THE HIRSCH CONJECTURE FOR DUAL TRANSPORTATION POLYHEDRA

M.L. Balinski

February 1983
CP-83-9

INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS
A-2361 Laxenburg, Austria

ABSTRACT

An algorithm is given that joins any pair of extreme points of a dual transportation polyhedron by a path of at most $(m-1)(n-1)$ extreme edges.

THE HIRSCH CONJECTURE
FOR DUAL TRANSPORTATION POLYHEDRA

M.L. Balinski

Laboratoire d'Econométrie de l'Ecole
Polytechnique, Paris, France

The *distance* between a pair of extreme points of a convex polyhedron P is the number of extreme edges in the shortest path that joins them. The *diameter* of P is the greatest distance between any pair of extreme points of P. The Hirsch conjecture (see [5], pp. 160 and 168, [7]) is that the diameter of a convex polyhedron defined by q halfspaces in p-dimensional space is at most q-p. In linear programming jargon it is that given r linearly independent equations in nonnegative variables it is possible to go from one feasible basis to any other in at most r pivots all the while staying feasible.

For unbounded polyhedra in dimension 4 or more the Hirsch conjecture is false [7], so it is false in general. However, it has been proven to be true for certain special cases: the polytopes arising from the shortest path problem [8], Leontief substitution systems [6], the assignment problem [4], and certain classes of transportation problems [1]. I will show it is true for the unbounded polyhedra arising from dual transportation problems. The approach will also establish a combinatorial characterization of extreme points that has proven to be very useful.

The *dual transportation polyhedron* for the m by n matrix $\underset{\sim}{c}$ is

$$D_{m,n}(\underset{\sim}{c}) = \{\underset{\sim}{u},\underset{\sim}{v}; u_i+v_j \leqq c_{ij}, i \in R, j \in C, u_1 = 0\}$$

where $|R| = m$ and $|C| = n$. Setting $u_1 = 0$ is an arbitrary choice that rules out lines of solutions $(u_i+\delta)$, $(v_j-\delta)$.

It is natural to study the extreme points of D in terms of a bipartite graph model. Let the set of m nodes R stand for the rows of $\underset{\sim}{c}$ and the set of n nodes C for the columns. The equations $u_1 = 0$, $u_i + v_j = c_{ij}$ for $(i,j) \in T$, $i \in R$, $j \in C$, form a maximal linearly independent set if and only if T is a spanning tree. The unique solution $\underset{\sim}{u}$, $\underset{\sim}{v}$ to T is an *extreme point* of D if and only if $u_i + v_j \leqq c_{ij}$ for $(i,j) \notin T$: I will say T *has* the extreme point $\underset{\sim}{u}$, $\underset{\sim}{v}$ and for clarity will sometimes write $T(\underset{\sim}{u},\underset{\sim}{v})$. Given a spanning tree T the unique solution to

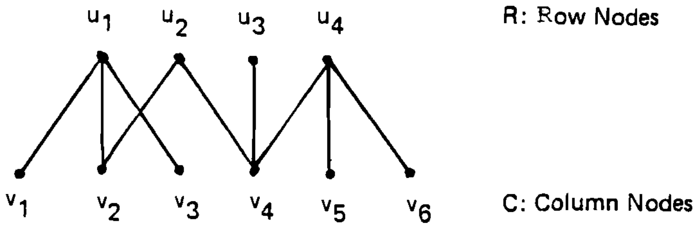

Figure 1. Spanning tree. Signature (3,2,1,3)

its equations is immediate. In the sequel *only* spanning trees that have extreme points $\underset{\sim}{u}$, $\underset{\sim}{v}$ are considered: to any T there corresponds exactly one extreme point.

To one extreme point, however, there can correspond many trees T: this happens when $u_i + v_j = c_{ij}$ for $(i,j) \notin T$ and is called "degeneracy". In this case any spanning tree chosen from $\{(i,j); u_i+v_j = c_{ij}\}$ has the same extreme point.

The (row) *signature* of a tree T is the uniquely defined vector of the degrees of its row nodes $\underset{\sim}{a} = (a_1,\ldots,a_m)$, $\sum a_i = m+n-1$, $a_i \geqq 1$.

Lemma. Two different trees T, T^1 with the same signatures have one and the same extreme point.

Proof. Let the extreme points of T and T^1 be $\underset{\sim}{u},\underset{\sim}{v}$ and $\underset{\sim}{u}^1,\underset{\sim}{v}^1$. I will show that $\underset{\sim}{u} = \underset{\sim}{u}^1$, $\underset{\sim}{v} = \underset{\sim}{v}^1$.

