ASSET ALLOCATION IN WEALTH MANAGEMENT USING STOCHASTIC MODELS

by

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Declaration

I declare that ASSET ALLOCATION IN WEALTH MANAGEMENT USING STOCHASTIC MODELS is my own work and that all the sources that I have used or quoted have been indicated and acknowledged by means of complete references.

I further declare that I have not previously submitted this work, or part of it, for examination at Unisa for another qualification or at any other higher education institution.

Signature: . . . .
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Date: 28th February 2016

Abstract

Modern financial asset pricing theory is a broad, and at times, complex field. The literature review in this study covers many of the asset pricing techniques including factor models, random walk models, correlation models, Bayesian methods, autoregressive models, moment-matching models, stochastic jumps and mean reversion models. An important topic in finance is portfolio optimisation with respect to risk and reward such as the mean variance optimisation introduced by Markowitz (1952). This study covers optimisation techniques such as single period mean variance optimisation, optimisation with risk aversion, multi-period stochastic programs, two-fund separation theory, downside optimisation techniques and multi-period optimisation such as the Bellman dynamic programming model.

The question asked in this study is, in the context of investing for South African individuals in a multi-asset portfolio, whether an active investment strategy is significantly different from a passive investment strategy. The passive strategy is built using stochastic programming with moment matching methods for non-Gaussian asset class distributions. The strategy is optimised in a framework using a downside risk metric, the conditional variance at risk. The active strategy is built with forward forecasts for asset classes using the time-varying transitional-probability Markov regime switching model. The active portfolio is finalised by a dynamic optimisation using a two-stage stochastic programme with recourse, which is solved as a large linear program. A hypothesis test is used to establish whether the results of two strategies are statistically different. The performance of the strategies are also reviewed relative to multi-asset peer rankings. Lastly, we consider whether the findings reveal information on the degree of efficiency in the market place for multi-asset investments for the South African investor.

Key terms: Asset allocation, active and passive investment strategy, multi-period portfolio optimisation, modern asset pricing, stochastic processes, conditional value at risk, time-varying transitional-probability Markov regime switching model, inter-temporal mean-reversion, integrated stochastic liability, two-stage problems with recourse, dynamic stochastic programme.

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Part I

Introduction and context

1 Introduction

1.1 Introduction

Passive and active investment strategies are central themes in the investment industry. Fama (1970, p. 416) concluded that "the evidence in support of the efficient markets models is extensive, and (somewhat uniquely in economics) contradictory evidence is sparse." Fabozzi et al. (2007b) explain that active investment management in markets where prices do not reflect all information (weak form market efficiency) can attain returns which outperform the return of the market. In contrast, in markets where prices reflect all information (strong form market efficiency), outperforming returns in the market is not possible once cost is taken into consideration.

Investment management strategies are considered active where, using the equities market as an example, a stock selection strategy is used to build a portfolio with net returns in excess of a market related benchmark, such as the Johannesburg Stock Exchange Allshare index. Achieving this with regularity is difficult. An investment manager must have the skill to harvest excess returns relative to a benchmark index that can be replicated passively at low cost. The active versus passive debate is often applied within an asset class and, as Fabozzi et al. above remind us, can be explained by the efficiency of that market. Notwithstanding, this study focuses on the multi-asset market, testing both passive and active asset allocation strategies. In this study the active asset allocation strategy is one that strategically varies over time, whereas the passive strategy is held constant and does not require any specific skill to manage. Both strategies are allocated into low cost funds that track benchmark indexes for each asset class. Any difference in investment performance will highlight the effectiveness of the respective asset allocation technique and potentially give the reader insight into the efficiency of the multi-asset market in South Africa. Can an active quantitative asset allocation technique achieve positive net returns greater than that of a passive quantitative technique or that of the market?

Fabozzi et al. (2007b) succinctly introduce a set of prominent quantitative techniques in asset allocation, positing that "before Markowitz's seminal article, the finance literature had treated the interplay between risk and return in a casual manner" (Fabozzi et al., 2007b, p. 2). Markowitz made a significant contribution to financial theory, by way of a formal framework published in the Journal of Finance in 1952 (Markowitz, 1952). His approach became known as the Modern Portfolio Theory (MPT). The original framework quantitatively expresses an optimal allocation of the investor's assets by making a trade-off between economic cost and opportunity. As is typical in a finance environment, the framework is based on an expression of financial risk and return, which is then optimised against a function of individual risk appetite.

The MPT makes use of the Mean Variance Optimisation (MVO) framework. In this approach, a core assumption is that an individual's utility function is quadratic, representing an Absolute Risk Aversion (ARA). Another key assumption is that return distributions are normal which, as we will see in this study, is not always the case. MVO per definition does not account for the changes in higher-order moments, namely skewness and kurtosis. The classic optimisation approach is a one-period model with no option to rebalance. MVO is often challenged for how well the assumptions reflect reality given that returns are not normally distributed and individuals do not hold identical expressions of utility. Scherer (2007) concludes that the appropriate use of the MVO should be one of empirical confirmation of the assumptions rather than one of dichotomous theory.

The MPT framework has its merits and much research has focussed on testing current approaches to solving the problem of portfolio choice and asset allocation. This study includes writings focused in the following areas relating to portfolio choice: utility functions; returns forecasting; volatility measures; Bayesian statistics applied to asset management; stochastic processes in asset pricing; downside risk measurements; investor behavioural studies; econometric time series methods; dynamic correlation; optimisation techniques and inter-temporal asset allocation; continuous time asset pricing models; asset-liability management; dynamic programming; multi-stage optimisation and stochastic programming. An area which has received special focus, in the aftermath of the financial crisis in 2008, is the method of quantifying risk measures (Scherer, 2007; Fabozzi et al., 2007b).

1.2 Aims and objectives of the study

This study is a quantitative test of passive and active investment philosophies and investigates high net worth private entities' invested assets where longer term contingent liabilities like consumption can be considered in the investment decision. This study does not involve leverage, thus portfolios are constrained to long only market positions. Portfolio rebalancing is yearly. Whilst the aim of the study is not to conclude on the completeness or efficiency of markets, reliance is placed on the economic theory in order to structure the use of different quantitative methods in portfolio construction and asset allocation. Establishing and applying these quantitative methods rooted in an economic theory is, however, central to the study.

Hens and Rieger (2010, p. 317) state that "a discrepancy between real life on financial markets and the theory is always an intriguing observation that helps us to improve at least one of them– either the model or the real life...".

As highlighted in the introduction, the debate of active and passive investment philosophy is often applied within a particular asset class. The application to the South African multi-asset market may shed light on the question of market efficiency and whether contemporary quantitative methods are effective for active asset allocation.

This study starts with a literature survey of financial asset pricing and classic methods for establishing optimal portfolios. Where necessary, the merits of these approaches are discussed. Contemporaneous approaches to portfolio construction, asset pricing and risk measures are then introduced and discussed. Candidate models are selected and are fully described, formalised and tested using an out of time sample data set. The aim is to discuss and conclude whether active quantitative portfolio strategies can capture a significantly different net return compared to passive optimal portfolio strategies. Both approaches are also compared to an equally weighted portfolio and a peer index representing the South African mutual funds over the period under consideration.

Goodwin (1998) reports that the information ratio is routinely used in investment management to gauge the performance of active investment managers. Goodwin (1998) shows the relationship between the information ratio and the t-stat used in a t-test as follows: $t - stat = \sqrt{T}(IR)$, where T is the number of years, and IR is the information ratio. The t-stat is a scaled information ratio. This study makes use of Welch's t-test, which is an unpaired t-test where the samples have unequal variances, to formally test whether there is a difference in returns between the proposed portfolio strategies. Florian Haberfelner (2013) uses the same approach to test whether the mean returns of a simulated portfolio is significantly different from the mean returns of comparable UCITS funds. The hypothesis states that an active investment management strategy, represented by the candidate active model, has higher real returns than the passive investment management strategy, represented by the candidate passive model. The testing framework uses a t-test to determine whether the performance of the active strategy is significantly different from the performance of the passive strategy. The hypothesis test is as follows:

- *The Null Hypothesis* states that the portfolio returns of the active investment strategy are not significantly different from the passive investment strategy;
- *Alternative Hypothesis* states that the portfolio returns of the active investment strategy are significantly different than that of the passive portfolio strategy.

The following asset classes, otherwise referred to as the asset class universe, are proposed for inclusion in the portfolio: equity; bonds; bills and commodities. With the exception of commodities which are represented globally only, each asset class is represented both locally and in the United States of America (US). The data is described in more detail where the data proxies and data sources used to represent the above asset classes are defined. The modelling reference period for the in-sample data for this study is from January 1900 up to and including December 2007. The out-of-sample period is from January 2008 up to and including December 2014.

1.3 Synopsis

The following synopsis is an overview that will highlight the theory in the proceeding literature review relevant to the models developed in this study. The literature review starts, in Section 2, by exploring the origins of utility and important probability theories, including Borel's law of large numbers. The concept of expected value is then explained. This is a key concept which is used in both the active and passive models.

Section 3 deals with modern asset pricing, which is a broad field of research. This study explores both discrete and continuous methods of pricing assets. The review explores financial models of risk, return and correlation as these parameters are central to the thesis. Section 3 is an important part that explains Fleishman, Vale and Maurelli's multivariate moment matching method that form a key part of the passive model. Stochastic processes are introduced, where the discrete time stochastic processes are used in the active model.

Section 4 introduces the all important theory of portfolio choice by Markowitz (1952). This theory formalises an optimisation programme for solving the asset weights in a portfolio, given a trade off between risk and return. The study goes further in assessing the appropriateness of the underlying assumptions. Based on these issues, we redefine an optimization programme that makes use of a coherent downside measure of risk and this is used in the passive model.

Section 4.5 details advances in financial econometric techniques, with focus on switching models. The most important of these econometric models is the time varying transitional probability Markov switching model, as this model is used to forecast asset prices as part of the active model.

Section 5 progresses to the multi-period portfolio choice models. The advantages of discrete stochastic portfolio models or stochastic programmes are explained in a lot of detail. This study formalises a multi-stage optimization model called the Two-stage recourse model. This model is utilized as the multi-stage portfolio model in the active model.

Part II

Literature survey

2 Overview of Modern Portfolio Theory: selected topics

This study starts by building the theory for an advanced quantitative analysis by briefly describing the mean variance optimisation (MVO), Capital Asset Pricing Model (CAPM), factor models, the two-fund separation theorem and econometric techniques adopted into portfolio management. This study does not have the aim to establish the theory, but rather makes use of the theory. For this purpose, descriptions are kept brief; more detailed and intuitive explanation can be in Fabozzi et al. (2007b), Fabozzi et al. (2010) and Fabozzi and Markowitz (2011). The original works are found in Markowitz (1952), Sharpe (1964), Lintner (1961), Treynor (1999), French (2003), Fama (1970), Samuelson (1969) and Tobin (1958).

2.1 Utility and choice theory

2.2 The origins of utility

Utility is an important notion in economics and many financial models including MVO. The theory started with the probability theory. Pierre de Fermat, Blaise Pascal and Christiaan Huygens developed the concept of probability in the 17th century. Probability theory seeks to describe seemingly random events and identify patterns in these events. The theories and models describing these random events are often used to predict future random events. Borel's law of large numbers is one interpretation where the probability of the event taking place is close to the average proportion of each event occurring over a large number of trials. Another result of probability theory, which is very important in portfolio theory, is the Central Limit Theorem. This theorem describes the average values, but also formalises the dispersion around the average or mean. The theory states that the dispersion of observed values can be described by way of a normal distribution. These concepts are both used in formulating the theory of choice between risky alternatives.

Until the middle of the 20th century, expected value was widely accepted as the basis for the theory of decisions under risk. Expected value can be explained by way of an experiment based on flipping coins. Each coin flip can be referred to as a lottery (A), with a chance or probability (p) of winning or losing (if it lands on heads you win, whereas tails you lose). There is also a value (x) attached to winning or losing (in this case the value of the coin). So how can one assess the value of the game? When using expected value, this is a simple calculation by taking the average value of each outcome by the probability of realising that outcome, after a large number of lotteries. In the case of the coin flipping, this is the fifty percent probability of each event multiplied by the value of each coin flipped. The expected value can be expressed as $E(A) = \sum x_i p_i$.

In 1738 Daniel Bernoulli challenged the idea that rational decisions should only be made based on expected return and he proposed that expected value may not be the best principle for choice and that a gain or loss of money may mean different things to different individuals, dependant on factors such as wealth. He explained his argument by making use of the St. Petersburg paradox (Jensen, 2012). Expected value may be better expressed in terms of utility, which may then be expressed in terms of monetary value. Jensen (2012, p. 409) writes "In the translation of Bernoulli's paper in Econometrica from 1954, *emolumentum* was translated as 'utility'." Jensen (2012, p. 409) explains that "*emolumentum*" has meaning similar to advantage or benefit. The value of an item to an individual is thus the degree to which it is in his interest, or how good it is for him. Bernoulli suggests the maxim that an individual should evaluate a gamble by its *emolumentum* medium, or its expected goodness to him (Hens and Rieger, 2010). Bernoulli's work forms the basis for what is known as Expected Utility Theory (EUT) and allows one to consider a utility function that assigns corresponding utility to increasing wealth levels. Choice is established by determining the maximum expected value of the utility (Hens and Rieger, 2010).

2.3 Utility functions and risk aversion coefficients

The Expected Utility Theory (EUT), allows one to set up functions for decision problems faced by a rational person; these are called utility functions. In reality each investor has her own utility and this needs to be matched to a "suitable" utility function from what are infinite possibilities. The tractability of a problem does depend on the form of the utility function. Once the representative utility function has been selected, it represents the rational investor's decisions and is maximised to establish the optimal utility. Naturally the choice of utility function is very important in quantitative portfolio management. What follows is the intuition of concepts of risk aversion and the main representatives of utility functions used in investment management. This allows us to establish the theoretical links to the mathematical formulations used in the MVO and alternative formulations optimisations. (Hens and Rieger, 2010; Scherer and Winston, 2012)

2.4 Graphical illustration of utility functions

A common requirement of a utility function proposed by Bernoulli's EUT, is that it adequately describes the investor's attitude towards risk. We illustrate three forms of utility curves below which would represent three very different investor attitudes to risk taken at varying wealth levels.

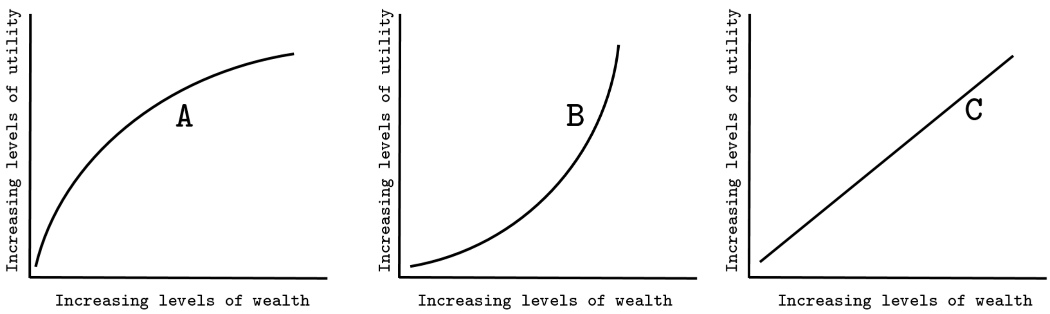


Figure 1: Utility Functions: Investor attitudes toward risk

In Figure 1, line A can represent an investor that is said to have a risk appetite which increases at a decreasing rate with increasing levels of wealth. Line B represents an investor having marginal increasing tolerance for risk with increasing wealth levels. Line C represents an investor who is said to be risk neutral in which the risk appetite increases at a constant rate with increasing levels of wealth. (Fabozzi et al., 2007b). An agreeable and tractable utility function in economics should be increasing and concave down for increasing levels of wealth (Varian, 1999).

2.4.1 Risk aversion coefficient

The standard risk aversion measure, or Arrow Pratt measure, or the measure of absolute risk aversion at levels of wealth represented by x:

$$r_A(x) = -\frac{u^{''}(x)}{u^{'}(x)},$$

The equivalent relative risk aversion measure is a follows:

$$r_B(x) = -\frac{xu^{''}(x)}{u^{'}(x)}.$$

Back (2010) explains that an investor is risk averse if her utility function is increasing for increasing levels of wealth and that function is concave. A risk averse investor is said to prefer a sure bet over a gamble (Back, 2010). This requires that a utility function for a risk averse investor will have a first derivative $u' > 0$ and the second derivative $u'' \leq 0$. The risk averse investor has an Arrow-Pratt measure that is decreasing for increasing levels of wealth (Varian, 1999).

The second derivative of the investor's utility curve measures the concavity of the function. The larger the value of r_B, the higher the level of the investor's relative risk aversion, or the curvature level of the utility function. Two popular forms of utility which are classified as hyperbolic absolute risk aversion (HARA) functions and constant absolute risk aversion (CARA) utility functions, which determine that as wealth increases the investor preference is to hold the same rand value in risky assets (Fabozzi et al., 2007b; Back, 2010; Amenc et al., 2011a). As the CARA utility function is characterised by the absence of wealth effects, economists consider it to be an unreasonable assumption for a utility function (Fabozzi et al., 2007b; Back, 2010; Amenc et al., 2011a).

The second group is constant relative risk aversion (CRRA) utility functions. CRRA functions are structured so that as wealth increases the investor would prefer to hold the same percentage of wealth in risky assets (Fabozzi et al., 2007b). These functions are therefore dependant on wealth levels. Examples of these utility functions include the logarithmic and the power utility function. Individuals can be said to be CRRA where relative risk aversion is the same at all levels of wealth (Fabozzi et al., 2007b; Back, 2010; Amenc et al., 2011a).

The more commonly used utility functions include Linear, Quadratic (MVO), Power and Logarithmic utility functions. It is common practice that the axioms of Von Neumann-Morgenstern are satisfied, implying that one's utility function is both concave and increasing. Fabozzi et al. (2011a) concur that selecting the correct utility function is difficult, but that this difficulty does not make the theory irrelevant. The portfolio selection problem depends both on the feasible portfolios and the utility function (Fabozzi et al., 2007b; Back, 2010; Amenc et al., 2011a).

3 Modern asset pricing

Fabozzi et al. (2007a) explain that an asset pricing model is set up to reproduce real market asset prices and this is done in a mathematical formulation. As MacKinlay et al. (1997) describe it, many prominent mathematicians and scientists have attempted to gain a competitive edge in the financial markets by applying significant skills and time in developing asset forecasting models. Fabozzi et al. (2007b) cite the origins of quantitative finance as being linked to the works of Theile (1880), Bachelier (1900) and Einstein (1905). They all wrote independently, knowing nothing of each other's work. Bachelier was referred to by many as the first quant as his doctoral thesis was the first to use advanced mathematics in finance.

An anecdote from Fabozzi et al. (2007a), reveals an important and highly relevant problem for modellers of economic phenomena. Mitchel Waldrop writes in his book *Complexity*, of a physicist attending a seminar in 1972, where economists and physicists presented on topics relating to *The economy as an evolving complex system*. He was surprised at the extent to which economists used highly sophisticated mathematics. He posed the question to Kenneth Arrow, 1972 Nobel prize winner in economics, why economists, who suffer the consequences of a lack of data, used such sophisticated mathematics? To which Arrow replied "It is just because we do not have enough data that we use sophisticated mathematics. We have to ensure the logical consistency of our arguments" (Fabozzi et al., 2007a, p. 408). In physics, empirical data is the best guarantee for logical consistency of a theory. If the theories fail empirically, no amount of logical subtlety will improve the theory. As economists, who work with a complex and evolving system with sparse and at times inaccurate data, it is tempting to explain phenomena using "clear reasoning only" (Fabozzi et al., 2007a). To aggravate matters further, high-performance computing is readily available at reasonable pricing. This has allowed for data patterns to be discovered and for models to be built with arbitrary levels of precision. These models can be meaningless, revealing no economic feature at all (Fabozzi et al., 2007a).

Fabozzi et al. (2007b) indicate that quantitative modelling of financial asset prices can broadly be broken down into theoretical models and econometric models. Theoretical models include Arbitrage Pricing Theory (APT), Capital Asset Pricing Model (CAPM) and General Equilibrium Theories (GET). Econometric models include random walk and multi-factor models (Fabozzi et al., 2007b). Any modelling approach has merit. Econometric models are empirically led and often not underpinned by a theoretical basis. Theoretical models, however, can be inflexible, which may lead to inaccuracies in describing the complexity found in asset pricing. Fabozzi et al. (2010) explain that theoretical models like APT and CAPM only consider events and actions over one period. In reality, investors can update their views dynamically and prices are ever-changing. There is potential to use more advanced approaches that deal with the dynamics encountered in financial assets over time.

Fabozzi et al. (2007b) explain that random walk models are based on the assumption of market efficiency, where market returns fluctuate around a mean value. These random fluctuations are viewed as uncertainty. This property of uncertainty when expressed through time is termed a stochastic process and can either be expressed in a discrete time or with continuous time variables (Hull, 2004). Munk (2011) explains that both discrete time models (where all actions can take place at finite points in time) and continuous time models (where actions can take place at any point in time) can capture dynamics of asset prices. Munk (2011) notes that both continuous and discrete methods are used to study real life and theoretical applications; it is reassuring that solutions from these approaches have identical or very similar conclusions.

3.1 Factor models

The law of one price in finance states an asset must have one price regardless of the instrument used in creating that asset (Fabozzi et al., 2007b). This implies that an asset can be synthetically created by a portfolio of assets in a process called replication (Fabozzi et al., 2007b). The law of one price was the basis for the arbitrage pricing theory (APT). APT was developed by Ross (1976) and the implication of this theory led to the design of factor models. Factor models rely on the underlying notion that the risk factors which affect the return of a security will require greater levels of reward. As Fabozzi et al. (2007b) points out, factors determine the return of a security.

Fabozzi et al. (2007b) continues to explain that the effects of rational investors trading upon opportunities, including arbitrage opportunities, will ensure the equilibrium prices are maintained. This implies that investors will only be rewarded for systematic risk and that unsystematic risk will not rewarded, this idea was introduced by Sharpe (1964) and is a key principle in the CAPM. Roll and Ross (1984) explain that a large enough portfolio that is sufficiently diversified will reduce the idiosyncratic risk (unsystematic risk) to zero.

3.1.1 APT formulation

The APT model is set out in Ross (1976) and further explored in Roll and Ross (1984). Following these readings, any asset (for instance a stock) the return R, can be broken down into constituents and expressed in a factor model:

$$E[R] = E + \beta f + \varepsilon \tag{1}$$

where E is the expected return of the security or asset. The return from the systematic factor is denoted as f. The sensitivity of the asset to changes in the systematic factor is denoted as β. Roll and Ross (1980) explain that the final term, e, the noise term, is the variability in returns due to idiosyncratic factors. The error term reduces as diversification increases. Roll and Ross (1984) report that between three and four factors are appropriate to account for the systematic risk. In the case of multi-factor models, the vector form of the APT model can be written as:

$$E[R] = \alpha + \beta F + \varepsilon \tag{2}$$

where

$$R = \begin{bmatrix} R_1 \\ \vdots \\ R_k \end{bmatrix},\ \alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{bmatrix},\ \varepsilon = \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_k \end{bmatrix},\ F = \begin{bmatrix} F_1 \\ \vdots \\ F_k \end{bmatrix},$$

and

$$\beta = \begin{bmatrix} \beta_{11} \cdots \beta_{1k} \\ \vdots \ddots \vdots \\ \beta_{k1} \cdots \beta_{kk} \end{bmatrix}.$$

Ross (1976) shows that in the absence of arbitrage, the following is true:

$$E(R) = R_f + \sum_{k=1}^{K} \beta_k (E(F_k) - R_f). \tag{3}$$

This relationship is referred to as the APT, where $(E(F_k) - R_f)$ is the excess return of the k-th systematic factor over the risk free rate, R_f, and is otherwise known as the price. As described above, the CAPM is a special case of the APT, where the APT has more general assumptions. APT provides the theoretical foundation for an asset pricing model that is extended to include multiple factors. These are known as multi-factor models, of which the well known Fama and French model is an example.

Broadly speaking there are three classes of multi-factor models (Fabozzi et al., 2007a):

- **Statistical factor models** use historical and cross sectional data on stock returns that are fed into a statistical model and tested for factors which significantly explain the observed stock returns. The statistical method of principal component analysis (PCA) is a good method to use with large volumes of data, where data reduction is needed. This could be the case when modelling a large number of stocks. One issue when using PCA is that it is a difficult method to interpret, which is an impediment when used for the purposes of valuation, risk control and economic interpretation. It is for this reason that the next two methods, that allow for greater levels of interpretation and intuition, are used more often.

- **Macroeconomic factor models** also use historical data of stock returns and observable macroeconomic factors. Again the goal of the model is to find factors, or raw descriptors, that persistently explain stock returns. Examples of factors include real interest rates, debt to GDP, interest rates, consumer price inflation, money supply, real business activity, investor confidence and market indices.

- **Fundamental factor models** or characteristic based factor models are the third class of multi-factor model. The most recognised model in this class is the Fama and French model. The original model had three factors, which were *high-minus-low (HML), small-minus-big (SMB)* and *market*. The model offers no explanation as to *why* it performs well, simply that it performs well from 1963−1993. Another well known example is the Barra MSCI model which is commercially available. Typical fundamental factor models include raw descriptors such as price-to-earnings ratio, credit rating, growth rate of earnings and stock liquidity as well as technical factors such as momentum, dividend yields and volatility of return and lastly a descriptor of the company such as size.

Munk (2011) points out that care must be taken in identifying factors. Without much thought, or understanding of the market, factors can easily be identified using historical data and a statistical package. Factor models set up purely through hunting for factors using large quantities of historical data may lead to data snooping. MacKinlay et al. (1997) provide a lot of detail around data snooping and the implications it may have for financial models. Munk (2011) argues that factors should be justified by a fundamentally sound asset pricing model or solid economic foundation. There are many instances in literature of factor models created by using data to identify factors without any real concern for the underlying mechanisms; the best known of these models is that of Fama (1970).

3.2 Random walk models

MacKinlay et al. (1997) categorise random walks into three categories:

- *Random walk I (RW1): Independent identically distributed (IID) increments.* This hypothesis requires that the increments are independent and identically distributed. The increments and any functions of the increments are also non-correlated. It is the simplest of the random walks.

- *Random walk II (RW2): Independent increments.* RW1 is not plausible over longer time periods which requires one to relax the assumptions of RW1. The more general assumptions include processes which have independent but not identically distributed increments. This means that the error term also may display heteroskedasticity (a phenomenon where a period of increased volatility is experienced followed by a period of reduced volatility).

- *Random walk III (RW3)*: The random walk in this category relaxes the assumption of independence. It is the most general category of the random walks. This, the weakest form of random walk, is one of the most often tested and this study utilises this version of RW.

3.3 Dependence and correlation in portfolio management

A standard assumption of a random walk is homoskedasticity or constant variance, which is often observed. Financial time series data features volatility clustering. This is a phenomenon where a period of increased volatility is experienced followed with a period of reduced volatility and is known as heteroskedasticity. Correlation of returns between asset classes is not always static, nor are the joint distributions normally distributed. Asset class returns are known to feature time-varying attributes and asymmetry in the distribution of returns (Thomaidis et al., 2011). The effect of correlation increasing during times of significant economic stress, is referred to as the leverage effect (Buraschi et al., 2010). Ledoit and Wolf (2002) also show that correlation is linked to the business cycle and varies over time. The period which included a run-up to and the financial crisis, from April 2005 to April 2008, was a notable time period in financial data, where the sample correlation of the weekly stock market returns for the Nikkei and the S&P500 was less than 20%. This relationship broke down and showed a significant increase in correlation to 80% by April 2008. This is otherwise referred to as *correlation breakdown* (Buraschi et al., 2010). For portfolio management, an accurate function to account for time-varying dependence between asset classes is key.

Fabozzi et al. (2007a) explains that covariance matrices may be viewed as a set of variable terms, that are time varying and can be forecast. Ledoit and Wolf (2002) argue that the simplest and most common approach, where the correlation is constant over the time period of investigation and is parametrised as a sample correlation matrix. This approach imposes too little structure. Given the maximum likelihood estimation of the matrix it can perform poorly under small samples. Small samples are often the case when working with financial data. Thomaidis et al. (2011) note that modelling expected returns is difficult, which implies that the improvement to portfolio returns can rather be achieved via improvements to the expected correlation matrix. Many studies have set out to construct models that quantify co-movements of financial assets with varying levels of success.

The autoregressive conditional heteroskedasticity model (ARCH) and the generalised ARCH model (GARCH) were developed in the articles by Bollerslev (1986) and Engle (1982) to deal with heteroskedasticity. These contributions to finance are significant and both models have the aim of capturing time-varying volatility. The GARCH models have been further extended to capture time-varying considerations in co-dependence (Rachev, 2007). Below follows a synopsis of the notable models that attempt to solve time varying correlation.

Historical covariance. This method, used to predict time varying covariances, computes the sample covariation between assets using a rolling window. We follow the notation in Thomaidis et al. (2011), where a proxy, H_t, for conditional variance-covariance is obtained at time, t:

$$H_t = (1/L) \sum_{i=t-L}^{t-1} e_i e_i' \tag{4}$$

where L is the length of the rolling window used to define the sample. e_i is the vector of residuals of a vector autoregressive model of the order of one (VAR(1)). A vector autoregressive model is a multivariate extension to an autoregressive model, which is detailed in section 3.5.2. Although this model is simple it has the potential to produce high standard errors and is very sensitive to outliers.

Exponentially moving weighted average model (EMWA). The EMWA model, which was made popular by RiskMetrics©, can assign greater weights to more recent observations in the sample period by using a decay factor. When the decay factor is set to one, the weights are equal and, therefore, the same as the sample correlation matrix. The conditional variance-covariance matrix, H_t, is estimated by the residuals of the mean VAR model in place of returns, as per Thomaidis et al. (2011):

$$H_t = (1 - \lambda)e_{t-1}e_{t-1}' + \lambda H_{t-1} \tag{5}$$

where λ is the decay factor bounded between $0 \leq \lambda \leq 1$. The value of λ is recommended by RiskMetrics© between 0.94 and 0.97. A greater weight will be given to more recent data for the lower values of λ. When λ is set to one, the values are weighted equally. Findings show that the EMWA model shows similar results to these more highly specified multivariate models (Thomaidis et al., 2011). The weakness in this model is the reliance on an arbitrary value assigned to decay, λ. This model is not capable of dealing with the effects of heteroskedasticity and would ensure a completely inappropriate estimate for expected correlation in the event of correlation breakdown.

Constant conditional correlation (CCC). The Bollerslev (1986) model is based on a constant correlation matrix:

$$H_t = D_t R D_t, \quad \text{where } D_t = \text{diag}\{\sqrt{h_{i,t}}\} \tag{6}$$

where R is a time invariant correlation matrix where $\rho_{ii} = 1$. The time varying correlation model is based on the conditional variance generated from a GARCH model, which is set in a diagonal matrix, $h_{i,t}$ and is the square root of the GARCH variance estimates (Bollerslev, 1986; Engle, 1982). It can fall into a univariate GARCH model category that allows for lead/lag relationships and exogenous variables to affect the correlation in a contemporaneous fashion. The basic notion is to hold the correlation constant whilst allowing for the risk (covariance) to vary in time (Thomaidis et al., 2011). The constant correlation is a drawback of the CCC where, as discussed, correlation is known to be time varying with features such as the leverage effect and correlation breakdown.

Dynamic Condition Correlation (DCC). Engle (2002) developed a model to attend to the drawback of constant correlation of the CCC. The DCC correlation matrix, H_t, follows:

$$H_t = D_t R_t D_t, \quad \text{where } D_t = \text{diag}\{\sqrt{h_{i,t}}\} \tag{7}$$

where D_t, follows the CCC above and a time varying condition correlation matrix is represented by $R_t = \{\rho_{ij}\}_t$. Where $h_{i,t}$ is a GARCH process that is detailed as follows:

$$h_{i,t} = \omega_i + \alpha_i \cdot \varepsilon^2_{i,t-1} + \beta_i \cdot h_{i,t-1} \tag{8}$$

where the conditional variance error term is ε_{it} and i indicates the ith equation in the VAR model. Standardised residuals are used in developing the DCC correlation:

$$Q_t = (1 - \theta_1 - \theta_2)\bar{Q} + \theta_1 \eta_{t-1}\eta'_{t-1} + \theta_2 Q_{t-1} \tag{9}$$

and

$$R_t = Q_t^{*-1} Q_t Q_t^{*-1} \tag{10}$$

where the standardised residuals of the unconditional covariance is $\bar{Q} = E[\eta\eta'_t]$ and $Q^*_t = (\text{diag}(Q_t))^{-1/2}$ is a matrix made up of the square root of elements of Q_t. The scalar parameters, θ_1 and θ_2, will introduce the shocks to the process and will be calibrated on previous shocks. When $\theta_1 = \theta_2 = 0$ then the DCC is the same as the CCC model. The limitation of that DCC is the inability of the model to capture effects of asymmetry in the conditional correlations.

Asymmetric Dynamic Condition Correlation (ADCC). The ADCC was developed by Cappiello et al. (2006) and is an extension of the DCC, that attends to effects of asymmetry in conditional correlations. Equation 9 is extended to allow for asymmetry as follows:

$$Q_t = (1 - \theta_1 - \theta_2)\bar{Q} - g\bar{\Xi} + \theta_1 \eta_{t-1}\eta^T_{t-1} + \theta_2 Q_{t-1} + g\xi_{t-1}\xi^T_{t-1} \tag{11}$$

where g is a parameter that governs the asymmetric effects into the model. When g is zero the model is reduced to a DCC; $g > 0$ implies that the positive shocks are not as significant as the negative shocks. ξ_t is a function returning 1 for a negative signal and 0 for a positive signal. The unconditional covariance matrix of ξ_t is represented by $\bar{\Xi} = E[\xi_t \xi^T_t]$.

The difference in risk adjusted returns can be seen between a set of portfolios constructed using different correlation models based on the same universe of assets that included bonds, stocks, real estate, commodities, and hedge funds from January 1996 to December 2008. The testing includes the equally weighted, historical, CCC, DCC, ADCC and EMWA models. Thomaidis et al. (2011) show that the portfolio built with time-varying asymmetric dynamic conditional correlation (ADCC) had the highest Sharpe ratio. The portfolio built using the ADCC attained a Sharpe ratio of 0.66. The portfolio built on historical correlation matrices attained a Sharpe ratio of 0.138. Interestingly, only the DCC and ADCC attained a Sharpe ratio higher than the equally weighted portfolio, which attained a Sharpe ratio of 0.27.

Other models Multivariate GARCH models have been used extensively, and the literature on approaches to GARCH models is quite voluminous (Bauwens et al. (2006) provides a review of (G)ARCH type modelling). There is evidence that volatilities are synchronised at times and a multivariate modelling framework gives recognition to this feature, more so than a univariate one. Fabozzi et al. (2007b) point out that a time-dependent variance-covariance matrix can be successfully modelled by an extension of a GARCH model. The parametrisation of these models is, however, difficult as the data requirements are significant and, therefore, this approach is often not practical.

Buraschi et al. (2010) find that a non-linear multivariate model in a stochastic framework can handle both the volatility and correlation dynamics. Buraschi et al. (2010) make use of the Wishart distribution within a stochastic process in the process of solving an inter-temporal portfolio problem. Buraschi et al. (2010) posits that there are only a few models other than the highly specified models using a Wishart distribution in stochastic processes, for dealing with mean reversion, persistence in volatilities and correlation with leverage effects. These include the use of stochastic jumps using a Poisson distribution or making use of a Markov regime switching model as considered in Ang and Bekaert (2002a). Markov regime switching models are considered a type of VAR model.

This study is primarily concerned with the development of a dependence measure as it pertains to portfolio management of assets (and liabilities). An interesting question central to this study concerns the use of time-varying methods for quantifying correlation as part of the greater optimal portfolio choice model. Recent work by Thupayagale and Molalapata (2002) conclude that accounting for time-varying effects in a portfolio model is especially important for emerging market interactions with the US (where South Africa forms part of the analysis). This is addressed in this study by way of the Markov regime switching model which includes endogenous factors for determining the current regime. This model is described in section 4.5.2 of this document.

3.4 Multivariate moment-matching methods

Real world distributions are often characterised by the first four moments of the distribution (these are the mean, variance, skewness and kurtosis) rather than just the first two. Random variables with non-normal distributions are therefore required to replicate the distributions observed in the real world. Fleishman (1978) developed a procedure in the psychometrics environment which allows one to generate data with non-normal distributions. The Fleishman (1978) moment-matching method has been challenged as it is not capable of creating distributions with **all** combinations of skewness and kurtosis, but it has been proven to be one of the simplest to implement and also speedy to execute (Vale and Maurelli, 1983). Fleishman's method has an added advantage that it can be extended easily to cater for multivariate data which is dependent.

3.4.1 Fleishman non-normal procedure

Fleishman's technique for simulated non-normal random variables, Y, is done by a power transformation of a standard normal random variable, X. We follow the notation of Vale and Maurelli (1983):

$$Y = a + bX + cX^2 + dX^3 \tag{12}$$

where Y is given a distributional form from the constants, a, b, c and d as they relate to a power transformation to the third power. The constants are determined by way of a system of simultaneous non-linear equations. The constants are used in generating data and results of this return distribution will match the first four moments of the original distribution. These constants have become known as the Fleishman coefficients.

In order to establish the first moment, the mean, we use the Fleishman coefficients supplied to equation 12. In order to establish the second moment, the variance of Y, we make use of the following equation:

$$E(Y^2) = a^2 + 2ac + b^2 + 6bd + 3c^2 + 15d^3. \tag{13}$$

For a standard normal distribution with a mean of zero and standard deviation of one, the constants b, c and d, for a desired skewness (γ_1) and desired kurtosis (γ_2) are found by solving for the parameters from the following equation set:

$$b^2 + 6bd + 2c^2 + 15d^2 - 1 = 0 \tag{14}$$

$$2c(b^2 + 24bd + 105d^2 + 2) - \gamma_1 = 0 \tag{15}$$

$$24\left[bd + c^2(1 + b^2 + 28bd) + d^2(12 + 48bd + 141c^2 + 225d^2)\right] - \gamma_2 = 0. \tag{16}$$

The constant a is simply equal to $a = -c$. Univariate numbers with a distribution with moments specified by γ_1 and γ_2 are generated by using the Fleishman coefficients substituted into equation 12 to transform the numbers of a standard normal random draw.

3.4.2 Fleishman, Vale and Maurelli multivariate procedure

Multivariate non-normal numbers with dependence can be simulated by using both Fleishman's method and a principal-components factorisation, or matrix decomposition procedure (Vale and Maurelli, 1983). The matrix decomposition is based on a correlation matrix which is observed from data or a desired correlation.

The procedure starts with the completed Fleishman procedure as in section 3.4.1, which we define in matrix notation following Vale and Maurelli (1983). The random data variables, X_1 and X_2, are generated from the standard normal distribution. Again we follow the notation of Vale and Maurelli (1983) for formula 17 through to 24. The vector x_i, where $i = 1 \ldots 2$; stores the power coefficients equation (12):

$$\boldsymbol{x_i}^T = [1, X_i^1, X_i^2, X_i^3]. \tag{17}$$

The vector w^T denotes the power coefficients for equation (12):

$$\boldsymbol{w_i}^T = [a, b, c, d]. \tag{18}$$

The non-normal data variable is then a product of vector x_i^T and w_i^T:

$$Y = \boldsymbol{w_i^T x_i}. \tag{19}$$

Letting $r_{Y_1Y_2}$ be the correlation between the two variables, the correlation between Y_1 and Y_2 is determined by the cross product:

$$\begin{aligned} r_{Y_1Y_2} &= E(Y_1Y_2) && (20)\\ &= E(\boldsymbol{w_1^T x_1 x_2^T w_2}) && (21)\\ &= \boldsymbol{w_1^T R w_2} && (22) \end{aligned}$$

where R is the expected value of the matrix product of $\boldsymbol{x_1}$ and $\boldsymbol{x_2^T}$:

$$R = E(\boldsymbol{x_1 x_2^T}) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \rho_{X_1X_2} & 0 & 3\rho_{X_1X_2} \\ 1 & 0 & 2\rho_{X_1X_2}^2 + 1 & 0 \\ 0 & 3\rho_{X_1X_2} & 0 & 6\rho_{X_1X_2}^3 + 9\rho_{X_1X_2} \end{bmatrix}. \tag{23}$$

By scalar algebra and collecting terms, the intermediate correlation of variables X_1X_2 is found by solving the polynomial function for $\rho_{X_1X_2}$:

$$r_{Y_1Y_2} = \rho_{X_1X_2}(b_1b_2 + 3b_1d_2 + 3d_1b_2 + 9d_1d_2) + \rho_{X_1X_2}^2(2c_1c_2) + \rho_{X_1X_2}^3(6d_1d_2). \tag{24}$$

3.5 Stochastic processes

3.5.1 Probability space

Prior to the specifying of stochastic models, we need to develop the framework for describing the probability space. This is made possible using a mathematical object defined in measure theory that will allow the full description of both the probability and events. The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a triple. We follow the definition in (Munk, 2011, page 25), where:

- Ω is the state space of possible outcomes. An element $\omega \in \Omega$ represents a specific realization from among all possible uncertain objects of the model. An event is a subset of Ω;
- $\mathcal{F}$ is a sigma-algebra on Ω, that is a collection of events of Ω with the properties:
  - $\Omega \in \mathcal{F}$;
  - For any set $F \in \mathcal{F}$, the compliment F^c is also in F;
  - If $F_1, F_2, ... \in \mathcal{F}$, then the union $\cup_{n=1}^{\infty} F_n$ is in $\mathcal{F}$.

  $\mathcal{F}$ is the collection of all events that can be assigned a probability.
- $\mathbb{P}$ is a probability measure that is a function $\mathbb{P} : \mathcal{F} \longrightarrow [0,1]$ with $\mathbb{P}(\Omega) = 1$ and the property that $\mathbb{P}(\cup_{m=1}^{\infty} A_m) = \sum_{m=1}^{\infty} \mathbb{P}(A_m)$ for any sequence $A_1, A_2, ...$ of disjoint events each in $\mathcal{F}$.

Munk (2011) explains in the case of multi-period models, for all periods in time defined $t \in T$, where $T = [0, T]$, the state space is a consideration of all factors and all possible combinations of events. The state space therefore becomes very large.

3.5.2 Discrete time stochastic processes

Munk (2011) explains that we have to understand that future events are not known with certainty when we establish an asset pricing process. Uncertainty in asset pricing can be dealt with by way of a stochastic process. As Fabozzi et al. (2010) explain, stochastic processes are a collection of random data variables that are indexed in time. Stochastic processes are made up of stochastic paths that are univariate functions of time. The collection of these paths is termed a stochastic process. Models where time increments are defined by discrete values are referred to as stochastic processes in discrete time (Fabozzi and Pachamanova, 2010). The discrete-time process for financial time series, or the one period model of portfolio management, is mostly constructed by a series of random variables, $\varepsilon_1, \varepsilon_2, \varepsilon_3....\varepsilon_T$, at the points in the defined time subset (Munk, 2011).

