$\mathcal{H}_2$ optimal vibration control using tuned mass dampers and inertial actuators: comparison of passive and active control effects

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Introduction

Vibration control using dynamic vibration absorbers is an interesting option for reducing vibrations of various types of mechanical structures. The attractiveness of the method is that a relatively small extra mass carefully suspended to the primary structure can lead to significantly reduced responsiveness of the primary structure to forcing. The mass is added to the primary structure via a coupling that can be approximately represented through a spring-dashpot pair.

Depending on whether the excitation of the primary structure is at a single frequency or it comes in a broad range of frequencies, the coupling is designed using different principles. For example, in case of simple harmonic excitations at a single frequency, the stiffness of the absorber coupling can be set such that the resonance frequency of the mass-spring system equals the excitation frequency. If the absorber damping can be set to be very low, then its addition to the primary structure generates virtually a zero in the primary structure input receptance at the excitation frequency. Thus the primary structure gets unresponsive to harmonic forces acting at the absorber mass-spring resonance frequency. This type of absorber is often referred to as vibration neutraliser.

The excitation can also cover a broad frequency range, such as the case is with aerodynamic loading, vehicle road excitations, earthquakes or impacts. In such cases the damping and the stiffness of the absorber could be tuned such that certain vibration metrics are minimised in the frequency band of interest. Devices with such tuned stiffness and damping are known as Tuned Mass Dampers (TMD) as well as damped vibration absorbers. The kinetic energy, which, given the constant mass of the primary structure, is proportional to the average squared vibration velocity, provides valid metrics for vibrations of the primary structure and can be used to optimise the system. This is in fact an $\mathcal{H}_2$ optimisation of the system composed by the TMD and the primary structure whose vibratory kinetic energy is thus minimised [1]. There are also other possible optimality criteria that can be of interest and some of the relevant studies can be found in [2].

Considering their mechanical layout, TMDs are rather similar to inertial actuators. However, in addition to the passive elastic and damping forces in the coupling, an active force can be generated between the inertial actuator proof mass and the primary structure. The force can be made proportional to the primary structure velocity via a fixed negative feedback gain in which case an amount of active

damping can be generated on the primary structure [3]. This is an attractive option as the active damping force is proportional to the absolute velocity of the primary structure only, instead of being proportional to the relative velocity between the primary structure and the proof mass of the inertial actuator. In other words, a force can be exerted which is proportional to the primary structure absolute velocity approximated by the integrated output of an accelerometer attached to the primary structure. This is a more attractive option as the principal concern could be vibrations of the primary structure rather than relative vibrations between the absorber and the primary structure.

This study is focused onto the described active damping approach where a damped inertial actuator is added to an otherwise undamped primary structure and its force generator is driven with a signal proportional to the primary structure absolute velocity in order to create an additional active damping effect. It is shown that there is an $\mathcal{H}_2$ optimal combination of the passive and the active damping which, when employed, minimise the kinetic energy of the primary structure. The passive and the active damping ratios are calculated in the closed form. Finally, the kinetic energy of the primary structure under $\mathcal{H}_2$ optimal active control is compared to that under $\mathcal{H}_2$ optimal passive control using tuned mass dampers having the same proof mass. The comparison of the two control effects reveals that the active control can significantly outperform the passive control provided that the resonance frequency of the inertial actuator is made very low. In fact, the optimal active control outperforms the optimally tuned TMD having equal mass by an amount which increases with the reduction of squared resonance frequency of the inertial actuator.

The paper is structured into 4 sections. In Section 2 the model problem studied and the mathematical formulation are given. In Section 3 the $\mathcal{H}_2$ optimal control using TMDs is reviewed, and finally, in Section 4 the $\mathcal{H}_2$ optimal active control parameters using inertial actuators are derived. Section 5, which gives the comparison of the passive and active control effects, is followed by conclusions.