$T \neq T^1$ means there is some node $i_1 \in R$, $(i_1,j_1) \in T$ but $(i_1,j_1) \notin T^1$ (see Figure 2 for what follows). In T^1 let (i_2,j_1) with $i_2 \neq i_1$ be the edge on the unique path that joins j_1 to i_1, and consider node i_2.

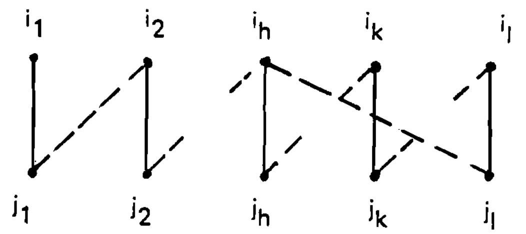

Figure 2. Solid lines in T; dashed lines in T^1.

It must have degree at least 2 in T^1 and so in T. Therefore there exists an edge $(i_2,j_2) \in T$, $j_2 \neq j_1$. Now, let (i_3,j_2) be the edge on the unique path that joins j_2 to i_1 in T^1. Continue to build this path until a node already on it is encountered again, forming a cycle: (i_h,j_h), $(i_{h+1},j_h),\ldots,(i_\ell,j_\ell)$, (i_h,j_ℓ). Call the edges of type (i_k,j_k) of the cycle odd, and of type (i_{k+1},j_k) and (i_h,j_ℓ) even. Then

$$u_i + v_j = c_{ij} \text{ for } (i,j) \text{ odd}, \qquad u_i + v_j \leqq c_{ij} \text{ for } (i,j) \text{ even}$$

and

$$u_i^1 + v_j^1 \leqq c_{ij} \text{ for } (i,j) \text{ odd}, \qquad u_i^1 + v_j^1 = c_{ij} \text{ for } (i,j) \text{ even.}$$

Summing,

$$\sum_{odd} c_{ij} = \sum_h^\ell (u_{i_k} + v_{j_k}) \leqq \sum_{even} c_{ij} = \sum_h^\ell (u_{i_k}^1 + v_{j_k}^1) \leqq \sum_{odd} c_{ij}$$

implying equality holds throughout and so

$$u_i + v_j = c_{ij} = u_i^1 + v_j^1 \text{ for all } (i,j) \text{ in the cycle.}$$

Transform T^1 by taking from it all even edges and putting in it all odd edges. The new T^1 has the same signature and the same $\underset{\sim}{u}^1, \underset{\sim}{v}^1$ but more edges in common with T. Repeat until $T = T^1$, showing $\underset{\sim}{u} = \underset{\sim}{u}^1$, $\underset{\sim}{v} = \underset{\sim}{v}^1$. □

Given $T(\underset{\sim}{u},\underset{\sim}{v})$, let (k,ℓ) be one of its edges with k and ℓ both of degree at least 2. A *pivot* on (k,ℓ) obtains $T^1(\underset{\sim}{u}^1,\underset{\sim}{v}^1)$ as follows (see Figure 3): drop (k,ℓ) from T to obtain two connected components, T^k containing k and T^ℓ containing ℓ. Let $\varepsilon = \min \{c_{ij} - u_i - v_j;\ i \in T^\ell, j \in T^k\} \geqq 0$ and (g,h) be some edge at which this minimum is obtained. Set $T^1 = T^k \cup T^\ell \cup (g,h)$ ((g,h) is the "incoming" edge). If row node $1 \in T^k$ define

$$u_i^1 = u_i + \varepsilon, \quad i \in T, \quad u_i^1 = u_i \text{ otherwise} \quad ,$$

$$v_j^1 = v_j - \varepsilon, \quad j \in T, \quad v_j^1 = v_j \text{ otherwise} \quad ;$$

and if row node $1 \in T^\ell$ define

$$u_i^1 = u_i - \varepsilon, \quad i \in T^k, \quad u_i^1 = u_i \text{ otherwise} \quad ,$$

$$v_j^1 = v_j + \varepsilon, \quad j \in T^k, \quad v_j^1 = v_j \text{ otherwise} \quad .$$

$\varepsilon \geqq 0$ because $\underset{\sim}{u},\underset{\sim}{v}$ satisfies all inequalities. The choice of ε guarantees that $\underset{\sim}{u}^1,\underset{\sim}{v}^1$ satisfies them all as well and that it belongs to T^1.

