mathcal{L}]^{1/2}\,. \tag{5.4}$$

By integrating Equation (5.3c), the density of electromagnetic O mode radiation seen by a satellite at the point x_o can be evaluated as

$$\mathcal{N}_O(\mathbf{k}'',x_o)=4\pi\int_{x_r}^{x_m}dx\int\frac{d\mathbf{k}\,d\mathbf{k}'}{(2\pi)^3}\frac{\mathcal{S}}{\partial_{k''_x}\Omega_O}\,. \tag{5.5}$$

Assuming that the main x dependence of the integrand comes from the variation of the group velocity we find

$$\mathcal{N}_O(\mathbf{k}'',x_o)=8\pi\frac{\omega\mathcal{L}}{\omega_{pe}^2c}[\omega_{ce}^2-{k''_{yz}}^2c^2]^{1/2}\int\frac{d\mathbf{k}\,d\mathbf{k}'}{(2\pi)^3}\mathcal{S}\,. \tag{5.6}$$

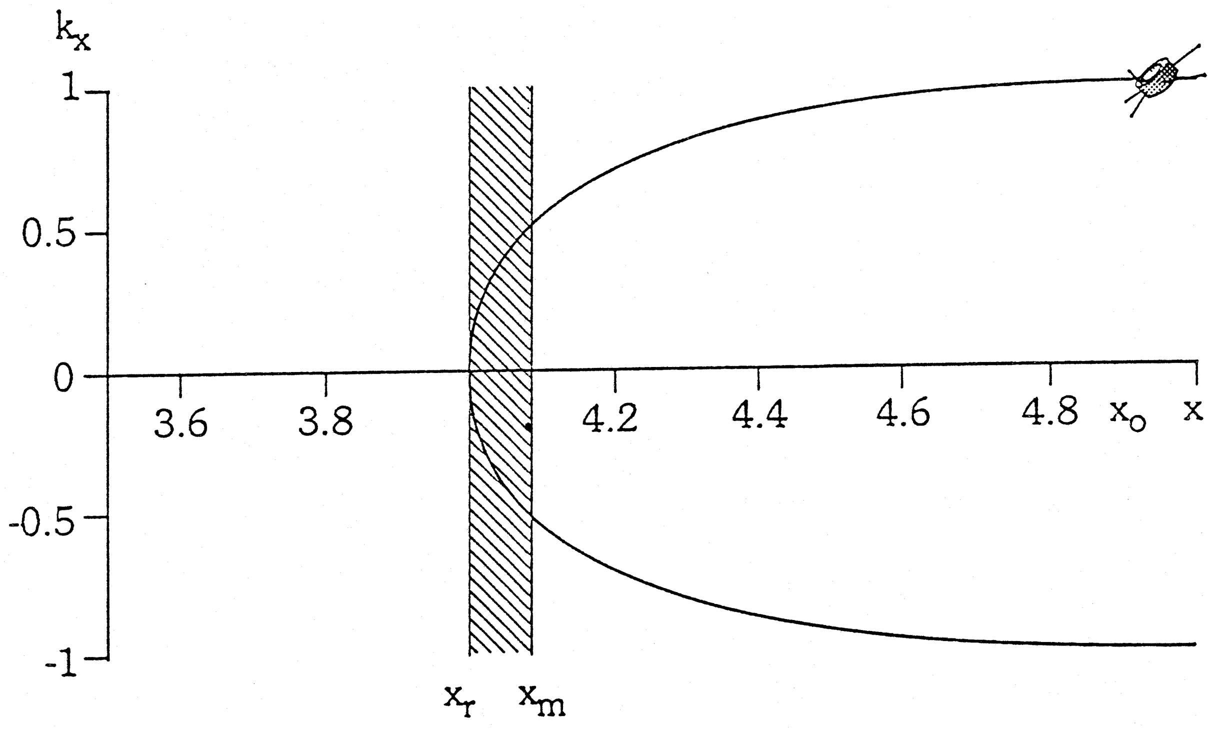

Figure 2: Near their reflection point, the O mode rays will pass through the layer of electrostatic turbulence. Within the shaded region, the frequency of the electromagnetic wave is below the upper hybrid frequency and the electrostatic waves can decay.

The electric field spectral density $G_O(\omega, x_o)$ of the radiation reaching the satellite outside the plasmapause can according to Equation (2.17) be calculated from the wave density as

$$\begin{aligned} G_O(\omega, x_o) &= Q_O \int \frac{d\mathbf{k}''}{(2\pi)^3} \delta(\omega - \Omega''_O)\, \mathcal{N}_O(\mathbf{k}'', x_o) \\ &= 8\pi Q_O \frac{\omega \mathcal{L}}{\omega_{pe}^2 c} \int \frac{d\mathbf{k}\, d\mathbf{k}'\, d\mathbf{k}''}{(2\pi)^6} \sqrt{\omega_{ce}^2 - {k''_{yz}}^2 c^2}\, \delta(\omega - \Omega''_O)\, \mathcal{S} \\ &\approx 4\pi Q_O \frac{\omega_{ce} \mathcal{L}}{\omega_{pe} c^2} \int \frac{d\mathbf{k}\, d\mathbf{k}'\, d\mathbf{k}''}{(2\pi)^6}\, \delta(k''_\perp - K_{\perp O}(\omega))\, \mathcal{S}\,. \end{aligned} \tag{5.7}$$

Since $\mathcal{N}_O(\mathbf{k}'', x_o)$, and hence $\mathcal{S}$, is significant only for $k''_{yz} \leq \omega_{ce}/c$, the square root has here been approximated by $\omega_{ce}/2$ and taken out of the integral. The delta function in ω has been transformed into a delta function in $k''_\perp$ using $K_{\perp O}(\omega)$ which is the solution for $k_\perp$ of the dispersion equation with given k_z and ω, that is $\mathcal{D}'_O(K_{\perp O}, k_z, \omega) = 0$.