Arithmetic walk with drift

The arithmetic random walk process with drift takes the assumption that the price movements follow a normal distribution with a mean return, μ and standard deviation, σ (Fabozzi et al., 2007b). Arithmetic random walks can be seen as RW1 random walks (defined in section 3.2) with shocks that are mutually independent.

We follow the notation in Fabozzi and Pachamanova (2010) to describe the arithmetic random walk. In general, an arithmetic random walk will describe any future price X_t, from some initial price, X_0 as follows:

$$X_t = X_0 + \mu \cdot t + \sigma \sum_{i=0}^{t-1} \varepsilon_t. \tag{25}$$

where the independent normally distributed random variables are denoted by ε_t. X_t have a mean price equal to the sum of standard deviations and the incremental means. The arithmetic random walk has a constant expected return otherwise referred to as drift, μ, and volatility, σ. This is often formulated by the closed-form expression:

$$X_t = X_0 + \mu \cdot t + \sigma \cdot \sqrt{t} \cdot \varepsilon \tag{26}$$

where ε is a standard normal random variable. The term $\sqrt{t}$ allows for simple scaling of volatility but only holds true when returns are driven by the geometric Brownian motion process. It is from this term where the expression *'square root of time'* stems.

Autoregressive process

Given the evidence of heteroskedasticity in financial markets, there has seen a surge in the use of conditional variance models. Among the most widely used, is the collection of autoregressive conditionally heteroskedasticity (ARCH) models. This family of models was introduced by Engle (1982). There are two broad classes of volatility models: the ARCH and generalised autoregressive conditionally heteroskedasticity (GARCH) models. GARCH models have conditional variance which is determined by a function of observable factors. The second class is related to models of latent volatility or stochastic volatility (Silvey, 2007). Stylised features of volatility in financial markets include volatility clustering, persistence, inter-temporal persistence, mean reversion and asymmetric innovations, referred to as the leverage effect and also known sometimes as the risk premium effect (Engle and Patton, 2007; MacKinlay et al., 1997).

The ARCH(l) process of $X = (X_t)_{t \in T}$ was introduced by Engle (1982). We follow the notation set out in Munk (2011):

$$X_{t+1} = \mu + \sigma_{t+1}\varepsilon_{t+1},$$

and

$$\sigma_{t+1}^2 = \delta + \sum_{i=1}^{l} \alpha_i \varepsilon_{t+1-i}^2$$

where the number of lagged variables is determined by l. The ARCH model is estimated by an ordinary least squares regression where the constant, δ and α are estimated parameters in the regression. The GARCH(l, m) process, is a generalisation of the ARCH process where m is the order of GARCH terms, σ^2, and l is the order of ARCH terms, ε^2. This model was introduced by Bollerslev (1986) is defined as follows:

$$X_{t+1} = \mu + \sigma_{t+1}\varepsilon_{t+1},$$

and

$$\sigma^2_{t+1} = \delta + \sum_{i=1}^{l} \alpha_i \varepsilon^2_{t+1-i} + \sum_{j=1}^{l} \beta_i \sigma^2_{t+1-j}$$

where σ^2_{t+1} is conditional variance at time $t + 1$. The model can be estimated by an ordinary least squares method where the constant δ, the parameter β and α are all parameters that require estimation, where $\delta \geq 0$, $\beta \geq 0$, $\alpha \geq 0$. A GARCH model is therefore a constant plus a weighted average of past squared innovations. This model is known to successfully account for the effects of volatility clustering (Fabozzi et al., 2011b).

As discussed there are many documented extensions to the (G)ARCH model family for modelling the volatility of financial asset prices. This includes successful extensions to multivariate models (Bauwens et al., 2006; Engle and Patton, 2007; Brownless and Gallo, 2010; Engle and Sokalska, 2012). The Ljung-Box statistic is a good test to detect the autocorrelation. A GARCH model applied to an autocorrelated series and then subject to the Ljung-Box test, is effective in removing the autocorrelation in the financial data series (Hull, 2004).

3.5.3 Geometric random walk

A noted issue of the arithmetic random walk is that the model allows for prices to become negative, which is not a reality and has the same random probability distribution at all price levels, this is a known issue of the arithmetic random walk. Fabozzi and Pachamanova (2010) suggest that the better model would be to assume returns are an IID sequence and hence not diminishing as the price level increases.

The geometric random walk formulation is essentially not very different from the arithmetic random walk. The difference is that the prior period price is a factor in the volatility and error part. It is clear from the GBM expression of return, $r_t = \mu + \sigma \cdot \varepsilon_t$, where the noise term, $\varepsilon_0, ..., \varepsilon_t$, is an independent normal variable. The return, r_t, is calculated by $r_t = \frac{S_{t+1} - S_t}{S_t}$. So, returns are normally distributed with mean, μ, and standard deviation, σ. To establish the future prices with equations, we suppose the price at $t + 1$, S_{t+1} can be written as (following the notation in

Fabozzi et al. (2010)):

$$\begin{aligned}
S_{t+1} &= S_t \cdot \frac{S_{t+1}}{S_t} \\
&= S_t \cdot \left(\frac{S_t}{S_t} + \frac{S_{t+1} - S_t}{S_t} \right) \\
&= S_t \cdot \left(1 + \frac{S_{t+1} - S_t}{S_t} \right) \\
&= S_t \cdot (1 + r_t) \\
&= S_t + \mu \cdot S_t + \sigma \cdot S_t \cdot \varepsilon_t
\end{aligned}$$

The geometric model is a multiplicative model, while as Fabozzi and Pachamanova (2010) point out, the arithmetic random walk is additive. Importantly, the geometric random walk does address the concerns of prices becoming negative. In reality nearly all assets are limited liabilities which are capped to a loss of 100 per cent of investment. The geometric random walk has issues, but due to its simplicity and ease in extension, it is one of the most widely used in quantitative finance (Fabozzi and Pachamanova, 2010). A key issue with geometric random walks is in the assumption that the log returns are IID. If the time series exhibits autocorrelation, geometric random walk technique is not a good representation of real world prices (Fabozzi and Pachamanova, 2010). Geometric random walk alone can be a poor representation for interest rates and commodities, which like other financial time series are often asymptotic or mean reverting. Key advantages of the GBM is that it is a flexible stochastic process which may be extended with relative ease. It is also a good foundation to model asset classes with features which are not adequately represented by a process, but only driven by drift and volatility.

Mean reversion

A feature of certain asset classes is the appearance of mean reversion where returns tend towards a long term mean. Mean reversion can be catered for in both the arithmetic random walk and the geometric walk. This process is also represented in the continuous case which is known as the Ornstein-Uhlenbeck process (Munk, 2011). The geometric mean reversion is an extension of the geometric mean random walk:

$$S_{t+1} = S_t + k \cdot (\mu - S_t) \cdot S_t + \sigma \cdot S_t \cdot \varepsilon_t \tag{27}$$

where k represents the speed of adjustment. The magnitude of k is positively related to the speed that the process takes to return to the long-term average.

Stochastic jump

A commonly observed feature of financial asset returns is the occurrence of spikes in returns, often asymmetrically. These can violate the assumption of normality in the returns distribution; negative return spikes are often more significant than positive return spikes. One way of attending to this issue is by incorporating stochastic jumps, representing a financial return spike, into the random walk process. This technique is often used in moment-matching exercises. The Poisson process is often used to configure the frequency of jumps, where the mean rate of arrival of a jump, is represented by λ. A separate process governs the magnitude of the jumps which may be described by another distribution. The normal and log-normal distribution are often used. The magnitude of the jumps can be modelled using any appropriate fitted distribution.

For a Poisson process of rate, λ, and for $t > 0$, the number of arrivals of a jump, J_t, in the case of N_t, is given by:

$$N_t = P_{N_t}(n) = \frac{(\lambda t)^n \exp(-\lambda t)}{n!} \tag{28}$$

where the magnitude of the jump of asset return is the magnitude of the shock in the return, M, multiplied by the shock arrival N;

$$J_t = M_t \cdot N_t. \tag{29}$$

This extension can be incorporated in the geometric random walk as follows (Fabozzi and Pachamanova, 2010):

$$X_{t+1} = X_t + \mu \cdot X_t + \sigma \cdot X_t \cdot \varepsilon_t + J_t. \tag{30}$$

3.5.4 Continuous time stochastic processes

MacKinlay et al. (1997) explain that as time increments get infinitesimally small, tending towards zero, the random walk emerges as a stochastic process in continuous time. Under this framework uncertainty is represented by a standard Brownian motion where stochastic process must satisfy the following properties (Munk, 2011; MacKinlay et al., 1997):

What's more, when we consider the multi-period case we have to consider a portfolio of assets over time, much like an inventory of a portfolio's asset values, dividends, consumption and income. When the asset dynamics use a time increment which is infinitely small, the process is a continuous time stochastic process (Fabozzi and Pachamanova, 2010).

- For all t such that $0 \leq t_0 \leq t_1 \leq ..... \leq t_n$, $X_{t+1} - X_t$ is $N\left(\mu, \sigma^2\right)$;
- For all t such that $0 \leq t_0 \leq t_1 \leq ..... \leq t_n$ the increment $X_{t+1} - X_t$ is statistically independent from $X_{t+2} - X_{t+1}$ and independent over all time periods;
- The sample paths, denoted by X_t, are continuous.

Continuous time models are very similar to discrete time models, developed in section 3.5.2. This study uses the language of stochastic differential equations (SDE). The SDE can be represented by way of a standard Brownian motion as follows:

$$dX_t = \mu dt + \sigma dW. \tag{31}$$

The notation for return increments changes from Δ to d as we move from discrete to infinitesimal time steps. In the Wiener process in the equation above, dW is an infinitesimal increment. The Wiener process is a model often used to study Brownian motion. This is the key element in classical asset pricing and it represents the random value or random element used to represent uncertainty (Fabozzi and Pachamanova, 2010).

4 Single-period optimisation models

The work of Markowitz (1952) was the first formal approach to optimising across asset classes and is the base for many of the single period optimisation models. The single period optimisation models are explored and defined below.

4.1 Portfolio optimisation

4.1.1 Definition of portfolio measures

The following common measures are defined for the portfolio optimisation formulations:

- Assets are denoted by S_i. The expected return of the assets is denoted by μ_i;
- The proportion of the total funds invested in each asset, the weight, is captured in the matrix w_i;
- The expected portfolio return is defined as $E[\mu_p] = \mu_1 w_1 + \mu_2 w_2 + \cdots + \mu_n w_n = \mu^T w$;
- In the case of a two asset portfolio (asset i and asset j), correlation is defined as $\rho_{ij} = \sigma_{ij}\sigma_i\sigma_j$, where the covariance is σ_{ij};
- Portfolio variance $\sigma_p^2 = \sum_{i=1}^{n} w_i^2\sigma_i^2 + \sum_{\substack{i=1\&j=1 \\ j\neq i}}^{n} x_i x_j \sigma_{ij} = x^T \rho x$.

In the case that more variables are required, the definitions for the variables will be specified in the formulation.

4.1.2 The Markowitz mean variance (MVO) optimisation

The original framework by Markowitz expresses an optimal allocation of the risky assets quantitatively by making a trade off between expected portfolio return and its variance. As it is related to the finance environment, the framework is based on an expression of financial risk and return, which is then optimised with a linear (or quadratic) programme against an investor utility function and/or risk tolerance index (Fabozzi et al., 2007b).

Linear programmes and quadratic programmes are versatile and powerful methods for addressing optimisation problems. A programme is set up with an objective function which is either a maximisation problem or a minimisation problem. The optimal solution to the objective function is the feasible solution which is part of the feasible region. Cornuejols and Tutuncu (2007) explain that the feasible region is a subset of $\mathbb{R}^n$ (n-dimensional real space). This solution can be constrained, the constraints are flexible expressions set up as equalities or inequalities. These are often used to add logical functions, for instance regulatory rules for funds in investment management, to objective function. There are different mathematical and numerical methods used to solve the problem, the best known of these is the interior-point methods and simplex (Cornuejols and Tutuncu, 2007).

The objective function for the MVO can be formulated in a number of different ways. We refer to the more commonly used approaches, the first being the classic MVO approach that Markowitz used. In this approach the objective function is the function representing an investor's utility and it is quadratic. Resolving the portfolio choice problem, or finding the optimal weights to each asset class, w, is done with the Cholesky decomposition in this study by Cornuejols and Tutuncu (2007). A requirement is to ensure that objective function is only set up with non-singular square matrices. The portfolio vector, w will fulfil the following relationship $\sum_i w_1 = 1$. We are also able to express the efficient frontier by solving the following problem at varying levels of required return, RR. We can describe the composition of the portfolios which are risk minimal, otherwise known as the efficient frontier, we follow the formulation set out in Cornuejols and Tutuncu (2007):

$$\begin{aligned} \min_{w} \; & w^T Q w \\ s.t. \quad & e^T w = 1 \\ & \mu^T w \geq RR \\ & w \geq 0 \end{aligned}$$

where $s.t.$ denotes *subject to*, which is the start of a list of constraints for the stated problem. The first constraint is a relationship which requires the sum of the components of vector e (which is a vector array of ones) to equal one. Therefore the weights in the portfolio will add to 100%. The second constraint specifies that expected return, based on the solved weights, is no less than a target return RR. The second constraint specifies that the optimisation is based on an a Cholesky decomposition (Fabozzi et al., 2007b). This is an efficient method that ensures a single local optimal solution (Cornuejols and Tutuncu, 2007). A consequence of using this formulation and a Cholesky decomposition, given that the covariance matrix, Q, is both symmetric (being a square matrix having elements $n \times n$) and positive definite (where eigenvalues are non-negative), is one optimal solution (Wicklin, 2013). Were it not for the constraints placed on the covariance matrix, there could be multiple solutions making it more difficult to solve the problem. The last constraint specifies no short sales, or that the asset weights must be greater than or equal to zero (Fabozzi and Pachamanova, 2010).

4.1.3 The risk aversion approach

An important alternative approach, which is central to many portfolio optimisation formulations, is based on the risk aversion coefficient. The objective function is developed by constructing portfolios that increase the risk-tolerance parameters against return. This objective function is created by using the absolute risk-aversion index r_A (or δ) in the following quadratic formulation found in Cornuejols and Tutuncu (2007):

$$\max_{w} \mu^T w - \frac{\delta}{2} w^T Q w$$
$$s.t. \quad Aw = b$$
$$w \geq 0$$

where the inclusion of the risk aversion coefficient, δ, sets up this formulation as a risk-adjusted function of returns (Cornuejols and Tutuncu, 2007). $Aw = b$ is a system of equality constraints. This study follows Fabozzi et al. (2007b) who explain that Q is an NxN matrix, μ is an N-dimensional vector, A is a JxN matrix and b is a J-dimensional vector. This formulation is important in finance and can be used for mean-variance optimisation and Sharpe ratio maximisation. As can seen, this formulation is the basis for two-fund separation theorem, where the solution is built by splitting into a speculative portfolio and a minimum-variance portfolio (Scherer, 2007; Fabozzi et al., 2007b; Sharpe, 1964; Cornuejols and Tutuncu, 2007; Fabozzi and Pachamanova, 2010).

4.1.4 Graphical representation of MVO results

We recapture and illustrate the MVO concepts graphically in figure 2.

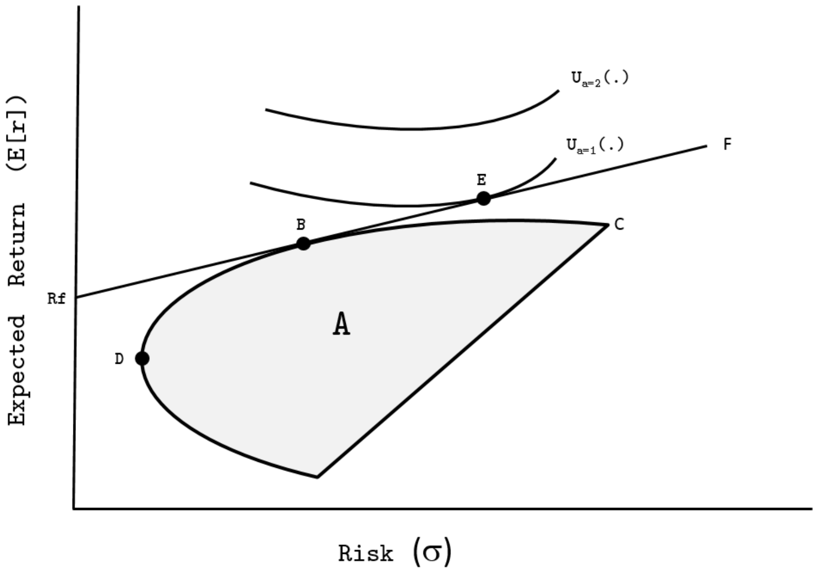


Figure 2: Mean-Variance Analysis

Figure 2 illustrates the results obtained from a mean-variance analysis. Risk is measured by standard deviation in returns (plotted on the x-axis), and expected return is measured by expected returns including any additional earnings, like dividends, over the time horizon (plotted y-axis). The enclosed area surrounding point A represents all the achievable combinations of assets (or feasible portfolios). The portfolio often referred to as the global minimum variance (GMV) portfolio, shown as point D, is the single portfolio which offers the least variance for an efficient return. There is no portfolio that has lower variance. The efficient frontier without a risk-free asset is illustrated by the curve between point D and point C. Point B is the point of tangency to the capital market line (CML) and is also called the market portfolio. The efficient frontier with a risk-free asset is illustrated by the line between point Rf and point F and is known as the CML. Curves $U_{a=2}(\cdot)$ and $U_{a=1}(\cdot)$ are curves on the chart to represent an investor's utility function. Each curve represents trade-offs between risk and return that the investor gains equal utility from, or where the investor has equal preference. Naturally the $U_{a=2}(\cdot)$ is a higher level of investor utility than $U_{a=1}(\cdot)$. Fabozzi et al. (2007b) explain that the theory of choice often involves the maximised value from a utility function for representing an investor's preference. The portfolio denoted by E on figure 2 is the geared portfolio that a risk averse individual, with utility curve $U_{a=1}(\cdot)$, would prefer.

Modern portfolio theory (MPT) uses the mean variance optimisation (MVO) framework to represent investor preference. The assumption that an individual's utility function is quadratic, representing an absolute risk aversion (ARA) has been questioned and as we have discussed, the selection of the utility function is very difficult to specify. Another key assumption is that return distributions are normal which, as we will see in this study, is not always the case. As the name indicates, MVO does not account for the effects of higher order moments, namely skewness and kurtosis. The classic model is based on a one-period model with no option to rebalance. Scherer (2007) explicitly queries how well MVO echoes reality given that returns are not normally distributed and individuals do not hold identical expressions of utility. He concludes that appropriate use of the MVO should be one of empirical confirmation of the assumptions rather than a debate between dichotomous theories.

4.2 Tobin's two fund separation theorem

The separation theorem was contributed by Professor James Tobin, of Yale University, in an article about liquidity preference (Guerard, 2010). Tobin (1958) developed the separation theorem, also known as *two fund separation theorem* or the *mutual fund separation theorem.* The premise of the theory relates to how an investor, given her risk appetite, can arrive at an optimal portfolio by holding different proportions of a risky asset and a risk free asset. This result is arrived at algebraically by using a standard argument from a quadratic programme. An implication of this result is that, in the presence of a risk-free asset, the optimal risky portfolio can be determined without any knowledge of investor preferences. The efficient frontier can, therefore, be achieved by two fund spanning, which involves varying the combinations of a risk-free asset and the tangency portfolio, where the exception is the tangency portfolio which consists of risky assets only.

An important implication of this theory is that it allows all investors to make an investment decision along the Capital Market Line (refer to the line between Rf and F on Figure 2) and this does not depend on the shape or form of the utility curve; this is known as separation. Campbell and Viceira (2002, page 2) write that "the striking conclusion of this analysis is that investors who care only about mean and standard deviation will hold the same portfolio of risk assets." In more sophisticated formulations of investor utility, one would not change the composition of the tangency portfolio (illustrated by B in Figure 2). The planning of investments over the longer horizon is inherently difficult for reasons such as an investor's risk tolerance changing over time. By utilising the two fund separation theorem, the investor can simply resolve this issue by varying the levels of risky portfolio as the risk tolerance changes over time. One of the key notions in the two fund separation theorem is that there is one efficient portfolio, this can be produced by maximising the Sharpe ratio given differing client risk tolerances through time (Amenc et al., 2011a; Back, 2010). With particular reference to institutional asset allocation, one of the greatest achievements in financial theory is the separation of valuation and utility (Scherer, 2007; Hens and Rieger, 2010).

4.3 Capital asset pricing model (CAPM)

The first capital asset pricing model (CAPM) was developed by Sharpe (1964) and Lintner (1961). An important contribution to the field of portfolio choice was made by Samuelson in 1969 in his paper 'Lifetime Portfolio Selection By Dynamic Stochastic Programming'. Markowitz laid the foundations through his work on the MVO, which was the basis for the economy wide implications of the CAPM. Here are list the of assumptions under which the Sharpe-Lintner version of CAPM was developed:

1. Investors base their decisions on expected return and variance of returns;
2. Investors are rational and risk-averse;
3. Investors subscribe to Markowitz method of portfolio diversification;
4. Investors all invest for the same period of time;
5. Investors have the same expectations about the expected return and variance of all assets;
6. There is a risk-free asset and investors can borrow and lend any amount at the risk-free rate;
7. Capital markets are (perfectly) competitive and frictionless.

The formulation of the CAPM, in its simple yet stylised elegance, describes the relationship of expected returns to a risk-free asset and the market portfolio. This relationship can be achieved as follows when the market is in equilibrium:

$$E[R_i] = R_f + \beta_i(E[R_M] - R_f)$$

where β_i is $\frac{cov(R_i, R_M)}{var(R_M)}$ which is determined by a linear regression. R_i is the return of the individual asset, R_M is the return of the market and R_f is the return of the risk-free asset. The beta (β_i) is estimated using a linear regression of the market portfolio and return on the asset. As one will note, there is no error term (ε_i) hence it is a deterministic model.

When this result is plotted on a chart of return and beta (see Figure 3) it is called the Security Market Line (SML). By contrast, individual assets do not lie on the Capital Market Line (CML), which is the line drawn between Rf and B on figure 2. The higher the established beta, the higher the expected return of the asset. Whereas the MVO model has been used to describe the idiosyncratic and systematic risks, the CAPM model shown by the beta of the individual assets implies that investors are rewarded not only for total risk, but also systematic risk (Fabozzi et al., 2007b; MacKinlay et al., 1997; Guerard, 2010).

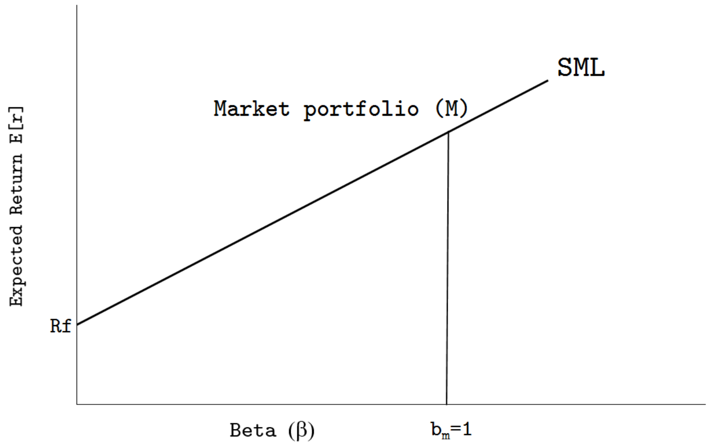


Figure 3: The security market line (SML)

4.4 Stochastic downside approach

4.4.1 Advances in downside risk measures

There has been much debate on the most appropriate formulation for optimal portfolios, hence the platter of the alternative formulations that are made available. Central to this debate is the risk measure used in the formulation. The risk measure used in the MVO formulation is portfolio variance and one of the underlying assumptions in the formulation is that asset returns are normally distributed. Variance is noted as a relatively poor measure of risk. Sortino (2001, page 4) points out that the MVO framework is based solely on the first two moments of the probability distribution and no one fully understands what "the true shape of uncertainty is, but we know what it isn't, and it isn't symmetric." Many market return distributions are not symmetrical and in particular exhibit a downside dispersion (or negative skewness). This study shows that fat tails have been detected in more than one financial returns series. In recognition that the shape of distribution in financial time series data is non-normal, there have been many developments and research on risk measures, with much focus on the down-side risk measures. Examples include semi-variance, which is like variance but there is no consideration given to returns above the expected return. Another category of risk measures are referred to as Roy's safety-first measures, these measures include the classic safety-first ratio. Other safety-first measures include Value at Risk (VAR), conditional Value at Risk (CVAR), expected tail loss (ETL), mean absolute deviation and the lower partial moment. Many of these have favourable properties, both as risk measures and that there are formulations allowing for direct solving of optimal solutions (Fabozzi et al., 2007b, 2011a; Haas and Pigorsch, 2009).

4.4.2 Conditional Value at Risk (CVAR)

Artzner et al. (1999) set out a list of properties or axioms for determining a coherent measure of risk. A financial risk measure ρ can be classified as coherent by fulfilling the following risk properties, where X and Y are random variables:

- Monotonicity: if $X \geq 0$, then $\rho(X) \leq 0$;
- Translation invariance: for a real number c, $\rho(X + c) \leq \rho(X) - c$;
- Sub-additivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$;
- Positive Homogeneity: for any positive real number c, $\rho(cX) = c\rho(X)$.

These four requirements are quite restrictive and in combination mean that many of the popular risk measures (for instance standard deviation which is not monotonic, or semi-variance which fails on sub-additivity) cannot be regarded as coherent (Fabozzi et al., 2007b).

Artzner et al. (1999) advance by setting up a framework for a coherent risk measure. Although VaR as a measure of risk, which Hull (2004) describes as a number for senior management which summarises the total risk in a portfolio, is almost ubiquitous as a risk measure in the financial industry (search "var" at www.bis.org), it has significant theoretical flaws. VaR is not a coherent measure of risk as it is not sub-additive and returns beyond the predefined confidence level are treated as equivalent, implying that users are impartial between a very small loss in excess of threshold and an extreme loss. These are not a realistic assumptions and therefore VaR fails to meet the coherent risk measure requirements. Lastly, Scherer (2007, page 222) points out the mathematical issues, as VaR is "a non-smooth, non-convex and multi-extremum (many local minima) function that makes it difficult to use in portfolio construction."

Despite the issues surrounding the VaR risk measure, a similar risk measure is conditional-value-at-risk (CVAR). CVAR is a coherent measure of risk which meets the sub-additivity axiom and, being convex, allows for optimisation using linear programming (Uryasev, 2000). Rockafellar and Uraysev (2002) indicated that, despite lots of research, efficient algorithms for VaR are still not available. As CVAR is swayed by the effects of the third and fourth moments of a distribution, skewness and kurtosis respectively, it serves as a reasonable risk measure for use in portfolio construction. CVAR will always be greater than or equal to VaR. CVAR and VaR are equivalent when a distribution is symmetric.

4.4.3 CVAR portfolio optimisation formulation

The CVAR optimisation formulation has the capability of dealing with fat-tail distributions; it is a stable and very favourable alternative to MVO. Chun et al. (2012) more recently optimise based on a CVAR risk measure using a method based on the least-squares estimation and another based on the M-estimation approach. We refer to the original definition of the CVAR measure and linear program for minimisation in Rockafellar and Uraysev (2000). Let $f(x, y)$ be a loss associated with the decision vector, x, with an underlying probability density $p(y)$. For a predefined confidence level of risk threshold which is expressed as α, the probability of $f(x, y)$ of a loss exceeding that threshold is given by:

$$\Psi(x, \alpha) = \int_{f(x,y) \geq \alpha} p(y) dy$$

where Ψ is the cumulative distribution function of loss of x at a level greater than a specified confidence level. Due to the difficulty in directly optimising CVAR based on density functions, an auxiliary function can be used which allows for a minimisation problem using a linear programme (Scherer and Winston, 2012). This formulation can also be set up to maximise returns given a maximum specified CVAR:

$$\max_{w} w^T \mu w$$
$$s.t. \quad \text{CVAR}_{\alpha}(w) \leq c_0.$$

where c_0, a constant, is the maximum acceptable CVAR value given the specified tolerance level α. This is a maximisation problem and effectively entails solving or calculating the GMV along the efficient frontier where the constituent asset classes are asymmetrical and have fat tails (Scherer and Winston, 2012; Uryasev, 2000; Rockafellar and Uraysev, 2000; Chun et al., 2012; Guerard, 2010; Fabozzi et al., 2007b).

CVAR is the mean of loss beyond a confidence level (1% in this case), as illustrated in chart 4, the skewness and the fat tails of the return distributions will affect the CVAR measure and in this case is between the maximum loss and VaR at the same level of confidence.

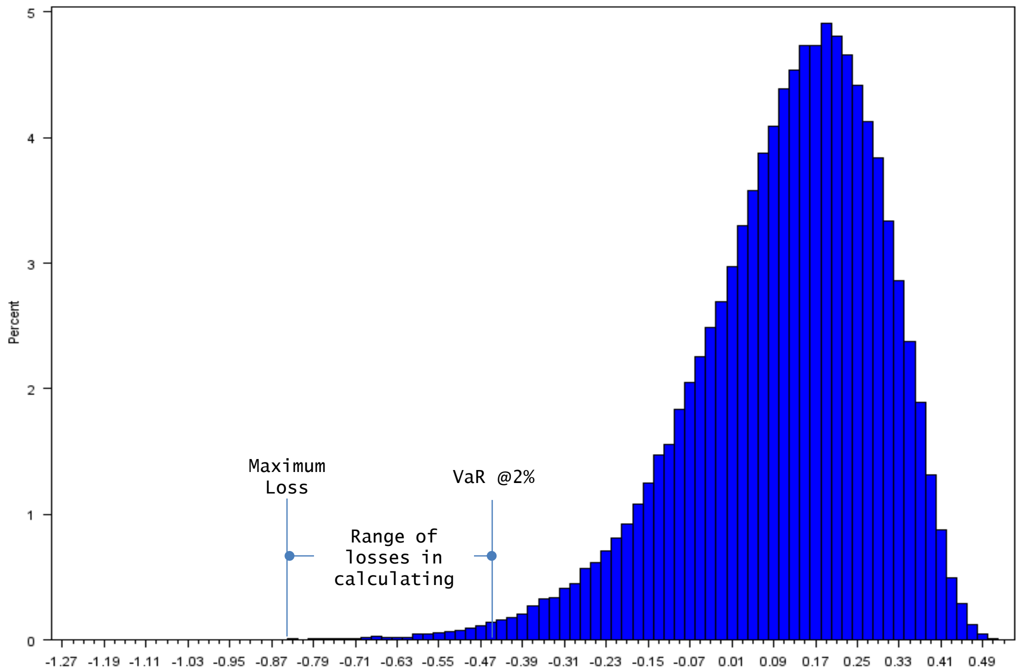


Figure 4: Risk measures illustrated on a simulated equity return distribution (SAS chart output)

4.5 Advances in econometric time series measures

Terasvirta et al. (2010); MacKinlay et al. (1997) explain that the method of inference for financial economists is model-based statistical inference. The starting point for models in finance models is uncertainty, without which problems in financial markets become basic routines in microeconomics (Terasvirta et al., 2010; MacKinlay et al., 1997). Many of the popular linear models are based on the assumption of multivariate normal distribution, which is not always realistic in financial markets. Often the relationships in economics and finance have proven to not be linear. This presents a challenge as the results and conclusions from simple ordinary least squared analysis can be spurious, but non-linear methods are generally a broad and challenging collection of models. As we have seen portfolio construction is based on time-series data at some point. Studies have shown that markets, and therefore asset prices, are characterised by issues relating to autocorrelation, serial correlation, volatility clustering, heteroskedasticity, time varying dependence (such as business cycles) and temporary fluctuations known as regimes (Ang and Bekaert, 2002a; Hamilton, 1989).

Non-linearity in financial time-series may arise from the structure and institutions of the economy. Time series that are known to be non-linear are mostly specified as non-linear in the mean or non-linear in variance. These are very seldom specified collectively (Terasvirta et al., 2010). The model developed by Engle (1982) is known as the Autoregressive Conditional Heteroskedastic (ARCH) model and it is non-linear in the variance but not the mean. We do not aim to give all types of non-linear models; rather give a brief account of non-linear methods required for the active portfolio models proposed in this study.

4.5.1 Standard switching models

Piger (2011) describes regime-switching models as time-series models in which parameters take on different values or regimes. These models can be broadly categorised into Threshold and Markov models. Regime-switching models are able to deal with issues related to financial markets like fat-tail distributions, skewness, ARCH effects and time-varying correlations. A regime model can mimic the persistence and thereby the time-varying correlation characteristics of financial markets. These are highly desirable features.

An early example, the Threshold Autoregressive (TAR) introduced by Tong (1983), is a piece-wise linear function of the first order which takes the following form:

$$x_t = \begin{cases} \alpha_1 + \beta_1 x_{t-1} + \varepsilon_t & \text{if } x_{t-1} < k; \\ \alpha_2 + \beta_2 x_{t-1} + \varepsilon_t & \text{if } x_{t-1} \geq k. \end{cases}$$

This simple specification of the model contains two regressions on the lagged value of x_{t-1} that are dependent on the value of the threshold parameter, k. The specification of k is in relation to an economic structural break. Terasvirta et al. (2010) point out that the TAR model has been criticised for its lack of smoothness in its transition mechanism, where abrupt transitions between regimes are not a desirable feature. The smooth transition regressive (STR) and autoregressive (STAR) models are advances on the TAR model in that they deal with the abruptness in the transition mechanism. TAR models have been widely used in economics and their flexibility allows for broad application (refer to Korenok (2011) for an extensive list of research).

4.5.2 Markov switching models

Markov processes in discrete time and space are referred to as Markov chains (Vassiliou, 2010). The Markov property is one where the future state of an asset price or phenomenon modelled by a Markov chain depends only on its present state, and not on history (Platen and Heath, 2006; Vassiliou, 2010). Leipold and Morger (2007) conclude that using a correctly specified Markov regime-switching portfolio model offers substantial economic reward.

4.5.3 Hidden Markov model (HMM)

The TAR model, introduced above, is very similar in its aims to the Markov switching model in Hamilton (1989). In contrast with the threshold models, with a lagged expression of the dependent variable, the Markov model is determined by a random variable (Korenok, 2011). The difference is that the observable indicator k is replaced by an unobservable discrete stochastic variable, a Markov chain. Due to the unobserved nature of the stochastic variable, these are often referred to as Hidden Markov Models (HMM). A simple example of the Markov switching model denoted in MacKinlay et al. (1997), is the following:

$$x_t = \begin{cases} \alpha_1 + \beta_1 x_{t-1} + \varepsilon_{1t} & \text{if } s_t = 1; \\ \alpha_2 + \beta_2 x_{t-1} + \varepsilon_{2t} & \text{if } s_t = 2. \end{cases}$$

Conditional probabilities are often used to analyse random variables and may be put to good use determining the current state or regime in the financial data series. The unobserved random state variable, s_t, takes the value of 1 or 2. In a discrete first-order Markov chain with s possible states, the one-step transitional probability that state i is followed by state j at time t is denoted by $Pr(s_t = j|s_{t-1} = i) = p_{ij}$. The transitional probabilities are assembled using the transition matrix below,

$$P = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix};$$

where the two state model can be simplified to:

$$\begin{pmatrix} P & 1-P \\ 1-Q & Q \end{pmatrix}.$$

The Markov chain in the above model is an unobserved (or hidden) two-state Markov chain where P and Q are the probabilities of staying in a given state. Markov switching models have been used to model financial crises, government changes, interest rates, business cycles and portfolio diversification (again refer to Korenok (2011) for examples of application of this technique).

4.5.4 Time-varying Markov-switching model

Transitional probabilities in the hidden Markov-switching models are models which do not rely on history. This has been extended so that the transitional probabilities are driven by observed variables (Diebold et al., 1994). The extended model relaxes the binding constraint of constant transitional probabilities in the Markov switching models and allows for transitional probabilities to vary through time. The time varying transitional probability (TVTP) model allows for transitional probabilities to be based on conditioning information. As Diebold et al. (1994) phrase it in their introduction, the “transitional probabilities are endogenous.” This property is particularly desirable given the task of extracting an active portfolio return from current market conditions.

Formulation of the TVTP We follow parts of the specification in Diebold et al. (1994), Starz (2015) and Ang and Bekaert (2002a) in the formulation of the TVTP. For the time-varying Markov chain with s possible states, the one-step transitional probability that state i is followed by state j at time t is denoted by $Pr(s_t = j|s_{t-1} = i) = p_{ij}(t-1)$. The transitional probabilities are assembled using the transition matrix below,

$$P_t = \begin{pmatrix} p_{00}(t-1) & p_{01}(t-1) \\ p_{10}(t-1) & p_{11}(t-1) \end{pmatrix};$$

Starz (2015) explains that the probability of transitioning from regime i, in period $t-1$, to regime j, in period t, is a two state model that is simplified to:

$$\begin{pmatrix} P(t-1) & 1-P(t-1) \\ 1-Q(t-1) & Q(t-1) \end{pmatrix}.$$

The regimes s_t are set out in a transition matrix which can vary through time, where $p_{11(t)} + p_{21(t)} = 1$ and $p_{12(t)} + p_{22(t)} = 1$. The probabilities of the matrix are each parameterised using a time-series multinomial Logit function. Diebold et al. (1994) define the function, with two transition probabilities which evolve over time, as logistic functions of $x_{t-1}\beta_i$. The endogenous variables are set in the conditioning vector, x_{t-1} and we know that $i = 1, 2$. The information in the conditioning vector will therefore affect the transition probabilities at each point in time.

$$p_{ij}(t) = \frac{\exp(\alpha_i + \beta_i x_{t-1})}{1 + \exp(\alpha_i + \beta_i x_{t-1})}, \tag{32}$$

as Ang and Bekaert (2002a) explain, if $\beta_i = 0$ then the transitional probabilities revert to a constant, which is the same as the model in section 4.5.3.

In the seminal paper of Hamilton (1989), a Markov-switching model is used to determine increasing and decreasing markets (bull and bear markets). Filardo (1994) works on business cycles but interestingly uses a model that varies in time; he considers the use of leading economic indicators, (Filardo, 1994; Moolman, 2004).

Isogai et al. (2004) model the Japanese stock market with the use of a TVTP with exogenous variables. Their findings, based on a relatively long dated time series (34 years), indicate a large improvement of the marginal likelihood when compared to conventional HMM with a constant transitional probability. They conclude that a bull and bear markets distinction is appropriate for the Japanese market.

In the South African context, Moolman (2003) models the turning points in the South African business cycle. Moolman (2003) used a Probit model with a series of indicators. She found that the model is statistically significant in its ability to predict turning points. In another analysis Moolman (2004) uses a first-order two-state TVTP with the spread between longer and shorter dated bonds, the yield spread, as an explanatory variable to forecast the turning points of the South African business cycle. Moolman (2004) tests the model accuracy and finds that the Markov model is favourable and accurately predicts the turning points of the business cycle.

Bosch and Ruch (2012) have more recently applied Markov-switching to both identifying and dating of business cycles. They showed that the Markov-switching results coincide with the South African Reserve Bank official turning points. They propose, that from a Markov-switching perspective, the mean and variance are sufficient determinants with extra autoregressive terms not necessary. Given the requirement to extract the contemporaneous impact, the lagged terms do not intuitively contribute.

Further evidence of South African regimes is shown in an interesting global perspective on business cycles by Altug and Bildirici (2010). Their study includes South Africa in a sample of 27 countries. The results indicate that the Markov-switching model will differentiate business cycles for both developing and developed countries rather accurately. In the case of South Africa, the model is specified with two regimes. This model is tested and it tracks recessionary periods well. The model is tested with the likelihood ratio test (as proposed in Ang and Bekaert (2002b)).

In the context of portfolio optimisation, Ang and Bekaert (2002a) make use of TVTP to seek optimal equity portfolios in the US, UK and Germany, which focussed on high volatility regimes. This approach specifies the transition probabilities by way of a logistic regression. The logistic regression depends on endogenous variables such as the log of the earnings yield and the short rate which are used to predict the states. They conclude that there is strong evidence that the high volatility-high correlation regime tends to coincide with a bear market, and failing to make use of regime switching is more incrementally costly than the use of international diversification in portfolio construction.

Guidolin (2011) surveys the literature pertaining to Markov-switching models in empirical finance. He concludes that the evidence supports that modelling Markov-switching model dynamics in asset returns makes a difference and enhances the ability to forecast. Another general conclusion is that stock returns have regimes in which the volatility is low and the risk premiums are elevated in a regime, with the opposite occurrence true in a second regime.

4.6 Bayesian portfolio approach

The Bayesian approach to probability, named after an English mathematician Thomas Bayes, contrasts with the classic or frequentist approach to probability and has been used with great effect in portfolio-choice models. Scherer (2007) explains that in order to understand the implication for portfolio choice, one needs to appreciate the difference between these approaches. Rachev (2007) elucidates that the classic approach makes use of a probability based on relative frequencies found in large samples, hence the *frequentist* interpretation of probability. For a Bayesian approach pure frequency is not a sound basis for probability and what is needed is a link between empirical relative frequency and probability. The classic approach to probability uses point estimates to generate parameters from a distribution. The Bayesian probability combines information or beliefs about the data (likelihood function) and the sample data (prior distribution) and is used to generate a density function (posterior density) (Fabozzi et al., 2007a). The Bayes Rule is used to formulate posterior beliefs based on both the observed sample and the informative prior beliefs. This provides the link between probability and the observed. To summarise the key difference is in the approach to probability, where the Bayes framework takes the subjectivity in the notion of probability into account and not just the objective notion of probability based on observations using data of the frequentist (O'Cinneide, 2012; Scherer, 2007).

A Bayesian framework allows, in addition to market data and other objective sources, for the incorporation of subjective views and insights. As an investment process is heavily dependent on decisions, it requires a rigorous methodology and therefore offers the potential for a Bayes framework to be incorporated.

4.6.1 Black-Litterman model

This popular approach to portfolio-asset allocation, adopted into an asset-allocation framework by Fischer Black and Robert Litterman of Goldman Sachs (Black and Litterman, 1992), is classed as a Bayesian approach. The approach deals with the often divergent outcomes of quantitative methods versus an expert's views on returns. Bevan and Winkelman (1998) explain that the central feature of the Black-Litterman model (BL) is that investors increase risk level, or tolerance, when they have a higher conviction in their view. The BL, by incorporating investor views, also deals with the challenges of input sensitivity of expected returns, which is a real challenge in frameworks such as the MVO. BL optimisation relies on Theil's mixed estimation technique (Theil, 1961). The approach is combined with the theoretical underpinning of the CAPM economic model to establish the market position as starting point (Brandt, 2010). The BL framework combines the implied returns using the CAPM (prior) with informative views and certainty levels to create the combined return vector (posterior). The typical Bayesian allocation strategy is a shrinkage, based on a confidence level, towards the investor's prior. Where the BL differs slightly is that the shrinkage takes place on the market distribution towards the investor's prior (Meucci, 2007).