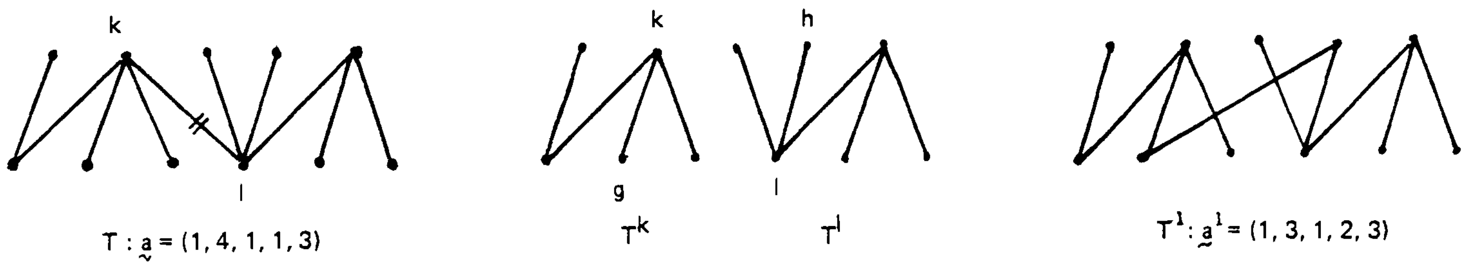

Figure 3. Pivot from T to T^1.

If $\varepsilon = 0$ then $(\underset{\sim}{u},\underset{\sim}{v}) = (\underset{\sim}{u}^1,\underset{\sim}{v}^1)$ and we have two different trees having the same extreme point (degeneracy). If $\varepsilon > 0$ then $\underset{\sim}{u},\underset{\sim}{v}$ and $\underset{\sim}{u}^1,\underset{\sim}{v}^1$ are *neighbors*, connected by an extreme edge of D. In either case, if $\underset{\sim}{a}$ is the signature of T then the signature $\underset{\sim}{a}^1$ of T^1 is the same except that $a_k^1 = a_k - 1$ and $a_g^1 = a_g + 1$.

Theorem 1. The diameter of $D_{m,n}(\underset{\sim}{c})$ *is at most* $(m-1)(n-1)$. *This bound is the best possible.*

Proof. I give a method that constructs a path of at most $(m-1)(n-1)$ extreme edges between any pair of extreme points. The idea is to begin at one extreme point (the "initial" one) and to pivot in order to obtain a tree T whose signature is equal to that of the other extreme point (the "destination"): for then, by the lemma, T has as its extreme point the desired one.

Let $\underset{\sim}{a}$ be the signature of the current tree T (e.g., the initial one), and $\underset{\sim}{a}^*$ the signature of the destination extreme point. If $a_i < a_i^*$, i is a *deficit node*. If there are d deficit nodes, m-d is the number of nondeficit nodes. If $a_i > a_i^*$, i is a *surplus* node. The *net deficit* is $\{\sum(a_i^* - a_i); a_i^* > a_i\}$. The method has the property that the number of surplus nodes never increases and within at most m-d pivots the net deficit must decrease by 1.

Choose some surplus node and designate it the *source* s and some deficit node and designate it the *target* t. Pivot on the edge (s,ℓ) incident to the source s that is on the unique path joining s to the target t (see Figure 4). Call Q the set of row nodes of the component of $T \sim (s,\ell)$ that contains t. $s \notin Q$. The degree of some $g \in Q$ increases by 1: either (i) it was not a deficit node of T or (ii) it was.

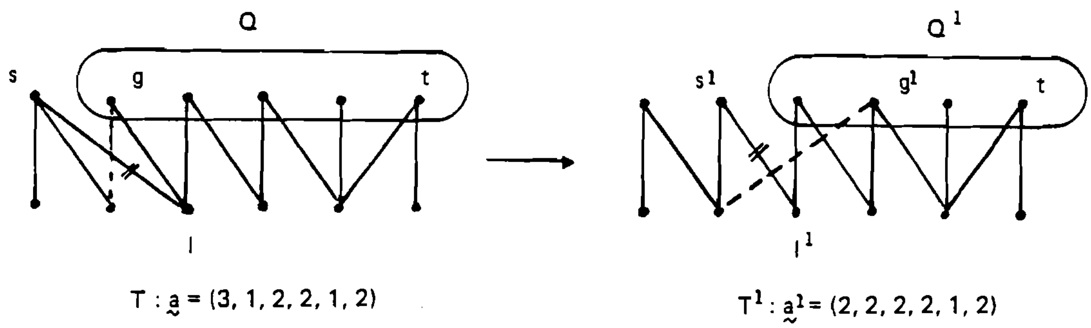

Figure 4.