Model problem studied

As stated in the introduction to the paper, two control approaches are studied and compared, passive and active. The passive is an addition of a TMD onto the primary structure, and the active is an addition of an Inertial Actuator (IA) onto the same primary structure with a velocity feedback control loop. The loop includes a vibration velocity sensor mounted onto the primary structure and a force actuator in parallel with the passive mount. The output of the

sensor is amplified by a negative gain and fed back to the actuator. The two control approaches are shown schematically in Fig. 1. As can be seen in Fig. 1, the primary structure is modelled as an undamped, lumped parameter one degree of freedom (dof) system, defined through the mass m_1 and the stiffness k_1. It is excited by the primary excitation force f_p. It is assumed that nothing a priori is known about this force, and throughout the paper it is thus considered to be a white noise random force with the flat spectral amplitude equal to unity.
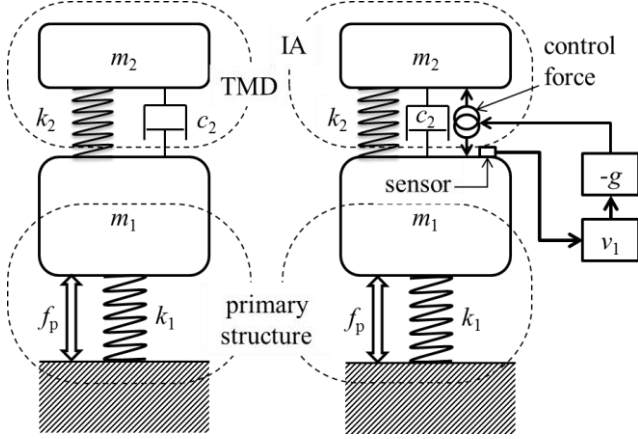

Fig. 1 The schematic of the passive vibration control using Tuned Mass Dampers (TMD - left hand side) and active vibration control using an Inertial Actuator and a velocity feedback loop (IA - right hand side)

As shown in the left hand side of Fig. 1 the TMD of mass m_2 is connected to the primary structure through a spring of stiffness k_2 and a dashpot with a damping coefficient c_2. On the other hand, the primary structure can be equipped with an inertial actuator, as shown on the right hand side of Fig. 1. In this case, in addition to the mass, spring and the dashpot, the inertial actuator has a reactive control force between the masses m_1 and m_2 which is proportional to the primary structure velocity v_1 through a feedback gain $-g$. Thus the feedback gain has the dimension of damping that is, Ns/m, and the feedback loop can thus deliver an active damping onto the primary structure in addition to the passive damping which is realised through the dashpot c_2. The following is assumed throughout this paper: the primary structure properties m_1 and k_1 are fixed, and the mass m_2, which in fact is added to the primary structure merely to control vibration, is constrained by requirements on the total weight of the structure and can also be considered as fixed. Therefore, the remaining parameters available for the optimisation are the spring stiffness and the damping coefficient (k_2 and c_2) in case that the tuned mass damper is used. On the other hand, if the active damping system with an inertial actuator is used then the parameters to optimise are the spring stiffness, the damping coefficient and the feedback gain (k_2, c_2 and g).

$\mathcal{H}_2$ optimal control using Tuned Mass Dampers

In order to generalise the study, non-dimensional parameters are introduced first.

$$\mu = \frac{m_2}{m_1}, \tag{1}$$

$$f = \frac{\Omega_2}{\Omega_1}, \tag{2}$$

$$\eta = \frac{c_2}{c_{2\text{crit}}}, \tag{3}$$

where $\Omega_1 = \sqrt{k_1/m_1}$ is the resonance frequency of the primary structure, $\Omega_2 = \sqrt{k_2/m_2}$ is the resonance frequency of the TMD, and $c_{2\text{crit}} = 2\sqrt{k_2 m_2}$ is the critical damping of the TMD. Thus the three non-dimensional parameters are the mass ratio μ, the frequency ratio f, and the damping ratio η such that the $\mathcal{H}_2$ norm of the primary structure velocity takes the form [1]:

$$I_4^{\text{TMD}} = \pi \frac{1 + f^4 \mu - 2f^2 + f^4 + 4\eta^2 f^2}{2\sqrt{k_1 m_1^3} \eta \mu f}. \tag{4}$$