The 'Prior' - implied excess returns

The theory established on the CAPM and Tobin's two-fund separation theorem is the basis for the reference portfolio in the BL model and is used to establish the implied excess returns vector (Π). The excess level of return for each asset class is established by inducing the MVO investor to hold assets at the observed market capitalisation. Walters (2009), explains that the theoretic link to BL is based on the Tobin two-fund separation theorem and the CAPM. The Tobin theory has the implication that an investor's risk aversion levels may be achieved by a combination of a risk-free asset and the CAPM market portfolio, with one notable exception of the tangent or optimal portfolio, which consists of risk assets only.

The BL optimisation, or intuitively the reverse optimisation, starts with the assumption that an investor holds the risky portfolio which is the market portfolio and it is optimal. Given that the market capitalisation weights are a known constant at t, Litterman and He (1999) derive the implied asset return formula by using a quadratic utility function to represent the investor's preferences, then by taking the first derivative with respect to the weights and setting the function to zero. Lastly, by solving for return, we arrive at the formula for implied excess returns (Meucci,

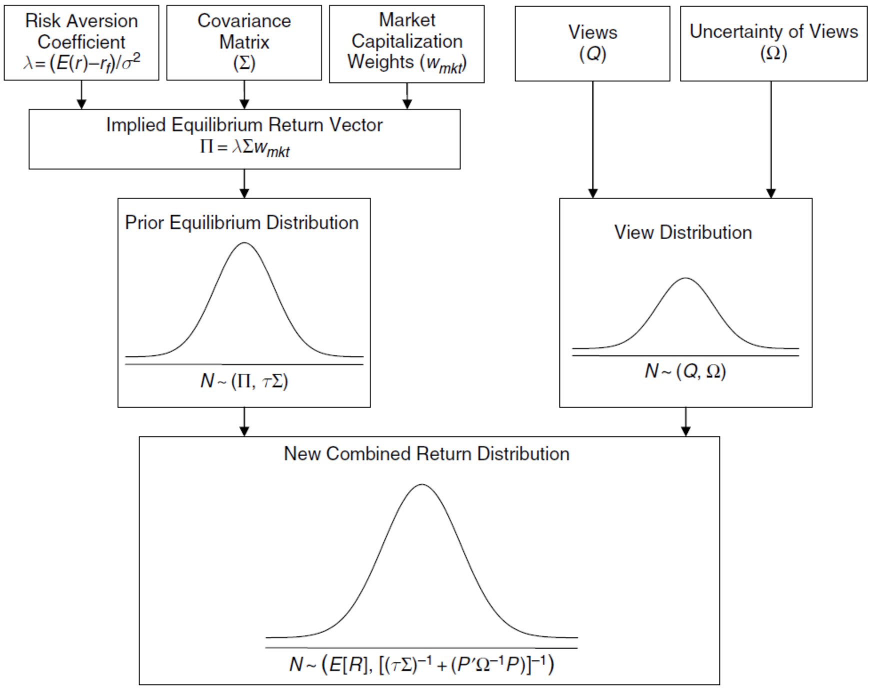


Figure 5: Deriving the combined return vector. This illustration is sourced from Idzorek (2007, page 27).

2007):

$$\Pi = \lambda \Sigma w,$$

where λ is the Sharpe ratio, w is the vector of weights proposed into each asset, and Σ is the covariance matrix of assets' excess return. As per the approach taken in Walters (2009), this is calculated as a constant correlation matrix of excess returns based on historical returns measured daily. We can see that with a simple rearranging of the maximisation problem (please refer to the risk-aversion objective function), we arrive at the implied optimal weights, using the following formula:

$$w = (\lambda \Sigma)^{-1} \Pi.$$

The risk aversion coefficient (λ) is the Sharpe ratio and estimated as follows:

$$\lambda = \frac{E(r) - r_f}{\sigma^2},$$

where $E(r)$ is the expected market total return, r_f is the risk free rate and σ^2 is the variance of r (Walters, 2009). The Sharpe ratio used by Litterman and He (1999) is 0.5 whilst Goldman Sachs Fixed Income use a Sharpe ratio of 1.

Investor views (Q)

Walters (2009) explains that investor views are expressed and assembled into matrices representing the position of the investor in terms of estimated excess implied returns. Where the investor does not present a view, the weights will not differ from the market portfolio and is therefore a passive investment strategy.

Uncertainty of views (ς)

The higher the level of certainty, or confidence in the investor's views, the closer the implied returns vector (Π) will be to the views and visa versa (Idzorek, 2007). The uncertainty of the view or the variance of the views (ς) is not explicitly defined in the framework. Walters (2009, page 7) lists several ways to calculate ς:

- Proportional to the variance of the prior;
- Use a confidence interval;
- Use the variance of residuals in a factor model;
- Use the method to specify confidence along the weight dimension.

Black and Litterman (1992) assume that this parameter should be close to zero, which is based on the assumption that historical variance is larger than the variance around the expected mean returns. Others, such as Koch (2004) feel it is plausible around 0.3.

4.6.2 The Black-Litterman optimisation formula

The optimal portfolio for the BL formula to determine the new combined return vector $E(BL)$ given the investors views is formulated as follows:

$$E(BL) = \left[(\tau\Sigma)^{-1} + \varsigma^{-1}\right]^{-1} \left[(\tau\Sigma)^{-1}\Pi + \varsigma^{-1}Q\right]$$

where τ is a scalar and the investor views are held in Q.

5 Multi-period portfolio choice models

Mulvey et al. (1997) list the features of a desirable and realistic multi-period investment planning framework:

- *Multi-periodicity:* to capture the dynamic aspects of problems such as a complex set of dynamic relationships, where in reality asset prices and cash flows are fluctuating.
- *Treatment of uncertainty:* this is especially important for uncertainty with regard to asset prices over time.
- *Flexible framework for risk attitudes:* clients or entities may have different risk bearing attitudes over time which could be linked to wealth and (or) age.
- *Transaction costs:* linked to trading costs or commission fees which also depend on rebalancing activities.
- *Integrated assets and liabilities* where the entire investment horizon is included in the financial planning.
- *Understandability:* this is important as the results of the models would need to be understood by investment managers who are the end users of the application.
- *Ability to account for practical decision making:* this includes, for instance, growth and budgetary constraints, as well as legal and institutional policies.

There are many forms of multi-period portfolio choice models. We review a few of the notable forms and discuss the merits of each with reference to the criteria in Mulvey et al. (1997) listed above. From the literature multi-stage optimisation models can be categorised broadly as follows, with key examples in each category:

	Deterministic	**Stochastic**
Discrete Time	Bellman Dynamic Programming	Stochastic programming
Continuous time	Hamilton Jacobi Bellman	Merton's Model

5.1 Discrete deterministic portfolio models

The dynamic approach to multi-stage optimal solutions is ostensibly related to Bellman's principle of optimality. This states that for a decision process that is sequential all the dependant decisions must be optimal with respect to state from the current state onwards. The decisions are based on optimal strategies for portfolio choice and consumption, where there is consideration for current and future decisions simultaneously (Cornuejols and Tutuncu, 2007). Dynamic programming is extensively used in literature, where the best known applications are the tree or lattice models. Examples are highlighted in the work of Hull (2004).

We follow the notation from Munk (2011) for the formulation of the Bellman dynamic programme. Munk (2011) explains that indirect utility, J, of an individual is very important and in this framework is used to maximise the expected utility of future and current consumption. This is linked to the similar dynamics captured in the stochastic liability model in this study that consider the dynamic value of liabilities given different consumption (expenditure) and contribution rates overtime. Let δ be the time preference rate and u be the utility function of an individual. The following is the Bellman equation applied to an optimal consumption plan, per Munk (2011):

$$J_t = \sup_{(c_t,\pi_t)} E_t\left[u(c_t) + e^{-\delta}J_{t+1}\right] = \sup_{c_t,\pi_t}\{u(c_t) + e^{-\delta}E_t[J_{t+1}]\} \tag{33}$$

where $E_t[J_{t+1}]$ will depend on the choice of consumption, c_t and optimal portfolio, π_t. Each decision in this formulation is broken down into a here and now decision of both consumption and allocation and thereafter a decision on future periods. The client's optimal strategy will be heavily dependent on the optimal level of consumption. The solving and maximisation of this programme is not in the scope of the study.

5.2 Continuous time models

5.2.1 Continuous deterministic portfolio models

The extension of the Bellman dynamic programme into a continuous time form is referred to as the Hamilton-Jacobi-Bellman (HJB) equation for a constant investment opportunity set (r, μ, Σ). For the formulation of the HJB we follow Back (2010), where the investor seeks to maximise the expected utility of terminal wealth. The investor's value at any date t is a function of his wealth, w:

$$V(t,w) = \max E_t\left[\int_t^T e^{-\delta s}u(C_s)\,ds + U(W_t)|W_t = w\right]. \tag{34}$$

$V(t,w)$ is the maximum attainable value at time, t, for the level of wealth, w. Solving this to establish an optimal solution is done by using Ito's formula and will result in solution that is a quadratic optimisation problem as follows:

$$\pi = -\frac{J_w}{wJ_{ww}}\Sigma^{-1}(\mu - r\mathbf{1}) \tag{35}$$

where $J(t,w) = e^{\delta t}V(t,w)$ and the vector $\Sigma^{-1}(\mu - r\mathbf{1})$ is an optimal portfolio for an investor with a log utility. J_w and J_{ww} are the partial derivatives with respect to the individual's level of wealth.

5.2.2 Continuous stochastic portfolio models

Merton (1975) again contributed, this time by using stochastic calculus in continuous finance literature. As Brandt (2010) explains it, when a solution is found for portfolio choice in the closed form it has the advantage of remaining tractable in discrete time. The objective function for the problem is formulated in continuous time and portfolios are maximised for the time period under consideration. Merton (1969) uses the Bellman equation and then applies Ito's Lemma to the value function to arrive at a formula which provides the optimal weights.

5.3 Discrete stochastic portfolio models - Stochastic programming

Stochastic programming techniques were developed as mathematical programmes by Dantzig (1955) and Beale (1955) independently. The computing hardware required to solve problems of this nature, or problems large enough to be meaningful, have not been adequate until recent times (Cornuejols and Tutuncu, 2007). Hilli et al. (2006) maintain that stochastic programmes allow for efficient designing of investment strategies, and the advantages include the ability to deal with multiple complex constraint functions. These programmes are not tied to a specific form of objective function; coupled with that is the ability to deal with dynamics found in financial planning problems of wealth management and Asset Liability Management (ALM).

Mulvey and Kim (2011) explain that multi-period planning models such as the ALM framework differ slightly from prominent models such as the classic single-period model of Markowitz (1952). They capture the gains as a consequence of portfolio rebalancing and deal with practical real world details such as transaction costs. They also incorporate dynamic measures such as return, risk and correlation and economic factors in a dynamic fashion. Other key benefits include the ability to set up triggers or stopping rules which are linked to client objectives or liability constraints.

Cornuejols and Tutuncu (2007) and Amenc et al. (2011b) also argue that a static single-stage investment model is a fairly restrictive framework that does not adequately deal with the multi-period nature of liabilities, or with transactions costs. Zenios and Kouwenberg (2011) note that the models of optimal control and dynamic programming, both in discrete time and continuous time, such as those introduced by Merton (1969, 1971), provide some insight into the issues of ALM in investments. These methods are limited in the practical application required for making portfolio choice decisions. This is owing to the simplifying assumptions required to solve complex problems of this nature in an appropriate period of time. Zenios and Kouwenberg (2011) list that restrictive assumptions around utility and asset prices, not dealing with non-normal return distributions, no accounting for transactions costs, nor with idiosyncratic regulatory requirements, are all known to contribute to strategies which show notable return fluctuations over time.

Lastly, Zenios and Kouwenberg (2011) point out that the practical issues are very difficult to overcome in closed-form solutions. Many of the same issues can be handled with an ALM making use of a multi-period stochastic programme. A well-documented example of a successful use of a multi-stage stochastic programming was to solve the problem of the Japanese insurance company, the Yasuda Fire and Marine InsuranceCo by the Frank consulting company (Cornuejols and Tutuncu, 2007).

Dempster and Thompson (2010) and Xi Yang and Jacek Gondzio and Andreas Grothey (2011) report that ALM techniques are widely used in finance: they are crucial in certain parts of banks, insurance companies and pension funds, and are applied in areas of portfolio insurance. The core investment philosophy espoused in ALM is a guarantee to meet the liability obligations whilst pursuing profit. This is done largely by configuring a portfolio which is set up to maximise an overall profit function and by recognising the uncertain value of a liability profile that evolves stochastically over the planning horizon. The portfolio is often made up of allocations to cash, bonds and stocks.

Consigli (2011) introduces a case for an individual ALM by pointing out that the personal finance planning problem faced by an individual is subject to the following considerations:

- Investment horizons over a longer period of time;
- Constraints such as variable earnings;
- Differences in consumption patterns;
- Differing life expectancies;
- Uncertainty in many variables;
- Differing cost considerations.

Consigli (2011) proposes that the proper framework to address these decisions under uncertainty is an ALM. Amenc et al. (2011b) discuss the successful use of ALM techniques for individual financial planning or wealth management, depending on whether the approach can handle the constraints related to the customers liabilities. Amenc et al. (2011b) highlight that in contrast to the ALM philosophies of institutions, a challenge in the adoption of an ALM for an individual is the specific nature of the timing of cash flows. This can be overcome by using stochastic programming with one additional state variable: the value of the liabilities, or liability portfolio. Having included the client liability values over the planning horizon, it makes the task of constructing a portfolio with specific constraints and objective functions possible. Cornuejols and Tutuncu (2007); Xi Yang and Jacek Gondzio and Andreas Grothey (2011) point out that the optimal management of investment problems, when the profile of liabilities and the return profile assets are uncertain, requires tools of optimisation under uncertainty and stochastic programming approaches.

Cornuejols and Tutuncu (2007) suggest a way to incorporate uncertain parameters into optimisation problems is by assuming that variables are random with known probability distributions. These distributions may be discretised by transformation of the stochastic programme into a deterministic equivalent.

5.4 Classes of stochastic programming

Cornuejols and Tutuncu (2007) state that stochastic programming falls into the category of mathematical programming. This mathematical method is one way to introduce uncertainty into problems of decision making. Cornuejols and Tutuncu (2007) explain these stochastic optimisation problems can be in the form of linear, non-linear or integer programmes. Gassmann and Ireland (1996) introduce the following as classes of stochastic programming, extended by Messina and Mitra (1997) with the addition of expected value models:

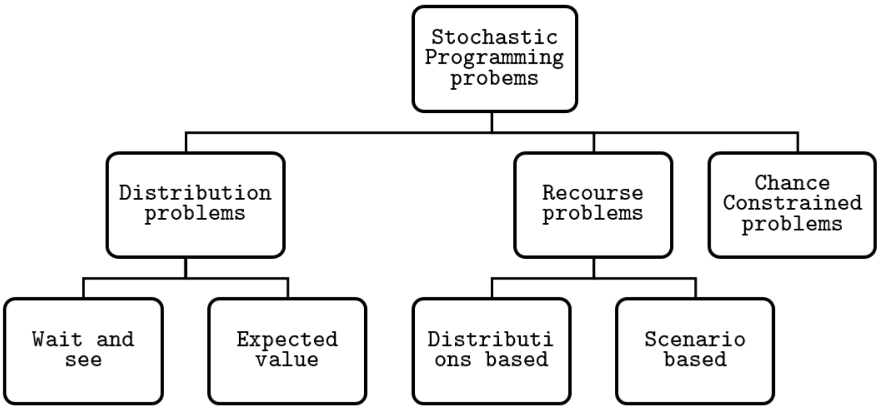


Figure 6: Categorisation of stochastic programs

Cornuejols and Tutuncu (2007) and Zenios and Kouwenberg (2011) explain that stochastic programmes are both anticipative and adaptive decisioning models. They explain that anticipative stochastic programming models include decision variables which do not depend on uncertain future expectations, or random events, where decisions are made immediately, or *here-and-now*. In contrast, when decisions are adaptive, they are made after the observation of a random event where the parameters are observed. These are often referred to as *wait-and-see* approaches.

Cornuejols and Tutuncu (2007) explain that when anticipative and adaptive models are combined into one mathematical framework, these models are referred to as recourse models. Recourse models are well positioned to address finance problems. Zenios and Kouwenberg (2011) cite the decision problem of a portfolio manager as an example of a recourse problem, where the portfolio manager needs to consider the uncertainty in stock prices as anticipative and the need to rebalance the portfolio as prices change as adaptive.

Dantzig (1955) describes linear programmes which are used to formulate multi-stage planning problems. The equality and inequality matrix appears in a cascading sparsely, populated structure or L-shape, with each of the cascading blocks corresponding to a different time period. As this formulation incorporates uncertainty, the coefficients on the right-hand side of the matrix are functions of several parameters with values that vary according to a discretised distribution.

Cornuejols and Tutuncu (2007) point out that financial planning and portfolio management problems may require rebalancing or trading at discrete points in time and, at times, require transactions which incur costs. The use of a multi-stage stochastic programming set-up, with recourse, may afford the modeller an environment that can deal with information progressively. Stochastic programming accepts new information as it is revealed, and is re-submitted into the decision framework, allowing for decisions to adapt dynamically in time.

5.4.1 Two-stage problems with recourse

We follow the notation of Cornuejols and Tutuncu (2007) in the formulation of a two-stage stochastic linear programme with recourse. The first stage decisions, which involve decisions which must be taken *here-and-now* and made prior to the random event w being observed, are represented by the decision vector x:

$$
\begin{array}{lrrl}
\max_x & a^T x & +E[\max_{y(w)} c(w)^T y(w)] & \\
 & Ax & & = b \\
 & B(w)x & +c(w)y(w) & = d(w) \\
 & x \geq 0, & y(w) & \geq 0.
\end{array}
$$

A and b define the constraints of the first stage decisions represented by x. E is an expectation and relates to the second stage decisions. The second stage decisions are only concluded after the random variable w is observed. The second stage decision vector is represented by $y(w)$. As Brandimarte (2006) explains, decisions for the second-stage are made after the random event, which is why w is a function of y. The second-stage decisions are recourse decisions which involve the adjustments to recourse variables w, as a function of y. There is a deterministic term $a^T x$ in the objective function and also the second-stage objective expectation, both are taken over the random event w (Brandimarte, 2006). The stochastic constraints $B(w)$, $C(w)$ and $d(w)$ are used to link the recourse decisions, $y(w)$, to the first stage decision (Cornuejols and Tutuncu, 2007).

An important feature when using the stochastic linear programming approach is how the scenario tree is constructed. As illustrated below, a scenario tree represents the asset return process in a discrete form, which is usable in a stochastic programme. The section on scenario trees (section 9.4.4) expands on the methods used in building scenario trees for asset-return processes. A scenario tree is set up to represent both the probability of a random event occurring and a result given that the random event takes place. Each tree has a root node, at node 1. Aside from the root node, each node has a father node; node 3 is the father node of node 6. The last nodes are terminal nodes such as nodes 4 to 8.

To illustrate and link the above formulation of a stochastic linear programme, we follow the approach of Cornuejols and Tutuncu (2007). The following formulation is based on a finite set of scenarios which are illustrated in the scenario tree in Figure 7. The structure of the matrix is set up with the subscripts matching the nodes in the tree in figure 7. Note how the terminal nodes correspond to the respective father nodes; for instance nodes 4 and 5 correspond to node 2. The probability of the terminal nodes, p_k, is the probability of each scenario, k. The earlier nodes, or father nodes, are the addition of the probability in the terminal nodes, for instance $q_2 = p_4 + p_5$, $q_3 = p_6 + p_7 + p_8$ and $q_4 = p_1$, $q_5 = p_2$, $q_6 = p_3$, $q_7 = p_4$ and $q_8 = p_5$. The block angular matrix relating to the tree in Figure 7, also labelled the L-shaped matrix from its distinctive sparsity pattern, goes as follows.

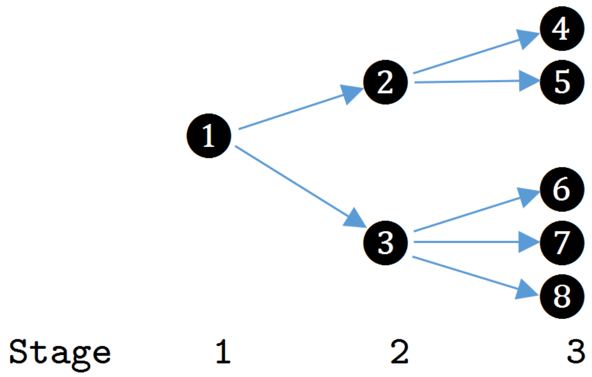


Figure 7: A scenario tree with three stages and eight scenarios

$$
\begin{array}{lllllllllll}
max & c_1^T x_i & + q_2 c_2^T & + q_3 c_3^T & + p_1 c_4^T & + p_2 c_5^T & + p_3 c_6^T & + p_4 c_7^T & + p_5 c_8^T & \\
 & A x_i & & & & & & & & = b \\
 & B_2 x_1 & + C_2 y_2 & & & & & & & = d_2 \\
 & B_3 x_1 & & + C_3 y_3 & & & & & & = d_3 \\
 & & B_4 x_2 & & + C_4 y_4 & & & & & = d_4 \\
 & & B_5 x_2 & & & + C_5 y_5 & & & & = d_5 \\
 & & & B_6 x_3 & & & + C_6 y_6 & & & = d_6 \\
 & & & B_7 x_3 & & & & + C_7 y_7 & & = d_7 \\
 & & & B_8 x_3 & & & & & + C_8 y_8 & = d_8 \\
 & & & & & & & & & x_i \geq 0
\end{array}
\tag{36}
$$

A key issue with regard to scenario trees is the exponential increase in the size of the scenario tree for a given increase in the number of nodes and/or stages. This issue can be greatly aided by an algorithm called Bender's decomposition. This technique is set aside for formulations with a block angular structure. Cornuejols and Tutuncu (2007) explain that the idea behind a Bender's decomposition is to take advantage of the two-stage problem with a block angular structure, such as the formulation in equation 36 above. Due to the structure of the matrix and how the constraints and blocks are related, the first stage can be used in representing the master problem and the second stage resolves the recourse problems. The stages are solved and are repeatedly added to the master problem as inequalities. This problem is solved iteratively, moving through each stage until the last stage. The nodes in the last stage are referred to as the terminal nodes (Cornuejols and Tutuncu, 2007).

Xi Yang and Jacek Gondzio and Andreas Grothey (2011) conclude that Bender's decomposition has been shown to be more efficient both in solution and hardware requirements when compared to the interior point simplex method.

5.5 Scenario tree generation

Zenios and Kouwenberg (2011) discuss that the computational requirement for solving stochastic trees is exponentially related to the addition of each node of a scenario tree. A tree with a large number of nodes quickly becomes unsolvable by the available hardware. This is, therefore, a natural boundary, or constraint, to the quantity of nodes in a scenario tree. This issue needs to be balanced with the discretisation error introduced when using fewer nodes at each stage. The number of nodes in a scenario tree affect the capacity of the scenario tree to represent the distribution appropriately.

Dempster and Thompson (2010) explain that the simplest trees are generated by way of random sampling of historical data. This forms a tree with few nodes and branches having the same number of nodes. On the other hand the more sophisticated trees are set up to fit the moments of a distribution of state variables at each of the nodes. King and Wallace (2011) provide a summary of the common ways to generate scenarios, which includes path and fan based methods, moment-matching, historical sampling and, lastly, an optimal discretisation. The approach adopted in this study for creating the scenario trees is per Hoyland and Wallace (2001), which is the moment-matching method with inputs of the key moments of the asset class distribution and the inter-asset correlation.

An important consideration when setting up scenario trees is the unintended introduction of arbitrage opportunities. This is where a trading profit can be made with no investment. This is tied to the method used to set up the scenario tree (Kouwenberg, 2001). To remain consistent with the *law of one price*, the arbitrage opportunities which would skew the optimisation algorithm are excluded. Klaassen (2002) points out that following the moment-matching procedure of Hoyland and Wallace (2001), arbitrage opportunities may become an issue. A procedure proposed to detect arbitrage by Klaassen (2002) solves this issue. This procedure is based on a review of the scenario tree for arbitrage opportunities and is completed *ex post*. A simple remedy to arbitrage opportunities, when detected in the scenario tree, is to start again with a scenario tree with more branches (Cornuejols and Tutuncu, 2007). Klaassen (2002) suggests that providing ex-ante constraints to the problem can resolve the arbitrage scenarios.

6 Discussion: active vs. passive

In concluding the literature survey, a brief link of the academic theory to the stated aims of this study is provided. Anson et al. (2011) propose that without inefficiency in markets there would be no long-term case for pursuing profits from the markets. This study pertains to the allocation of investments across multiple asset classes where the risk return profile of these asset classes is very different. The top and bottom return quartiles in stock and bond asset classes can be separated by as little as one percent, whereas alternative asset classes can have as much as 25 percent return differential. Different investment philosophies may be more aggressive in seeking excess return, which implies that there is deviation from the security market line and, therefore, a more "active" investment strategy. Hens and Rieger (2010) argue that alpha, return that is in excess of the defined benchmark, is not stable through time and the investor can be more active depending on the investment skill, the smaller the costs of being active, the efficiency of the market and, finally, the less risk averse the investor is. With this in mind, the multi-asset models are developed to capture net excess active returns and test whether the time varying nature of excess returns in an active investment philosophy can significantly exceed a passive approach. If cost and risk aversion parameters are held constant, which in this instance is assumed, then we may argue that markets are efficient and underlying excess returns are insignificant. Potentially, the investor skill is high (maybe by a highly specified model).

Part III

Portfolio model specification and analysis

7 Common portfolio specifications

The following model specifications are common elements for both of the portfolio models in this study:

- *Return requirement:* A portfolio return requirement is based on the client requirement of the hypothetical private-banking wealth-management book. The final return requirement is an aggregate of the clients on the book (further detail is provided in section 9.4);
- *Constraints:* A long-only position in assets are permitted. International exposure of less than 30% is incorporated in the models. This is to ensure the portfolio respects local regulatory constraints for multi-asset funds;
- *Optimisation principles:* The principle needs to be considered largely in conjunction with the client objectives. The client base is comprised of higher-income individuals who are serviced in a private banking segment and who would like to minimise risk against a return target.

7.1 Asset class universe and proxies

The following asset classes are defined as the common universe from which all portfolios can be constructed:

- Equity for RSA and US;
- Bonds for RSA and US;
- Bills for RSA and US;
- Global commodities in USD.
  The commodity index is set up as an equally weighted index of metals, non-energy and energy indexes which are part of the World Bank Commodity Price Data (The Pink Sheet). The pink sheet forms part of the The World Bank (2015) database.

With the exception of commodities data, the asset class data, foreign exchange and CPI is sourced courtesy of Dimson et al. (2002). The data is expressed as yearly total returns and starts from 1900. Please refer to the data methodology in *Triumph of the Optimists* by Dimson et al. (2002). The proxy data for commodities is sourced from the The World Bank (2015) from January 1960. This index is an index comprised of the major commodity markets in equal proportion, which is made up of agriculture, metals and minerals and lastly energy.

Other data proxies are:

- US CPI;

- RSA CPI;
- United States Dollar (USD) South African Rand (ZAR) exchange rate (where this is expressed as x number of rands to purchase a USD dollar). The change from pounds to rand is reflected in this index.

7.2 Holding costs and trading costs

The holding cost assumptions are observed directly from available exchange traded funds (ETF) which can be found on www.ETFSA.co.za for all but US equity and US bonds which have been taken from www.ishares.com. The cost of switching is assumed to be 5 basis points for purchasing and 4 basis points for the sale. The industry practice is to express the costs of holding the ETF as a total expense ratio (TER). This is a yearly figure charged against the asset value. This can be, therefore, directly subtracted from the return in a given year period.

Category	Asset	TER
Local	Bills	0.20%
	Bonds	0.24%
	Equity	0.40%
Global	Bills	0.23%
	Bonds	0.24%
	Equity	0.49%
	Commodities	0.78%
Switching fee	Buy	0.05%
	Sell	0.04%

Table 1: A South African investor's passive costs

7.3 Use of tools in the modelling

The tools and code sets used in this study are summarised in table 2. The area of application is listed by referencing each section in this document.

Tool	**Area of application**	**Section**
SAS	Inferential statistics and summaries	8.2.1
	Stochastic liability model	9.4
	Asset path simulation	9.3
Code set	Vale and Maurelli (1983) method of	8.1.1
	simulation using code set in Fan et al. (2001)	8.2.2
Matlab	Uryasev (2000) CVAR optimisation method	8.1.2
	using Matlab portfolio optimisation	8.2.3
Eviews	TVTP regime switching modelling	9.2
AMPL DEV	Two stage recourse modelling and optimisation	9.5

Table 2: A summary of the use of tools in modelling and data analysis

Part IV

Passive single-period model

8 The passive coherent risk model

8.1 Model description

This portfolio is constructed as a passive portfolio, with a fixed-mix policy which is rebalanced yearly. This portfolio is set up to take advantage of the properties of a coherent risk measure and makes use of the CVAR optimisation framework. This is implemented as part of a long-run passive investment philosophy. This method is formulated with downside sensitivity to avoid the unexpected large losses in financial markets. Xiong and Idzorek (2011) point out that ignoring asset class skewness and kurtosis may severely impact the accuracy of optimal asset allocations. They further show that constructing portfolios that take skewness and kurtosis into account made a significant difference to portfolio returns during the 2008 financial crisis.

Asset allocation is determined by using the optimisation techniques of Rockafellar and Uraysev (2002). Their method for portfolio optimisation can be achieved by linear programming when using a CVAR risk measure. This technique is a single period, discrete time, minimum variance portfolio optimisation. Given that this is based on a VAR model, we have to express a loss threshold, which has been calibrated against the real world notion of frequency of a negative event in financial markets and extent of economic losses.

In order to complete the optimisation of Rockafellar and Uraysev (2002), which may be classified as a chance constrained stochastic programming technique, we require a simulated reference data set of total net returns against which the optimisation will be performed. To simulate the data, we make use of moment-matching techniques prescribed by Fleishman (1978) for the simulation of non-normal data with specific degrees of skewness and kurtosis. Given that financial data exhibits dependence structures as the asset class descriptive statistics summary shows (refer to Appendix A), we use the techniques prescribed in Vale and Maurelli (1983) in the data simulation to address inter-variable correlation.

As this portfolio is constructed from both local and international asset classes, we need a framework to deal with the added risk due to foreign exchange fluctuations. This portfolio is a longer-term passive portfolio, therefore, we take the logic in Leipold and Morger (2007) who argue that real exchange rates are mean reverting and fluctuations are ultimately explained by purchasing power parity over the horizon of the investment. Therefore, given that the investment period is relatively long, we need not explicitly deal with the impact of foreign exchange. Should there be impacts due to foreign exchange fluctuation, it will be captured in the risk measure CVAR and will therefore have an impact on the results of the CVAR optimisation.

8.1.1 Return distribution simulation

For the set of all assets in the defined universe in section 7.1, total returns are simulated to represent the full distribution of possible events. A data set of 100 000 scenarios is constructed to represent

$T = t + 1$ for each variable. The simulation method will capture any non-Gaussian features found in finance asset classes by using the techniques of Fleishman (1978) in section 3.4, whilst accounting for the correlation effects found in financial assets using the techniques specified by Vale and Maurelli (1983) in section 3.4.2. The original works of Vale and Maurelli and Fleishman have been codified in Fan et al. (2001).

Simulated data testing framework

In order to ensure that the data simulation correctly replicates the historical asset class distribution, each simulated distribution is assessed for correctness with the following tools:

- *Comparison of the moments of the distribution:* We perform careful review to ensure that the first four moments of the distributions and correlation are matched appropriately;
- *Visual inspection of histograms and box-plots:* By inspecting the histogram we can be sure that the general shape of the historical and simulated are matched;
- *Visual inspection of the probability Q-Q plots:* The Q-Q plots are relative to the normal distribution; this comparison allows one to review the shape of the modelled and historical distribution relative to the normal;
- *T-test:* The test has a tolerance of 5% and is set up to ensure that the distributions are not statistically different ($H_0 : \mu_{\text{hist}} = \mu_{\text{MCMC}}$).

For detailed testing results, refer to Appendix A.

8.1.2 CVAR portfolio optimisation

The optimisation is done using the methods prescribed by Rockafellar and Uraysev (2002) as described in section 4.4.3. This modelling technique has been enabled in Matlab 2014 and used in this study. When using the CVAR risk measure, the modeller needs to decide on a VAR tolerance level. As the CVAR measure is set up to represent significant loss events in the market, the tolerance level is set at 2% which implies a risk tolerance of a one-in-50-year-event. The following constraints have been put in place. So as to conform to local and regulatory guidelines a restriction of international exposures to a maximum of 30% has been put in place. The second constraint is to allow for long-only positions.

8.2 Analysis

The following section is an analysis of both the historical data and the resultant simulated data. Given that this is a moment matching technique, we are interested that the moments of the simulated data and the historical data match. The Pearson correlation matrix of the historical data and the simulated data match.

8.2.1 Input parameters

The historical statistics are drawn from the in-sample data which spans from January 1900 to December 2007, using data supplied by Dimson, Marsh and Staunton (2002).

Real Returns	**Arithmetic average**	**Standard deviation**	**Skewness**	**Kurtosis**	**CVAR2%**
RSA bill	0.9%	6.8%	-1.74	12.71	-21.3%
RSA bond	1.9%	11.2%	-0.42	3.17	-30.0%
RSA equity	9.5%	24.1%	0.44	2.48	-47.0%
US bill USD	0.9%	4.9%	-0.76	3.67	-15.4%
US bond USD	2.0%	10.5%	0.54	0.81	-19.2%
US equity USD	8.2%	20.8%	-0.25	-0.31	-38.6%
Commodities USD	2.6%	18.7%	1.30	2.95	-28.5%
USD ZAR	2.61%	14.9%	0.93	3.37	-34.4%

Table 3: Historical descriptive statistics: key inputs for the moment-matching simulation

Correlation matrix	**RSA bills**	**RSA bonds**	**RSA equity**	**US bills**	**US bonds**	**US equity**	**Commo -dities**	**USD ZAR**
RSA bill	100%	66%	25%	42%	37%	18%	-28%	6%
RSA bond	66%	100%	46%	22%	37%	21%	-2%	-12%
RSA equity	25%	46%	100%	-1%	2%	45%	36%	-12%
US bill	42%	22%	-1%	100%	59%	18%	-60%	4%
US bond	37%	37%	2%	59%	100%	21%	-40%	20%
US equity	18%	21%	45%	18%	21%	100%	-12%	-12%
US commodities	-28%	-2%	36%	-60%	-40%	-12%	100%	-52%
USD ZAR	6%	-12%	-12%	4%	20%	-12%	-52%	100%

Table 4: Correlation inputs for simulation based on historical data - Pearson correlation matrix

8.2.2 Results of the moment-matching simulation

The moment matching simulation is based on a sample of 100 000.

Real Returns	Arithmetic average	Standard deviation	Skewness	Kurtosis	CVAR2%
RSA bill	0.9%	6.9%	-1.70	12.70	-23.2%
RSA bond	2.0%	11.3%	-0.40	3.60	-29.9%
RSA equity	9.6%	24.1%	0.50	2.40	-46.2%
US bill USD	0.9%	4.9%	-0.80	3.80	-13.7%
US bond USD	2.1%	10.5%	0.50	0.70	-19.0%
US equity USD	8.4%	20.8%	-0.30	-0.30	-40.4%
Commodities USD	2.6%	18.6%	1.30	2.80	-25.4%
USD ZAR	2.65%	15.0%	0.95	3.6	-28.5%

Table 5: Descriptive statistics of simulated data: key inputs for the moment-matching simulation

Correlation matrix	RSA bills	RSA bonds	RSA equity	US bills	US bonds	US equity	Commo -dities	USD ZAR
RSA bill	100%	63%	23%	39%	32%	17%	-25%	4%
RSA bond	63%	100%	45%	20%	36%	20%	0%	-12%
RSA equity	23%	45%	100%	0%	1%	44%	35%	-12%
US bill	39%	20%	0%	100%	53%	18%	-57%	3%
US bond	32%	36%	1%	53%	100%	19%	-36%	20%
US equity	17%	20%	44%	18%	19%	100%	-11%	-13%
US commodities	-25%	0%	35%	-57%	-36%	-11%	100%	-48%
USD ZAR	4%	-12%	-12%	3%	20%	-13%	-48%	100%

Table 6: Pearson correlation matrix of the simulated data

Table 7 reports the comparison of the simulated data to the historical data set. The average absolute difference is measured as the average of the difference between historical and simulated data across asset classes. The maximum absolute difference is measured as the maximum absolute difference between historical and simulated data across asset classes. With all simulations differences are expected, and this experiment is no different. As table 7 reports these differences are not material and the simulation is regarded as accurate.

History vs simulation	Average absolute difference	Max absolute difference
Return mean	0.06%	0.18%
Return std deviation	0.03%	0.07%
Return skewness	0.04	0.07
Return kurtosis	0.15	0.44
CVAR @ 2%	1.98%	6.28%

Table 7: Comparison of historical and simulated data

8.2.3 Optimisation results

The efficient frontier represents all the feasible combinations of risk and return which are optimal. The efficient frontier in Figure 8 illustrates these combinations for the universe of asset, and reports the CVAR risk measure on the x-axis and expected return on the y-axis.

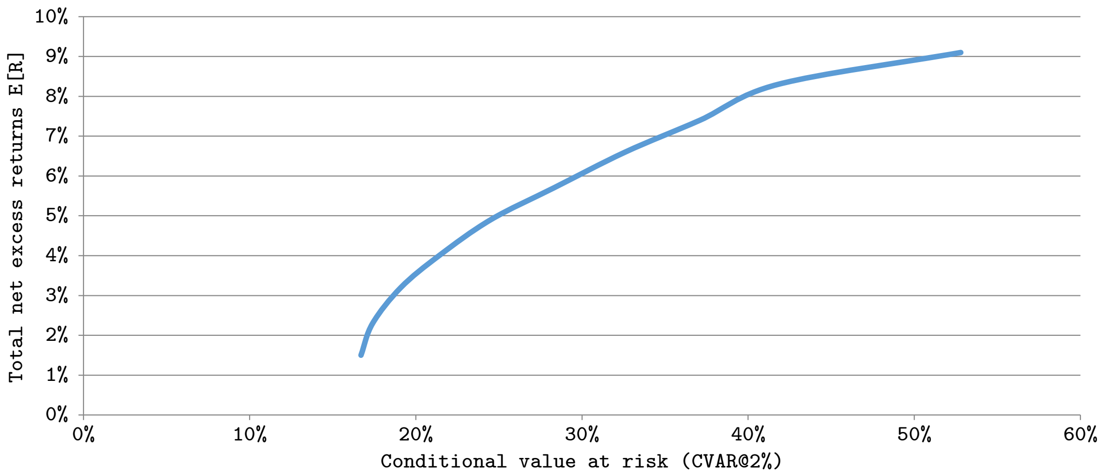
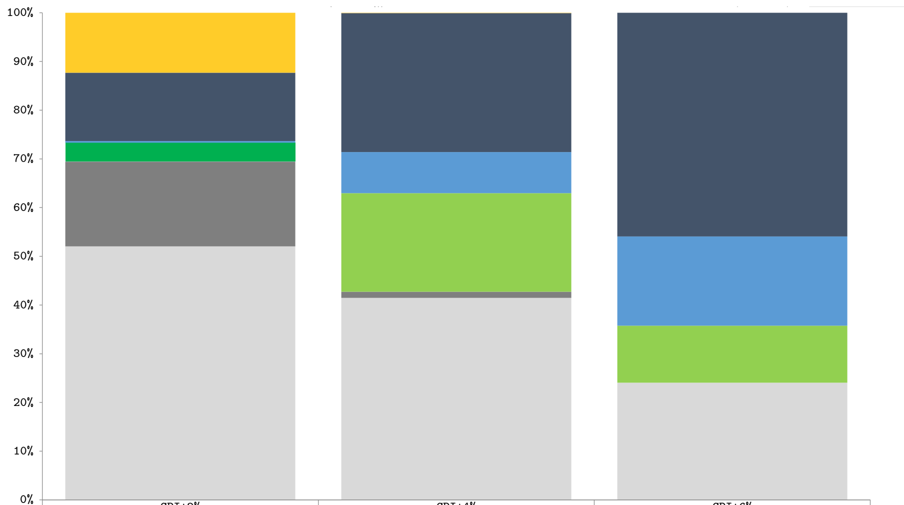


Figure 8: CVAR Efficient Frontier

Three points on the efficient frontier have been selected to match the client risk return requirements. The risk appetite detailed in section 9.4.1 provides further detail. The optimal asset allocation at these points on the efficient frontier are illustrated in figure 9.



	CPI+2%	CPI+4%	CPI+6%
Commodities ZAR	12%	0%	0%
RSA Equity	14%	29%	46%
US Equity ZAR	0%	8%	18%
US Bonds ZAR	0%	20%	12%
RSA bond	4%	0%	0%
US Bills ZAR	17%	1%	0%
RSA bill	52%	41%	24%

Figure 9: Optimal CVAR portfolios (CPI is in ZAR)

Dynamic CVAR portfolio allocation

Per the liability model specified in this study, each client is linked to a return requirement that is driven by risk appetite function linked to age (please refer to 9.4.1 for the set of heuristic rules). The total portfolio asset allocation, C, over time is calculated as follows:

$$C_{at} = \sum_{r=1}^{3} W_{ar} P_{rt} \tag{37}$$

where:

a	denotes the asset classes;
t	is the time period of the analysis, between 2005 and 2014;
r	denotes the risk based portfolios: low (1), medium (2), and high (3);
W_{ar}	is the asset allocation of each portfolio in r;
P_{rt}	is the proportion of the book applied to each portfolio, r, at each point in time, t.

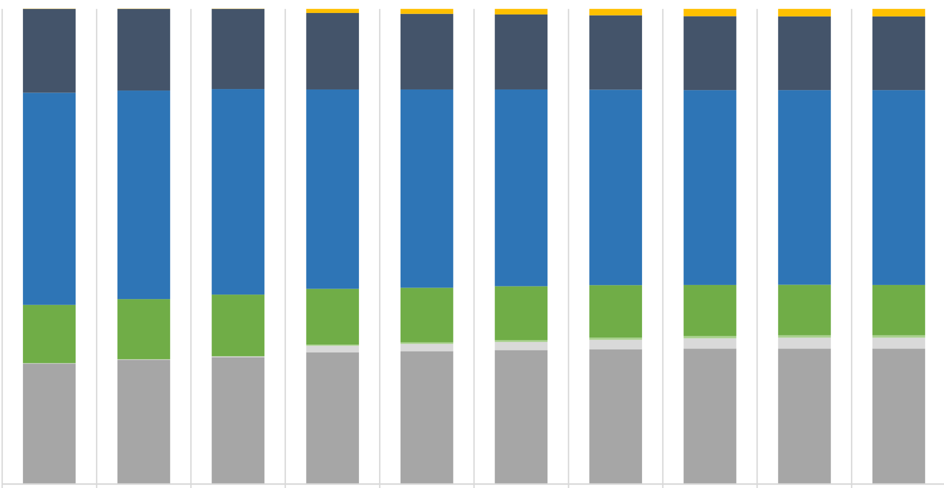
The allocation of the total investment amount to each risk category is reported in table 8. This is determined in the client simulation in section 9.4.1.