Using Equation (5.3a), the spectral density of the upper hybrid waves can be expressed as

$$\begin{aligned} G_U(\omega, x_r) &= Q_U \int \mathcal{N}_U(\mathbf{k}, x_r) \delta(\omega - \Omega_U) \frac{d\mathbf{k}}{(2\pi)^3} \\ &= \frac{\pi Q_U}{\gamma_U} \int \frac{d\mathbf{k}\, d\mathbf{k}'\, d\mathbf{k}''}{(2\pi)^6} \delta(\omega - \Omega_U) \mathcal{S}\,, \end{aligned} \tag{5.8}$$

which implies

$$\int \frac{d\mathbf{k}\, d\mathbf{k}'\, d\mathbf{k}''}{(2\pi)^6} \delta(k_\perp - K_{\perp U}(\omega))\mathcal{S} \approx \frac{\gamma_U |\partial_{\mathbf{k}} \Omega_U|}{\pi Q_U} G_U(\omega, x_r)\,. \tag{5.9}$$

Recalling the resonance condition $\delta(\mathbf{k} - \mathbf{k}' - \mathbf{k}'')$ contained in $\mathcal{S}$ we conclude that $\delta(k''_\perp - K_{\perp O}) = \delta(k_\perp - k'_\perp - K_{\perp O})$ is essentially the same as $\delta(k_\perp - K_{\perp U})$, and hence we can substitute (5.9) for the last integral in (5.7) to find

$$\begin{aligned} G_O(\omega, x_o) &\approx 4\frac{Q_O \omega_{ce}}{Q_U \omega_{pe}} \frac{\gamma_U \mathcal{L} |\partial_{\mathbf{k}} \Omega_U|}{c^2} G_U(\omega, x_r) \\ &\approx 4\frac{\omega_{ce}^2}{\omega^2} \frac{\gamma_U \mathcal{L} V_c}{c^2} G_U(\omega, x_r) \\ &\sim 5 \times 10^{-5} G_U(\omega, x_r)\,. \end{aligned} \tag{5.10}$$

In the last step we have also used the estimate $|\partial_{\mathbf{k}} \Omega_U| \sim V_c\, \omega_{ce}/\omega$ for the group velocity of the upper hybrid waves, and the parameter values $V_c = 10^6$ m/s, $\gamma_U = 10^{-3}\omega$, $\omega/\omega_{ce} = 5$, and $\mathcal{L} = 100c/\omega$. This result indicates that nonlinear conversion can produce magnetospheric continuum radiation with a spectral density of $10^{-13}\mathrm{V}^2\,\mathrm{m}^{-2}\,\mathrm{Hz}^{-1}$ when $G_U \sim 10^{-8}\mathrm{V}^2\,\mathrm{m}^{-2}\,\mathrm{Hz}^{-1}$.

Several simplifying assumptions have been introduced above, in order to obtain a solution by simple, analytical methods. We have assumed a homogeneous magnetic field, and the only inhomogeneity in the plasma is a density gradient perpendicular to $\mathbf{B}_o$. This prevents us from discussing the spatial distribution of the radiation in the magnetosphere and other details, but it should not affect the efficiency estimates much. Another important simplification is the assumption of a steady state. For consistency, this requires that the coupling is strong enough to allow the upper hybrid instability to saturate nonlinearly at a level corresponding to less than $10^{-8}\mathrm{V}^2\,\mathrm{m}^{-2}\,\mathrm{Hz}^{-1}$. A preliminary estimate based on the coupling coefficient found by Rönnmark [1985] and a reasonable model for the distribution $\mathcal{N}_U$ indicates that the stationary state described by (5.1) can be maintained at the observed spectral density.

6 Conclusions

As was first shown by Rönnmark [1989], the efficiency of linear mode conversion is three orders of magnitude too small to explain the observed levels of continuum radiation. This result was noted by Horne [1990], but he did not explain the difference between his results and ours. Horne used a sophisticated ray tracing program to integrate the dissipation of the upper hybrid waves along the rays, and assumed that '*wave growth (or damping) corresponds to an increase (or decrease) in wave amplitude.*' This last sentence (quoted from p. 3928 in Horne [1990]) looks almost self–evident, and yet it contains the error that causes the discrepancy. The quoted sentence implies that wave amplitude should be constant along rays in the absence of dissipative growth or damping, but by considering for example a radiating antenna in vacuum we immediately see that this is wrong—the amplitude decreases in proportion to the distance from the antenna. The assumption that wave amplitude is conserved along rays seems to be implicit in many ray tracing studies, but it should clearly be replaced by conservation of wave density in phase space.

The efficiency of the nonlinear conversion of upper hybrid waves to electromagnetic radiation is estimated from the set of nonlinear Equations (5.1). These equations are at first sight very similar to those normally found in weak turbulence theory. However, the wave density $\mathcal{N}(\mathbf{k}, \mathbf{r}, t)$ appearing in (5.1) is unambiguously defined as a function on phase space, and it can via the transformation (2.1) be calculated from an arbitrary wave field by a well specified procedure. We can hence be confident that convective effects are properly described, and by integrating (5.3*c*) along the rays we find a stationary solution to the set of nonlinear equations. Measured frequency spectra can be used as normalizing input, and the predicted spectral densities are consistent with observations.

The results presented in this study illustrate that phase space methods provide a conceptually and computationally simple description of the nonlinear as well as linear conversion processes. The efficiencies of these mechanisms can be quantitatively estimated, and the results are obtained in the form of spectral densities that directly can be compared to measured frequency spectra.

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