(i) If not, name it the new source s^1 and repeat: pivot on (s^1,ℓ^1) the edge on the path joining s^1 to t in T^1. The set of row nodes Q^1 of the component $T^1 \sim (s^1,\ell^1)$ containing t belongs to Q but must be smaller: $s^1 \notin Q^1$. Each time a nondeficit node's degree goes up it is immediately brought down and cannot again increase unless the target node is changed. Therefore, in at most m-d pivots a case (ii) must occur.

(ii) The net deficit decreases by 1. If the net deficit is zero, the desired tree is found. Otherwise, name new source and target nodes and continue.

The net deficit can be at most n-1; the number of nondeficit nodes at most m-1: this gives the upper bound (m-1)(n-1) on the number of pivots and so on the distance.

The bound is best possible. Consider the polyhedron $D_{m,n}(\underset{\sim}{c})$ with $c_{ij} = (m-i)(j-1)$. Suppose $i_1 < i_2$ and $j_1 < j_2$: it is impossible to have both (i_1,j_2) and (i_2,j_1) in a tree $T(\underset{\sim}{u},\underset{\sim}{v})$. This implies that the trees T of this $D_{m,n}(\underset{\sim}{c})$ are characterized as all those that have "no crossings" (see Figure 5) -- it being understood that the row and column nodes are drawn in their natural orders. In particular, this polyhedron admits no degeneracy. In pivoting from one tree to a neighbor if node i's degree decreases by 1 then the degree of either node i+1 or node i-1 must increase by 1. Therefore, to decrease the degree of node 1 by 1 and increase that of node m by 1 it takes m-1 steps. This shows that to go from the extreme point with signature (n,1,...,1) to that with (1,...,1,n) it takes (m-1)(n-1) steps. □

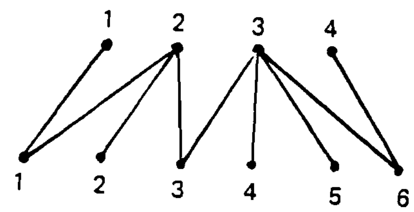

Figure 5. A tree with "no crossings".

An immediate result of the foregoing is:

Theorem 2. *To every integer vector* $\underset{\sim}{a}$, $a_i \geq 1$, $\sum a_i = m+n-1$ *there corresponds an extreme point* $\underset{\sim}{u},\underset{\sim}{v}$.

Proof. Given any such vector $\underset{\sim}{a}^*$ the method of the above proof finds an extreme point having $\underset{\sim}{a}^*$ as its signature. □

So for nondegenerate polyhedra $D_{m,n}(\underset{\sim}{c})$ there is a one-to-one correspondence between extreme points and signatures. This characterization enables one to describe and count all faces of $D_{m,n}(\underset{\sim}{c})$ [3]. It has also motivated a new algorithm for the assignment problem that is guided entirely by the signatures and terminates in at most $(n-1)(n-2)/2$ pivots [2].

Acknowledgment

It is a pleasure to acknowledge the fact that part of this work was done during my visit to I.I.A.S.A. during the summer of 1982.

References

[1] M.L. Balinski, "On two special classes of transportation polytopes," in M.L. Balinski (ed.), *Pivoting and Extensions*, Mathematical Programming Study 1, North-Holland Publishing Co., Amsterdam, (1974), 43-58.

[2] M.L. Balinski, "Signature methods for the assignment problem," research report, Laboratoire d'Econométrie de l'Ecole Polytechnique, (November, 1982).

[3] M.L. Balinski and A. Russakoff, "Faces of dual transportation polyhedra," research report, Laboratoire d'Econométrie de l'Ecole Polytechnique, (November, 1982).

[4] M.L. Balinski and A. Russakoff, "On the assignment polytope," *SIAM Review*, 16(1974), 516-525.

[5] G.B. Dantzig, *Linear Programming and Extensions*, Princeton University Press, Princeton, N.J. (1963).

[6] R.C. Grinold, "The Hirsch conjecture in Leontief substitution systems," working paper, Center for Research in Management Science, University of California at Berkeley, (March, 1970).

[7] V. Klee and D.W. Walkup, "The d-step conjecture for polyhedra of dimension $d < 6$," *Acta Mathematica*, 117(1967), 53-78.

[8] R. Saigal, "A proof of the Hirsch conjecture on the polyhedra of the shortest route problem," *SIAM Journal of Applied Mathematics* 17(1969), 1232-1238.