Single-period asset allocation over time

Figure 10 indicates that the asset allocation does not change significantly over the time period of this study.

	Risk Appetite		
Year	**Low**	**Med**	**High**
2005	0.0%	7.3%	92.7%
2006	0.0%	11.7%	88.3%
2007	0.0%	15.1%	84.9%
2008	7.0%	9.9%	83.1%
2009	8.2%	9.2%	82.6%
2010	9.3%	8.8%	82.0%
2011	10.9%	7.4%	81.8%
2012	12.3%	5.7%	82.1%
2013	12.8%	5.0%	82.2%
2014	12.8%	5.0%	82.2%

Table 8: Aggregate allocation to risk levels over time.



	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014
Commodities ZAR	0%	0%	0%	1%	1%	1%	1%	2%	2%	2%
US equity ZAR	18%	17%	17%	16%	16%	16%	16%	16%	16%	16%
RSA equity	45%	44%	43%	42%	42%	41%	41%	41%	41%	41%
US bonds ZAR	12%	13%	13%	12%	12%	11%	11%	11%	11%	11%
RSA bond	0%	0%	0%	0%	0%	0%	0%	0%	1%	1%
US bills ZAR	0%	0%	0%	1%	2%	2%	2%	2%	2%	2%
RSA bills	25%	26%	27%	28%	28%	28%	28%	28%	29%	29%

Figure 10: Aggregate asset allocation of over time.

Part V

Active multi-period framework

9 The dynamic multi-stage integrated asset-liability model

Dermine (2011) introduces asset liability management (ALM) as a risk and value control process. It is known for its traditional application in interest rate risk and liquidity risk on banking balance sheets, insurance books and institutional pension funds. Zenios and Kouwenberg (2011) describe ALM problems as dealing with the planning of financial resources under uncertainty, in particular actuarial risk, demographic conditions, capital market and economic environments. The validity of stochastic programming (the mathematical tools for dealing with uncertainty) when used in an ALM setting, is enhanced and is able to address multiple correlated sources of information from both the liabilities and asset classes, spanning multiple time horizons in a dynamic fashion whilst dealing with risk aversion (Zenios and Kouwenberg, 2011).

In imparting financial advice to wealthy individual investors as part of wealth management, an important part of this advice relates to asset allocation. Amenc et al. (2011b) observe that the recommendations of wealth management for portfolio asset allocation mostly rely on single-period mean-variance portfolio optimisation techniques, which cannot yield correct asset allocation for a number of reasons:

- The parameters used in the optimisation, which include expected return, volatility and correlation, are defined as static in the optimisation. This is not supported empirically;
- The optimisation technique does not take into account differing lengths of investment horizons;
- Perhaps the most important issue is that the client commitments or liability constraints and risk factors affecting the client position at a given time are not explicitly modelled, nor directly taken into account, in the asset allocation process.

In a wealth management process client information is collected and formalised, including the client's objectives, expressions of risk appetite, hard and soft constraints and investment horizon. Amenc et al. (2011b) explain that although the information is often detailed, the portfolio construction tends to have very little to do with the clients specific needs. Amenc et al. (2011b) argue that ALM optimisation techniques, including stochastic programming, used by institutional investors can be neatly transposed to a wealth management problem. This is because the original institutional methods have been designed to meet individual investors' specific constraints, horizons and objectives. They can all be summarised into a single-state variable: the commitments that represent the liabilities of a client or the *liability portfolio*.

An ALM model for wealth management requires a model for the future value of both the assets that will be allocated to the liability portfolio and its future value. The liability portfolio will hereafter be referred to as the liability. Finally, the ALM model will require a model for the optimal asset allocation conditional on the requirements defined by the liability.

9.1 Model construct and formulation

In this section we describe and formalise a model to generate the dynamic asset return paths, scenario trees, liability profiles and the multi-stage ALM portfolio choice model. The schema below breaks down, what is a complex portfolio model with multiple processes, into three parts. The first part of the model, A in figure 11, is a parameter-forecasting model. The model is a time-varying regime-switching model. The model provides forward-looking asset class risk and return estimates and forward looking estimates of the economic drivers of the liability model, namely the CPI and GDP. The risk and return parameters are used in the simulation engine for asset pricing in B in figure 11. The estimates of CPI and GDP are direct inputs in the stochastic liability simulation process in C in figure 11. Since the key input parameters in the dynamic modelling of asset and liabilities are based on the same underlying economic data, the simulation processes of asset and liabilities are thus integrated in a dynamic setting. The simulations are summarised into an asset-pricing scenario tree per each asset class in B and an expected liability profile for each sub-category of client in C. Along with model constraints, both A and B are key inputs for the multi-stage ALM portfolio choice model, D in figure 11.

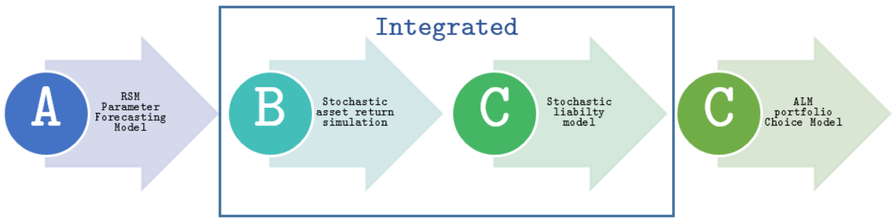


Figure 11: Model schema: multi-period dynamic stochastic optimal ALM portfolio model

The parameter-forecasting model is based on the time-varying Markov regime-switching model. This solves for the non-linearities found in financial markets, providing the dynamic return, risk and dependence parameters. The asset-pricing model is a key input into the portfolio choice model. This stochastic simulation takes into account uncertainty of the customer liability. There are functions built to represent inter-relationships based on economic variables and asset prices in a dynamic process. Both the liability and asset return paths are generated using models which address risk and return dynamics over the horizon of the investment period. A multi-stage optimisation portfolio-choice model will be used for dynamic portfolio decisions.

9.2 Dynamic return forecast: regime switching model

Leipold and Morger (2007) conclude that the correctly specified regime-switching model offers substantial economic reward. Piger (2011) explains that the TVTP model is varying based on conditioning information over time and is a particularly desirable feature in the task of extracting active net positive portfolio return from markets. In the South African context Moolman (2003);

Bosch and Ruch (2012) built a TVTP model using endogenous information in the South African context. The model is statistically significant in predicting turning points in the South African business cycle. Thupayagale and Molalapata (2002) conclude that accounting for time-varying effects in the context of portfolio management is important, especially in emerging market interactions with the US (where South Africa is part of the analysis). In the context of portfolio optimisation, Ang and Bekaert (2002a) make use of TVTP to establish regimes for optimal portfolios. The results are in favour of using a TVTP.

Hull (2004) points out that volatility in equity markets tends to increase more when the return decreases than it does when the return increases. This is otherwise known as the leverage effect (Hull, 2004). Objectives of the model include capturing asset-pricing dynamics by allowing the model to receive exogenous information over time and update key parameters. Put differently, a dynamic portfolio model requires a predictive ability and a translation of how the forward-looking information relates to a portfolio position at a point in time. The portfolio-choice model must be able to react to changes in parameter through time (anticipative and adaptive). The model in this study is anticipative in the predictive asset pricing model and adaptive in the portfolio choice model. The portfolio choice model can adapt given evolving conditions. Firstly, for a portfolio model to be predictive, we require a model to best estimate asset returns in the next time period and this is done with a TVTP regime-switching model. The model construct of Mulvey and Zhao (2010) is largely used in the formulation of a dynamic asset-pricing model in this study. A regime-switching model is used to characterise the dynamics of the economic indicators. A second regime-switching model predicts the asset class returns based on the regime switching economic indicators. The transition-probabilities are time varying and are modelled using the economic variables as endogenous variables (which is in contrast to Mulvey and Zhao (2010)). The regime switching models are fitted using a time varying transitional probability regime-switching model. The model output is a $t+1$ return model for the return and risk expectation. These parameters are used as the key parameters of the stochastic simulation process.

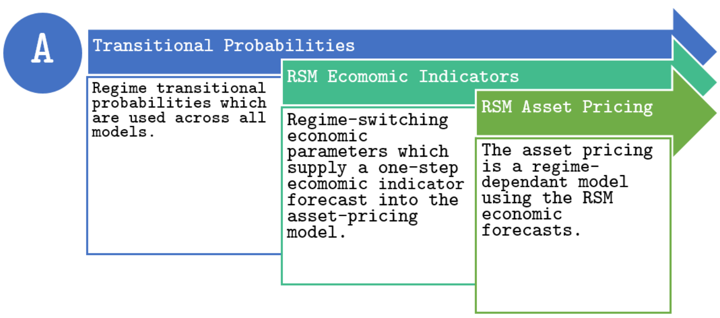


Figure 12: Regime-switching model construct

In this context the aim of a regime-switching processes is to be able to accept and adapt to sudden market phenomena found in the market information flow (Mulvey and Zhao, 2010). The TVTP regime switching model follows the notation set out in section 4.5.4.

Stochastic volatility (SV) is an area of interest in academic research. Numerous studies have revealed clustering of asset-price volatility and it is observed in financial market as being time-varying (Barndorf-Nielsen and Veraart, 2013; Fabozzi et al., 2007b). SV models consider volatility as a variable term which should be forecast (Fabozzi et al., 2007a). The regime-switching model construct can deal with risk, as it does with return in a time-varying fashion. The clustering and time varying nature is catered for by the drivers of the endogenous variables of the regime. The regime-switching model is critical to capture heteroskedasticity across the increasing and decreasing markets (bull and bear markets). Longin and Solnik (1995) findings that the market states or regimes affect correlations supports the use of an regime-switching model for risk modelling.

The modelling of TVTP regime switching models is done in Eviews version 9. The results of modelling are covered below.

9.2.1 Universe of predictive variables

The explanatory variables used for the modelling of asset classes, CPI and RSA GDP growth are drawn from macro-economic indicators and capital market indicators. The underlying data was attained from The World Bank development indicator database (The World Bank, 2015) and Dimson et al. (2002) asset class data. Section A.2 the Appendix lists the variables considered in the modelling.

The selection of explanatory variables was based on simple ranking of explanatory power of each variable to the dependent variable. Factors were introduced to the TVTP model in a stepwise fashion. The criteria for model selection is covered in section 9.2.2. Given the data constraints the number of factors utilised in the modelling was a limiting factor.

9.2.2 Model selection criteria

The logic follows that if a model forecast cannot improve on using a simple average based on historical data, the model won't be utilised. Given that a model converges in the modelling process, the model selection procedure is based on the following criteria. Firstly, each model is optimal in terms of its ranking based on an information criterion. Secondly, standard deviation of the model residuals is less than the standard deviation of returns. This is a test of the variance around the historical average versus the variance around the TVTP model prediction.

The TVTP models are formulated with total return regimes. Mulvey and Zhao (2010) note that regimes are not directly observable and establishing the number of regimes is not an easy task.

Guidolin (2011) in a review of the regime-switching models in empirical finance argues that ranking based information criteria are more favourable than formal hypothesis testing for model selection. Following the approach of Altug and Bildirici (2010) and Mulvey and Zhao (2010), the Bayesian Information Criterion (BIC) is used to select the most appropriate model. The BIC measure is sensitive to both the number of regimes and parameters used in the model. The BIC selection criterion allows for a balance between a parsimonious model and goodness-of-fit of the model.

$$BIC = -2 \cdot \ln \hat{L} + k \cdot \ln(n) \tag{38}$$

where $\hat{L}$ is the maximised value of the likelihood function for the estimated model. k is the number of parameters estimated, which we know is also determined by the number of regimes and n is the sample size. The sample size of the observed data is T. The optimal model is that model which minimises the BIC, subject to the number of regimes and model parameters.

9.2.3 TVTP BIC test results

The regime-switching model modelling results are listed in Table 9. The table includes the final Log-likelihood and BIC measures, the number of regimes and parameters used in the modelling (all models are ZAR based):

Model name	Variable	# Regimes	Log likelihood	BIC
RSA bill	Bond-bill spread	2	817.4	-5.30
RSA bonds	Bond-bill spread	2	123.14	-3.39
RSA equity	Bond-bill spread	2	78.69	-0.69
US bills	Bond-bill spread	2	1001.1	-6.52
US bonds	Bond-bill spread	2	897.1	-7.27
US equity	Bond-bill spread	2	244.1	-1.24
Commodities	%Δ in USDZAR	2	281.3	-0.86
RSA inflation	Bond-bill spread	2	797.8	-4.45
RSA GDP	RSA consumption	2	479.0	-6.95

Table 9: TVTP BIC testing results.

All asset classes have models which converged and thus eligible for further assessment. The models reported in Table 9 have the highest BIC and are the final optimal models. From the BIC results that are reported in Table 9, the US and RSA bills models and the RSA GDP model report the highest BIC results. The lowest BIC results are reported for equity and commodities which is not an unexpected result.

Table 10 reports the comparison between the in-sample standard deviation and in-sample standard deviation of residuals. The criteria requires that in order for the model output to be applied in the asset pricing simulation, the standard deviation of residuals must be less than the standard deviation of the sample. This is true for all models, as the comparison in Table 10 shows:

Asset class	**Historical returns**	**Model residual**
RSA bills	3.50%	1.52%
RSA bonds	4.93%	3.58%
RSA equity	22.0%	14.96%
US bills	2.20%	0.81%
US bonds	1.36%	0.58%
US equity	17.65%	11.69%
Commodities	16.2%	10.63%
RSA inflation	4.68%	2.30%
RSA GDP	0.94%	0.65%

Table 10: Comparison of standard deviation of the sample and standard deviation of residuals.

This is a favourable modelling result and making use of the TVTP models is expected to be an improvement in model forecast versus a simple average of historical returns.

9.2.4 Coefficients of regime-switching model TVTP model

The coefficient of the time-varying regime-switching models are captured in Table 11:

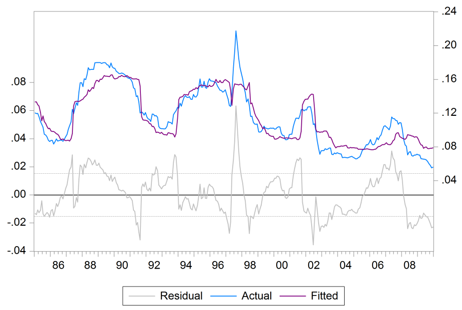
	RSA bills	**RSA bonds**	**RSA equity**	**RSA inflation**	**RSA GDP**	**US bills**	**US bonds**	**US equity**	**US commodities**
				Regime-specific coefficients					
Regime 1									
μ	0.052	0.048	-0.087	0.037	0.024	-0.001	0.011	0.179	-0.068
σ	0.004	0.006	0.021	0.0031	0.001	0.001	0.002	0.010	0.008
Log(sigma)							-5.025		
Regime 2									
μ	0.109	0.030	0.265	-0.017	0.038	0.024	-0.001	-0.129	0.192
σ	0.004	0.006	0.015	0.004	0.001	0.002	0.002	0.019	0.011
Log(sigma)							-5.557		
				Non-switching terms					
Log(sigma)	-4.21	-3.512	-1.957	-3.854	-5.143	-4.801		-2.204	-2.300
RSA Bills	0.368								
RSA CPI				0.813					
US Bills						0.604			
US Bonds							0.870		
				Transitional probabilities					
P11	0.982	0.934	0.957	0.951	0.930	0.982	0.920	0.974	0.971
P21	0.036	0.065	0.026	0.066	0.210	0.026	0.062	0.089	0.043
P12	0.018	0.060	0.043	0.049	0.068	0.018	0.074	0.025	0.028
P22	0.964	0.935	0.974	0.934	0.790	0.974	0.930	0.911	0.957

Table 11: Regime-switching model TVTP model coefficients (modelling outputs from Eviews)

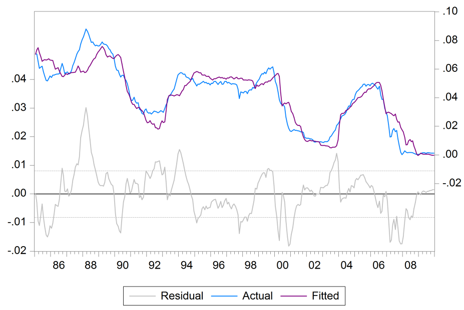
9.2.5 Modelling residuals

The in-sample model residuals, which includes data from 2004 to 2010, are shown in Figures 13 and 14. The charts illustrate the actuals, model forecast and model residuals for each model. As expected there are time periods in which the residual error is large, and other periods with high forecasting accuracy.

It is important to highlight that in the modelling process there is no use of more sophisticated econometric modelling techniques, such as co-integration and vector error correction. The use of these techniques has the potential to improve the model forecasts, but these methods are outside of the scope of this study.

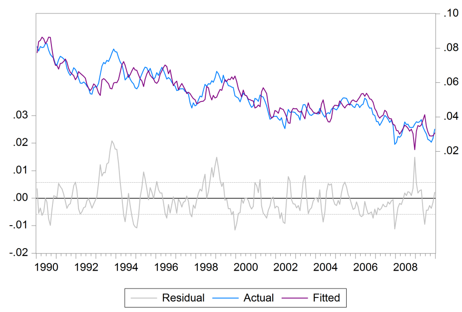


(a) Model results of RSA bills.



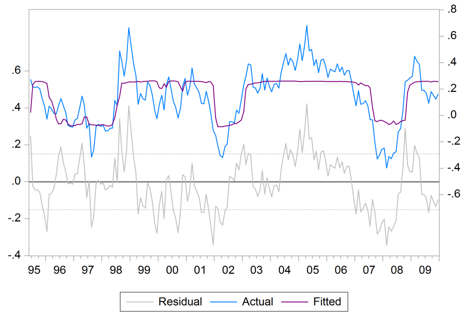
(b) Model results of US bills.

(c) Model results of RSA bonds.

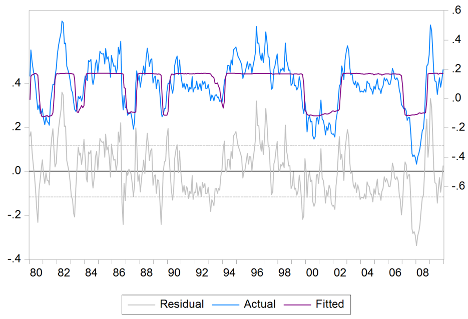


(d) Model results of US bonds.

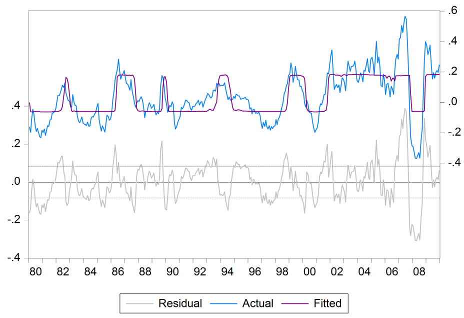
Figure 13: Actuals, model results and residual error (charts are an Eviews output).



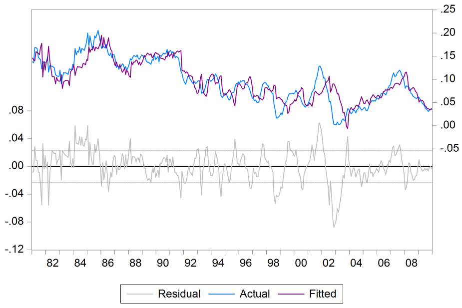
(a) Model results of RSA equity.



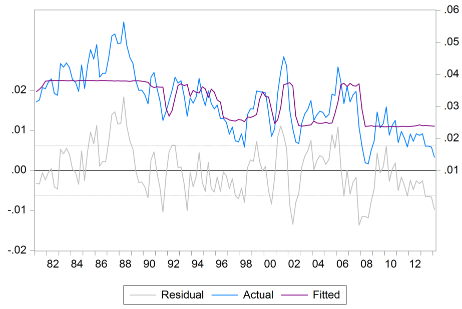
(b) Model results of US bills.



(c) Model results of Commodities.



(d) Model results of RSA CPI.



(e) Model results of RSA GDP.

Figure 14: Actuals, model results and residual errors (charts are an Eviews output).

9.2.6 TVTP model forecasts

Table 13 report the one-period ahead model forecasts for return.

	Return forecast for the period				
From	Dec-2004	Dec-2005	Dec-2006	Dec-2007	Dec-2008
To	Dec-2005	Dec-2006	Dec-2007	Dec-2008	Dec-2009
RSA Cash	7.9%	7.7%	8.4%	9.1%	9.2%
RSA Bonds	0.2%	-4.4%	-4.2%	-3.0%	-3.9%
RSA Equity	25.9%	25.8%	25.2%	-3.7%	23.2%
US Cash ZAR	3.7%	4.6%	2.9%	1.8%	-0.03%
US Bonds ZAR	4.7%	4.9%	4.5%	3.5%	2.1%
US Equity ZAR	25.9%	16.9%	15.6%	-10.9%	17.2%
RSA GDP	2.8%	2.9%	3.2%	3.1%	2.7%
RSA Inflation	5.6%	6.2%	8.2%	10.6%	6.3%
Commodities	18.1%	17.8%	17.6%	-6.1%	17.0%

Table 12: In-sample regime-switching TVTP model return forecast

	Return forecast for the period				
From	Dec-2009	Dec-2010	Dec-2011	Dec-2012	Dec-2013
To	Dec-2010	Dec-2011	Dec-2012	Dec-2013	Dec-2014
RSA Cash	7.9%	7.3%	7.3%	7.1%	7.2%
RSA Bonds	2.8%	2.5%	2.9%	-4.5%	2.8%
RSA Equity	25.7%	21.0%	22.3%	26.0%	25.7%
US Cash ZAR	-0.01%	0.04%	-0.04%	0.00%	0.00%
US Bonds ZAR	3.1%	2.8%	1.7%	2.5%	2.5%
US Equity ZAR	17.2%	15.3%	17.1%	17.1%	17.1%
RSA GDP	2.5%	2.7%	2.7%	2.6%	2.6%
RSA Inflation	3.8%	6.2%	5.9%	7.2%	7.4%
Commodities	18.4%	18.3%	-6.0%	-6.1%	-6.1%

Table 13: Out-of-sample regime-switching TVTP model return forecast

9.3 Stochastic asset-pricing model

The asset pricing model expresses the future pricing paths for each asset-class, GDP and CPI. The model is built using stochastic differential equations. The asset pricing simulation model was designed as part of this study and is formulated as follows:

$$X_t = I_{S(t)} \left[L_{(t)} - \mu_{(t)} X_t \right] + I_{S(t)} \left[\gamma_{(t)} - V_{(t)} \right] \varepsilon_t, \tag{39}$$

where $I_s = \text{Diag}[S_1 \cdots S_n]$ and

- I_s is the matrix of mean return reversion speeds (the rates of mean return reversion) and is based on a standard Ornstein-Uhlenbeck model where the parameters are calculated using a regression;
- L is mean reversion drift levels (long-run expected mean return);
- X_t is the state vector of process variables;
- μ is the expected instantaneous rate of return matrix drawn from the regime-switching TVTP model, refer to Table 13;
- $S_{(t)}$ is the vector of mean risk reversion speeds (the rates of mean risk reversion) and is based on a standard Ornstein-Uhlenbeck model where the parameters are calculated using a regression;
- γ is the mean reversion volatility levels (long-run expected volatility of return), is standard deviation of returns over the full horizon of investment;
- V is standard deviation of returns over a period of one year;
- ε_t is a standard Brownian motion vector, the random numbers are generated using the Mersenne Twister algorithm which is enabled by SAS 9.4 software. The number generator is seeded using the SAS time based function 'datetime'.

Where ε_t is modelled using Brownian motion processes which are conditionally correlated in each regime such that $E\left(dW_t^i dW_t^j\right) = \Gamma_{i,j} dt$ where $\Gamma_{i,i} = 1$ for $\Gamma_{i,i} = 0$ if $i \neq j$. The returns are assumed to be uncorrelated in this analysis.

The return is calculated as a geometric return in each time period, t. This can be seen as the expected return over the full holding period.

Results of the model simulation

Figure 15 and 16 are boxplots of a 1000 scenarios for both RSA equity and commodities for the year 2007 and 2008. The x-axis plots 10 holding periods into the future. The boxplots are specified as follows:

- The median of scenarios in each holding period is represented by the line joining the boxes;
- The box around the median indicates the return level of 25th and 75th percentiles;
- The whisker which extends from top of the box illustrates the returns between the 25th and 1st percentile. The bottom whisker illustrates returns between the 75th and 99th percentile.
- The bubbles indicate the scenarios which lie outside of the 1st and 99th percentile.

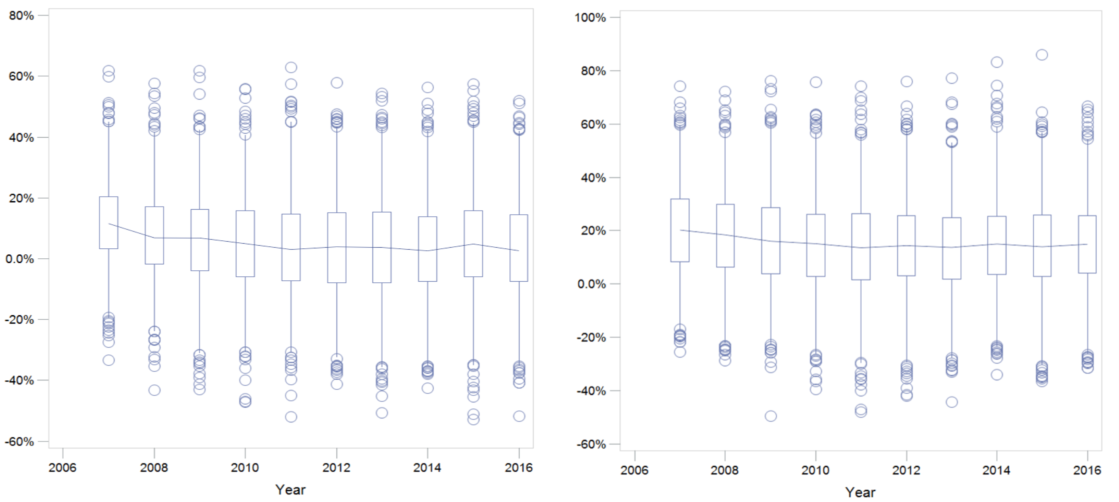


Figure 15: Asset return simulation for commodities (left) and RSA equity (right) in ZAR, based on the 2007 forecast (SAS output)

Figure 15 illustrates an example of the asset return simulation over the investment horizon for RSA equity and commodities in ZAR based on the TVTP forecast from 2007. The instantaneous equity return (T_0), the forecast from the regime-switching model, for the first period is lower than the long term average. The same is true for commodities where the regime-switching model forecast for the first period is higher than the long term average.

Figure 16 illustrates an example of an asset return simulation over the investment horizon for RSA equity and commodities in ZAR based on the TVTP forecast from 2008. The TVTP forecast is indicating that one could expect an equity return in the short run which is higher than the long-run average, this is illustrated by the downward trending median line. The dispersion of returns decrease over the investment holding period that is due to the mean reversion of standard deviation.

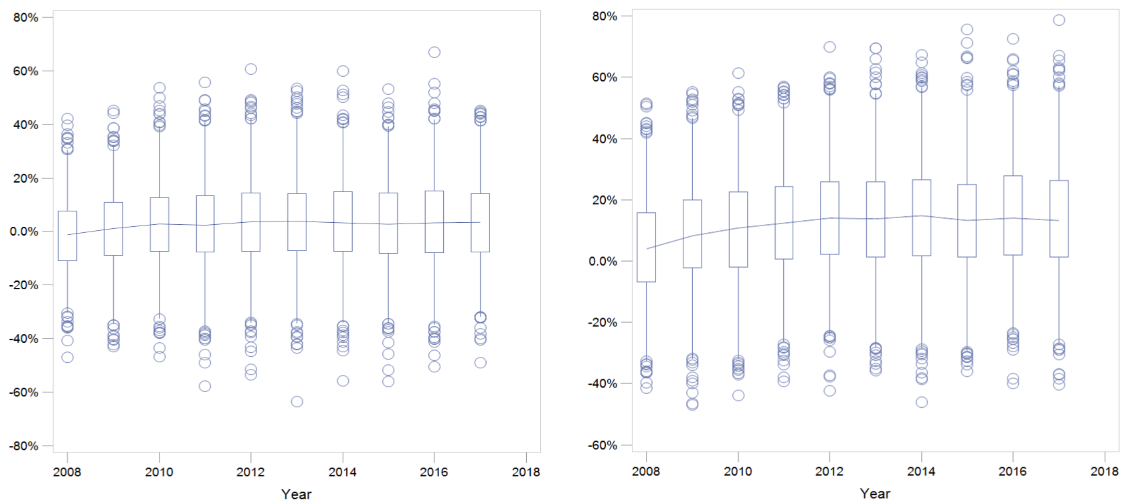


Figure 16: Asset return simulation for commodities in ZAR (left) and RSA equity (right), based on the 2008 forecast (SAS output)

9.4 Integrated stochastic liability model

As Martellini and Milhau (2008) point out, ALM distinguishes itself from an asset only investment management framework. ALM does not focus on the total terminal wealth but rather the total terminal wealth relative to the terminal value of future liabilities. The term used in Mulvey et al. (1997) is *integrated financial risk management* where liabilities and assets are integrated in a dynamic stochastic setting. The liability which is the return expectation in this exercise where the key drivers of the relevant risk factors or the drivers of value are inflation, GDP and risk appetite.

The liability profile, which determines the return requirement in a dynamic way, is created from simulating a sample of clients by simulation with characteristics aimed at broadly representing a South African private bank's wealth management practice. The clients are grouped into private banking segments. For the client segments and distribution of assets between segments we refer to McKinsey Global Private Banking Survey of 2013 (McKinsey & Company, 2013). The distribution of assets between client segments is based on countries with strong inequality, which is typified in the report by a relatively small middle-income class and significant assets held by the wealthy and high income class. The proxy for the asset distribution among segments is based on South African peers that includes India, Brazil and Mexico. These countries are selected as they all have unequal wealth and asset distribution. The asset value of ultra high net worth which are clients with asset values of more than $30 million makes up 54% and high net worth, for clients with asset value between $1 million and $30 million, makes up 24% of the private banking assets. The remainder is taken up by the affluent segment (clients with asset values between $100 thousand and $1 million), which meets the definition of a private banking client in this study.

Each 'client' in the simulation experiment is classified into one of four segments to align with the private banking client segmentation principles in McKinsey & Company (2013). Please refer to Table 14 to review the empirical assumptions for each of the following segments:

UHNW: As is typical for a private banking book, a small segment of clients (300 or 5% of the number of clients) account for a significant proportion of the book. The UHNW clients is set up to represent that client segment. They require no withdrawals or contributions; they just have a large upfront investment. These clients are assumed to have the highest risk appetite.

HNW: Customers with an average age of 55 with relatively high upfront contributions and withdrawals. Some of these clients will be expected to retire over the investment horizon.

Affluent – middle aged: Customers with an average age of 45 with smaller upfront contributions than HNW but have the highest monthly contributions.

Affluent – young: Customers with an average age of 35, with no upfront contribution, it is the segment with the smallest on-going contributions varying between R3 000 and R5 000.

9.4.1 Liability value model

The liability model has stochastic inputs and the data is generated using simulation. The model will establish the expected value of the liabilities for each point on the investment horizon for each client in the sample. The valuation model is repeated for each year between 2005 and 2014:

$$L_t = \sum_{i=1}^{I} \left(U_{it(0)} + L_{it-1} + C_{it} e^{g_t t} (1 - \mathring{R}_{it}) - W_{it} e^{x_t t} \mathring{R}_{it} \right) e^{\mathring{A}_{it} t} \left(1 - \mathring{E}_{it} \right) \tag{40}$$

For every individual ($i = 1..I$) across each time period, t, the value of the total liability, (L) is the future value of the liability. Following the formulation above, L is determined by the following factors:

- *Retirement age indicator ($\mathring{R}$):* The indicator is a dummy value and is forecast to the expected date of retirement. For an age greater than retirement age, the indicator is set to one. The retirement age is assumed to follow a normal distribution.
- *Starting age*: Starting age is a key assumption from which client age is calculated. Client age incrementally increases for every year and forecast year in the model. The starting age of clients are assumed to be normally distributed. The current age of client at t is a natural function of the date at t and start date.
- *Upfront amount invested (U):* expressed in ZAR at $t = 0$. This is conditional on the client category, and represents the existing wealth the client adds to the total book at starting age.
- *Contributions (C):* Expressed in ZAR at $t = 0$, contributions are conditional on the client segment and retirement status. Contributions are made yearly at the beginning of each period. Increases to contributions are based on GDP growth rates (g_t) and therefore vary according to the forecast GDP value (see the asset pricing simulation model which simulates GDP expectation in section 9.3 for further detail).
- *Withdrawals (W):* Expressed in ZAR at $t = 0$, withdrawals are conditional on the client category and retirement status. Withdrawals are made yearly at the beginning of each period. The withdrawals are set up to vary according to levels of inflation. Increases in withdrawals are based on CPI (x_t) and therefore vary according to the forecast inflation value (see the asset-pricing simulation which drives the inflation expectation model in section 9.3 for further detail).
- *Expected date of exit ($\mathring{E}$)*: This indicator is a dummy value which is forecast. The value is based on a standard actuarial life expectancy models making using the PA90 table for South Africa published by The SOA (2015). The clients who exit, or die, are assumed to withdraw their holdings and therefore carry a dummy value of 1. Bell and Miller (2013) define the actuarial conversion from the PA90 table to life expectancy at every age. The results are illustrated in Figure 17.

- *Investment term* ($\mathring{T}$): The investment term is a measure of time in year from the current year to the expected date of maturity of investment. For instance if the current year is 2008 and the expected date of maturity is 2014, then investment term is 6.

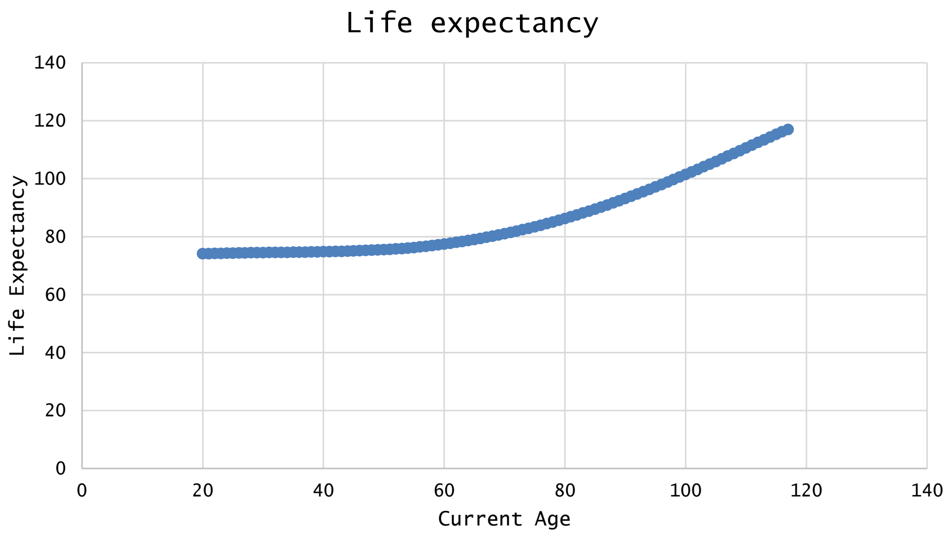


Figure 17: Actuarial life expectancy.

- *Risk appetite* ($\mathring{A}$) is the required return and is determined by a simple deterministic model where the risk aversion is assumed to increase with age. The model applies different risk appetite levels which results in different real return requirements, of CPI+2%, CPI+4% or CPI+6%. These are linked to client age and the retirement indicator. The assumed highest risk appetite is up to and including age 55 (CPI+6%). The lowest risk appetite is set aside for retired individuals (CPI+2%). The remainder are assumed a moderate risk appetite which implies a required return of CPI+4%. UHNW clients are assumed to have the highest risk appetite which is set at a return expectation of CPI+6%.

The summary of liability model simulation assumptions (with standard deviation in brackets) in Table 14 :

The results show that the UHNW clients which make up 6% of the book by frequency and account for 54% of the value of the book. In contrast the young and middle affluent segments combined, constitutes 85% of the book by frequency yet only 22% of the value of the book. The simulated book is reflective of the peer countries displaying wealth inequality in the study by McKinsey & Company (2013). The results of the simulation exercise to build a the representative book are reported in Table 15.

	UHNW	**HNW**	**Affluent –middle**	**Affluent –young**
Risk appetite	High	By age	By age	By age
Upfront assets*	10 000k(2 000k)	2 000k(1 000k)	100k(75k)	0
Yearly contribution	0(0)	60k(12k)	45k(4 5k)	24k (2 4k)
Yearly expenses	500k(5k)	300k(3k)	300k(3k)	300k (1k)
Contributions increase	GDP	GDP	GDP	GDP
Expenses increase	CPI	CPI	CPI	CPI
Age at T_0	50(5)	55(3)	45(3)	35(3)
Retirement age	60(2)	60(2)	60(2)	60(2)

*k = denotes thousands of rands.

Table 14: Liability book breakdown

	UHNW	**HNW**	**Aff-middle**	**Aff-young**
Number of clients	300	465	1000	3500
% of book by frequency	6%	9%	19%	66%
% of book by value	54%	24%	11%	11%

Table 15: Liability model assumptions

9.4.2 Stochastic liability

The liability value model as we have explained thus far is largely a deterministic system that reports the liability value of the simulated population, which is broken down by segments of the book. In this study we are interested in the uncertainty inherent in the value of liabilities due to the random process observed in CPI (x_t), and GDP (g_t), in equation 40. To that end, the entire liability model is looped in a simulation exercise which recasts the book by allowing long-run GDP and long-run inflation as part of the Ornstein-Uhlenbeck model to follow a stochastic process using a normal distribution. The long-run parameters are determined empirically using the historical data.

9.4.3 Model Results

In a similar finding to Martellini and Milhau (2008), values in the liability model are directly affected by inflation. We incorporate time-varying forecasts (TVTP) that address changing inflation rates for the remaining term of the liability. Figure 18 to 21 are illustrations of the distribution of liability values for one client from each of the specified segments. This illustrates the different sensitivity in value to the same change in inflation rate and GDP.

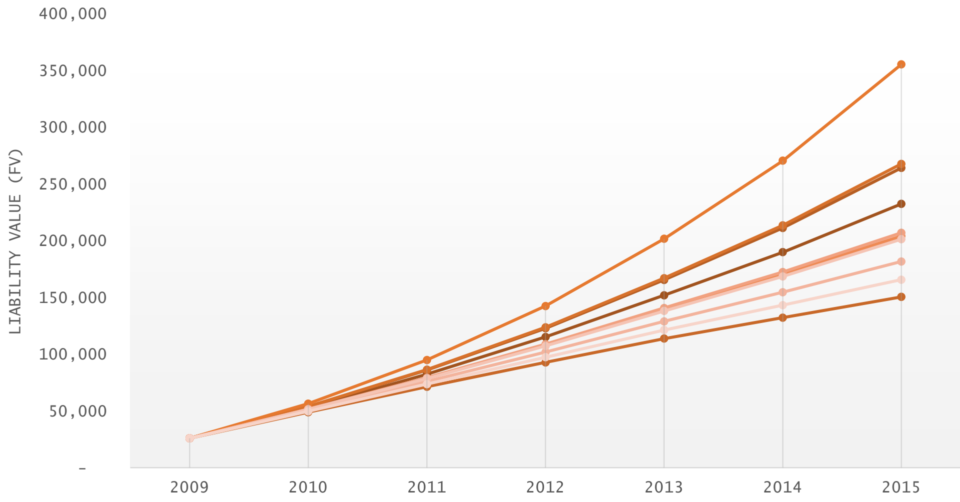


Figure 18: The liability values for an affluent young client.

Figure 18 illustrates the sensitivity of a client in the affluent-young segment to changes in the GDP and inflation assumptions over the term of investment horizon. Figure 21 illustrates that the UHNW segment is the most sensitive. This is due to the cash flow that is all in an upfront contribution, effectively an endowment. The other segments all contribute over the investment horizon which lowers the liability value sensitivity to inflation rate and GDP.

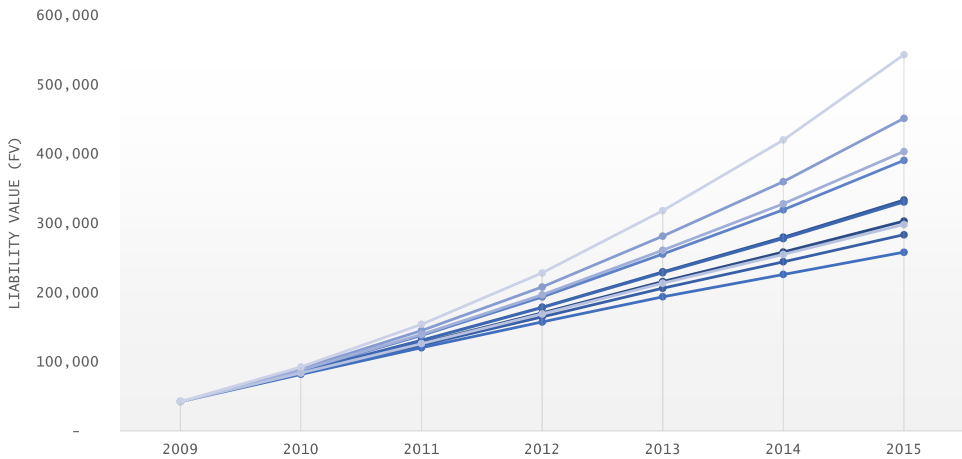


Figure 19: Liability profile for an affluent middle client.

9.4.4 Liability target

The liability target is set-up as an expression of the future portfolio return requirements. The return target calculated as the internal rate of return of the aggregate cash flows required to match the target liability at the terminal node, which in this case is December 2014. The mechanism is illustrated in the figure 22.

Figure 22 is a simplified illustration of the mechanism for establishing the inputs for IRR

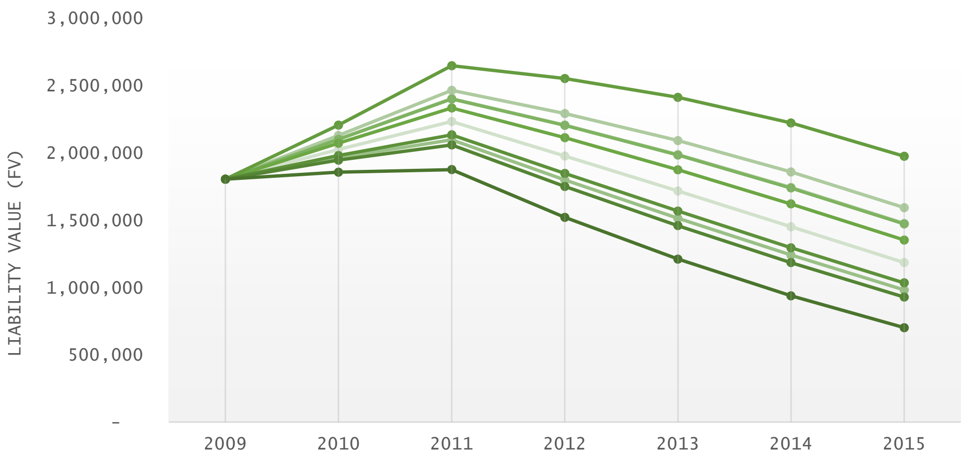


Figure 20: HNW client liability values. This client retires in 2011 and starts making withdrawals.