As can be seen in Eq. (4) the part $\sqrt{k_1 m_1^3}$ determines the scale of the problem, and the remaining part is a varying one and can be minimised through adjusting the damping ratio η and the frequency ratio f. Now the expression in Eq. (4) can be partially differentiated with respect to η and f, the partial derivatives equated to zero, and solved for η and f. Using this procedure, the optimal damping and frequency ratios are obtained as [1]:

$$\eta_{opt,f}^{TMD} = \frac{\sqrt{\mu}}{2}, \tag{5}$$

$$f_{opt}^{TMD} = \frac{1}{\sqrt{\mu+1}}, \tag{6}$$

whereas the $\mathcal{H}_2$ norm of the velocity of the primary structure under the optimal setting is [1]:

$$I_{4,opt}^{\text{TMD}} = \frac{2\pi}{\sqrt{k_1 m_1^3 \mu(\mu+1)}}. \tag{7}$$

$\mathcal{H}_2$ optimal control using inertial actuators

In this section the focus is put onto the $\mathcal{H}_2$ optimal active control using inertial actuators, and the optimal control parameters are derived in the closed form. Again, non-dimensional parameters are included which help to generalise the study. Besides to those previously defined in Eqs. (1) to (3) an additional one related to the feedback gain g is introduced:

$$\zeta = \frac{g}{c_2}, \tag{8}$$

which defines the ratio between the active and passive damping coefficients. The $\mathcal{H}_2$ norm of the primary structure velocity is then given by:

$$I_4^{\mathrm{IA}} = \frac{\pi\left(1+\left(\mu\zeta+1+\mu\right)f^4+\left(-\mu\zeta+4\eta^2-2\right)f^2\right)}{2\sqrt{k_1 m_1^{\,3}}\,\eta\mu f\left(\zeta+1-\zeta\left(\mu\zeta+1+\mu\right)f^2\right)} \tag{9}$$

The right hand side in Eq. (9) can be partially differentiated with respect to the passive and active damping ratios and the numerators of the two derivatives equated to zero. These operations yield a set of equations:

$$1+\left(\mu\zeta+\mu+1\right)f^4+\left(-4\eta^2-2-\mu\zeta\right)f^2=0 \tag{10}$$

$$\begin{aligned}
&\left[\left(\zeta+1\right)\mu+1\right]^2 f^6+\\
&+\left[\begin{array}{l}-\mu^2\zeta^2+\\+\left(-4\zeta-2+8\zeta\eta^2+4\eta^2\right)\mu-3+4\eta^2\end{array}\right]f^4+\\
&+\left(2\mu\zeta-4\eta^2+3\right)f^2-1=0
\end{aligned} \tag{11}$$

If the system defined in Eqs. (10) and (11) is solved for η and ζ, the optimal passive and active damping coefficients can be calculated as follows:

$$\eta_{opt}^{\mathrm{IA}} = \frac{\sqrt{6\left[\left(\mu+1\right)f^4+\left(\mu-2\right)f^2+\sqrt{p}+1\right]}}{12f}, \tag{12}$$

$$\zeta_{opt}^{\mathrm{IA}} = \frac{\sqrt{p}-5-\left(5+5\mu\right)f^4+\left(10+\mu\right)f^2}{6\mu f^2\left(f^2-1\right)}, \tag{13}$$

where p is a substitution to shorten the expressions:

$$\begin{aligned}
p = &1+\left(\mu+1\right)^2 f^8+\left(14\mu^2-2\mu-4\right)f^6+\\
&+\left(\mu^2-2\mu+6\right)f^4+\left(2\mu-4\right)f^2
\end{aligned} \tag{14}$$

The $\mathcal{H}_2$ norm under optimal setting can be expressed as:

$$I_{4,opt}^{\mathrm{IA}} = \frac{3\pi\sqrt{6}f^2\left(f+1\right)^2\left(f-1\right)^2\sqrt{\begin{array}{l}\left(\mu+1\right)f^4+\\+\left(\mu-2\right)f^2+\\+p+1\end{array}}}{\sqrt{k_1 m_1^{\,3}}\left\{\begin{array}{l}\left(\mu+1\right)^2 f^8-\left(4+2\mu+10\mu^2\right)f^6+\\+\left[\mu^2+\left(p-2\right)\mu+6+p\right]f^4+\\+\left(\mu-2\right)\left(p+2\right)f^2+p+1\end{array}\right\}} \tag{15}$$

As can be seen in Eq. (15) the optimal active and passive damping ratios are still a function of the frequency ratio f. In other words, the optimisation has been performed without considering the frequency ratio as an optimisation parameter. However, as shown in Fig. 2, where the $\mathcal{H}_2$ norm defined in Eq. (15) is plotted against the mass and the frequency ratio (with the unit scaling factor, $\sqrt{k_1 m_1^{\,3}}=1$), there is no optimal

frequency ratio. On the contrary, there is a frequency ratio that is to be avoided, especially for small mass ratios, where the kinetic energy under optimal setting increases. This frequency ratio is about one. The lowest kinetic energy levels are in the range where the frequency ratio is below one. In this range the trend is that a decrease in the kinetic energy comes with a decrease in the frequency ratio. The mass ratio, as discussed earlier is also well below unity for a lightweight inertial actuator. Thus, the lower the frequency ratio, the better is the reduction in kinetic energy. The lower limit on the frequency ratio can be imposed by practical problems related to designing very compliant springs k_2 combined with small size of the actuator [5].
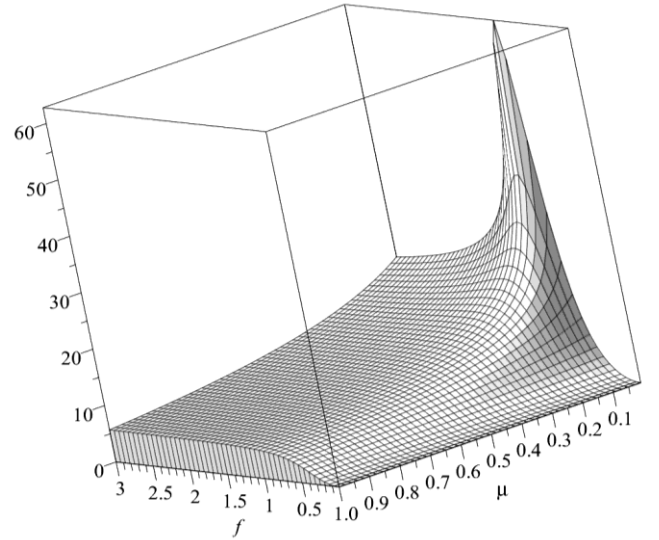

Fig. 2 The primary structure velocity $\mathcal{H}_2$ norm under optimal active and passive damping ratios, plotted against the mass ratio and the frequency ratio. Unit mass and stiffness of the primary structure are assumed.

To conclude, the frequency ratio f should be set to a values as small as possible. In the following section, the $\mathcal{H}_2$ optimal vibration control performance using an Inertial Actuator is compared to that of using a Tuned Mass Damper having the same mass. It is assumed that the frequency ratio for the active approach can be set to a value below one, and the results are expressed as a function of a small frequency ratio f.

Comparison of the control effects using Tuned Mass Dampers and Inertial Actuators

It is now possible to compare the vibration control effects obtained with Inertial Actuators to those obtained with Tuned Mass Dampers. In order to do that, the Control Performance Ratio is defined as:

$$CPR = \frac{I_{4,opt}^{\mathrm{IA}}}{I_{4,opt}^{\mathrm{TMD}}}, \tag{16}$$

which, taking into account $\mu \ll 1$ can be developed into Taylor series around $f=0$ which yields:

$$CPR = (3\sqrt{3})/2f^2\sqrt{(\mu(\mu+1))} + O(f^4) , \qquad (17)$$

Thus, the Control Performance Ratio expressed in dBs is:

$$CPR \approx 4,15 + 5\log_{10}[\mu(\mu+1)] + 20\log_{10} f , \qquad (18)$$

It is possible to see from Eq. (18) that halving the frequency ratio gives an extra 6 dB vibration reduction over the optimally tuned passive absorber, and decimating it gives an extra 20 dB reduction. This is illustrated in Fig. 3, where the control performance ratio is plotted against the frequency ratio for various mass ratios.
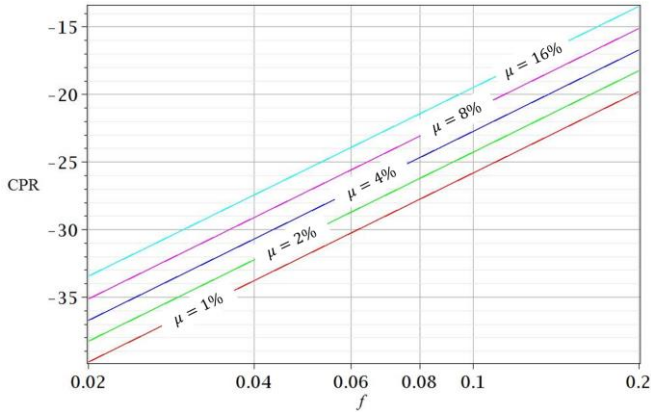

Fig. 3 The ratio of the control performance (CPR) between $\mathcal{H}2$ optimally tuned Inertial Actuators and TMDs for different mass ratios, as a function of the frequency ratio.

It can be noted that the improvement achieved by using active control over the passive control is more evident for smaller added mass of the device.

Finally the two control approaches are compared on an example system, and the effects of adding either an optimally tuned TMD or an optimally tuned IA onto an undamped primary structure are illustrated. The example system has a unit mass of the primary structure, the resonance frequency of the primary structure of 100 s^{-1} and the absorber mass of 10 per cent of the primary mass ($\mu = 0.1$). Fig. 4 shows the amplitude of primary structure driving point mobility plotted against frequency for the two cases.
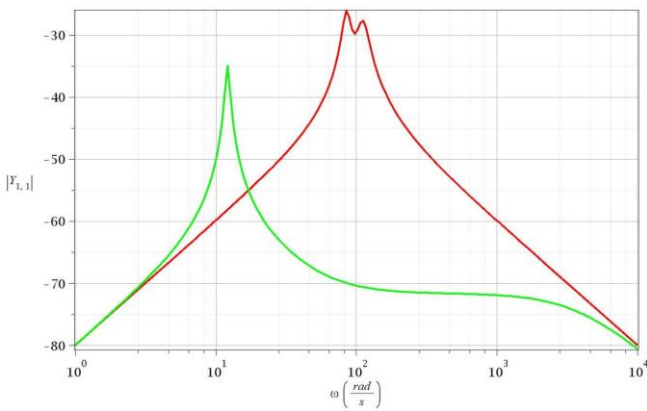

Fig. 4 The amplitude of the primary structure driving point mobility plotted against frequency for two cases. Red line is for an $\mathcal{H}_2$ optimally tuned mass damper, and the green line is for an optimally tuned inertial actuator having the resonance frequency of 1/10 the resonance frequency of the primary structure.

It can be seen that the amplitude of driving point mobility in case of the passive control is higher than the it amplitude of driving point mobility in case of the active control at all frequencies except around the resonance frequency of the inertial actuator (in this case at about 10 Hz). This is due to the fact that the active control generates control spillover at frequencies around the actuator resonance. Thus the optimal active control is a trade-off between the vibration reduction at around the resonance frequency of the actuator and around the resonance frequency of the primary structure. This trade-off however can be achieved with significant improvement in vibration reduction over the passive control approach.

Conclusions

The optimal tuning parameters for the active control using inertial actuators are derived. These are the passive damping of the actuator dashpot and the active damping determined through the velocity feedback gain. There is no optimal resonance frequency for the active control using inertial actuators; instead it should be as below the resonance frequency of the primary structure as practically possible. Depending on how low the frequency ratio can be, the active control can outperform the passive control for the same added mass. Halving the frequency ratio gives an extra 6 dB vibration reduction, decimating it gives an extra 20 dB reduction. The improvement over the passive control is more evident for smaller added mass. It is noted that the optimal active control is a trade-off between the vibration reduction at around the resonance frequency of the actuator and around the resonance frequency of the primary structure.

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