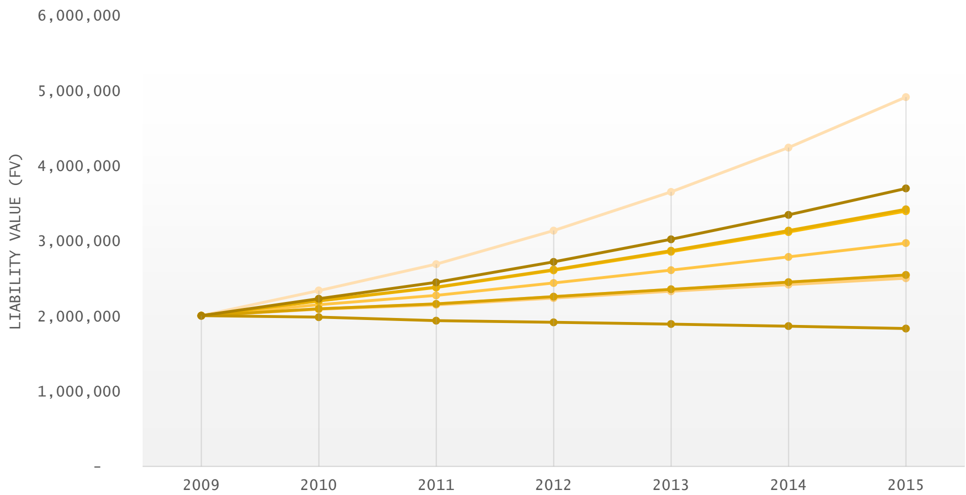


Figure 21: UHNW client liability values.

calculation. IRR solves the required return to meet the liability value at the terminal node. The IRR calculation is performed for all segments of the wealth book. This process is repeated for each year over the investment horizon and again for each stochastic scenario. This allows one to formulate the liability as an index value over time. Figure 22 illustrates the liability indices over the investment horizon for each segment, with scenarios plotted. This illustration again highlights the different sensitivity to changes in assumptions using a stochastic process. The economic environment over the period in this study is relatively stable by comparison to the period between 2001 and 2005, where the expected liability value of the UHNW book (as an example) can vary in terminal value by as much 25% from one year to the next, using the stochastic liability value model developed in this study. These analyses are akin to understanding the effects of bond duration in an interest rate sensitivity analysis.

The expected liability value is calculated as a sampled expected value over the horizon of the investment period. The risk target is expressed as the liability value at the 20th percentile based

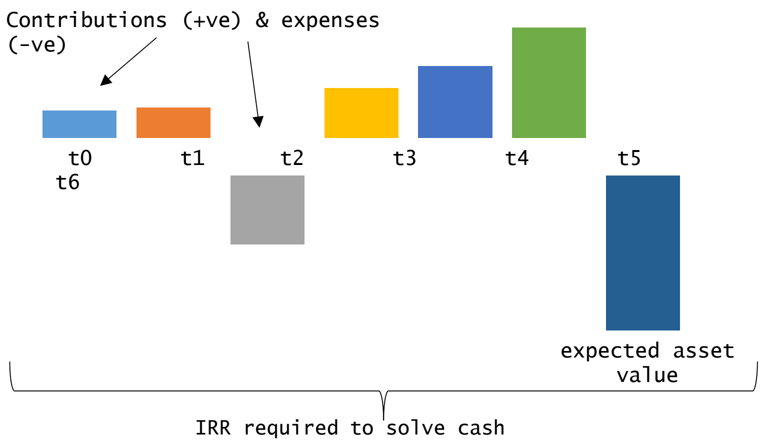


Figure 22: IRR calculation illustrated.

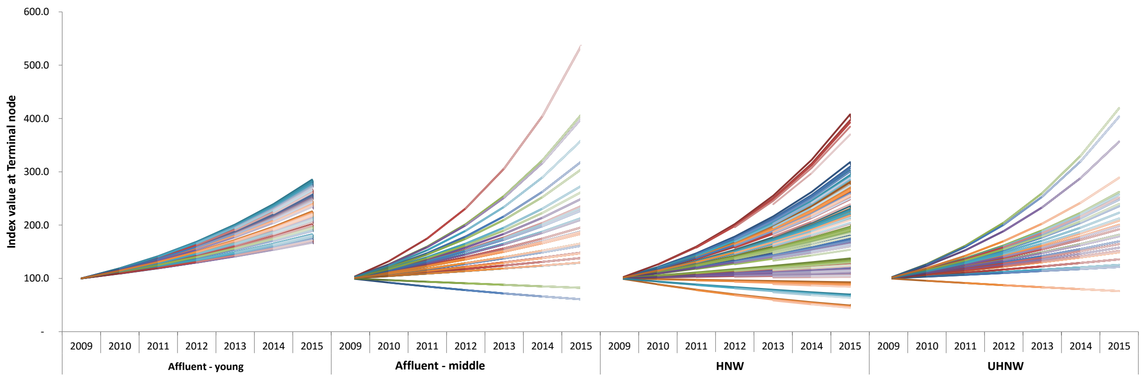


Figure 23: Index value over investment horizon per each segment.

on the stochastic liability distribution for each segment. Note that the value attributed to the 20th percentile will vary through time as we understand the stochastic programme in a dynamic setting. The economic interpretation is that the programme will need to solve for the expected liability value whilst ensuring that it achieves all but the top 20% of potential inflation scenarios. Each segment will be different as it has a different sensitivity to the scenarios. As the stochastic programme is based on the risk target using probability the programme can be categorised as a chance-constrained stochastic programme.

Scenario Tree

In order to solve a dynamic-stochastic programmes the input data must take on a particular structure. These structures, not unique to stochastic programming, are known as scenario trees. A scenario tree is used to represent the expected returns and liabilities simulated in sections 9.3 and 9.4.4. With regard to the structure of the scenario tree, Dempster and Thompson (2010) in testing dynamic strategies for ALM problems conclude that most of the branching should be done at the root node. The style of tree structure resembles a fan.

As the stochastic programme is dynamic, the arrival of a new point in time (for instance rolling forward from 2009 to 2010), requires the casting of a new scenario tree. The new information is introduced into the programme making it adaptive.

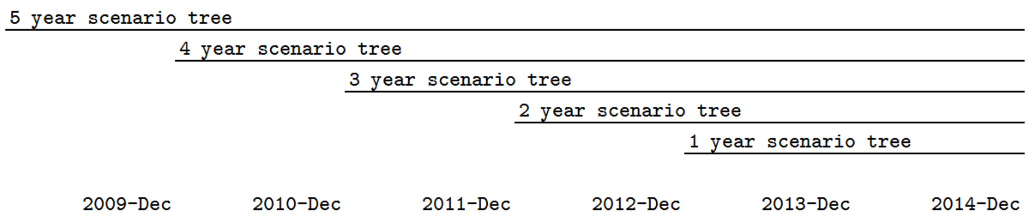


Figure 24: Scenario tree scope.

The dynamic nature of the stochastic programme can be seen in the illustration in Figure 25 as it was illustrated in Dempster et al. (2011). Two scenario trees at separate points in time are plotted (light blue and dark blue) with the actual price of the asset in grey. Following the specification of a fan, the first node in a tree is where all the branching is. The tree then becomes a set of asset paths. The asset pricing process uses outputs from the TVTP model forecast in the first node (A in Figure 25), thereafter the mean reverting simulations are generated up to the terminal node (B in Figure 25). This is an important feature in this study where the TVTP one period forecast is used coupled with a stylised model of asset return paths tending to a long-term expected average. The fan is therefore set up with the first time step contain all the branching and thereafter each path will be a stochastic process. Even though the illustration in figure 25 shows fans with paths which do not appear to cross, these pathways will cross as each path is a random process.

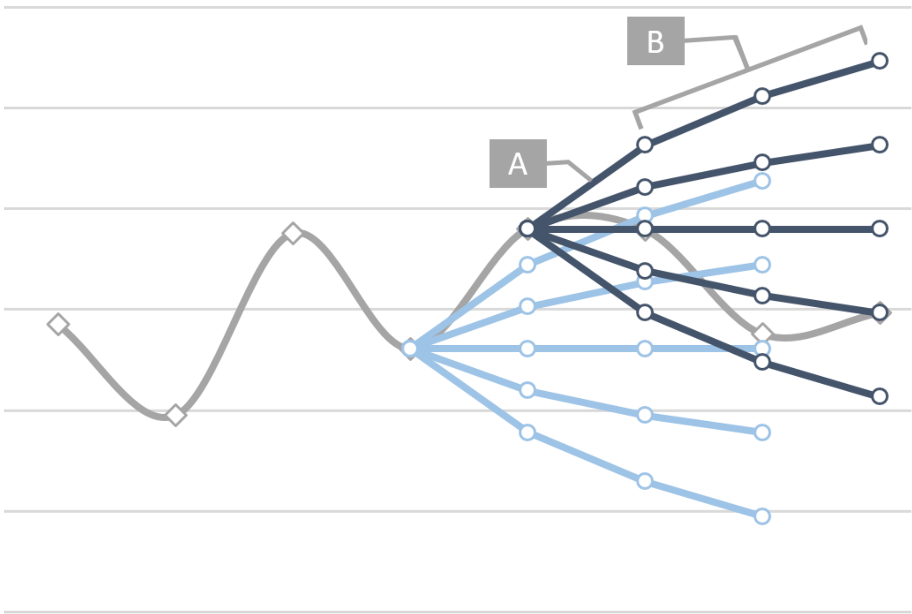


Figure 25: Asset returns in a dynamic stochastic setting.

9.5 Multi-stage portfolio choice model

The use of an ALM model for a private-banking wealth management-book liability profile is used in solving the multi-period portfolio choice model. The upfront contribution is defined as the starting value of the liability thereafter intermediate contributions and withdrawals can be made. The liability value is known to be an uncertain over time, it is a treated as a stochastic process. The out-of-sample and in-sample investigation periods are exactly the CVAR experiment. The objective of the model is to meet the liability at the terminal point, whilst accounting properly for the risk constraint and transaction costs. Other assumptions built into the model include:

- Rebalancing can take place at all time periods or nodes on the scenario tree;
- Transaction costs are taken into account during the rebalancing. These transaction costs are a linear function of traded value. The cost of purchasing an asset is different from the cost of the sale of the asset;
- Holding costs of each asset class differ;
- A stream of uncertain liabilities are accounted for in the modelling process;
- We do not consider the possibility of borrowing money and all available resources are invested fully at each period;
- The position in each asset is long only;
- International exposures are limited to a maximum 30% of the portfolio.

9.5.1 Model formulation

Notation and constraints

The model is a multi-stage stochastic programme which makes use of a split-variable scenario tree representation per Messina and Mitra (1997) and Brandimarte (2006). The modelling is completed by using the modelling platform AMPL DEV by OptiRisk Systems. The model notation and constraints are as follows:

Time sets:

X_{its}	is the expected price of each asset, i, at each time period, t, in each scenario, s;
T	is the number of time periods in the time horizon;
$t = 1 \dots T$	denotes the time period within the time horizon;
$i = 1 \dots I$	is the list of available assets;
$s = 1 \dots S_c$	indicates a scenario;

Asset class indices and macro variables:

$J_t(w)$	South African equity total return at time t in scenario w ($i_1 = 1$);
$D_t(w)$	South African bond total return at time t in scenario w ($i_2 = 2$);
$T_t(w)$	South African treasury bill total return at time t in scenario w ($i_3 = 3$);
$C_t(w)$	Global commodities total return at time t in scenario w ($i_4 = 4$);
$E_t(w)$	United States equity total return at time t in scenario w ($i_5 = 5$);
$G_t(w)$	United States bond total return at time t in scenario w ($i_6 = 6$);
$M_t(w)$	United States treasury bill total return at time t in scenario w ($i_7 = 7$);

Parameters:

p_s	is the probability associated with scenario s;
L_t	is the expected liability at time period t;
$F_t \geq 0$	is the funding available at time period t;
$A_t \geq 0$	is the predefined target at time period t;
$C_s \geq 0$	is the fee charged for sell transactions when portfolio is rebalanced;
$C_b \geq 0$	is the fee charged for buy transactions when portfolio is rebalanced;
$C_h \geq 0$	is the fee charged per asset class for holding assets in portfolio when portfolio is rebalanced;

Decision variables:

$H_{its} \geq 0$	is the amount of asset of type i held in time period t under scenario s;
$B_{its} \geq 0$	is the amount of asset of type i bought in time period t under scenario s;
$S_{its} \geq 0$	is the amount of asset of type i sold in time period t under scenario s.

Objective function:

The asset allocation which maximises the expected value of the final portfolio wealth:

$$\max \sum_{s=1}^{S_c} p_s \sum_{i=1}^{I} X_{iTs} H_{iTs} (1 - C_{ha})$$

subject to:

Asset holding constraint:

$$H_{its} = H_{i(t-1)s} + B_{its} - S_{its} \qquad t = 1..T, i = 1 \ldots I, s = 1 \ldots S_c$$

Fund balance constraint:

$$(1 - C_{sa}) \sum_{i=1}^{I} X_{its} S_{its} - L_t + F_t = (1 - C_{ba}) \sum_{i=1}^{I} X_{its} B_{its} \qquad t = 1, i = 1..I, s = 1..S_c$$

Long only constraint:

$$H_{its} \geq 0$$

Tracking error and risk constraint:

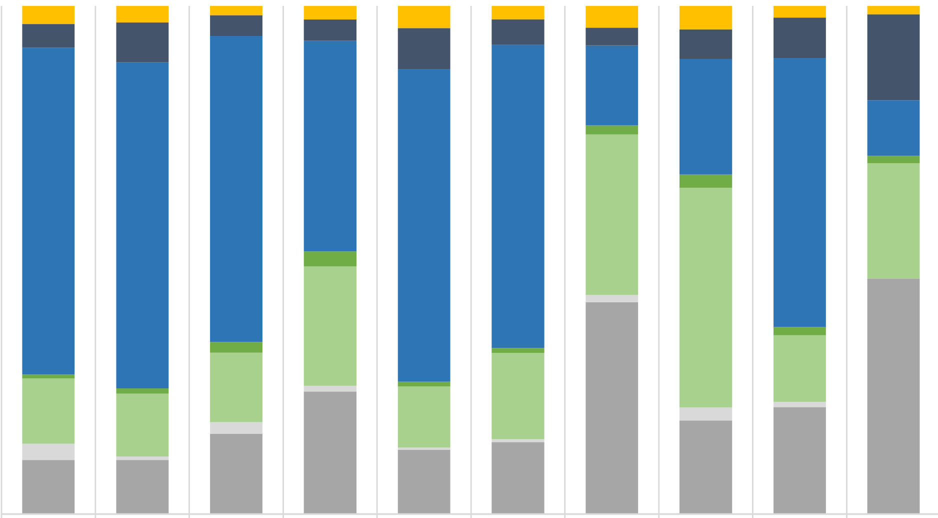
$$0.95 \leq X_T P_T / L_T \leq 1.05$$

International exposure constraint:

$$C_t + G_t + E_t + M_t \leq 0.30$$

9.5.2 Multi-period asset allocation

Solving for dynamic weights with the tracking error constraints was successful with the exception of the year 2008, where the downside constraint needed to dropped to 70%. The results of the multi-period asset allocation are captured in Figure 26. This allocation is dynamic and re-allocates over time.



	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014
Commodities	4%	3%	2%	3%	4%	3%	4%	5%	2%	2%
US equity	5%	8%	4%	4%	8%	5%	4%	6%	8%	17%
RSA equity	64%	64%	60%	41%	62%	60%	16%	23%	53%	11%
US bonds	1%	1%	2%	3%	1%	1%	2%	3%	2%	1%
RSA bonds	13%	12%	14%	23%	12%	17%	32%	43%	13%	23%
US bills	3%	1%	2%	1%	0%	1%	1%	3%	1%	0%
RSA bill	11%	11%	16%	24%	13%	14%	42%	18%	21%	46%

Figure 26: Asset allocation of the ALM stochastic linear program over time

Part VI

Results and Analysis

10 Results of the investment strategies

The testing framework consists of a review of the portfolio results, a comparison of the results to market peer groups and a formal hypothesis test. The review includes the performance of a portfolio which is equally weighted and another which is made up of equity only. The performance of both the active and passive strategy are assessed by considering portfolio risk and portfolio return using the Sharpe ratio. If the null hypothesis is rejected therefore the ALM strategy is significantly different, we may query whether the market is not efficient. The following parameters are analysed and also used in the testing:

- Returns that are calculated as an average yearly compound return in excess of inflation;
- Standard deviation used in the hypothesis test is calculated based on real returns. The standard deviation reported in the out-of-sample and in-sample analysis is a simple standard deviation of real returns;
- The Sharpe ratio is utilized, which is the ratio of return and standard deviation.

The first analysis, the in-sample period, covers the years from 2005 to 2009. The second analysis, the out-of-sample period from 2010 to 2014, is the sample set used in hypothesis testing. The high growth levels followed by a significant crash which occur in the in-sample set provides for an interesting performance analysis.

CVAR portfolio: attribution of returns

Figure 27 illustrates the total real return of the passive CVAR model. Each bar reports the decomposed asset class returns for a given year. Inflation is plotted so as to illustrate the performance of the strategy relative to inflation.

ALM portfolio: attribution of returns

Figure 28 illustrates the return of the active ALM model. Each stacked bar reports the decomposed asset class returns for a given year. Inflation is plotted so as to illustrate the performance of the strategy relative to inflation.

10.1 In-sample analysis

The results in table 16 show that the ALM model is superior to the CVAR model as the Sharpe ratio is higher. The second section of table 16 reports the results of the ZAR multi-asset fund categories from Morningstar Direct. The ALM model again reports the best risk adjusted performance with the highest Sharpe ratio.

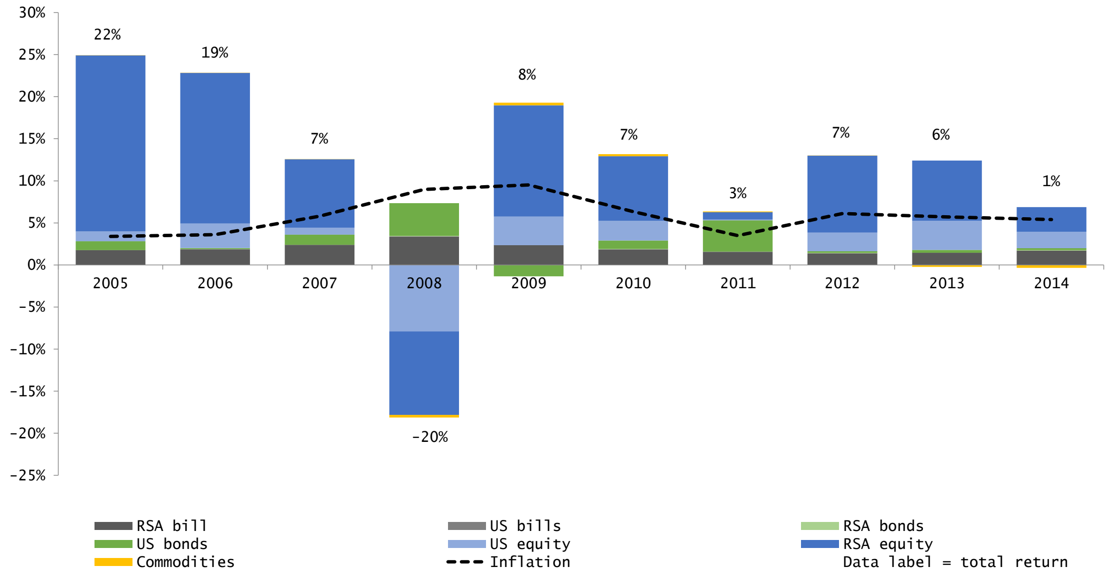


Figure 27: attribution of returns for CVAR portfolio

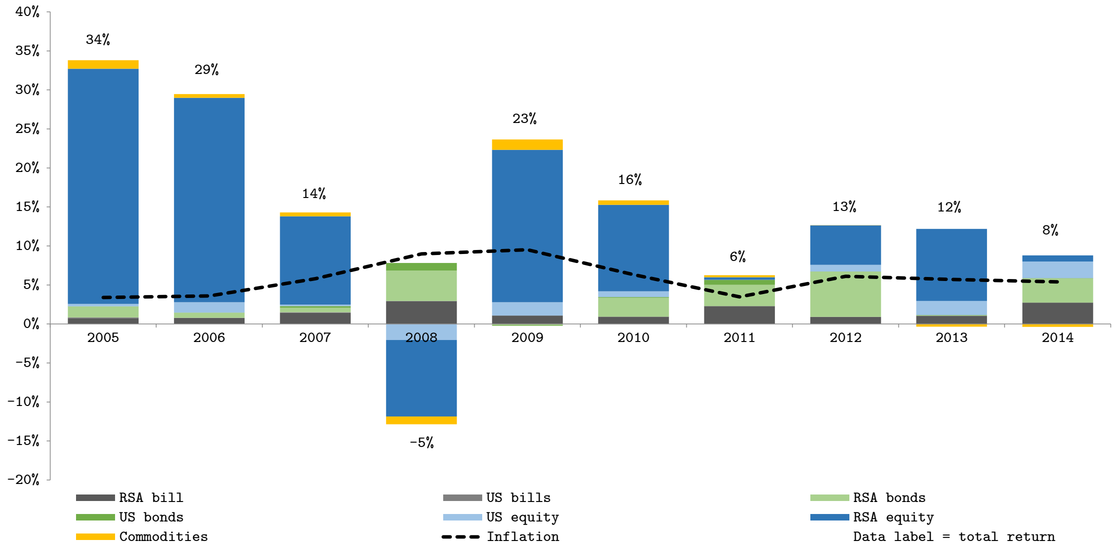


Figure 28: Attribution of returns for ALM portfolio

Figure 29 shows the cumulative return trend over the investigation period. The results show a high level of dispersion between the strategies. The peer groups; Morningstar cautious allocation, Morningstar moderate allocation and Morningstar aggressive allocation have all fared better than the passive CVAR model in the in-sample analysis, but the ALM model attains the highest cumulative return over the investigation period.

Strategy	Real return (geometric return)	Risk (standard deviation)	Sharpe ratio
CVAR Model	6.1%	16.4%	0.37
ALM model	11.8%	17.5%	0.67
RSA 100% equity	12.7%	30.2%	0.42
Equally weighted portfolio	2.2%	11.1%	0.20
ZAR Cautious allocation*	10.5%	23.8%	0.44
ZAR Moderate allocation*	6.9%	14.7%	0.47
ZAR Aggressive allocation*	11.9%	31.6%	0.38

* Morningstar categories, data supplied courtesy of Morningstar direct. Indices are capital weighted.

Table 16: In-sample (2005-2009) performance results

10.2 Out-of-sample analysis

The out-of-sample analysis is based on data that is not used to build the models and is therefore an important review. The out-of-sample data period is characterised by a period of lower volatility and average return. Table 17 reports the performance of the strategies in out-of-sample period:

Strategy	Real return (geometric return)	Risk (standard deviation)	Sharpe ratio
CVAR Model	4.8%	2.7%	1.81
ALM model	5.6%	2.8%	1.99
RSA 100% equity	8.4%	8.9%	0.94
Equally weighted portfolio	1.7%	3.6%	0.47
ZAR Cautious allocation*	6.9%	4.6%	1.49
ZAR Moderate allocation*	4.5%	2.5%	1.80
ZAR Aggressive allocation*	8.2%	5.5%	1.48

* Morningstar categories, data supplied courtesy of Morningstar direct. Indices are capital weighted.

Table 17: Out-of-sample (2010-2014)

Both the ALM model and the CVAR model perform well, achieving the highest Sharpe ratios; the ALM model reports the highest Sharpe ratio.

As Figure 30 illustrates the dispersion of outcomes between the strategies is a lot lower than the in-sample analysis. The Morningstar aggressive category and the equity strategy achieve the highest compound return.

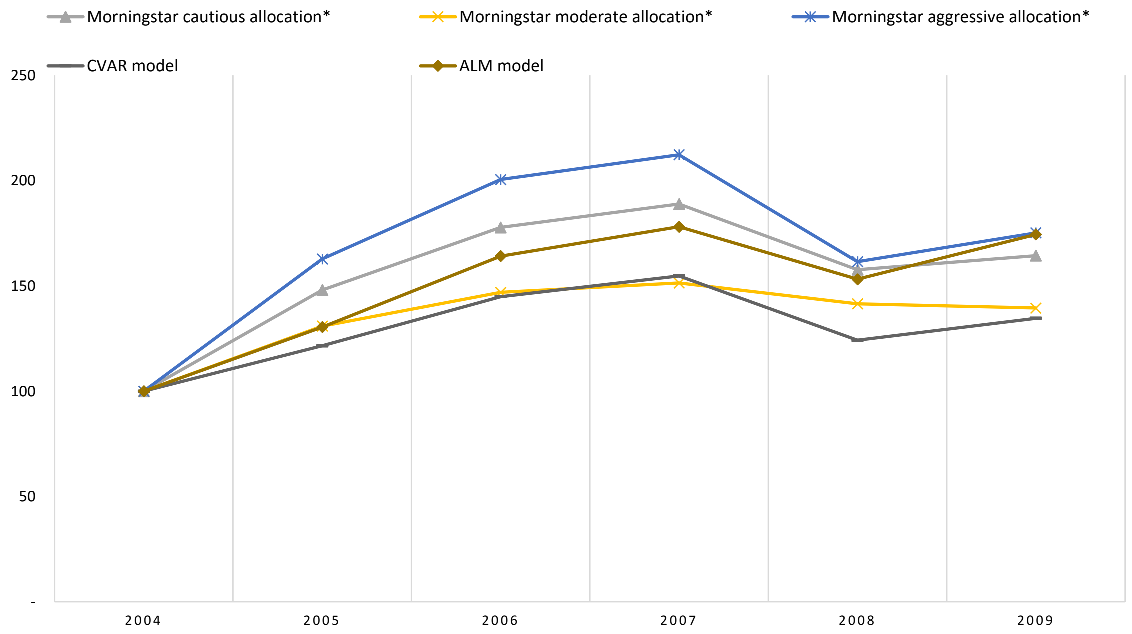


Figure 29: In-sample performance tracking

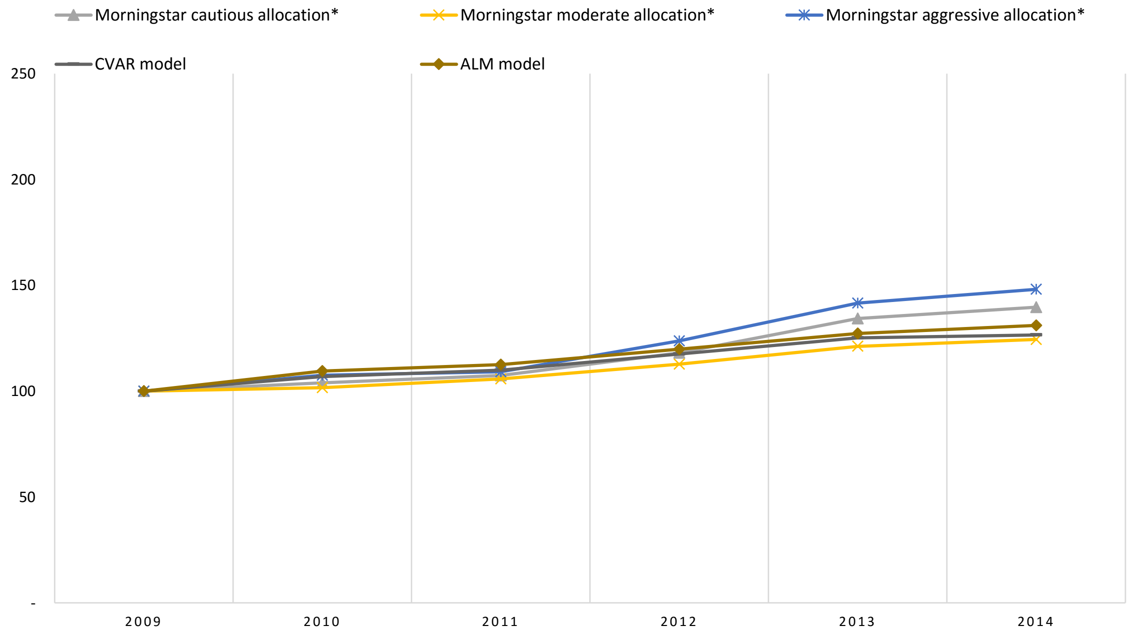


Figure 30: Out-of-sample performance tracking

10.3 Hypothesis testing

The testing framework is based on a two sided t-test [1]. We make use of Welch's t-test, which is an unpaired t-test where the samples have unequal variances, to formally test whether there is a difference in returns between the proposed portfolio strategies. The T-testing module in SAS has been utilised and based on the real returns of the passive CVAR model and the active ALM model:

- *The Null Hypothesis* states that the Sharpe ratios of the passive investment strategies are not significantly different from active investment strategies;
- *Alternative Hypothesis* states that the Sharpe ratio of the active investment strategy is significantly different than that of the passive portfolio strategy.

Although the risk adjusted returns of the ALM model are greater than the CVAR model, the results of the t-test indicated that the null hypothesis cannot be rejected. The F value is 2.27, which equates to a probability of 38.96%. This cannot be rejected at the 5% level and, therefore, the ALM model returns are not different from the CVAR model returns at a statistically significant level.

[1] In this experiment the assumption is that portfolio returns are normally distributed. Improvement to the testing framework is possible by making use of the Wilcoxon-Mann-Whitney test.

Part VII

Conclusion

11 Conclusion

Investing in financial markets provides both individuals and institutions access to significant financial reward. The financial market crisis of 2008 is a reminder that this is not without its risks. In this study, the economic efficiency of multi-asset market has been discussed. Is the multi-asset market for a South African investor inefficient, whereby additional investment management efforts provide increased economic reward relative to the inputs, or are markets efficient in that any additional efforts will not carry an economic reward?

Academic approaches to financial markets have been reviewed with specific focus on the quantitative models for pricing of financial assets and models of portfolio choice. This study explores various methods of asset pricing using quantitative models. As this study has shown in this study the quantitative pricing models are varied and have provided interesting areas for research. Asset pricing models often focus on one of the key parameters being returns or risk or dependence amongst the assets. These parameters are enhanced when time variation of the parameters is incorporated. This allows for asset prices to be surmised by cycles or regimes. Understanding dependence and correlation between assets is interesting and at times challenging to formulate. Correlation, in particular a time varying construct, can be very difficult to formulate and model accurately. The research shows that these models provide an attributable improvement to portfolio efficiency.

In the review of portfolio choice models, the work of Markowitz (1952) has had an indelible influence in the way portfolios are constructed. Some of the underlying assumptions for the MVO framework have been challenged over time. The study has focussed on the relevance of two of the assumptions. First, the assumption that asset returns follow a normal distribution. Second, the assumption that portfolio choice can be solved appropriately by a single period model.

The assumption of normally distributed returns has been shown to not always be the case, in particular during the most recent financial crisis. This consideration is resolved by making use of the CVAR downside risk measure. The portfolio choice model in this construct is determined by an optimisation procedure using a chance constrained stochastic linear programming model.

In reviewing the literature for multi-period portfolio models, we recognise a notable method for dynamic multi-period portfolio optimisation is Bellman's dynamic programming. The final selected model is formulated by stochastic programming methods to generate asset pricing paths, coupled with a large linear program to solve the portfolio choice. This multi-period model is based on the ALM techniques used in banking and insurance in the ALM setting. The multi-period model is enhanced with forward forecasts of the time varying regime switching model. This classes the model as an active model that is used for financial planning of wealth management investments.

The models have been tested using a formal hypothesis and an analytical peer comparison. The results of the peer group analysis show that ALM model is superior with the highest Sharpe ratio. Both the in-sample and out-of-sample analysis show that the ALM has a higher Sharpe ratio, but the hypothesis test indicates that the returns of the ALM model are not statistically different to the returns of the passive model.

One could argue that making use of active models introduces reward in excess of the additional cost of extracting the gains, but not significantly. This could indicate that the market for multi-asset investments is not inefficient.

11.1 Areas for further research

Finally, there is recognition that improvements are possible and the following are areas for further research:

- The client commitment, which can be driven from the Wealth management relationship, to a time horizon for the investment would allow for the introduction of a Liability driven investment (LDI). By introducing the matching of cash flows, the LDI investment paradigm has the potential to reduce the portfolio risk over the investment horizon and hence improve the Sharpe ratio of the strategy;
- Introducing a robust volatility modelling technique will enhance the certainty in decisions driven by the stochastic linear programming technique;
- Improvements to the econometric forecasts will always improve the results of any strategy.

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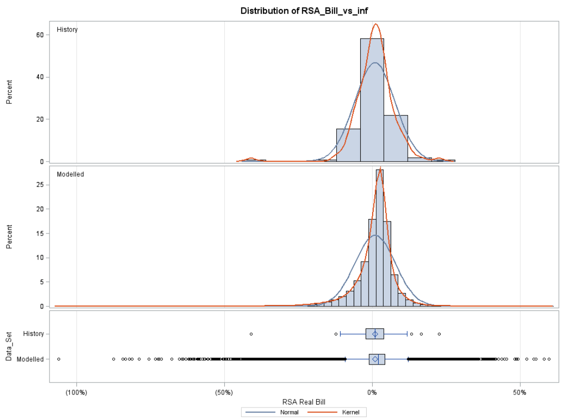
Part VIII

Appendix

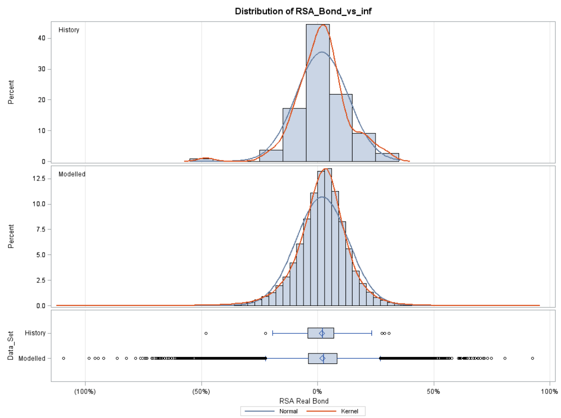
A Asset class simulation testing

The following summary tables refer to the testing of the historical data to the simulated data.

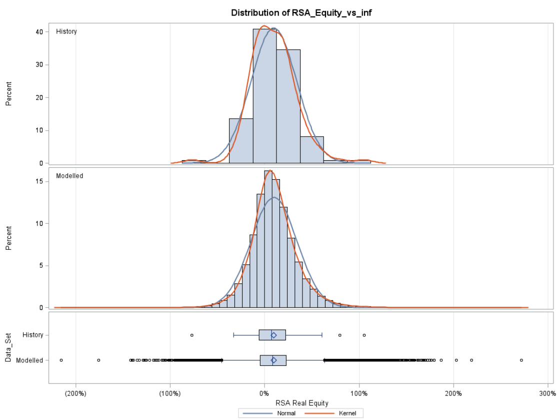
A.1 t-test and histograms



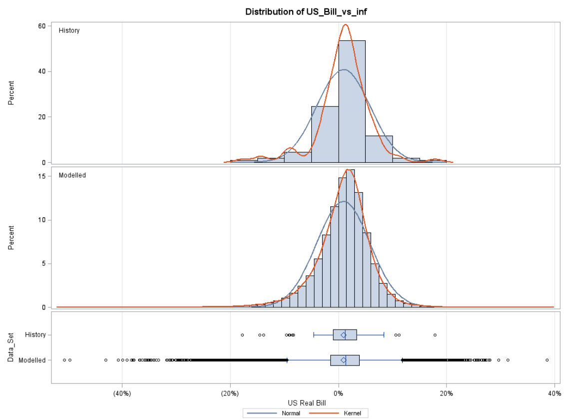
(a) t-Test and Histogram: RSA Bills



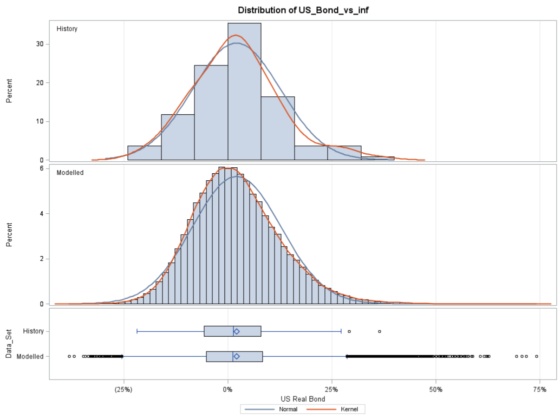
(b) t-Test and Histogram: RSA Bonds



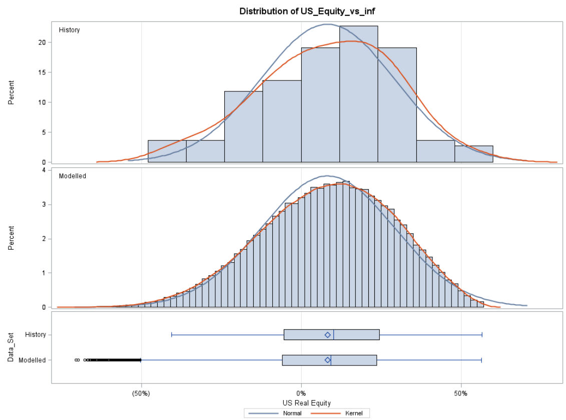
(c) t-Test and Histogram: RSA Equity



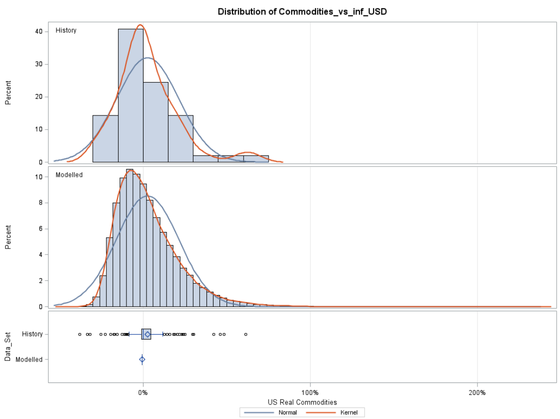
(d) t-Test and Histogram: US Bills



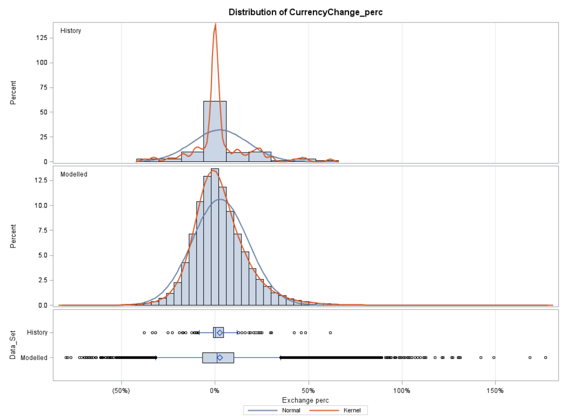
(a) t-Test and Histogram: US Bonds



(b) t-Test and Histogram: US Equity



(c) t-Test and Histogram: US Commodities



(d) t-Test and Histogram: USD ZAR exchange Rate

A.2 TVTP data parameters

The following data variables where analysed for the modelling of asset class TVTP models:

Table 18: Add caption

Asset class total returns	Gross domestic and national product (YOY)
Currency strength - exchange rate	Gross private domestic investment % of GDP
Equity Dividend yield and Dividend Growth rate	Gross private domestic investment (YOY)
Treasury yield spread, Bond cash spread	Brent oil price index
Gross domestic product	Change (YOY) in oil price index, Brent
Interest rate, both real and nominal	Change (YOY) in Money supply, M3
Inflation increases	Individual savings % of GDP
Inflation rate	Individual savings change (YOY)
Roll yield	foreign direct investment into RSA YOY %
Wage inflation	Net Government borrowing increases (YOY)
Change (YOY) in private inventory	Net government borrowing % of GDP
Change (YOY) in consumption	Unemployment rate
Budget deficit as a percentage of GDP	Population increase (YOY)
Exports, imports and trade balance % of GDP	Business leading indicators
Exports, imports and trade balance change (YOY)	Research and development as % of GDP
Final sales of domestic goods	Private consumption changes (YOY)
Fixed Investment % of GDP	PPI inflation
Fixed investment change (YOY)	Private rentals increases (YOY)
Government consumptions % of GDP	Volume of manufacturing, % change YOY
change in government consumption (YOY)	Personal savings rates

Part IX

Addendum

B Single-period model

B.1 SAS Code: inferential statistics

```
/* \section{SAS Code} */
/* Code to generate inferential statistics */

/*Folder setup - specific to each machine */

LIBNAME CVAR  '...\1. CVAR Model\1. Data';
x 'cd  ...\1. CVAR Model\1. Data\Output'; RUN;

%LET Date =  '31Dec09'd;
Proc Sort DATA = CVAR.Dms_v1;
BY  Date;
WHERE Date le  &Date;
RUN;

PROC TRANSPOSE DATA = CVAR.Dms_v1
OUT=CVAR.Trans_CVAR_Data_V5
NAME=IndexCode
LABEL=IndexLabel;
BY Decade Date;
VAR
USD_ZAR
Inv_USD_ZAR
RSA_Bill_TR_ZAR
RSA_Bond_TR_ZAR
RSA_Equity_TR_ZAR
RSA_inf_ZAR
US_Bill_TR_USD
US_Bond_TR_USD
US_Equity_TR_USD
US_inf_USD
US_Bill_TR_ZAR
US_Bond_TR_ZAR
US_Equity_TR_ZAR
US_inf_ZAR
```

```
RSA_Bill_vs_inf
RSA_Bond_vs_inf
RSA_Equity_vs_inf
US_Bill_vs_inf
US_Bond_vs_inf
US_Equity_vs_inf
CurrencyChange_perc
CurrencyChange_perc_Index
RSA_Bill_TR_ZAR_Index
RSA_Bond_TR_ZAR_Index
RSA_Equity_TR_ZAR_Index
RSA_inf_ZAR_Index
US_Bill_TR_USD_Index
US_Bond_TR_USD_Index
US_Equity_TR_USD_Index
US_inf_USD_Index
US_Bill_TR_ZAR_Index
US_Bond_TR_ZAR_Index
US_Equity_TR_ZAR_Index
US_inf_ZAR_Index
RSA_Bill_vs_inf_Index
RSA_Bond_vs_inf_Index
RSA_Equity_vs_inf_Index
US_Bill_vs_inf_Index
US_Bond_vs_inf_Index
US_Equity_vs_inf_Index
Commodities_USD
Commodities_ZAR
Commodities_USD_Index
Commodities_ZAR_Index
commodities_vs_inf_usd;

LABEL
decade                          = "Decade"
date                            = "Date"
usd_zar                         = "USD ZAR"
inv_usd_zar                     = "Inv USD ZAR"
```

```
currencychange_perc              = "Exchange perc"
rsa_bill_tr_zar                  = "RSA Bill ZAR"
rsa_bond_tr_zar                  = "RSA Bond ZAR"
rsa_equity_tr_zar                = "RSA Equity ZAR"
rsa_inf_zar                      = "RSA Inf ZAR"
us_bill_tr_usd                   = "US Bill USD"
us_bond_tr_usd                   = "US Bond USD"
us_equity_tr_usd                 = "US Equity USD"
us_inf_usd                       = "US inf USD"
us_bill_tr_zar                   = "US Bill ZAR"
us_bond_tr_zar                   = "US Bond ZAR"
us_equity_tr_zar                 = "US Equity ZAR"
us_inf_zar                       = "US Inf ZAR"
rsa_bill_vs_inf                  = "RSA Real Bill"
rsa_bond_vs_inf                  = "RSA Real Bond"
rsa_equity_vs_inf                = "RSA Real Equity"
us_bill_vs_inf                   = "US Real Bill"
us_bond_vs_inf                   = "US Real Bond"
us_equity_vs_inf                 = "US Real Equity"
commodities_vs_inf_usd           = "US Real Commodities"

/* Label the data series */
Commodities_USD                  = "Commodities USD"
Commodities_ZAR                  = "Commodities ZAR"
Commodities_USD_Index            = "Commodities USD"
Commodities_ZAR_Index            = "Commodities ZAR"
CurrencyChange_perc_Index        = "Exchange perc"
RSA_Bill_TR_ZAR_Index            = "RSA Bill ZAR"
RSA_Bond_TR_ZAR_Index            = "RSA Bond ZAR"
RSA_Equity_TR_ZAR_Index          = "RSA Equity ZAR"
RSA_inf_ZAR_Index                = "RSA Inf ZAR"
US_Bill_TR_USD_Index             = "US Bill USD"
US_Bond_TR_USD_Index             = "US Bond USD"
US_Equity_TR_USD_Index           = "US Equity USD"
US_inf_USD_Index                 = "US inf USD"
US_Bill_TR_ZAR_Index             = "US Bill ZAR"
US_Bond_TR_ZAR_Index             = "US Bond ZAR"
```

```
US_Equity_TR_ZAR_Index          = "US Equity ZAR"
US_inf_ZAR_Index                = "US Inf ZAR"
RSA_Bill_vs_inf_Index           = "RSA Real Bill"
RSA_Bond_vs_inf_Index           = "RSA Real Bond"
RSA_Equity_vs_inf_Index         = "RSA Real Equity"
US_Bill_vs_inf_Index            = "US Real Bill"
US_Bond_vs_inf_Index            = "US Real Bond"
US_Equity_vs_inf_Index          = "US Real Commodities";
RUN;

Data CVAR.Trans_CVAR_Data_V5;
SET CVAR.Trans_CVAR_Data_V5;
RENAME COL1 = Return;
LABEL IndexCode =  'Asset Class Code';
LABEL IndexLabel = 'Asset Class';
FORMAT Return  PERCENT10.4;
WHERE COL1 NE . and Date <  &Date;
RUN;

proc Sort data = CVAR.Trans_CVAR_Data_V5;
BY IndexCode Date;
RUN;

/* Setting up filters for exclusions and categories for reporting*/
Data CVAR.Trans_CVAR_Data_V5;
SET CVAR.Trans_CVAR_Data_V5;
Year = Year(date);
Return_G = 1+ Return;
IF Return <= -0.850   THEN Return= -0.85;
If INDEXCODE = 'CurrencyChange_perc'    THEN Include_Ind =  '1';
If INDEXCODE = 'Inv_USD_ZAR'            THEN Include_Ind =  '0';
If INDEXCODE = 'RSA_Bill_tr_ZAR'        THEN Include_Ind =  '0';
If INDEXCODE = 'RSA_Bill_vs_inf'        THEN Include_Ind =  '1';
If INDEXCODE = 'RSA_Bill_TR_ZAR'        THEN Include_Ind =  '0';
If INDEXCODE = 'RSA_Bond_vs_inf'        THEN Include_Ind =  '1';
If INDEXCODE = 'RSA_Bond_TR_ZAR'        THEN Include_Ind =  '0';
If INDEXCODE = 'RSA_Equity_TR_ZAR'      THEN Include_Ind =  '0';
```

```
If INDEXCODE = 'RSA_Equity_vs_inf'       THEN Include_Ind =  '1';
If INDEXCODE = 'RSA_inf_ZAR'             THEN Include_Ind =  '0';
If INDEXCODE = 'US_Bill_TR_USD'          THEN Include_Ind =  '0';
If INDEXCODE = 'US_Bill_TR_ZAR'          THEN Include_Ind =  '0';
If INDEXCODE = 'US_Bill_vs_inf'          THEN Include_Ind =  '1';
If INDEXCODE = 'US_Bond_TR_USD'          THEN Include_Ind =  '0';
If INDEXCODE = 'US_Bond_TR_ZAR'          THEN Include_Ind =  '0';
If INDEXCODE = 'US_Bond_vs_inf'          THEN Include_Ind =  '1';
If INDEXCODE = 'US_Equity_TR_USD'        THEN Include_Ind =  '0';
If INDEXCODE = 'US_Equity_TR_ZAR'        THEN Include_Ind =  '1';
If INDEXCODE = 'US_Equity_vs_inf'        THEN Include_Ind =  '1';
If INDEXCODE = 'US_inf_USD'              THEN Include_Ind =  '0';
If INDEXCODE = 'US_inf_ZAR'              THEN Include_Ind =  '1';
If INDEXCODE = 'USD_ZAR'                 THEN Include_Ind =  '0';
If INDEXCODE = 'Commodities_USD'         THEN Include_Ind =  '0';
If INDEXCODE = 'Commodities_ZAR'         THEN Include_Ind =  '1';
If INDEXCODE = 'Commodities_vs_inf_USD'  THEN Include_Ind =  '1';
If INDEXCODE = 'CurrencyChange_perc_Index'THEN Include_Ind= '0';
If INDEXCODE = 'RSA_Bill_TR_ZAR_Index'   THEN Include_Ind =  '0';
If INDEXCODE = 'RSA_Bill_vs_inf_Index'   THEN Include_Ind =  '0';
If INDEXCODE = 'RSA_Bond_TR_ZAR_Index'   THEN Include_Ind =  '0';
If INDEXCODE = 'RSA_Bond_vs_inf_Index'   THEN Include_Ind =  '0';
If INDEXCODE = 'RSA_Equity_TR_ZAR_Index'THEN Include_Ind =  '0';
If INDEXCODE = 'RSA_Equity_vs_inf_Index'THEN Include_Ind =  '0';
If INDEXCODE = 'RSA_inf_ZAR_Index'       THEN Include_Ind =  '0';
If INDEXCODE = 'US_Bill_TR_USD_Index'    THEN Include_Ind =  '0';
If INDEXCODE = 'US_Bill_TR_ZAR_Index'    THEN Include_Ind =  '0';
If INDEXCODE = 'US_Bill_vs_inf_Index'    THEN Include_Ind =  '0';
If INDEXCODE = 'US_Bond_vs_inf_Index'    THEN Include_Ind =  '0';
If INDEXCODE = 'US_Bond_TR_USD_Index'    THEN Include_Ind =  '0';
If INDEXCODE = 'US_Bond_TR_ZAR_Index'    THEN Include_Ind =  '0';
If INDEXCODE = 'US_Bond_vs_inf_Index'    THEN Include_Ind =  '0';
If INDEXCODE = 'US_Equity_TR_USD_Index'  THEN Include_Ind =  '0';
If INDEXCODE = 'US_Equity_TR_ZAR_Index'  THEN Include_Ind =  '0';
If INDEXCODE = 'US_Equity_vs_inf_Index'  THEN Include_Ind =  '0';
If INDEXCODE = 'US_inf_USD_Index'        THEN Include_Ind =  '0';
If INDEXCODE = 'US_inf_ZAR_Index'        THEN Include_Ind =  '0';
```

```
If INDEXCODE = 'Commodities_USD_Index'  THEN Include_Ind =  '0';
If INDEXCODE = 'Commodities_ZAR_Index'  THEN Include_Ind =  '0';
If INDEXCODE = 'Inv_USD_ZAR' THEN AssetMarket =  'F. Macros  ';
If INDEXCODE = 'CurrencyChange_perc' THEN AssetMarket = 'F. Macros
   ';
If INDEXCODE = 'RSA_Bill_TR_ZAR'THEN AssetMarket = 'A. Local';
If INDEXCODE = 'RSA_Bill_vs_inf'THEN AssetMarket='D.Local relative
   ';
If INDEXCODE = 'RSA_Bond_TR_ZAR'THEN AssetMarket = 'A. Local';
If INDEXCODE = 'RSA_Bond_vs_inf'THEN AssetMarket='D.Local relative
   ';
If INDEXCODE ='RSA_Equity_TR_ZAR'THEN AssetMarket ='A. Local';
If INDEXCODE = 'RSA_Equity_vs_inf'THEN AssetMarket='D.Local
   relative';
If INDEXCODE = 'RSA_inf_ZAR' THEN AssetMarket = 'F. Macros';
If INDEXCODE = 'US_Bill_TR_USD' THEN AssetMarket = 'B. US USD';
If INDEXCODE = 'US_Bill_TR_ZAR' THEN AssetMarket = 'C. US ZAR';
If INDEXCODE = 'US_Bill_vs_inf' THEN AssetMarket = 'E. US relative
   ';
If INDEXCODE = 'US_Bond_TR_USD' THEN AssetMarket = 'B. US USD';
If INDEXCODE = 'US_Bond_TR_ZAR' THEN AssetMarket = 'C. US ZAR';
If INDEXCODE = 'US_Bond_vs_inf' THEN AssetMarket = 'E. US relative
   ';
If INDEXCODE = 'US_Equity_TR_USD' THEN AssetMarket = 'B. US USD';
If INDEXCODE = 'US_Equity_TR_ZAR' THEN AssetMarket = 'C. US ZAR';
If INDEXCODE = 'US_Equity_vs_inf' THEN AssetMarket='E. US relative
   ';
If INDEXCODE = 'US_inf_USD'     THEN AssetMarket = 'F. Macros';
If INDEXCODE = 'US_inf_ZAR'     THEN AssetMarket = 'F. Macros';
If INDEXCODE = 'USD_ZAR'        THEN AssetMarket = 'F. Macros';
If INDEXCODE = 'Commodities_USD'THEN AssetMarket = 'B. US USD';
If INDEXCODE = 'Commodities_ZAR'THEN AssetMarket = 'C. US ZAR';
If INDEXCODE = 'Commodities_vs_inf_USD'THEN AssetMarket='E. US
   relative';
If INDEXCODE ='CurrencyChange_perc_Index'THEN AssetMarket='G.Index
   ';
If INDEXCODE = 'RSA_Bill_TR_ZAR_Index'  THEN AssetMarket='G.Index';
```

```
If INDEXCODE = 'RSA_Bill_vs_inf_Index'  THEN AssetMarket='G.Index';
If INDEXCODE = 'RSA_Bond_TR_ZAR_Index'  THEN AssetMarket='G.Index';
If INDEXCODE = 'RSA_Bond_vs_inf_Index'  THEN AssetMarket='G.Index';
If INDEXCODE = 'RSA_Equity_TR_ZAR_Index'THEN AssetMarket='G.Index';
If INDEXCODE = 'RSA_Equity_vs_inf_Index'THEN AssetMarket='G.Index';
If INDEXCODE = 'RSA_inf_ZAR_Index'      THEN AssetMarket='G.Index';
If INDEXCODE = 'US_Bill_TR_USD_Index'   THEN AssetMarket='G.Index';
If INDEXCODE = 'US_Bill_TR_ZAR_Index'   THEN AssetMarket='G.Index';
If INDEXCODE = 'US_Bill_vs_inf_Index'   THEN AssetMarket='G.Index';
If INDEXCODE = 'US_Bond_TR_ZAR_Index'   THEN AssetMarket='G.Index';
If INDEXCODE = 'US_Bond_TR_USD_Index'   THEN AssetMarket='G.Index';
If INDEXCODE = 'US_Bond_vs_inf_Index'   THEN AssetMarket='G.Index';
If INDEXCODE = 'US_Equity_TR_USD_Index' THEN AssetMarket='G.Index';
If INDEXCODE = 'US_Equity_TR_ZAR_Index' THEN AssetMarket='G.Index';
If INDEXCODE = 'US_Equity_vs_inf_Index' THEN AssetMarket='G.Index';
If INDEXCODE = 'US_inf_USD_Index'       THEN AssetMarket='G.Index';
If INDEXCODE = 'US_inf_ZAR_Index'       THEN AssetMarket='G.Index';
If INDEXCODE = 'Commodities_ZAR_Index'  THEN AssetMarket='G.Index';
If INDEXCODE = 'Commodities_USD_Index'  THEN AssetMarket='G.Index';

FORMAT date Date9.;
FORMAT Return Percent9.2;
RUN;

Proc Sort data = CVAR.Trans_CVAR_data_v5;
BY Date IndexCode  ;
RUN;

proc transpose data= CVAR.Trans_CVAR_data_v5
out= CVAR.CVAR_data_transpose
name=column_that_was_transposed;
id  IndexCode;
BY Year Date;
VAR Return;
WHERE Include_Ind = '1';
RUN;
```

```
/* Organising data for Correlation analysis */
DATA CVAR.CVAR_data_transpose (Drop = Date Year   _LABEL_);
SET CVAR.CVAR_data_transpose;
RUN;

Proc Corr Data = CVAR.CVAR_data_transpose COV NOSIMPLE NOPRINT
OUT = CVAR.CVAR_data_Corr  (WHERE= (UPCASE(_Type_) IN ("CORR","COV
   ", "MEAN","VAR","STD"))) ;
RUN;

TITLE 'CORR History Data Set';
PROC CORR DATA= CVAR.Dms_v1  NOSIMPLE NOPROB OUT = CVAR.
   CVAR_data_Corr  (WHERE= (UPCASE(_Type_) IN ("CORR","COV", "MEAN
   ","VAR","STD"))) ;
VAR
commodities_vs_inf_usd
currencychange_perc
rsa_bill_vs_inf
rsa_bond_vs_inf
rsa_equity_vs_inf
us_bill_vs_inf
us_bond_vs_inf
us_equity_vs_inf;
RUN;
QUIT;


/* Create GEOMETRIC MEANS */
Proc Sort data = CVAR.Trans_CVAR_data_v5;
BY IndexCode;
RUN;

proc surveymeans data=CVAR.Trans_CVAR_data_v5   Geomean ;
ODS output GeometricMeans   = CVAR.GeometricMeans;
BY IndexCode;
var Return_G;
WHERE Return_G > 0 and Include_Ind = '1';
```

```
RUN;

Data CVAR.GeometricMeans (DROP = GMStdErr);
SET CVAR.GeometricMeans;
GeoMean = GeoMean -1;
RUN;

Proc Sort data= CVAR.GeometricMeans;
BY VarName;
RUN;

proc transpose data= CVAR.GeometricMeans
out= CVAR.GeometricMeans_transpose
name=column_that_was_transposed;
id  IndexCode;
BY VarName;
RUN;

proc means
data=CVAR.Trans_CVAR_data_v5   noprint nway;
class IndexLabel /      ORDER=UNFORMATTED ASCENDING;
class IndexCode /       ORDER=UNFORMATTED ASCENDING;
var Return;
output out=CVAR.CVAR_Inferential_1  mean= STD=  P99 =   MIN= MAX=
          P1=  Skewness=           Kurtosis= PROBT= / autoname;
WHERE Include_Ind = '1';
run;

/*
    -------------------------------------------------------------------

Code to calculated CVAR @ 2% tolerance level
-------------------------------------------------------------------
    */

DATA CVAR.CVAR_1;
SET CVAR.Trans_cvar_data_v5;
```

```
RUN;

PROC SORT DATA = CVAR.CVAR_1;
BY IndexCode DATE;
WHERE Return <> .;
RUN;

PROC SORT
DATA=CVAR.CVAR_1
OUT=WORK.SORTTempTableSorted;
BY IndexCode;
RUN;

PROC RANK DATA = WORK.SORTTempTableSorted
GROUPS=100
TIES=MEAN
OUT=WORK.RANKRankedUNIVERSE;
BY IndexCode;
VAR Return;
RANKS rank_Return;
RUN;

DATA WORK.RANKRANKEDUNIVERSE;
SET WORK.RANKRANKEDUNIVERSE;
IF Return <= -.85 THEN Return = - 0.85;
RUN;

PROC TABULATE
DATA=WORK.RANKRANKEDUNIVERSE
OUT=CVAR.cvar2perc      ;
WHERE( rank_Return <= 2);
VAR Return;
CLASS IndexCode /       ORDER=UNFORMATTED MISSING;
TABLE
IndexCode={LABEL=""},
Return={LABEL="CVAR @ 2%"}*
Mean*F=PERCENTN8.2 ;
```

```
RUN; QUIT;

/*
   -----------------------------------------------------------------

Final Infernential summary table
------------------------------------------------------------------
    */
PROC SQL;
CREATE TABLE CVAR.All_Inferences AS
SELECT t2.IndexLabel,
t2.IndexCode,
t2.Return_Mean                  FORMAT=PERCENTN8.2,
t4.GeoMean                      FORMAT=PERCENTN8.2,
t2.Return_StdDev                FORMAT=PERCENTN8.2,
t3.Return_Mean AS cvar2Perc     FORMAT=PERCENTN8.2,
t2.Return_Skew                  FORMAT=PERCENTN8.2,
t2.Return_Kurt                  FORMAT=PERCENTN8.2
FROM  CVAR.CVAR_INFERENTIAL_1 t2, CVAR.cvar2PERC t3, CVAR.
   GEOMETRICMEANS t4
WHERE (t2.IndexCode = t3.IndexCode AND t2.IndexCode = t4.IndexCode)
   ;
QUIT;

/************** Inferences Reporting *************/
/*Set report destination and PDF options */
options orientation=landscape;
ODS PDF FILE = "...\1. CVAR Model\1. Data\Output\
SAS_Base_1Inferences.pdf" STYLE=Journal3 startpage=yes;
options orientation=landscape;
options papersize=(12in 10in);

/* Setup HTML report */
ODS HTML;
goptions reset=all;

/* Configure Graphics */
```

```
ods graphics on /
width=10in
imagefmt=gif
imagemap=on
imagename="Inferential Stats"
border=off;

/*Report 1: Basic summary of moments */
proc report data=CVAR.ALL_INFERENCES2 nowd;
column IndexLabel IndexCode Return_Mean /*GeoMean*/ Return_StdDev
Return_Skew Return_Kurt cvar2Perc/*DD_Min DD_plus_Min Duration_Max
   */;
define IndexLabel / group 'Asset Class' missing;
compute IndexLabel;
if IndexLabel ne ' ' then hold1=IndexLabel;
if IndexLabel eq ' ' then IndexLabel=hold1;
endcomp;
define IndexCode / group 'Asset code' missing;
compute AssetClass;
if IndexCode ne ' ' then hold2=IndexCode;
if IndexCode eq ' ' then IndexCode=hold2;
endcomp;
define Return_Mean / analysis SUM 'Return Mean' format=PERCENTN8.2
   missing;
define Return_StdDev / analysis SUM 'Return StdDev' format=
   PERCENTN8.2 missing;
define Return_Skew / analysis SUM 'Return Skew' format=COMMA8.2
   missing;
define Return_Kurt / analysis SUM 'Return Kurt' format=COMMA8.2
   missing;
define cvar2Perc / analysis SUM 'CVAR@2%' format=PERCENTN8.2
   missing;
run;
quit;
ODS HTML CLOSE;

/* Chart: Normalised density plots per asset class */
```

```
ODS PDF STYLE=Plateau ;
Title1 'Comparitive: Normalised Return Distribution';
title2;
FOOTNOTE;
proc sgpanel data=CVAR.CVAR_data_transpose;
panelby column_that_was_transposed;
DENSITY CurrencyChange_perc     / legendlabel= "Exchange perc";
DENSITY RSA_Bill_vs_inf         / legendlabel= "RSA Real Bill";
DENSITY RSA_Bond_vs_inf         / legendlabel= "RSA Real Bond";
DENSITY RSA_Equity_vs_inf       / legendlabel= "RSA Real Equity";
DENSITY US_Bill_vs_inf          / legendlabel= "US Real Bill";
DENSITY US_Bond_vs_inf                  / legendlabel= "US Real
   Bond";
DENSITY US_Equity_vs_inf                / legendlabel= "US Real
   Equity";
DENSITY Commodities_vs_inf_USD  / legendlabel= "US Real Commodities
   ";

LABEL column_that_was_transposed = 'Parameter Plotted';
RUN;

Title1 'Comparitive: OffShore Normalised Return Distribution';
title2;
FOOTNOTE;
/* Chart: Boxplot Trend per asset class */
ODS PDF STYLE=Plateau ;
Proc sort data=CVAR.Trans_CVAR_data_v5;
BY  IndexLabel Decade year;
RUN;

ODS HTML;
option gstyle;
ods listing style=statistical;
goptions reset=all  ;

TITLE j=l "Monthly Returns Box Plot of #BYVAL(IndexCode)";
proc sgplot data=CVAR.Trans_CVAR_data_v5;
```

```
vbox return / category= Decade;
BY  IndexLabel;
WHERE Include_Ind = '1';
run;

/************** Histograms ****************/
PROC SORT
DATA=CVAR.TRANS_CVAR_DATA_V5(WHERE=(Include_Ind = '1')
KEEP=Return IndexCode AssetMarket Include_Ind)
OUT=WORK.SORTTempTableSorted;
BY IndexLabel;
FORMAT Return PERCENTN9.1;
RUN;

Title1;
title2;
goptions cback=White;
ods graphics on /
width=15in
imagefmt=gif
imagemap=on
imagename="Asset Class Distributions"
border=off;

PROC UNIVARIATE DATA = work.Sorttemptablesorted NOPRINT;
BY  IndexCode;
CLASS IndexCode;
HISTOGRAM Return  / CFILL = lightblue NORMAL
(COLOR = gray NOPRINT) NOFRAME ;
INSET
MEAN
SKEWNESS
KURTOSIS
STD
MIN MAX
P1
P99
```

```
PNORMAL
/ POSITION=NW
CTEXT=Gray
CFILL=WHITE
CFILLH=WHITE
NOFRAME
CHEADER=Gray
HEIGHT=2
FORMAT=PERCENTN9.1 ;
LABEL Return = '% Yearly Returns';
WHERE Include_Ind = '1';
RUN;

/*Close report 1*/
ODS HTML CLOSE;
ODS PDF CLOSE;

/******** Report 2: Comparitive Box and whiskers *********/
Proc Sort Data = CVAR.Trans_CVAR_data_v5;
BY  IndexCode Date ;
RUN;

goptions reset=all;
ods graphics on /
width=15in
imagefmt=gif
imagemap=on
imagename="Box Plots"
border=off;
options orientation=landscape;
ODS PDF FILE = "...\1. CVAR Model\1. Data\Output\SAS_Base_2BoxPlot.
   pdf"
STYLE=Statistical startpage=yes;
options orientation=landscape;
options papersize=(15in 10in);
goptions rotate=landscape;
pattern v=me;
```

```
axis1 minor=none color=black label=(angle=180 rotate=0);
title ;
title1 'Box Plot : Full Universe (YOY Returns)';
proc boxplot data=CVAR.Trans_CVAR_data_v5;
Plot Return*IndexCode  / boxstyle = schematicid  vaxis = axis1
   TURNHLABELS ;
inset  mean stddev /  header = 'Overall Statistics' FORMAT =
   Percent9.2  pos = tm;
insetgroup mean  Q1 Q3   stddev /  header = 'Per Asset Type' FORMAT
    = Percent9.2 ;
ID year;
FORMAT Return mean  Q1 Q3   stddev Percent6.0;
FORMAT IndexCode $20.;
WHERE Include_Ind = '1';
run;

ODS PDF CLOSE;

/************ Report 3:  Correlation Matrix ************/
goptions reset=all;
ods graphics on /
width=12in
imagefmt=gif
imagemap=on
imagename="Correlation Tables"
border=off;
options orientation=landscape;
options papersize=(15in 15in);
goptions rotate=landscape;
pattern v=me;
options orientation=landscape;
ODS PDF FILE = "...\1. CVAR Model\1. Data\Output\
   SAS_Base_3Correlation.pdf"
STYLE=Plateau startpage=yes;

/* Covariance Matrix */
TITLE1 "Correlation Matrix";
```

```
Proc Print data =  CVAR.CVAR_data_corr NOOBS ;
WHERE _TYPE_ = 'CORR';
LABEL _NAME_ = 'AssetClass';
LABEL _TYPE_ = 'Measure';
FORMAT  CurrencyChange_perc            COMMA8.2;
FORMAT  RSA_Bill_vs_inf                COMMA8.2;
FORMAT  RSA_Bond_vs_inf                COMMA8.2;
FORMAT  RSA_Equity_vs_inf              COMMA8.2;
FORMAT  US_Bill_vs_inf                 COMMA8.2;
FORMAT  US_Bond_vs_inf                 COMMA8.2;
FORMAT  US_Equity_vs_inf               COMMA8.2;
FORMAT  Commodities_vs_inf_USD         COMMA8.2;
RUN;

/* Scatterplot Matrix Plot */
Proc Sort data = CVAR.Trans_CVAR_data_v5;
BY Date AssetMarket ;
RUN;

options orientation=portrait;
ODS HTML style = Journal;
ods graphics on;
TITLE1 'Log Monthly Means (ZAR)';
Footnote ;Footnote1 ;Footnote2 ;
proc corr data=CVAR.CVAR_data_transpose
nomiss plots(maxpoints = 100000)
=matrix(histogram nvar = 12) NOCORR NOSIMPLE;
FOOTNOTE 'Log Monthly Means (ZAR)' ;
RUN;QUIT;
ODS graphics off;
ODS PDF CLOSE;
```

B.2 SAS Code: Fleishman simulation

The original works of Vale and Maurelli and Fleishman have been codified in Fan et al. (2001). The following code has been adapted from Fan et al. (2001), SAS for Monte Carlo studies: a guide for quantitative researchers.

```
LIBNAME CVAR  '...\1. Data';
x 'cd  ...\1. Data\Output'; RUN;

/* Select AssetClasses required */
Data CVAR.Cvar_inferential_1_Fleishman;
Set CVAR.Cvar_inferential_1;
If indexcode = 'Commodities_ZAR'         THEN Include_ind = 0;
If indexcode = 'CurrencyChange_perc'     THEN Include_ind = 1;
If indexcode = 'RSA_Bill_vs_inf'         THEN Include_ind = 1;
If indexcode = 'RSA_Bond_vs_inf'         THEN Include_ind = 1;
If indexcode = 'RSA_Equity_vs_inf'       THEN Include_ind = 1;
If indexcode = 'US_Equity_TR_ZAR'        THEN Include_ind = 0;
If indexcode = 'US_inf_ZAR'              THEN Include_ind = 0;
If indexcode = 'US_Bill_vs_inf'          THEN Include_ind = 1;
If indexcode = 'US_Bill_vs_inf_ZAR'      THEN Include_ind = 0;
If indexcode = 'US_Bond_vs_inf'          THEN Include_ind = 1;
If indexcode = 'US_Bond_vs_inf_ZAR'      THEN Include_ind = 0;
If indexcode = 'Commodities_vs_inf_USD'  THEN Include_ind = 1;
If indexcode = 'US_Equity_vs_inf'        THEN Include_ind = 1;
If indexcode = 'US_Equity_vs_inf_ZAR'    THEN Include_ind = 0;
RUN;

ODS HTML;
PROC SORT Data = CVAR.CVAR_inferential_1_Fleishman;
BY indexCode;
WHERE Include_ind = 1;
RUN;

/* Flieschmans Power Transform */
PROC IML;
USE CVAR.CVAR_inferential_1_Fleishman;
```

```
READ  ALL VAR{Return_Skew Return_Kurt}  INTO SKEWKURT  [ROWNAME =
   IndexCode COLNAME = _NAME_]; PRINT SKEWKURT ;
READ  ALL VAR{Return_Mean Return_StdDev}  INTO Stats  [ROWNAME =
   IndexCode COLNAME = _NAME_]; PRINT Stats ;
READ  ALL VAR{IndexCode}  INTO IndexCode  [ COLNAME = _NAME_];
   PRINT IndexCode ;
START NEWTON;
RUN FUN;
DO ITER = 1 TO MAXITER
WHILE(MAX(ABS(F))>CONVERGE);
RUN DERIV;
DELTA=-SOLVE(J,F);
COEF=COEF+DELTA;
RUN FUN;
END;
FINISH NEWTON;
MAXITER=25;
CONVERGE=.000001;
START FUN;
X1=COEF[1];
X2=COEF[2];
X3=COEF[3];
F=(X1**2+6*X1*X3+2*X2**2+15*X3**2-1)//
(2*X2*(X1**2+24*X1*X3+105*X3**2+2)-SKEWNESS)//
(24*(X1*X3+X2**2*(1+X1**2+28*X1*X3)+X3**2*
(12+48*X1*X3+141*X2**2+225*X3**2))-KURTOSIS);
FINISH FUN;
START DERIV;
J=((2*X1+6*X3)|| (4*X2) ||(6*X1+30*X3))//
((4*X2*(X1+12*X3))|| (2*(X1**2+24*X1*X3+105*X3**2+2))|| (4*X2*(12*
   X1+105*X3)))//
((24*(X3+X2**2*(2*X1+28*X3)+48*X3**3)) || (48*X2*(1+X1**2+28*X1*X3
   +141*X3**2)) ||  (24*(X1+28*X1*X2**2+2*X3*(12+48*X1*X3+141*X2
   **2+225*X3**2)
+X3**2*(48*X1+450*X3))));
FINISH DERIV;
DO;
```

```
NUM = NROW(SKEWKURT);
DO VAR=1 TO NUM;
SKEWNESS=SKEWKURT[VAR,1];
KURTOSIS=SKEWKURT[VAR,2];
COEF={1.0, 0.0, 0.0};
RUN NEWTON;
COEF=COEF`;
SK_KUR=SKEWKURT[VAR,];
COMBINE=SK_KUR || COEF;
IF VAR=1 THEN RESULT=COMBINE;
ELSE IF VAR>1 THEN RESULT=RESULT // COMBINE ;
END; TITLE;
PRINT "COEFFICIENTS OF B, C, D FOR FLEISHMAN'S POWER TRANSFORMATION
   ";
PRINT "Y = A + BX + CX^2 + DX^3";
PRINT " A = A";
MATTRIB RESULT COLNAME=({SKEWNESS KURTOSIS B C D}) FORMAT=12.9;
PRINT RESULT;
FLIESCHMANS = Stats  || RESULT;
MATTRIB FLIESCHMANS
COLNAME=({MEAN STD SKEWNESS KURTOSIS B C D})
FORMAT=12.9; PRINT FLIESCHMANS  [ROWNAME =IndexCode ];
CREATE CVAR.FLIESCHMANS FROM FLIESCHMANS
[COLNAME=({ MEAN STD SKEWNESS KURTOSIS B C D})];
APPEND FROM FLIESCHMANS ;
CREATE CVAR.IndexCode FROM IndexCode
[COLNAME=({IndexCode})];APPEND FROM IndexCode ;
END;
QUIT;

PROC IML;
USE CVAR.FLIESCHMANS;
READ  ALL VAR{MEAN}  INTO MEAN;
READ  ALL VAR{STD}  INTO STD;
READ  ALL VAR{SKEWNESS}  INTO SKEWNESS;
READ  ALL VAR{KURTOSIS}  INTO KURTOSIS;
READ  ALL VAR{B}  INTO B;
```

```
READ  ALL VAR{C}  INTO C;
READ  ALL VAR{D}  INTO D;
USE CVAR.IndexCode; READ  ALL VAR{IndexCode}  INTO IndexCode;
create CVAR.FLIESCHMANS_2;
append var {IndexCode MEAN STD SKEWNESS KURTOSIS B C D};
close CVAR.FLIESCHMANS_2;
QUIT;

DATA CVAR.Flieschmans_2;
SET CVAR.Flieschmans_2;
A = -1*C;
RUN;

PROC SQL;
CREATE TABLE CVAR.FLIESCHMANS_3 AS
SELECT t1.IndexCode,
t1.B,
t1.C,
t1.D,
t2.IndexCode AS IndexCode1,
t2.B AS B1,
t2.C AS C1,
t2.D AS D1
FROM CVAR.FLIESCHMANS_2 t1,
CVAR.FLIESCHMANS_2 t2; *ON (t1.ASSETCLASS <> t2.ASSETCLASS);
QUIT;

/* 1  */
PROC SORT DATA = CVAR.CVAR_data_corr;
BY _TYPE_ _NAME_;
RUN;

proc transpose data=CVAR.CVAR_data_corr
out=CVAR.CVAR_Long_corr prefix= CORR
name=AssetClassB;
BY _NAME_;
WHERE _TYPE_ = 'CORR';
```

```
LABEL AssetClassB = 'AssetClassB';
LABEL _NAME_ = 'AssetClassA';
run;

PROC SQL;
CREATE TABLE CVAR.FLIESCHMANS_4 AS
SELECT t1.*, t2.Corr1
FROM CVAR.FLIESCHMANS_3 t1
INNER JOIN CVAR.CVAR_Long_corr t2 ON (t1.IndexCode = t2._NAME_ and
   t1.IndexCode1 = t2.ASSETCLASSB );
QUIT;

DATA CVAR.Flieschmans_5 (DROP = B C D);
SET CVAR.Flieschmans_4;
B2= B1;
C2= C1;
D2= D1;
B1= B;
C1= C;
D1= D;
TARGET=  CORR1;
R=.5;
* starting value for iteration;

DO I=1 TO 5;
FUNCTION=(R**3*6*D1*D2+R**2*2*C1*C2+R*(B1*B2+3*B1*D2+3*D1*B2+9*D1*
   D2)-TARGET);
DERIV=(3*R**2*6*D1*D2+2*R*2*C1*C2+(B1*B2+3*B1*D2+3*D1*B2+9*D1*D2));
RATIO=FUNCTION/DERIV;
R_TEMP = R - RATIO;
IF ABS(R_TEMP - R)>.00001 THEN R = R_TEMP;
IF IndexCode = IndexCode1 THEN R = 1;
OUTPUT;
END;

FORMAT R comma12.9;
```

```
FORMAT Ratio comma12.9;
RUN;

ODS HTML;
TITLE 'Intermediate Correlation';
PROC PRINT Data =  CVAR.Flieschmans_5; WHERE I=5;
VAR IndexCode IndexCode1 I RATIO R Target;
WHERE  I = 5;
RUN;
ODS HTML CLOSE;

Proc SORT data = CVAR.Flieschmans_5 ;
BY IndexCode1 IndexCode ;
RUN;

proc transpose data= CVAR.Flieschmans_5
out= CVAR.Flieschmans_6
name=column_that_was_transposed;
ID IndexCode;
VAR R;
BY  IndexCode1;
WHERE I = 5;
RUN;

/* Set up Factor */
DATA Cvar_data_corr;
SET CVAR.Cvar_data_corr;
WHERE _TYPE_ = 'CORR';
RUN;

proc sql noprint;
select count(*)
into :OBSCOUNT
from Cvar_data_corr;
quit;
%put Count=&OBSCOUNT.;
%let Count=&OBSCOUNT.;
```

```
/* This correlation table is taken directly from CVAR.
   Cvar_data_corr */
DATA A (TYPE=CORR);  _TYPE_='CORR';
INPUT Col1-Col&Count;
CARDS;
1.000   -0.520  -0.275  -0.004  0.360   -0.594  -0.377  -0.112
-0.520  1.000   0.054   -0.123  -0.129  0.039   0.210   -0.131
-0.275  0.054   1.000   0.662   0.252   0.415   0.355   0.184
-0.004  -0.123  0.662   1.000   0.455   0.210   0.371   0.206
0.360   -0.129  0.252   0.455   1.000   -0.006  0.010   0.449
-0.594  0.039   0.415   0.210   -0.006  1.000   0.553   0.186
-0.377  0.210   0.355   0.371   0.010   0.553   1.000   0.196
-0.112  -0.131  0.184   0.206   0.449   0.186   0.196   1.000
;


/* Create Lower Triangle */
data A;
set A;
array yy{&Count} Col1-Col&Count;
DO _i=1 to &Count;
if (_i>_n_) then yy[_i]=.;
END;
drop _i _NAME_ AssetClass1;
run;

DATA A (type=corr DROP =  AssetClass1 _Name_);
SET A;
RUN;

ODS HTML;
PROC FACTOR Data = A Nobs = &Count  N = &Count MINEIGEN = 0
NFACTORS= &Count OUTSTAT=FACOUT ; RUN;
DATA PATTERN; SET FACOUT;
IF _TYPE_='PATTERN';
DROP _TYPE_ _NAME_;
```

```
RUN;
ODS HTML CLOSE;

/* --------------------------------------------------------------
SIMULATION
---------------------------------------------------------------*/
PROC SORT Data = CVAR.Flieschmans_2;
BY INDEXCODE;
RUN;

ODS HTML;TITLE ;
PROC IML;
USE PATTERN;
READ ALL VAR _NUM_ INTO F; ODS HTML; *PRINT F;
USE CVAR.Flieschmans_2;
READ ALL VAR {A} INTO A  [COLNAME = _NAME_];
READ ALL VAR {B} INTO B  [COLNAME = _NAME_];
READ ALL VAR {C} INTO C  [COLNAME = _NAME_];
READ ALL VAR {D} INTO D  [COLNAME = _NAME_];
READ ALL VAR {MEAN} INTO  MEAN [COLNAME = _NAME_];
READ ALL VAR {STD} INTO STD  [COLNAME = _NAME_];
A= A‘;
B= B‘;
C= C‘;
D= D‘;
MEAN = MEAN‘;
Print MEAN;
STD = STD‘; PRINT STD;
F=F‘; PRINT F;
ODS HTML; PRINT F;
DATA=RANNOR(J(100000,&Count,INT(datetime())));   * PRINT Data;
ODS HTML CLOSE;
DATA=DATA‘;
Z = F*DATA;
Z = Z‘;
CREATE Z FROM Z ;
APPEND FROM Z   ;
```

```
/* These are the Fleishman coefficients - A B C D */
X1 =   A[,1]   + B[,1]#Z[,1] + C[,1]#Z[,1]##2 + D[,1]#Z[,1]##3;
X2 =   A[,2]   + B[,2]#Z[,2] + C[,2]#Z[,2]##2 + D[,2]#Z[,2]##3;
X3 =   A[,3]   + B[,3]#Z[,3] + C[,3]#Z[,3]##2 + D[,3]#Z[,3]##3;
X4 =   A[,4]   + B[,4]#Z[,4] + C[,4]#Z[,4]##2 + D[,4]#Z[,4]##3;
X5 =   A[,5]   + B[,5]#Z[,5] + C[,5]#Z[,5]##2 + D[,5]#Z[,5]##3;
X6 =   A[,6]   + B[,6]#Z[,6] + C[,6]#Z[,6]##2 + D[,6]#Z[,6]##3;
X7 =   A[,7]   + B[,7]#Z[,7] + C[,7]#Z[,7]##2 + D[,7]#Z[,7]##3;
X8 =   A[,8]   + B[,8]#Z[,8] + C[,8]#Z[,8]##2 + D[,8]#Z[,8]##3;

commodities_vs_inf_usd  = X1#STD[,1]    + MEAN[,1];
currencychange_perc             = X2#STD[,2]    + MEAN[,2];
rsa_bill_vs_inf                 = X3#STD[,3]    + MEAN[,3];
rsa_bond_vs_inf                 = X4#STD[,4]    + MEAN[,4];
rsa_equity_vs_inf               = X5#STD[,5]    + MEAN[,5];
us_bill_vs_inf                  = X6#STD[,6]    + MEAN[,6];
us_bond_vs_inf                  = X7#STD[,7]    + MEAN[,7];
us_equity_vs_inf                = X8#STD[,8]    + MEAN[,8];

CREATE CVAR.MCMC var {
commodities_vs_inf_usd
currencychange_perc
rsa_bill_vs_inf
rsa_bond_vs_inf
rsa_equity_vs_inf
us_bill_vs_inf
us_bond_vs_inf
us_equity_vs_inf

};
APPEND;       CLOSE CVAR.MCMC;
QUIT;

ODS HTML CLOSE;

DATA CVAR.MCMC;
```

```
SET CVAR.MCMC;
us_bill_vs_inf_ZAR      = us_bill_vs_inf * (1+currencychange_perc);
us_bond_vs_inf_ZAR      = us_bond_vs_inf *  (1+currencychange_perc)
   ;
us_equity_vs_inf_ZAR    = us_equity_vs_inf *  (1+
   currencychange_perc);
commodities_vs_inf_ZAR  = commodities_vs_inf_usd *  (1+
   currencychange_perc);
RUN;

/* ------------------------------------------
REPORTING
---------------------------------------------*/

ODS HTML Style = Journal;
*Proc Print Data = A;
*Proc Print Data = Facout;
TITLE   'History Data Set';
FOOTNOTE1 "Generated by the SAS System
on %TRIM(%QSYSFUNC(DATE(),NLDATE20.))
at %TRIM(%SYSFUNC(TIME(),TIMEAMPM12.))";

PROC MEANS DATA=CVAR.Trans_CVAR_data_v5
FW=7
PRINTALLTYPES
CHARTYPE
QMETHOD=OS
NWAY
VARDEF=DF
MEAN
STD
Skewness
Kurtosis;
VAR  Return;
CLASS IndexCode /       ORDER=UNFORMATTED ASCENDING;
WHERE IndexLabel NE 'Macros & Benchmarks' and Include_Ind = '1';
RUN;
```

```
TITLE 'Fleishman Data Set';
PROC PRINT Data = CVAR.CVAR_inferential_1_flieshman
NOBS    ;
VAR IndexLabel IndexCode Return_Mean Return_StdDev Return_Skew
   Return_Kurt;
RUN;

TITLE 'MCMC Data Set';
PROC MEANS DATA= CVAR.MCMC  N MEAN STD SKEWNESS KURTOSIS;
VAR

commodities_vs_inf_usd
currencychange_perc
rsa_bill_vs_inf
rsa_bond_vs_inf
rsa_equity_vs_inf
us_bill_vs_inf
us_bond_vs_inf
us_equity_vs_inf
US_Bills_ZAR
US_Bonds_ZAR
US_Equity_ZAR
US_commodities_ZAR;
FORMAT  commodities_vs_inf_usd          PERCENTN12.3;
FORMAT  currencychange_perc             PERCENTN12.3;
FORMAT  rsa_bill_vs_inf                 PERCENTN12.3;
FORMAT  rsa_bill_vs_inf                 PERCENTN12.3;
FORMAT  rsa_equity_vs_inf               PERCENTN12.3;
FORMAT  us_bill_vs_inf                  PERCENTN12.3;
FORMAT  us_bond_vs_inf                  PERCENTN12.3;
FORMAT  us_equity_vs_inf                PERCENTN12.3;
RUN;

TITLE 'CORR MCMC Data Set';
PROC CORR DATA= CVAR.MCMC  NOSIMPLE NOPROB ;
VAR
```

```
commodities_vs_inf_usd
currencychange_perc
rsa_bill_vs_inf
rsa_bond_vs_inf
rsa_equity_vs_inf
us_bill_vs_inf
us_bond_vs_inf
us_equity_vs_inf;
FORMAT  commodities_vs_inf_usd          PERCENTN12.3;
FORMAT  currencychange_perc             PERCENTN12.3;
FORMAT  rsa_bill_vs_inf                 PERCENTN12.3;
FORMAT  rsa_bill_vs_inf                 PERCENTN12.3;
FORMAT  rsa_equity_vs_inf               PERCENTN12.3;
FORMAT  us_bill_vs_inf                  PERCENTN12.3;
FORMAT  us_bond_vs_inf                  PERCENTN12.3;
FORMAT  us_equity_vs_inf                PERCENTN12.3;

RUN; QUIT;
TITLE ;

PROC SORT DATA =  CVAR.CVAR_data_Corr;
BY _NAME_;
RUN;

TITLE 'CORR History Data Set';
PROC PRINT DATA = CVAR.CVAR_data_Corr
(WHERE= (UPCASE(_Type_) IN ("CORR"))) NOOBS;
VAR
_NAME_
commodities_vs_inf_usd
currencychange_perc
rsa_bill_vs_inf
rsa_bond_vs_inf
rsa_equity_vs_inf
us_bill_vs_inf
us_bond_vs_inf
us_equity_vs_inf;
```

```
FORMAT  commodities_vs_inf_usd  PERCENTN12.3;
FORMAT  currencychange_perc             PERCENTN12.3;
FORMAT  rsa_bill_vs_inf                 PERCENTN12.3;
FORMAT  rsa_bill_vs_inf                 PERCENTN12.3;
FORMAT  rsa_equity_vs_inf               PERCENTN12.3;
FORMAT  us_bill_vs_inf                  PERCENTN12.3;
FORMAT  us_bond_vs_inf                  PERCENTN12.3;
FORMAT  us_equity_vs_inf                PERCENTN12.3;
RUN; QUIT;

ODS HTML CLOSE;

/* Prepare data for Comparison and exporting */
DATA T_Modelling_Data1
(Drop = Date Year column_that_was_transposed _LABEL_);
SET CVAR.CVAR_data_transpose;
Data_Set = 'History';
WHERE column_that_was_transposed ='Return'  ;
RUN;

Data T_Modelling_Data2;
SET CVAR.MCMC;

Data_Set = 'Simulated';
RUN;

PROC SQL;
CREATE TABLE CVAR.T_Modelling_Data3 AS
SELECT * FROM T_Modelling_Data1
OUTER UNION CORR
SELECT * FROM T_Modelling_Data2;
Quit;


DATA CVAR.T_Modelling_Data3;
SET CVAR.T_Modelling_Data3;
LABEL
```

```
decade                          = "Decade"
date                            = "Date"
usd_zar                         = "USD ZAR"
inv_usd_zar                     = "Inv USD ZAR"
currencychange_perc             = "Exchange perc"
rsa_bill_tr_zar                 = "RSA Bill ZAR"
rsa_bond_tr_zar                 = "RSA Bond ZAR"
rsa_equity_tr_zar               = "RSA Equity ZAR"
rsa_inf_zar                     = "RSA Inf ZAR"
us_bill_tr_usd                  = "US Bill USD"
us_bond_tr_usd                  = "US Bond USD"
us_equity_tr_usd                = "US Equity USD"
us_inf_usd                      = "US inf USD"
us_bill_tr_zar                  = "US Bill ZAR"
us_bond_tr_zar                  = "US Bond ZAR"
us_equity_tr_zar                = "US Equity ZAR"
us_inf_zar                      = "US Inf ZAR"
rsa_bill_vs_inf                 = "RSA Real Bill"
rsa_bond_vs_inf                 = "RSA Real Bond"
rsa_equity_vs_inf               = "RSA Real Equity"
us_bill_vs_inf                  = "US Real Bill"
us_bond_vs_inf                  = "US Real Bond"
us_equity_vs_inf                = "US Real Equity"
commodities_vs_inf_usd          = "US Real Commodities";
RUN;

DATA CVAR.MCMC_trans;
SET CVAR.MCMC;
MCMC = _N_;
RUN;

PROC SORT
DATA=CVAR.MCMC_TRANS(KEEP=COMMODITIES_VS_INF_USD
   CURRENCYCHANGE_PERC
RSA_BILL_VS_INF RSA_BOND_VS_INF RSA_EQUITY_VS_INF US_BILL_VS_INF
   US_BOND_VS_INF US_EQUITY_VS_INF MCMC)
OUT=WORK.TMP0TempTableInput;
```

```
BY MCMC;
RUN;

PROC SQL;
CREATE VIEW WORK.TMP1TempTableWork AS
SELECT SRC.*, "StackedValues" AS _EG_IDCOL_
FROM WORK.TMP0TempTableInput AS SRC;
QUIT;

PROC TRANSPOSE DATA = WORK.TMP1TempTableWork
OUT=CVAR.MCMC_trans
NAME=ValueSource
LABEL=ValueDescription;
BY MCMC;
ID _EG_IDCOL_;
VAR COMMODITIES_VS_INF_USD CURRENCYCHANGE_PERC RSA_BILL_VS_INF
RSA_BOND_VS_INF RSA_EQUITY_VS_INF US_BILL_VS_INF US_BOND_VS_INF
   US_EQUITY_VS_INF;
RUN;

DATA CVAR.MCMC_Trans;
SET CVAR.MCMC_Trans;
If Return < -1 THEN Return = -1;
If Return > 1 THEN Return = 1;
FORMAT  commodities_vs_inf_usd  PERCENTN12.5;
FORMAT  currencychange_perc     PERCENTN12.5;
FORMAT  rsa_bill_vs_inf PERCENTN12.5;
FORMAT  rsa_bill_vs_inf PERCENTN12.5;
FORMAT  rsa_equity_vs_inf       PERCENTN12.5;
FORMAT  us_bill_vs_inf  PERCENTN12.5;
FORMAT  us_bond_vs_inf  PERCENTN12.5;
FORMAT  us_equity_vs_inf        PERCENTN12.5;

RUN;


PROC SORT
```

```
DATA=CVAR.MCMC_TRANS(KEEP=Return Asset_Class MCMC)
OUT=CVAR.MCMC_TRANS;
BY MCMC;
RUN;
PROC TRANSPOSE DATA=CVAR.MCMC_TRANS
OUT=CVAR.MCMC
NAME=Source
LABEL=Label;
BY MCMC;
ID Asset_Class;
VAR Return;
RUN; QUIT;

PROC SQL; DROP VIEW WORK.TMP1TempTableWork;
QUIT;

PROC DELETE DATA=WORK.TMP0TempTableInput;RUN;
Data CVAR.MCMC_trans;
RENAME ValueSource = Asset_Class;
RENAME StackedValues = Return;
Set CVAR.MCMC_trans;
LABEL ValueSource = 'Asset Class';
LABEL StackedValues = 'Simulated Return';
RUN;

/*
   -------------------------------------------------------------------

Create CVAR - RANKING
-------------------------------------------------------------------
    */
DATA CVAR.CVAR_1MCMC;
SET CVAR.MCMC_trans;
RUN;

PROC SORT
DATA=CVAR.CVAR_1MCMC
```

```
OUT=WORK.SORTTempTableSorted;
BY Asset_Class MCMC;
RUN;

PROC RANK DATA = WORK.SORTTempTableSorted
GROUPS=100
TIES=MEAN
OUT=WORK.RANKRankedUNIVERSE;
BY Asset_Class;
VAR Return;
RANKS rank_Return;
RUN;

PROC TABULATE
DATA=WORK.RANKRANKEDUNIVERSE
OUT=CVAR.CVAR2percMCMC;
WHERE( rank_Return <= 2);
VAR Return;
CLASS Asset_Class /     ORDER=UNFORMATTED MISSING;
TABLE
Asset_Class={LABEL="Asset Class"},
Return={LABEL="CVAR @ 2%"}*
Mean*F=PERCENTN8.2;
RUN;QUIT;

PROC MEANS DATA=CVAR.Trans_CVAR_data_v5
FW=7
PRINTALLTYPES
CHARTYPE
QMETHOD=OS
NWAY
VARDEF=DF
MEAN
STD
Skewness
Kurtosis;
VAR  Return;
```

```
CLASS IndexCode / ORDER=UNFORMATTED ASCENDING;
WHERE IndexLabel NE 'Macros & Benchmarks' and Include_Ind = '1';
output out=CVAR.CVAR_Inferential_1HISTORY  mean= STD=  P99 =
MIN= MAX=       P1=  Skewness=          Kurtosis= PROBT= / autoname
   ;
RUN;

PROC MEANS DATA=CVAR.MCMC_trans
FW=7
PRINTALLTYPES
CHARTYPE
QMETHOD=OS
NWAY
VARDEF=DF
MEAN
STD
Skewness
Kurtosis;
VAR  Return;
CLASS Asset_Class / ORDER=UNFORMATTED ASCENDING;
LABEL Return = 'Simulated Return';
LABEL Asset_Class = 'Asset Class';
output out=CVAR.CVAR_Inferential_1MCMC mean= STD=  P99 =
MIN= MAX=       P1=  Skewness=          Kurtosis= PROBT= / autoname
   ;
RUN;

/*Add the CVAR measure...*/
PROC SQL;
CREATE TABLE CVAR.All_Inferences2MCMC AS
SELECT
t2.Asset_Class,
t1.Return_Mean AS Return_Mean                     FORMAT=PERCENTN8.2,
t1.Return_StdDev AS Return_StdDev                 FORMAT=PERCENTN8.2,
t1.Return_Skew  AS Return_Skew                    FORMAT=COMMA8.1,
t1.Return_Kurt  AS Return_Kurt                    FORMAT=COMMA8.1,
t2.Return_Mean  AS CVAR2Perc                      FORMAT=PERCENTN8.2
```

```
FROM CVAR.Cvar2percmcmc t2
RIGHT JOIN CVAR.Cvar_inferential_1mcmc t1
ON (t1.Asset_Class = t2.Asset_Class);
QUIT;

DATA CVAR.All_Inferences2MCMC;
SET CVAR.All_Inferences2MCMC;
LABEL   Return_Mean = 'Return Mean';
LABEL   Return_StdDev = 'Return Std Dev';
LABEL   Return_Skew = 'Return Skew';
LABEL   Return_Kurt = 'Return Kurt';
LABEL   CVAR2Perc = 'Return CVAR@2%';
IF Asset_Class = 'CURRENCYCHANGE_PERC' THEN IndexLabel = 'Exchange
   perc';
IF Asset_Class = 'COMMODITIES_VS_INF_USD' THEN IndexLabel = 'US
   Real Commodities';
IF Asset_Class = 'RSA_BILL_VS_INF' THEN IndexLabel = 'RSA Real Bill
   ';
IF Asset_Class = 'RSA_BOND_VS_INF' THEN IndexLabel = 'RSA Real Bond
   ';
IF Asset_Class = 'RSA_EQUITY_VS_INF' THEN IndexLabel = 'RSA Real
   Equity';
IF Asset_Class = 'US_BILL_VS_INF' THEN IndexLabel = 'US Real Bill';
IF Asset_Class = 'US_BOND_VS_INF' THEN IndexLabel = 'US Real Bond';
IF Asset_Class = 'US_EQUITY_VS_INF' THEN IndexLabel = 'US Real
   Equity';
IF Asset_Class = 'CurrencyChange_perc' THEN IndexLabel = 'Exchange
   perc';
IF Asset_Class = 'Commodities_vs_inf_USD' THEN IndexLabel = 'US
   Real Commodities';
IF Asset_Class = 'RSA_Bill_vs_inf' THEN IndexLabel = 'RSA Real Bill
   ';
IF Asset_Class = 'RSA_Bond_vs_inf' THEN IndexLabel = 'RSA Real Bond
   ';
IF Asset_Class = 'RSA_Equity_vs_inf' THEN IndexLabel = 'RSA Real
   Equity';
IF Asset_Class = 'US_Bill_vs_inf' THEN IndexLabel = 'US Real Bill';
```

```
IF Asset_Class = 'US_Bond_vs_inf' THEN IndexLabel = 'US Real Bond';
IF Asset_Class = 'US_Equity_vs_inf' THEN IndexLabel = 'US Real
   Equity';
RUN;

/* ----------------------------------------------------
REPORTING
----------------------------------------------------*/
ODS PDF FILE = "...\1. CVAR Model\1. Data\Output\SAS_Base_5MCMC.pdf
   "
STYLE=Journal3 startpage=yes;
ODS HTML STYLE=JOURNAL;
*Proc Print Data = A;
*Proc Print Data = Facout;
TITLE  'History Data Set';
FOOTNOTE1 "Generated by the SAS System on
%TRIM(%QSYSFUNC(DATE(),NLDATE20.))
at %TRIM(%SYSFUNC(TIME(),TIMEAMPM12.))";

proc report data=CVAR.ALL_INFERENCES2 nowd;
column IndexLabel IndexCode Return_Mean Return_StdDev
Return_Skew Return_Kurt CVAR2Perc;
define IndexLabel / group 'Asset Class' missing;
compute IndexLabel;
if IndexLabel ne ' ' then hold1=IndexLabel;
if IndexLabel eq ' ' then IndexLabel=hold1;
endcomp;
define IndexCode / group 'Asset code' missing;
compute AssetClass;
if IndexCode ne ' ' then hold2=IndexCode;
if IndexCode eq ' ' then IndexCode=hold2;
endcomp;
define Return_Mean / analysis SUM 'Return Mean' format=PERCENTN8.2
   missing;
define Return_StdDev / analysis SUM 'Return StdDev' format=
   PERCENTN8.2 missing;
```

```
define Return_Skew / analysis SUM 'Return Skew' format=COMMA8.2
   missing;
define Return_Kurt / analysis SUM 'Return Kurt' format=COMMA8.2
   missing;
define CVAR2Perc / analysis SUM 'CVAR@2%' format=PERCENTN8.2
   missing;
RUN;
QUIT;

TITLE 'MCMC Data Set';
proc report data=CVAR.ALL_INFERENCES2MCMC nowd;
column IndexLabel Asset_Class Return_Mean   Return_StdDev
Return_Skew Return_Kurt CVAR2Perc;
define IndexLabel / group 'Asset Class' missing;
compute IndexLabel;
if IndexLabel ne ' ' then hold1=IndexLabel;
if IndexLabel eq ' ' then IndexLabel=hold1;
endcomp;
define Asset_Class / group 'Asset Class' missing;
compute Asset_Class;
if Asset_Class ne ' ' then hold1=Asset_Class;
if Asset_Class eq ' ' then Asset_Class=hold1;
endcomp;
define StackedValues_Mean / analysis SUM 'Return Mean' format=
   PERCENTN8.2 missing;
define Return_StdDev / analysis SUM 'Return StdDev' format=
   PERCENTN8.2 missing;
define Return_Skew / analysis SUM 'Return Skew' format=COMMA8.2
   missing;
define Return_Kurt / analysis SUM 'Return Kurt' format=COMMA8.2
   missing;
define CVAR2Perc / analysis SUM 'CVAR@2%' format=PERCENTN8.2
   missing;
run;
quit;

TITLE 'CORR History Data Set';
```

```
PROC CORR DATA= CVAR.Dms_v1  NOSIMPLE NOPROB ;
VAR
commodities_vs_inf_usd
currencychange_perc
rsa_bill_vs_inf
rsa_bond_vs_inf
rsa_equity_vs_inf
us_bill_vs_inf
us_bond_vs_inf
us_equity_vs_inf;
RUN; QUIT;

ODS HTML;
TITLE 'CORR MCMC Data Set';
PROC CORR DATA= CVAR.MCMC  NOSIMPLE NOPROB ;
VAR
commodities_vs_inf_usd
currencychange_perc
rsa_bill_vs_inf
rsa_bond_vs_inf
rsa_equity_vs_inf
us_bill_vs_inf
us_bond_vs_inf
us_equity_vs_inf;
RUN; QUIT;
TITLE ;

TITLE ;
ODS HTML CLOSE;
ODS PDF CLOSE;
/* Prepare data for Comparison and exporting */
DATA T_Modelling_Data1
(Drop = Date Year column_that_was_transposed _LABEL_);
SET CVAR.CVAR_data_transpose;
Data_Set = 'History';
WHERE column_that_was_transposed ='Return'  ;
RUN;
```

```
Data T_Modelling_Data2;
SET CVAR.MCMC;
Data_Set = 'Modelled';
RUN;

PROC SQL;
CREATE TABLE
CVAR.T_Modelling_Data3 AS
SELECT * FROM T_Modelling_Data1
OUTER UNION CORR
SELECT * FROM T_Modelling_Data2;
Quit;

DATA CVAR.T_Modelling_Data3;
SET CVAR.T_Modelling_Data3;
LABEL
usd_zar                         = "USD ZAR"
inv_usd_zar                     = "Inv USD ZAR"
currencychange_perc             = "Exchange perc"
rsa_bill_tr_zar                 = "RSA Bill ZAR"
rsa_bond_tr_zar                 = "RSA Bond ZAR"
rsa_equity_tr_zar               = "RSA Equity ZAR"
rsa_inf_zar                     = "RSA Inf ZAR"
us_bill_tr_usd                  = "US Bill USD"
us_bond_tr_usd                  = "US Bond USD"
us_equity_tr_usd                = "US Equity USD"
us_inf_usd                      = "US inf USD"
us_bill_tr_zar                  = "US Bill ZAR"
us_bond_tr_zar                  = "US Bond ZAR"
us_equity_tr_zar                = "US Equity ZAR"
us_inf_zar                      = "US Inf ZAR"
rsa_bill_vs_inf                 = "RSA Real Bill"
rsa_bond_vs_inf                 = "RSA Real Bond"
rsa_equity_vs_inf               = "RSA Real Equity"
us_bill_vs_inf                  = "US Real Bill"
us_bond_vs_inf                  = "US Real Bond"
```

```
us_equity_vs_inf                = "US Real Equity"
commodities_vs_inf_usd          = "US Real Commodities";
RUN;

DATA CVAR.MCMC_Export;
SET CVAR.MCMC ;
FORMAT  US_Bills_ZAR                            PERCENTN12.5;
FORMAT  US_Bonds_ZAR                            PERCENTN12.5;
FORMAT  US_Equity_ZAR                           PERCENTN12.5;
FORMAT  US_commodities_ZAR                      PERCENTN12.5;
FORMAT  commodities_vs_inf_usd                  PERCENTN12.5;
FORMAT  currencychange_perc                     PERCENTN12.5;
FORMAT  rsa_bill_vs_inf                         PERCENTN12.5;
FORMAT  rsa_bill_vs_inf                         PERCENTN12.5;
FORMAT  rsa_equity_vs_inf                       PERCENTN12.5;
FORMAT  us_bill_vs_inf                          PERCENTN12.5;
FORMAT  us_bond_vs_inf                          PERCENTN12.5;
FORMAT  us_equity_vs_inf                        PERCENTN12.5;
FORMAT  US_Bills_ZAR                            PERCENTN12.5;
FORMAT  US_Bonds_ZAR                            PERCENTN12.5;
FORMAT  US_Equity_ZAR                           PERCENTN12.5;
FORMAT  US_commodities_ZAR                      PERCENTN12.5;
RUN;

/* ---------------------------------------------------
EXPORTING
Check Exported Distributions
-----------------------------------------------------*/

PROC SQL;
CREATE VIEW WORK.MCMC_Export AS
SELECT *
FROM CVAR.MCMC_Export as T
;
QUIT;

ODS HTML STYLE = JOURNAL;
```

```
TITLE; TITLE1 "Distribution analysis of: Assets in Universe";
FOOTNOTE;
FOOTNOTE1 "Generated by the SAS System
on %TRIM(%QSYSFUNC(DATE(), NLDATE20.))
at %TRIM(%SYSFUNC(TIME(), TIMEAMPM12.))";
ODS EXCLUDE EXTREMEOBS MODES MOMENTS QUANTILES;
GOPTIONS htext=1 cells;
SYMBOL v=SQUARE c=BLUE h=1 cells;
PATTERN v=SOLID;
PROC UNIVARIATE DATA = WORK.MCMC_Export
CIBASIC(TYPE=TWOSIDED ALPHA=0.05) MU0=0;
VAR
commodities_vs_inf_usd
currencychange_perc
rsa_bill_vs_inf
rsa_bond_vs_inf
rsa_equity_vs_inf
us_bill_vs_inf
us_bond_vs_inf
us_equity_vs_inf
US_Bills_ZAR
US_Bonds_ZAR
US_Equity_ZAR
US_commodities_ZAR;
HISTOGRAM /     CFRAME=GRAY CAXES=BLACK WAXIS=1
CBARLINE=BLACK CFILL=BLUE PFILL=SOLID ;
RUN; QUIT;

/*-----------------------------------------------------
Export Distributions
------------------------------------------------------*/
proc export
data=CVAR.MCMC
dbms=xlsx
outfile="...\1. CVAR Model\1. Data\MCMC.xlsx"
replace;
run;
```

B.3 Matlab code: portfolio optimisation

The following section is a report detailing the optimisation program in Matlab. The report is generated by the Matlab publishing function for latex documents where, both the underlying code and results are included.

Import simulated data set

```
[~, ~, raw] = xlsread('...\2. Matlab\MCMC.xlsx','MCMC','A2:L100001');
raw(cellfun(@(x) ~isempty(x) && isnumeric(x) && isnan(x),raw)) = {''};
R = cellfun(@(x) ~isnumeric(x) && ~islogical(x),raw);
raw(R) = {NaN};
% Create output Matrix
MCMC = reshape([raw{:}],size(raw));
% Create Table
MCMC_Table  = table;
MCMC_Table.COMMODITIES_VS_INF_USD  = MCMC(:,1);
MCMC_Table.CURRENCYCHANGE_PERC  = MCMC(:,2);
MCMC_Table.RSA_BILL_VS_INF = MCMC(:,3);
MCMC_Table.RSA_BOND_VS_INF  = MCMC(:,4);
MCMC_Table.RSA_EQUITY_VS_INF  = MCMC(:,5);
MCMC_Table.US_BILL_VS_INF  = MCMC(:,6);
MCMC_Table.US_BOND_VS_INF  = MCMC(:,7);
MCMC_Table.US_EQUITY_VS_INF  = MCMC(:,8);
MCMC_Table.US_Bills_ZAR  = MCMC(:,9);
MCMC_Table.US_Bonds_ZAR  = MCMC(:,10);
MCMC_Table.US_Equity_ZAR  = MCMC(:,11);
MCMC_Table.US_commodities_ZAR  = MCMC(:,12);
clearvars data raw R;
```

Subtract the costs of holding the passive

```
ETF_Cost = [0,0,0.002,0.0024,0.004,0,0,0,0,0.0024,0.0049,0.0078,]';


MCMC_net = MCMC - repmat(ETF_Cost',size(MCMC,1),1);
```

Report statistics of simulated return data (net of fees)

```
z = [mean(MCMC_net);std(MCMC_net);min(MCMC_net); max(MCMC_net);
skewness(MCMC_net);kurtosis(MCMC_net)];


'Total net return relative to inflation'
```

```
dataset({z 'COMMODITIES','USDZAR','RSABILL','RSABOND',
'RSAEQUITY','USBILL','USBOND','USEQUITY','USBills_ZAR','USBonds_ZAR',
'USEquity_ZAR','Commodities_ZAR'}, ...
'obsnames', {'mu','std','min','max','skew','kurt'})
```

```
ans =  Total net return relative to inflation


ans =

COMMODITIES    USDZAR      RSABILL      RSABOND     RSAEQUITY
mu        0.02638       0.026208    0.0073687    0.016357    0.091255
std       0.18796        0.14937     0.068115     0.11208     0.24192
min      -0.43346       -0.82845      -0.9773     -1.0114     -2.3511
max        1.9057         1.3882      0.68608     0.90001      2.6515
skew       1.3192        0.89516      -1.6467    -0.41131     0.45997
kurt       6.0465         6.1697        15.23      5.8804      5.8079


USBILL       USBOND       USEQUITY    USBills_ZAR    USBonds_ZAR
mu      0.0095329    0.020637    0.081645    0.010049        0.022026
std       0.04882     0.10501     0.20814    0.050654          0.1112
min      -0.51752    -0.42878    -0.72322    -0.53772          -0.437
max       0.46315     0.70513      0.5645     0.48073         0.98965
skew     -0.72548     0.54469    -0.25274    -0.68536         0.82792
kurt       6.4688      3.7968      2.6821      7.0383          4.8794


USEquity_ZAR    Commodities_ZAR
mu      0.074994          0.0058359
std      0.21563            0.18359
min      -1.0308           -0.73401
max      0.87762             1.5408
skew    -0.31662            0.83579
kurt      3.0056             4.6168
```

Set up portfolio CVAR object

```
p = PortfolioCVaR('Scenarios', MCMC_net, ...
```

```
'LowerBound', 0.00, 'LowerBudget', 1, 'UpperBudget', 1, ...
'ProbabilityLevel', 0.98, 'NumAssets', 12, 'InitPort', []);
p = p.setAssetList({'COMMODITIES','USDZAR','RSABILL','RSABOND',
'RSAEQUITY','USBILL','USBOND','USEQUITY','USBills_ZAR',
'USBonds_ZAR','USEquity_ZAR','Commodities_ZAR'});
disp(p);
```

```
  PortfolioCVaR with properties:

BuyCost: []
SellCost: []
RiskFreeRate: []
ProbabilityLevel: 0.9800
Turnover: []
BuyTurnover: []
SellTurnover: []
NumScenarios: 100000
Name: []
NumAssets: 12
AssetList: {1x12 cell}
InitPort: []
AInequality: []
bInequality: []
AEquality: []
bEquality: []
LowerBound: [12x1 double]
UpperBound: []
LowerBudget: 1
UpperBudget: 1
GroupMatrix: []
LowerGroup: []
UpperGroup: []
GroupA: []
GroupB: []
LowerRatio: []
UpperRatio: []
```

Set international exposure constraint

```
I = 0.30;

G = [0 0 0 0 0 0   0   0   1   1 1 1];

p = p.setGroups(G, 0.0, I);
disp(p.NumAssets);
disp(p.GroupMatrix);
disp(p.LowerGroup);
disp(p.UpperGroup);
```

```
    12

0     0     0     0     0     0     0     0     1     1     1     1

0

0.3000
```

Restrict to asset classes to universe

```
G = [1 1 0 0 0   1 1 1   0 0   0   0];

p = p.addGroups(G, 0.0, 0.0);
disp(p.NumAssets);
disp(p.GroupMatrix);
disp(p.LowerGroup);
disp(p.UpperGroup);
```

```
    12

0     0     0     0     0     0     0     0     1     1     1     1
1     1     0     0     0     1     1     1     0     0     0     0

0
0
```

```
0.3000
0
```

Optimisation: plot frontier and report weights of 10 portfolios

```
p.plotFrontier;
[prsk, pret] = p.plotFrontier(10);
pwgt = p.estimateFrontier;

International_Allocation =  I;
Time = fix(clock)

Results = dataset({[(round(prsk*1000)/1000)'; (round(pret*1000)/1000)';round(pwgt*1000)/1000]
'P1','P2','P3','P4','P5','P6','P7','P8','P9','P10'}, ...
'obsnames', {'Risk','Return','COMMODITIES','USD_ZAR','RSA_BILL','RSA_BOND','RSA_EQUITY',
'US_BILL','US_BOND','US_EQUITY', 'US_Bills_ZAR','US_Bonds_ZAR','US_Equity_ZAR',
'US_commodities_ZAR'})

Feasibility = p.checkFeasibility(pwgt)
```

```
Time =

2015          10          13          12          34          54


Results =

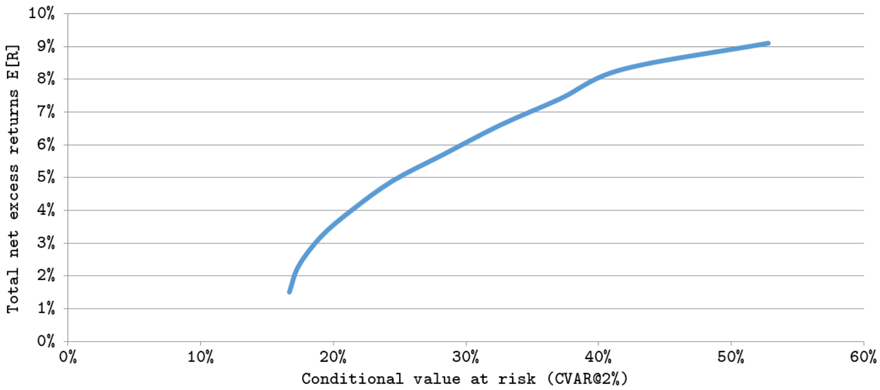
P1       P2       P3       P4       P5       P6
Risk                   0.167    0.174    0.191    0.214    0.245    0.283
Return                 0.015    0.023    0.032     0.04    0.049    0.057
COMMODITIES                0        0        0        0        0        0
USD_ZAR                    0        0        0        0        0        0
RSA_BILL                0.56    0.511    0.474    0.413     0.34    0.264
RSA_BOND               0.059    0.023        0        0        0        0
RSA_EQUITY             0.082    0.166    0.226    0.287     0.36    0.436
US_BILL                    0        0        0        0        0        0
US_BOND                    0        0        0        0        0        0
US_EQUITY                  0        0        0        0        0        0
```

US_Bills_ZAR	0.136	0.156	0.08	0.007	0	0
US_Bonds_ZAR	0	0.025	0.114	0.207	0.172	0.132
US_Equity_ZAR	0	0.018	0.055	0.086	0.128	0.168
US_commodities_ZAR	0.164	0.101	0.051	0	0	0

P7 P8 P9 P10

Risk	0.326	0.371	0.418	0.528
Return	0.066	0.074	0.083	0.091
COMMODITIES	0	0	0	0
USD_ZAR	0	0	0	0
RSA_BILL	0.187	0.11	0.033	0
RSA_BOND	0	0	0	0
RSA_EQUITY	0.513	0.59	0.667	1
US_BILL	0	0	0	0
US_BOND	0	0	0	0
US_EQUITY	0	0	0	0
US_Bills_ZAR	0	0	0	0
US_Bonds_ZAR	0.092	0.055	0.016	0
US_Equity_ZAR	0.208	0.245	0.284	0
US_commodities_ZAR	0	0	0	0

Feasibility = 1 1 1 1 1 1 1 1 1 1



Optimisation: allocation to specific portfolios (CPI+2%, CPI+4%, CPI+6%) * Matlab output

```
pwgt = p.estimateFrontierByReturn([0.02, 0.04, 0.06]);
```

```
prsk = p.estimatePortRisk(pwgt);
pret = p.estimatePortReturn(pwgt);


Results = dataset({[(round(prsk*1000)/1000)';
(round(pret*1000)/1000)';round(pwgt*1000)/1000] 'CPI_plus_2','CPI_plus_4','CPI_plus_6'}, ...
'obsnames', {'Risk','Return','COMMODITIES','USD_ZAR','RSA_BILL','RSA_BOND','RSA_EQUITY',
'US_BILL','US_BOND','US_EQUITY', 'US_Bills_ZAR','US_Bonds_ZAR','US_Equity_ZAR',
'US_commodities_ZAR'})

Results =

CPI_plus_2    CPI_plus_4    CPI_plus_6
Risk                   0.17          0.213          0.296
Return                 0.02           0.04           0.06
COMMODITIES               0              0              0
USD_ZAR                   0              0              0
RSA_BILL              0.521          0.415          0.241
RSA_BOND              0.039              0              0
RSA_EQUITY             0.14          0.285          0.459
US_BILL                   0              0              0
US_BOND                   0              0              0
US_EQUITY                 0              0              0
US_Bills_ZAR          0.174          0.013              0
US_Bonds_ZAR              0          0.202          0.117
US_Equity_ZAR         0.003          0.084          0.183
US_commodities_ZAR    0.123          0.001              0
```

C Multi-period model

C.1 Eviews Code: regime-switching model

Please note that if following Eviews code is used in Eviews with the seed provided in the code, the results are expected to be the same.

```
RSA Bills - Estimation Command:
=========================
SWITCHREG(TYPE=MARKOV, SEED=1340617527, RNG=KN)
RSA_BILLS_LEAD C @NV RSA_BILLS @PRV C GOV_BOND_CASH_SPREAD

Estimation Equation:
=========================
1: RSA_BILLS_LEAD = C(1) + C(3)*RSA_BILLS

2: RSA_BILLS_LEAD = C(2) + C(3)*RSA_BILLS

SIGMA = @EXP(C(4))

Forecasting Equation:
=========================
1: RSA_BILLS_LEAD = C(1) + C(3)*RSA_BILLS

2: RSA_BILLS_LEAD = C(2) + C(3)*RSA_BILLS

SIGMA = @EXP(C(4))

Substituted Coefficients:
=========================
1: RSA_BILLS_LEAD = 0.0518177893781 + 0.368382228435*RSA_BILLS

2: RSA_BILLS_LEAD = 0.101850439727 + 0.368382228435*RSA_BILLS

SIGMA = @EXP(-4.20699055065)
```

```
US Bills - Estimation Command:
=========================
SWITCHREG(TYPE=MARKOV, SEED=1340617527, RNG=KN)
US_BILL_LEAD C @NV US_BILLS @PRV C US_BOND_CASH_SPREAD

Estimation Equation:
=========================
1: US_BILL_LEAD = C(1) + C(3)*US_BILLS

2: US_BILL_LEAD = C(2) + C(3)*US_BILLS

SIGMA = @EXP(C(4))

Forecasting Equation:
=========================
1: US_BILL_LEAD = C(1) + C(3)*US_BILLS

2: US_BILL_LEAD = C(2) + C(3)*US_BILLS

SIGMA = @EXP(C(4))

Substituted Coefficients:
=========================
1: US_BILL_LEAD = -0.000893960176328 + 0.604592885028*US_BILLS

2: US_BILL_LEAD = 0.0235100832969 + 0.604592885028*US_BILLS

SIGMA = @EXP(-4.80189816299)


RSA Bonds - Estimation Command:
=========================
SWITCHREG(TYPE=MARKOV, SEED=364078019, RNG=KN)
RSA_BONDS_YOY_LEAD C @PRV C GOV_BOND_CASH_SPREAD

Estimation Equation:
```

```
=========================
1: RSA_BONDS_YOY_LEAD = C(1)

2: RSA_BONDS_YOY_LEAD = C(2)

SIGMA = @EXP(C(3))

Forecasting Equation:
=========================
1: RSA_BONDS_YOY_LEAD = C(1)

2: RSA_BONDS_YOY_LEAD = C(2)

SIGMA = @EXP(C(3))

Substituted Coefficients:
=========================
1: RSA_BONDS_YOY_LEAD = -0.0478748084774

2: RSA_BONDS_YOY_LEAD = 0.0303359707471

SIGMA = @EXP(-3.51665588571)


US Bonds - Estimation Command:
=========================
SWITCHREG(TYPE=MARKOV, HETERR, SEED=364078019, RNG=KN)
US_BONDS_LEAD C @NV US_BONDS @PRV C  US_BOND_CASH_SPREAD

Estimation Equation:
=========================
1: US_BONDS_LEAD = C(1) + C(5)*US_BONDS

1: SIGMA = @EXP(C(2))

2: US_BONDS_LEAD = C(3) + C(5)*US_BONDS
```

```
2: SIGMA = @EXP(C(4))

Forecasting Equation:
=========================
1: US_BONDS_LEAD = C(1) + C(5)*US_BONDS

1: SIGMA = @EXP(C(2))

2: US_BONDS_LEAD = C(3) + C(5)*US_BONDS

2: SIGMA = @EXP(C(4))

Substituted Coefficients:
=========================
1: US_BONDS_LEAD = 0.0111179994955 + 0.870195717533*US_BONDS

1: SIGMA = @EXP(-5.02451369544)

2: US_BONDS_LEAD = -0.00124117007037 + 0.870195717533*US_BONDS

2: SIGMA = @EXP(-5.55715275911)

RSA Equity - Estimation Command:
=========================
SWITCHREG(TYPE=MARKOV, SEED=364078019, RNG=KN)
RSA_EQUITY_YOY_LEAD C @PRV C GOV_BOND_CASH_SPREAD

Estimation Equation:
=========================
1: RSA_EQUITY_YOY_LEAD = C(1)

2: RSA_EQUITY_YOY_LEAD = C(2)

SIGMA = @EXP(C(3))

Forecasting Equation:
=========================
```

```
1: RSA_EQUITY_YOY_LEAD = C(1)

2: RSA_EQUITY_YOY_LEAD = C(2)

SIGMA = @EXP(C(3))

Substituted Coefficients:
=========================
1: RSA_EQUITY_YOY_LEAD = -0.0874317781116

2: RSA_EQUITY_YOY_LEAD = 0.264580069823

SIGMA = @EXP(-1.95736090418)

US Equity - Estimation Command:
=========================
SWITCHREG(TYPE=MARKOV, SEED=364078019, RNG=KN)
US_EQUITY_YOY_LEAD C @PRV C US_BOND_CASH_SPREAD

Estimation Equation:
=========================
1: US_EQUITY_YOY_LEAD = C(1)

2: US_EQUITY_YOY_LEAD = C(2)

SIGMA = @EXP(C(3))

Forecasting Equation:
=========================
1: US_EQUITY_YOY_LEAD = C(1)

2: US_EQUITY_YOY_LEAD = C(2)

SIGMA = @EXP(C(3))

Substituted Coefficients:
=========================
```

```
1: US_EQUITY_YOY_LEAD = 0.178518971855

2: US_EQUITY_YOY_LEAD = -0.128583785994

SIGMA = @EXP(-2.20438638687)


RSA GDP - Estimation Command:
=========================
SWITCHREG(TYPE=MARKOV, HETERR, SEED=364078019, RNG=KN)
RSA_GDP_YOY_LEAD C @NV RSA_GDP_YOY @PRV C CONSUMPTION_GDP

Estimation Equation:
=========================
1: RSA_GDP_YOY_LEAD = C(1) + C(5)*RSA_GDP_YOY

1: SIGMA = @EXP(C(2))

2: RSA_GDP_YOY_LEAD = C(3) + C(5)*RSA_GDP_YOY

2: SIGMA = @EXP(C(4))

Forecasting Equation:
=========================
1: RSA_GDP_YOY_LEAD = C(1) + C(5)*RSA_GDP_YOY

1: SIGMA = @EXP(C(2))

2: RSA_GDP_YOY_LEAD = C(3) + C(5)*RSA_GDP_YOY

2: SIGMA = @EXP(C(4))

Substituted Coefficients:
=========================
1: RSA_GDP_YOY_LEAD = 0.0213893699292 + 0.213461176663*RSA_GDP_YOY

1: SIGMA = @EXP(-5.11761934534)
```

```
2: RSA_GDP_YOY_LEAD = 0.0331388309161 + 0.213461176663*RSA_GDP_YOY

2: SIGMA = @EXP(-5.19771807566)


RSA CPI - Estimation Command:
=========================
SWITCHREG(TYPE=MARKOV, SEED=364078019, RNG=KN) RSA_CPI_LEAD C @NV
RSA_CPI @PRV C GOV_BOND_CASH_SPREAD

Estimation Equation:
=========================
1: RSA_CPI_LEAD = C(1) + C(3)*RSA_CPI

2: RSA_CPI_LEAD = C(2) + C(3)*RSA_CPI

SIGMA = @EXP(C(4))

Forecasting Equation:
=========================
1: RSA_CPI_LEAD = C(1) + C(3)*RSA_CPI

2: RSA_CPI_LEAD = C(2) + C(3)*RSA_CPI

SIGMA = @EXP(C(4))

Substituted Coefficients:
=========================
1: RSA_CPI_LEAD = 0.0364779268051 + 0.812650428258*RSA_CPI

2: RSA_CPI_LEAD = -0.01713572292 + 0.812650428258*RSA_CPI

SIGMA = @EXP(-3.85430705495)


US Commodities - Estimation Command:
```

```
=========================
SWITCHREG(TYPE=MARKOV, SEED=1931791586, RNG=KN)
   COMMODITIES_YOY_LEAD
C @PRV C US_BOND_CASH_SPREAD

Estimation Equation:
=========================
1: COMMODITIES_YOY_LEAD = C(1)

2: COMMODITIES_YOY_LEAD = C(2)

SIGMA = @EXP(C(3))

Forecasting Equation:
=========================
1: COMMODITIES_YOY_LEAD = C(1)

2: COMMODITIES_YOY_LEAD = C(2)

SIGMA = @EXP(C(3))

Substituted Coefficients:
=========================
1: COMMODITIES_YOY_LEAD = -0.06841275529

2: COMMODITIES_YOY_LEAD = 0.191919422542

SIGMA = @EXP(-2.29905039142)
```

C.2 SAS Code: asset price simulation

```
/*
Asset pricing simulation Program
By Stuart Royden-Turner
Relating to section 9.3
*/
```

```
/* Drives*/
LIBNAME Asset '....\4. Asset Pricing Simulation';
x 'cd ....\4. Asset Pricing Simulation\SAS Output'; RUN;

/* General assumptions and simulation inputs */
%LET Scenarios       = 1000;
%LET Holding_periods = 10;
```

```
/* Short term return forecast */
DATA Asset.Assumptions_1;
input Date Measure $    RSA_Cash    RSA_Inflation    RSA_Bonds    RSA_equity  US_Cash     US_Bonds
     US_equity   Commodities RSA_GDP;
datalines;

2005  LRX   0.0631      0.0482      0.0726      0.1400      0.0510      0.0614      0.1518      0.0381
        0.0320
2006  LRX   0.0631      0.0482      0.0726      0.1400      0.0510      0.0614      0.1518      0.0381
        0.0320
2007  LRX   0.0631      0.0482      0.0726      0.1400      0.0510      0.0614      0.1518      0.0381
        0.0320
2008  LRX   0.0631      0.0482      0.0726      0.1400      0.0510      0.0614      0.1518      0.0381
        0.0320
2009  LRX   0.0631      0.0482      0.0726      0.1400      0.0510      0.0614      0.1518      0.0381
        0.0320
2010  LRX   0.0631      0.0482      0.0726      0.1400      0.0510      0.0614      0.1518      0.0381
        0.0320
2011  LRX   0.0631      0.0482      0.0726      0.1400      0.0510      0.0614      0.1518      0.0381
        0.0320
2012  LRX   0.0631      0.0482      0.0726      0.1400      0.0510      0.0614      0.1518      0.0381
        0.0320
2013  LRX   0.0631      0.0482      0.0726      0.1400      0.0510      0.0614      0.1518      0.0381
        0.0320
2014  LRX   0.0631      0.0482      0.0726      0.1400      0.0510      0.0614      0.1518      0.0381
        0.0320

2005  LDA   0.4447      0.5602      0.6620      0.5962      0.4356      0.6631      0.4431      0.6039
        0.5106
2006  LDA   0.4447      0.5602      0.6620      0.5962      0.4356      0.6631      0.4431      0.6039
        0.5106
2007  LDA   0.4447      0.5602      0.6620      0.5962      0.4356      0.6631      0.4431      0.6039
        0.5106
2008  LDA   0.4447      0.5602      0.6620      0.5962      0.4356      0.6631      0.4431      0.6039
        0.5106
2009  LDA   0.4447      0.5602      0.6620      0.5962      0.4356      0.6631      0.4431      0.6039
        0.5106
2010  LDA   0.4447      0.5602      0.6620      0.5962      0.4356      0.6631      0.4431      0.6039
        0.5106
2011  LDA   0.4447      0.5602      0.6620      0.5962      0.4356      0.6631      0.4431      0.6039
        0.5106
2012  LDA   0.4447      0.5602      0.6620      0.5962      0.4356      0.6631      0.4431      0.6039
        0.5106
2013  LDA   0.4447      0.5602      0.6620      0.5962      0.4356      0.6631      0.4431      0.6039
        0.5106
2014  LDA   0.4447      0.5602      0.6620      0.5962      0.4356      0.6631      0.4431      0.6039
        0.5106

2005  STX   0.0793      0.0563      0.0025      0.2588      0.0367      0.0469      0.2588      0.1813
        0.0282
2006  STX   0.0773      0.0619      -0.0435     0.2582      0.0463      0.0493      0.1686      0.1776
        0.0292
2007  STX   0.0836      0.0821      -0.0420     0.2524      0.0290      0.0449      0.1564      0.1760
        0.0324
2008  STX   0.0912      0.1058      -0.0296     -0.0368     0.0177      0.0353      -0.1090     -0.0611
        0.0315
2009  STX   0.0919      0.0631      -0.0387     0.2321      -0.0003     0.0210      0.1715      0.1698
        0.0270
2010  STX   0.0788      0.0381      0.0281      0.2574      -0.0001     0.0308      0.1723      0.1839
        0.0252
2011  STX   0.0734      0.0619      0.0247      0.2095      0.0004      0.0281      0.1529      0.1826
        0.0271
2012  STX   0.0731      0.0591      0.0289      0.2231      -0.0004     0.0168      0.1706      -0.0598
       0.0273
2013  STX   0.0713      0.0721      -0.0448     0.2605      0.0000      0.0253      0.1707      -0.0610
       0.0260
2014  STX   0.0722      0.0738      0.0283      0.2574      0.0000      0.0248      0.1714      -0.0606
       0.0265



2005  STV   0.0153      0.0226      0.0359      0.1489      0.0082      0.0048      0.1181      0.1080
        0.0062
```

2006	STV	0.0157	0.0239	0.0360	0.1492	0.0076	0.0054	0.1209	0.1105	0.0064
2007	STV	0.0152	0.0240	0.0361	0.1492	0.0084	0.0056	0.1163	0.1074	0.0065
2008	STV	0.0155	0.0246	0.0363	0.1491	0.0084	0.0057	0.1182	0.1046	0.0060
2009	STV	0.0153	0.0243	0.0362	0.1488	0.0081	0.0065	0.1195	0.1077	0.0060
2010	STV	0.0156	0.0229	0.0360	0.1493	0.0082	0.0048	0.1210	0.1030	0.0060
2011	STV	0.0145	0.0245	0.0360	0.1489	0.0082	0.0061	0.1171	0.1031	0.0060
2012	STV	0.0145	0.0242	0.0359	0.1490	0.0080	0.0063	0.1198	0.1030	0.0060
2013	STV	0.0147	0.0246	0.0359	0.1491	0.0077	0.0066	0.1165	0.1043	0.0060
2014	STV	0.0146	0.0228	0.0363	0.1493	0.0081	0.0048	0.1195	0.1120	0.0060
2005	LRV	0.0599	0.0601	0.0805	0.1790	0.0877	0.0970	0.2080	0.1664	0.0200
2006	LRV	0.0599	0.0601	0.0805	0.1790	0.0877	0.0970	0.2080	0.1664	0.0200
2007	LRV	0.0599	0.0601	0.0805	0.1790	0.0877	0.0970	0.2080	0.1664	0.0200
2008	LRV	0.0599	0.0601	0.0805	0.1790	0.0877	0.0970	0.2080	0.1664	0.0200
2009	LRV	0.0599	0.0601	0.0805	0.1790	0.0877	0.0970	0.2080	0.1664	0.0200
2010	LRV	0.0599	0.0601	0.0805	0.1790	0.0877	0.0970	0.2080	0.1664	0.0200
2011	LRV	0.0599	0.0601	0.0805	0.1790	0.0877	0.0970	0.2080	0.1664	0.0200
2012	LRV	0.0599	0.0601	0.0805	0.1790	0.0877	0.0970	0.2080	0.1664	0.0200
2013	LRV	0.0599	0.0601	0.0805	0.1790	0.0877	0.0970	0.2080	0.1664	0.0200
2014	LRV	0.0599	0.0601	0.0805	0.1790	0.0877	0.0970	0.2080	0.1664	0.0200

;

```
/* Transpose and organise data */

PROC SORT
DATA=ASSET.ASSUMPTIONS_1
(KEEP=RSA_Cash RSA_Inflation RSA_Bonds
RSA_equity US_Cash US_Bonds US_equity
Commodities RSA_GDP Date Measure)
OUT=WORK.TMP0TempTableInput
;
BY Date Measure;
RUN;

PROC SQL;
CREATE VIEW WORK.TMP1TempTableWork AS
SELECT SRC.*, "StackedValues" AS _EG_IDCOL_
FROM WORK.TMP0TempTableInput AS SRC;
QUIT;

PROC TRANSPOSE DATA = WORK.TMP1TempTableWork
OUT=Asset.Assumptions_2
NAME=ValueSource
LABEL=ValueDescription
;
BY Date Measure;
ID _EG_IDCOL_;
VAR RSA_Cash RSA_Inflation RSA_Bonds
RSA_equity US_Cash US_Bonds US_equity
Commodities RSA_GDP;

RUN;

PROC DATASETS LIB=  Asset NOLIST;
MODIFY Assumptions_2;
LABEL StackedValues = "Measure";
LABEL ValueSource = "Asset_Class";
RUN;
```

```
PROC SQL; DROP VIEW WORK.TMP1TempTableWork;
QUIT;
PROC DELETE DATA=WORK.TMP0TempTableInput;RUN;

PROC SORT
DATA=Asset.Assumptions_2
(KEEP=StackedValues Measure Date ValueSource)
OUT=WORK.SORTTempTableSorted
;
BY Date ValueSource;
RUN;

PROC TRANSPOSE DATA=WORK.SORTTempTableSorted
OUT=ASSET.ASSUMPTIONS_3
NAME=Source
LABEL=Label
;
BY Date ValueSource;
ID Measure;
VAR StackedValues;

RUN; QUIT;

Data ASSET.ASSUMPTIONS_3 (DROP = Label Source );
SET ASSET.ASSUMPTIONS_3;
RUN;

/* Generate Data */
DATA Asset.Scenarios_1;
SET Asset.Assumptions_3;

DO  Scenario = 1 to &Scenarios by 1;
DO Holding_period = 1 to &Holding_periods by 1;

Year_Term = Holding_period + date -1;
OUTPUT;
```

```
END;
END;
RUN;


/* Asset price simulation model */
Data Asset.Scenarios_2;
RENAME ValueSource = AssetClass;
SET Asset.Scenarios_1;

Return_deterministic = STX + (LRX - STX) * (1-EXP(-LDA*
   Holding_period));
Lambda_function = (1-EXP(-LDA*Holding_period));
Vol = + (STV + ((LRV - STV) * (1-EXP(-LDA*Holding_period))));
call streaminit(INT(datetime()));
Return = STX + (LRX - STX) * (1-EXP(-LDA*Holding_period))
+ (STV + ((LRV - STV) * (1-EXP(-LDA*Holding_period)))) * RAND('
   NORMAL',0,1);
RUN;

Data Asset.AMPL_Dev_v1 (KEEP = Date T A AssetClass S P);
SET Asset.Scenarios_2;

T = Holding_period;
IF  AssetClass = 'RSA_equity'          THEN A = 1;
IF  AssetClass = 'RSA_Bonds'           THEN A = 2;
IF  AssetClass = 'RSA_Cash'            THEN A = 3;
IF  AssetClass = 'Commodities'         THEN A = 4;
IF  AssetClass = 'US_equity'           THEN A = 5;
IF  AssetClass = 'US_Bonds'            THEN A = 6;
IF  AssetClass = 'US_Cash'             THEN A = 7;
S = Scenario;
P = Return;
WHERE Scenario < 11 and Date = 2005 and
AssetClass NOT IN ('RSA_GDP' 'RSA_Inflation');
RUN;
```

```
PROC SORT Data = Asset.AMPL_Dev_v1;
BY   Date A S T;
RUN;

DATA Asset.AMPL_Dev_v2;
SET Asset.AMPL_Dev_v1;
BY  Date A S;
retain  P_index 0;
if T = 1 then P_index = 100;
if T > 1 then P_index = P_index *(1+P);

OUTPUT;
RUN;
```

C.3 SAS Code: liability model

```
/*LIBNAME LIFE '....\1. Models\2. Stochastic Model\2. Code\1.
   Liability Model';*/
/*x 'cd  ....\1. Models\2. Stochastic Model\2. Code\1. Liability
   Model'; RUN;*/

/* General assumptions and simulation inputs */
%LET Scenarios  = 1000;
%LET Early_CustomerNumber = 3500;
%LET Middle_CustomerNumber = 1000;
%LET Late_CustomerNumber = 465;
%LET UHNW_CustomerNumber = 300;
%LET seed = 1+ROUND(10000*INT(Datetime()));
%LET Year_Start = 2005;
%LET Year_End = 2015;
%LET Start_Term = 0;
%LET End_Term = 11;
%Let Liability_Start = 100;
%LET Customer_Number = 1;

/* Key assumptions */
%LET L_CPI = 0.06;
```

```
%LET L_CPI_std = 0.0466;
%LET L_GDP = 0.0324;
%LET L_GDP_std = 0.0129;
%LET Lambda_GDP = 0.511;
%LET Lambda_CPI = 0.561;
%LET RR_High = 0.06;
%LET RR_Med = 0.04;
%LET RR_Low = 0.02;
%LET Retirement_Age = 60;
%LET Retirement_Age_std = 2;
%LET Terminal_Age = 100;
%LET Terminal_Age_std = 0;

/*First Loop: Affluent Young (early Stage)*/
%LET Upfront_Contribution = 1000;
%LET Upfront_Contribution_std   = 0;
%LET Start_Age  = 35;
%LET Start_Age_std = 3;
%LET Current_Age = 35;
%LET Current_Age_std = 3;
%LET Contribution = 24000;
%LET Contribution_std = 2400;
%LET Expenses = 300000;
%LET Expenses_std = 1000;

DATA LIFE.Assumptions_early;
DO Customer_Number = 1 to &Early_CustomerNumber by 1;
call streaminit(INT(datetime()));
Category     = '1. Early';
Terminal_Age = round(&Terminal_Age
+ &Terminal_Age_std*RAND('NORMAL',0,1));
Retirement_Age  = MIN (Terminal_Age -3, round(&Retirement_Age
+ &Retirement_Age_std*RAND('NORMAL',0,1)));
Start_Age  = MAX(25,MIN (Retirement_Age -3, round(&Start_Age
+ &Start_Age_std*RAND('NORMAL',0,1))));

Expenses  = MAX(0,round(&Expenses
```

```
+ &Expenses_std*RAND('NORMAL',0,1)));
Upfront  = MAX(0,round(&Upfront_Contribution
+ &Upfront_Contribution_std*RAND('NORMAL',0,1)));
Contribution  = MAX(0,round(&Contribution
+ &Contribution_std*RAND('NORMAL',0,1)));

DO Scenarios = 1 to &Scenarios by 1;
call streaminit(INT(datetime()));
L_CPI = &L_CPI + &L_CPI_std*RAND('NORMAL',0,1);
L_GDP = &L_GDP + &L_GDP_std*RAND('NORMAL',0,1);

FORMAT  L_CPI 10.3;
FORMAT  L_GDP 10.3;

OUTPUT;
END;
END;
RUN;


DATA LIFE.Assumptions_early;
SET LIFE.Assumptions_early;
DO Age = Start_Age TO Terminal_Age by 1;
dAge = 'Yr'||Put(Age,3.);
Year = Age - Start_Age + &Year_Start;
IF Age GE Retirement_Age THEN Retirement_Ind = 1;
IF Age LT Retirement_Age THEN Retirement_Ind = 0;
IF Age = Retirement_Age THEN Retirement_date_Ind = 1;
IF Age NE Retirement_Age THEN Retirement_date_Ind = 0;
IF Age = Terminal_Age THEN Terminal_date_Ind = 1;
IF Age NE Terminal_Age THEN Terminal_date_Ind = 0;
Investment_Horizon = Terminal_Age - Age;
Investment_Horizon_Total = Retirement_Age - Start_Age;
Retirement_Horizon = Terminal_Age - Retirement_Age;
Retirement_Time = Age - Retirement_Age -2;
Time_on_Book = Age - Start_Age;
Analysis_Term = &Year_End - year;
```

```
OUTPUT;
END;
RUN;

/*First Loop: Affluent - middle (Middle Stage)*/
%LET Upfront_Contribution = 100000;
%LET Upfront_Contribution_std = 75000;
%LET Start_Age = 45;
%LET Start_Age_std = 3;
%LET Current_Age = 45;
%LET Current_Age_std = 3;
%LET Contribution = 45000;
%LET Contribution_std = 4500;
%LET Expenses = 300000;
%LET Expenses_std = 3000;

DATA LIFE.Assumptions_middle;
DO Customer_Number = 1 to &middle_CustomerNumber by 1;
call streaminit(INT(datetime()));

Category  = '2. Middle';
Terminal_Age = round(&Terminal_Age + &Terminal_Age_std*RAND('NORMAL
    ',0,1));
Retirement_Age = MIN (Terminal_Age -3, round(&Retirement_Age +
&Retirement_Age_std*RAND('NORMAL',0,1)));
Start_Age = MAX(25,MIN (Retirement_Age -3, round(&Start_Age  +
&Start_Age_std*RAND('NORMAL',0,1))));
Expenses  = MAX(0,round(&Expenses + &Expenses_std*RAND('NORMAL
    ',0,1)));
Upfront  = MAX(0,round(&Upfront_Contribution
+ &Upfront_Contribution_std*RAND('NORMAL',0,1)));
Contribution = MAX(0,round(&Contribution
+ &Contribution_std*RAND('NORMAL',0,1)));

DO Scenarios = 1 to &Scenarios by 1;
call streaminit(INT(datetime()));
```

```
L_CPI = &L_CPI + &L_CPI_std*RAND('NORMAL',0,1);
L_GDP = &L_GDP + &L_GDP_std*RAND('NORMAL',0,1);

FORMAT  L_CPI 10.3;
FORMAT  L_GDP 10.3;

OUTPUT;
END;
END;
RUN;



DATA LIFE.Assumptions_middle;
SET LIFE.Assumptions_middle;
DO Age = Start_Age TO Terminal_Age by 1;
dAge = 'Yr'||Put(Age,3.);
Year  = Age - Start_Age + &Year_Start;
IF Age GE Retirement_Age THEN Retirement_Ind = 1;
IF Age LT Retirement_Age THEN Retirement_Ind = 0;
IF Age = Retirement_Age THEN Retirement_date_Ind = 1;
IF Age NE Retirement_Age THEN Retirement_date_Ind = 0;
IF Age = Terminal_Age THEN Terminal_date_Ind = 1;
IF Age NE Terminal_Age THEN Terminal_date_Ind = 0;
Investment_Horizon = Terminal_Age - Age;
Investment_Horizon_Total = Retirement_Age - Start_Age;
Retirement_Horizon = Terminal_Age - Retirement_Age;
Retirement_Time = Age - Retirement_Age -2;
Time_on_Book = Age - Start_Age;
Analysis_Term = &Year_End - year;

OUTPUT;
END;
RUN;



/*First Loop: High net worth (Late Stage)*/
```

```
%LET Upfront_Contribution = 2000000;
%LET Upfront_Contribution_std = 1000000;
%LET Start_Age = 55;
%LET Start_Age_std = 3;
%LET Current_Age = 55;
%LET Current_Age_std = 3;
%LET Contribution = 60000;
%LET Contribution_std = 12000;
%LET Expenses = 300000;
%LET Expenses_std = 3000;

DATA LIFE.Assumptions_late;
DO Customer_Number = 1 to &Late_CustomerNumber by 1;
call streaminit(INT(datetime()));
Category  = '3. Late';
Terminal_Age  = round(&Terminal_Age
+ &Terminal_Age_std*RAND('NORMAL',0,1));
Retirement_Age = MIN (Terminal_Age -3, round(&Retirement_Age
+ &Retirement_Age_std*RAND('NORMAL',0,1)));
Start_Age = MAX(25,MIN (Retirement_Age -3,
round(&Start_Age+ &Start_Age_std*RAND('NORMAL',0,1))));
Expenses = MAX(0,round(&Expenses + &Expenses_std*RAND('NORMAL',0,1)
   ));
Upfront = MAX(0,round(&Upfront_Contribution
+ &Upfront_Contribution_std*RAND('NORMAL',0,1)));
Contribution = MAX(0,round(&Contribution
+ &Contribution_std*RAND('NORMAL',0,1)));

DO Scenarios = 1 to &Scenarios by 1;
call streaminit(INT(datetime()));

L_CPI = &L_CPI + &L_CPI_std*RAND('NORMAL',0,1);
L_GDP = &L_GDP + &L_GDP_std*RAND('NORMAL',0,1);
FORMAT  L_CPI 10.3;
FORMAT  L_GDP 10.3;
OUTPUT;
END;
```

```
END;
RUN;


DATA LIFE.Assumptions_late;
SET LIFE.Assumptions_late;
DO Age = Start_Age TO Terminal_Age by 1;
dAge = 'Yr'||Put(Age,3.);
Year = Age - Start_Age + &Year_Start;
IF Age GE Retirement_Age THEN Retirement_Ind = 1;
IF Age LT Retirement_Age THEN Retirement_Ind = 0;
IF Age = Retirement_Age THEN Retirement_date_Ind = 1;
IF Age NE Retirement_Age THEN Retirement_date_Ind = 0;
IF Age = Terminal_Age THEN Terminal_date_Ind = 1;
IF Age NE Terminal_Ag THEN Terminal_date_Ind = 0;

Investment_Horizon = Terminal_Age - Age;
Investment_Horizon_Total = Retirement_Age - Start_Age;
Retirement_Horizon = Terminal_Age - Retirement_Age;
Retirement_Time = Age - Retirement_Age -2;
Time_on_Book = Age - Start_Age;
Analysis_Term = &Year_End - year;

OUTPUT;
END;
RUN;

/*UHNW Stage*/
%LET Upfront_Contribution  = 10000000;
%LET Upfront_Contribution_std= 2000000;
%LET Start_Age = 50;
%LET Start_Age_std = 3;
%LET Current_Age = 50;
%LET Current_Age_std = 3;
%LET Contribution = 0;
%LET Contribution_std = 0;
%LET Expenses = 500000;
```

```
%LET Expenses_std = 5000;
DATA LIFE.Assumptions_UHNW;
DO Customer_Number = 1 to &UHNW_CustomerNumber by 1;
call streaminit(INT(datetime()));
Category  = '4. UHNW';
Terminal_Age  = round(&Terminal_Age   + &Terminal_Age_std*RAND('
   NORMAL',0,1));
Retirement_Age  = MIN (Terminal_Age -3, round(&Retirement_Age
+ &Retirement_Age_std*RAND('NORMAL',0,1)));
Start_Age  = MAX(25,MIN (Retirement_Age -3, round(&Start_Age +
&Start_Age_std*RAND('NORMAL',0,1))));
Expenses  = MAX(0,round(&Expenses + &Expenses_std*RAND('NORMAL
   ',0,1)));
Upfront= MAX(1000000,round(&Upfront_Contribution
+ &Upfront_Contribution_std*RAND('NORMAL',0,1)));
Contribution = MAX(0,round(&Contribution + &Contribution_std*RAND('
   NORMAL',0,1)));

DO Scenarios = 1 to &Scenarios by 1;
call streaminit(INT(datetime()));
L_CPI = &L_CPI + &L_CPI_std*RAND('NORMAL',0,1);
L_GDP = &L_GDP + &L_GDP_std*RAND('NORMAL',0,1);
FORMAT  L_CPI 10.3;
FORMAT  L_GDP 10.3;
OUTPUT;
END;
END;
RUN;


DATA LIFE.Assumptions_UHNW;
SET LIFE.Assumptions_UHNW;
DO Age = Start_Age TO Terminal_Age by 1;
dAge = 'Yr'||Put(Age,3.);
Year = Age - Start_Age + &Year_Start;
IF Age GE Retirement_Age THEN Retirement_Ind = 1;
IF Age LT Retirement_Age THEN Retirement_Ind = 0;
```

```
IF Age = Retirement_Age THEN Retirement_date_Ind = 1;
IF Age NE Retirement_Age THEN Retirement_date_Ind = 0;
IF Age = Terminal_Age THEN Terminal_date_Ind = 1;
IF Age NE Terminal_Age THEN Terminal_date_Ind = 0;
Investment_Horizon = Terminal_Age - Age;
Investment_Horizon_Total = Retirement_Age - Start_Age;
Retirement_Horizon = Terminal_Age - Retirement_Age;
Retirement_Time = Age - Retirement_Age -2;
Time_on_Book = Age - Start_Age;
Analysis_Term = &Year_End - year;
OUTPUT;
END;
RUN;

PROC SQL;
CREATE TABLE      LIFE.ASSUMPTIONS_2            AS
SELECT * FROM     LIFE.ASSUMPTIONS_EARLY        OUTER UNION CORR
SELECT * FROM     LIFE.ASSUMPTIONS_LATE         OUTER UNION CORR
SELECT * FROM     LIFE.ASSUMPTIONS_MIDDLE       OUTER UNION CORR
SELECT * FROM     LIFE.ASSUMPTIONS_UHNW
;
QUIT;

/*Add Interest rate and CPI data (Actuals) */
DATA Life.Endogenous;
input Year  CPI   GDP;
datalines;

1980  0.1577      0.044
1981  0.1398      0.044
1982  0.1368      0.041
1983  0.1107      0.006
1984  0.132       0.045
1985  0.1848      0.005
1986  0.1811      0.000
1987  0.147       0.042
1988  0.1254      0.041
```

```
1989  0.1535      0.040
1990  0.1464      0.028
1991  0.1624      0.019
1992  0.0961      0.000
1993  0.0951      0.001
1994  0.0989      0.044
1995  0.0686      0.042
1996  0.094       0.042
1997  0.0608      0.038
1998  0.0901      0.032
1999  0.0224      0.005
2000  0.0699      0.042
2001  0.0459      0.038
2002  0.1241      0.038
2003  0.0033      0.029
2004  0.0339      0.041
2005  0.0427      0.0437
2006  0.0436      0.0440
2007  0.0559      0.0441
2008  0.0745      0.0317
2009  0.0782      0.0052
2010  0.0587      0.0071
2011  0.0432      0.0445
2012  0.0578      0.0309
2013  0.0563      0.0400
2014  0.0535      0.0227;

DATA Life.Appetite;
LENGTH Risk_Appetite $8;
input Age   Risk_Appetite $;
datalines;
20    High
21    High
22    High
23    High
24    High
25    High
```

```
26     High
27     High
28     High
29     High
30     High
31     High
32     High
33     High
34     High
35     High
36     High
37     High
38     High
39     High
40     High
41     High
42     High
43     High
44     High
45     High
46     High
47     High
48     High
49     High
50     High
51     High
52     High
53     High
54     High
55     High
56     Med
57     Med
58     Med
59     Med
60     Med
61     Med
62     Med
```

63	Med
64	Med
65	Med
66	Med
67	Med
68	Med
69	Med
70	Med
71	Med
72	Med
73	Med
74	Med
75	Med
76	Med
77	Med
78	Med
79	Med
80	Med
81	Med
82	Med
83	Med
84	Med
85	Med
86	Med
87	Med
88	Med
89	Med
90	Med
91	Med
92	Med
93	Med
94	Med
95	Med
96	Med
97	Med
98	Med
99	Med

```
;

/* Life expectancy data */
DATA Life.Life_expectancy_import;
input Age   Expected_mortality_age;
datalines;

20        74.1
21        74.2
22        74.3
23        74.3
24        74.4
25        74.4
26        74.4
27        74.5
28        74.5
29        74.5
30        74.6
31        74.6
32        74.6
33        74.6
34        74.7
35        74.7
36        74.7
37        74.7
38        74.8
39        74.8
40        74.9
41        74.9
42        75.0
43        75.0
44        75.1
45        75.1
46        75.2
47        75.3
48        75.4
49        75.5
```

50	75.6
51	75.7
52	75.8
53	76.0
54	76.1
55	76.3
56	76.5
57	76.7
58	77.0
59	77.2
60	77.5
61	77.8
62	78.1
63	78.4
64	78.7
65	79.1
66	79.5
67	79.8
68	80.2
69	80.6
70	81.1
71	81.5
72	82.0
73	82.4
74	82.9
75	83.5
76	84.0
77	84.5
78	85.1
79	85.7
80	86.3
81	86.9
82	87.5
83	88.2
84	88.9
85	89.6
86	90.3

```
87        91.0
88        91.7
89        92.5
90        93.2
91        94.0
92        94.8
93        95.6
94        96.4
95        97.2
96        98.1
97        98.9
98        99.8
99        100.6
100 101.5
101 102.4
102 103.3
103 104.2
104 105.1
105 106.0
106 106.9
107 107.8
108 108.8
109 109.7
110 110.7
111 111.6
112 112.6
113 113.5
114 114.5
115 115.4
116 116.3
117 117.0
;

/* Join Tables*/
PROC SQL;
CREATE TABLE LIFE.Assumptions_3 AS
SELECT   t1.*, t3.Risk_Appetite, t2.CPI, t2.GDP,
```

```
t4.Expected_mortality_age
FROM LIFE.ASSUMPTIONS_2 t1
LEFT JOIN LIFE.ENDOGENOUS t2   ON (t1.Year = t2.Year)
LEFT JOIN LIFE.APPETITE t3     ON (t1.Age = t3.Age)
LEFT JOIN LIFE.Life_expectancy_import t4 ON (t1.Age = t4.Age)
WHERE t1.Year LE &Year_End  AND t4.Expected_mortality_age > t1.Age;
QUIT;

/* Cash Flows */
PROC SORT Data = LIFE.Assumptions_3;
BY Category Scenarios Customer_Number  Age;
RUN;

DATA Life.Cash_Flows;
SET LIFE.Assumptions_3;
BY Category Scenarios Customer_Number Year;
IF Category NE '4. UHNW' AND Retirement_ind = 1
THEN Risk_Appetite = 'Low';
IF Category NE '4. UHNW' AND Risk_Appetite = 'High'
THEN Required_Return    = &RR_High + CPI;
IF Category NE '4. UHNW' AND Risk_Appetite = 'Med'
THEN Required_Return    = &RR_Med + CPI;
IF Category NE '4. UHNW' AND Risk_Appetite = 'Low'
THEN Required_Return    = &RR_Low + CPI;
IF Category =  '4. UHNW'
THEN Required_Return    = &RR_High + CPI;
IF Category =  '4. UHNW'
THEN Risk_Appetite      = 'High';
Expenses = -Expenses;
Discount_Factor = 1/((1+ L_CPI)**(Age-Start_Age));
Expected_Contribution
=  (Contribution)*((1+GDP)**(Age-Start_Age));

/* Future Value */
IF age = Start_Age THEN Upfront = Upfront;
IF age > Start_Age THEN Upfront = 0;
Consumption_Requirement       = Retirement_Ind *(Expenses)
```

```
*((1+CPI)**(Age-Start_Age));
Expenses_Forecast             = (Expenses)*((1+CPI)**(Age-Start_Age
   ));
IF Category = '4. UHNW'    THEN Consumption_Requirement = 0;
IF Retirement_ind = 1   THEN Contribution = 0;
IF Retirement_ind = 1   THEN Expected_Contribution = 0;
IF age = Start_Age            THEN Asset_Total_T0 = 0;
RETAIN Asset_Total_T0 0;IF FIRST.Customer_Number THEN
   Asset_Total_T0
= Upfront*(1+Required_Return) + Expected_Contribution +
Consumption_Requirement;
IF age > start_age            THEN    Asset_Total_T0
= sum(Asset_Total_T0*(1+Required_Return), Expected_Contribution,
Consumption_Requirement);
RETAIN Contribution_Total 0;
IF FIRST.Customer_Number THEN Contribution_Total =
   Expected_Contribution;
IF age > start_age THEN      Contribution_Total
= sum(Contribution_Total, Expected_Contribution);
RETAIN Consumption_Total 0;    IF FIRST.Customer_Number
THEN Consumption_Total      = Consumption_Requirement;
IF age > start_age THEN      Consumption_Total
= sum(Consumption_Total, Consumption_Requirement);
InCome_Generated = Asset_Total_T1 - Asset_Total_T0;
InCome_Generated_PV = (Asset_Total_T0 - Contribution_Total
+ Consumption_Requirement)* Discount_Factor;

/* Present Value */
Consumption_Requirement_PV  = Consumption_Requirement *
   Discount_Factor;
Asset_Total_T0_PV           = Asset_Total_T0 * Discount_Factor;
OUTPUT;
FORMAT Expected_Contribution 10.2;
FORMAT Income_Generated 10.2;
FORMAT Asset_Total_T0 10.2;
FORMAT Consumption_Requirement 10.2;
RUN;
```

```
/* Create Retirement future foreast indicator mapping table */
DATA LIFE.Retirement_Mapping (KEEP = Category Customer_Number
Scenarios Age Age_Forecast Term Retirement_ind);
SET LIFE.Cash_Flows;
DO Term = &Start_Term TO &End_Term  BY 1;
Age_Forecast = Age + Term;
OUTPUT;
END;
RUN;

PROC SQL;
CREATE TABLE LIFE.RETIREMENT_MAPPING AS
SELECT t1.Customer_Number,
t1.Scenarios,
t1.Category,
t1.Age,
t1.Retirement_Ind,
t1.Term,
t1.Age_Forecast,
t2.Retirement_Ind AS Retirement_Ind_Term
FROM LIFE.RETIREMENT_MAPPING t1
LEFT JOIN LIFE.Cash_flows t2 ON (t1.Customer_Number = t2.
   Customer_Number)
AND (t1.Age_Forecast = t2.Age)AND (t1.Scenarios = t2.Scenarios)
AND (t2.Category = t1.Category);
QUIT;

/* Cash Flow forecast &  Calculate Term structures */
DATA LIFE.Cash_Flows_Forecast (DROP = CPI GDP);
SET LIFE.Cash_Flows;
DO Term = &Start_Term TO &End_Term  BY 1;
Effective_Year = Term + Year;
Effective_Age = Term + Age;
GDP_Term = GDP +(L_GDP-GDP)*(1-exp(-&Lambda_GDP*Term));
CPI_Term = CPI +(L_CPI-CPI)*(1-exp(-&Lambda_CPI*Term));
IF Category NE '4. UHNW' AND Risk_Appetite = 'High'
```

```
THEN Required_Return    = &RR_High + CPI_Term;
IF Category NE '4. UHNW' AND Risk_Appetite = 'Med'
THEN Required_Return    = &RR_Med + CPI_Term;
IF Category NE '4. UHNW' AND Risk_Appetite = 'Low'
THEN Required_Return    = &RR_Low + CPI_Term;
IF Category =  '4. UHNW'
THEN Required_Return    = &RR_High + CPI_Term;
IF Category =  '4. UHNW'
THEN Risk_Appetite      = 'High';
OUTPUT;
END;

RUN;

PROC SQL;
CREATE TABLE LIFE.Cash_Flows_Forecast AS
SELECT t1.Customer_Number,
t1.Scenarios,
t1.Category,
t1.Age,
t1.Year,
t1.Required_Return,
t1.Discount_Factor,
t1.Upfront,
t1.Expected_Contribution,
t1.Expenses_Forecast,
t1.Asset_Total_T0,
t1.Contribution_Total,
t1.Asset_Total_T0_PV,
t1.Term,
t1.Analysis_Term,
t2.Retirement_Ind_Term,
t1.Effective_Year,
t1.Effective_Age,
t3.Risk_Appetite AS Effective_Appetite,
t1.GDP_Term,
t1.CPI_Term,
```

```
t1.L_GDP,
t1.L_CPI
FROM LIFE.CASH_FLOWS_FORECAST t1
LEFT JOIN LIFE.RETIREMENT_MAPPING t2 ON
(t1.Customer_Number = t2.Customer_Number)
AND (t1.Scenarios = t2.Scenarios) AND (t1.Category = t2.Category)
AND (t1.Age = t2.Age) AND (t1.Term = t2.Term)
LEFT JOIN LIFE.Appetite t3 ON (t3.Age = t1.Effective_Age)
WHERE t1.Term LE Analysis_Term;
QUIT;

DATA LIFE.Cash_Flows_Forecast;
SET LIFE.Cash_Flows_Forecast;
IF Category NE '4. UHNW' AND Retirement_Ind_Term = 1
THEN Effective_Appetite       = 'Low';
IF Category NE '4. UHNW' AND Effective_Appetite = 'High'
THEN Effective_Return         = &RR_High  + CPI_Term;
IF Category NE '4. UHNW' AND Effective_Appetite = 'Med'
THEN Effective_Return         = &RR_Med   + CPI_Term;
IF Category NE '4. UHNW' AND Effective_Appetite = 'Low'
THEN Effective_Return         = &RR_Low   + CPI_Term;
IF Category =  '4. UHNW'
THEN Effective_Return         = &RR_High  + CPI_Term;
IF Category =  '4. UHNW'
THEN Effective_Appetite       = 'High';
RUN;

PROC SORT Data = LIFE.Cash_Flows_Forecast;
BY Category Scenarios Customer_Number Age  Term;
RUN;

DATA LIFE.Cash_Flows_Forecast2 (DROP = Expected_Contribution
Expenses_Forecast Asset_Total_T0  Contribution_Total
   Asset_Total_T0_PV);
SET LIFE.Cash_Flows_Forecast;
BY Category Scenarios Customer_Number Age Term;
IF Retirement_Ind_Term = '.' THEN Retirement_Ind_Term = 1;
```

```
/*Expected Contributions */
Incremental_Cont_Forecast = Expected_Contribution * (1+GDP_Term)**
   Term;
RETAIN Contribution_Forecast 0;
IF FIRST.Age THEN Contribution_Forecast = Expected_Contribution;
IF Term > 0 THEN     Contribution_Forecast   =
sum(Contribution_Forecast * (1+GDP_Term), Incremental_Cont_Forecast
   );

/*Expected Expenses */
Incremental_Exp_Forecast = Expenses_Forecast * (1+CPI_Term)**Term;
RETAIN Expenses_Forecast_term 0; IF FIRST.Age
THEN Expenses_Forecast_term = Incremental_Exp_Forecast;
IF Term > 0 THEN  Expenses_Forecast_term =
sum(Expenses_Forecast_term * (1+CPI_Term), Incremental_Exp_Forecast
   );
Contribution_Forecast  = (1-Retirement_Ind_Term) *
   Contribution_Forecast;
Incremental_Cont_Forecast=(1-Retirement_Ind_Term) *
   Incremental_Cont_Forecast;
Expenses_Forecast = Retirement_Ind_Term * Expenses_Forecast;
Incremental_Exp_Forecast = Retirement_Ind_Term *
   Incremental_Exp_Forecast;
Expenses_Forecast_term   = Retirement_Ind_Term *
   Expenses_Forecast_term;

/*Asset Value Term*/
RETAIN Asset_Value_Forecast 0;
IF FIRST.Age THEN Asset_Value_Forecast =  Asset_Total_T0;
IF Term > 0 THEN Asset_Value_Forecast =
MAX(0,sum(Asset_Value_Forecast*(1+Effective_Return),
Incremental_Cont_Forecast,Incremental_Exp_Forecast));

/* Present Value */
Discount_Factor_Forecast            = 1/((1+ (CPI_Term))**(Term));
```

```
Contribution_Forecast_PV              = Contribution_Forecast *
   Discount_Factor_Forecast;
Expenses_Forecast_PV                  = Expenses_Forecast *
   Discount_Factor_Forecast;
Asset_Value_Forecast_PV               = (Asset_Total_T0_PV * (1+
   Effective_Return)**Term)
* Discount_Factor_Forecast;
IF Asset_Value_Forecast < 0           THEN Asset_Value_Forecast = 0;
IF Asset_Total_T0 < 0                       THEN Asset_Value_Forecast
    = 0;
IF TERM > 0                           THEN Upfront = 0;
WHERE Year <= &Year_End;
RUN;

/*Reporting*/
ODS SELECT NONE;
PROC TABULATE
DATA=LIFE.CASH_FLOWS_FORECAST2
OUT=LIFE.CASH_FLOWS_SUMMARY1 ;
VAR Incremental_Exp_Forecast Incremental_Cont_Forecast Upfront
Asset_Value_Forecast L_CPI CPI_Term Effective_Return;
CLASS Category /                      ORDER=UNFORMATTED MISSING;
CLASS Year /                          ORDER=UNFORMATTED MISSING;
CLASS Term /                          ORDER=UNFORMATTED MISSING;
CLASS Scenarios /                     ORDER=UNFORMATTED MISSING;
CLASS Customer_Number /               ORDER=UNFORMATTED MISSING;
CLASS Retirement_Ind_Term /           ORDER=UNFORMATTED MISSING;
TABLE
Category*
Year*
Term*
Customer_Number*
Scenarios,
N
Upfront * Sum
Incremental_Cont_Forecast * Sum
Incremental_Exp_Forecast * Sum
```

```
Asset_Value_Forecast *  Sum
L_CPI * Mean *F=PERCENTN9.3
CPI_Term *  Mean *F=PERCENTN9.3
Effective_Return * Mean * F=PERCENTN9.3;
;
RUN; RUN; QUIT;

TITLE; FOOTNOTE;
ODS SELECT ALL;
Data LIFE.Cash_flows_summary2 (DROP = _TYPE_ _PAGE_ _TABLE_);
SET LIFE.Cash_flows_summary1;
Analysis_Term = &Year_End - year;
IF Term > 0 THEN CashFlow = Incremental_Exp_Forecast_Sum
+ Incremental_Cont_Forecast_Sum;
IF Term = 0 THEN CashFlow = Asset_Value_Forecast_Sum;
IF Term = Analysis_Term THEN CashFlow = -1*Asset_Value_Forecast_Sum
+ Incremental_Exp_Forecast_Sum + Incremental_Cont_Forecast_Sum;
IF Term > 0                          THEN Asset =
   Asset_Value_Forecast_Sum;
IF Term = 0                          THEN Asset =
   Asset_Value_Forecast_Sum;
IF Term = Analysis_Term              THEN Asset = 0;
IF Term = Analysis_Term              THEN Liability =
-1*Asset_Value_Forecast_Sum + Incremental_Exp_Forecast_Sum
+ Incremental_Cont_Forecast_Sum;
RUN;

/* Final Summary */
ODS SELECT NONE;
PROC TABULATE
DATA=LIFE.CASH_FLOWS_SUMMARY2
OUT=LIFE.CASH_FLOWS_SUMMARY3;
WHERE( Year < 2015);
VAR CashFlow;
CLASS Category /  ORDER=UNFORMATTED MISSING;
CLASS Year /      ORDER=UNFORMATTED MISSING;
CLASS Scenarios / ORDER=UNFORMATTED MISSING;
```

```
CLASS Term /      ORDER=UNFORMATTED MISSING;
TABLE /* Row Dimension */
Year*
Category*
Scenarios,
/* Column Dimension */
Term*
CashFlow*
Sum;RUN;

ODS SELECT ALL;
PROC SORT
DATA=LIFE.CASH_FLOWS_SUMMARY3
OUT=LIFE.CASH_FLOWS_SUMMARY3;
BY Category Year Scenarios;
RUN;

PROC TRANSPOSE DATA= LIFE.CASH_FLOWS_SUMMARY3
OUT=LIFE.CASH_FLOWS_SUMMARY4
PREFIX=Term
NAME=Term
LABEL=Term;
BY Category Year Scenarios;
ID Term;
VAR CashFlow_Sum;
RUN; QUIT;

Data LIFE.CASH_FLOWS_SUMMARY4;
SET LIFE.CASH_FLOWS_SUMMARY4;
IF Term0 = . THEN Term0 = 0;
IF Term1 = . THEN Term1 = 0;
IF Term2 = . THEN Term2 = 0;
IF Term3 = . THEN Term3 = 0;
IF Term4 = . THEN Term4 = 0;
IF Term5 = . THEN Term5 = 0;
IF Term6 = . THEN Term6 = 0;
IF Term7 = . THEN Term7 = 0;
```

```
IF Term8 = . THEN Term8 = 0;
IF Term9 = . THEN Term9 = 0;
IF Term10 = . THEN Term10 = 0;
IRR = IRR(1,Term0, Term1, Term2, Term3, Term4,
Term5, Term6,Term7, Term8, Term9, Term10)/100;
RUN;

ODS SELECT NONE;
PROC TABULATE
DATA=LIFE.CASH_FLOWS_SUMMARY2 (KEEP = Year Analysis_Term)
OUT=LIFE.Analysis_Term;
VAR Analysis_Term;
CLASS Year /      ORDER=UNFORMATTED MISSING;
TABLE /* Row Dimension */
Year*
Analysis_Term ,
/* Column Dimension */
Mean={LABEL="Analysis_Term"};
RUN;
ODS SELECT ALL;

PROC SQL;
CREATE TABLE LIFE.IRR_1 AS
SELECT t1.Category,
t1.Year,
t1.Scenarios,
t2.Analysis_Term_Mean AS Analysis_Term,
t1.IRR
FROM LIFE.CASH_FLOWS_SUMMARY4 t1
INNER JOIN LIFE.ANALYSIS_TERM t2 ON (t1.Year = t2.Year);
QUIT;

DATA LIFE.IRR_2;
SET LIFE.IRR_1;
DO Term = 0 to Analysis_Term by 1;
Expected_Liability = &Liability_Start * EXP(IRR * (Term + year - &
    Year_Start));
```

```
Dynamic_Term = Year - &Year_End  +  (&Year_End -  &Year_Start) +1;
IF Analysis_Term = Term THEN Final_Term_ind = 1;
/*IF Analysis_Term <> Term THEN Final_Term_ind = 0;*/
Forecast_year = Year + Term;
OUTPUT;
END;
RUN;
```

C.4 AMPL DEV code: multi-period optimisation

```
option model solver CPLEX;

#PARAMETERS: SCALARS
param NA :=7;
param NT :=11;
param NS :=1000;
param cbuy  := 0.005;
param csell  := 0.004;
param LC := 0.95;
param UC := 1.1;

#SETS
set ASSETS  := 1..NA;
set TIME  := 1..NT;
set SCENARIO  := 1..NS;

#PARAMETERS VECTORS
param Liabilities {TIME};
param Target   {TIME};
param Initialholdings {ASSETS};
param Income {TIME};
param chold {ASSETS};
param International {ASSETS};

#Stochastic parameters
param prob {SCENARIO} := 1/NS;
param price {TIME, ASSETS, SCENARIO};

#VARIABLES
var hold {t in TIME, a in ASSETS, s in SCENARIO} >=0;
var buy {t in TIME, a in ASSETS, s in SCENARIO} >=0;
var sell {t in TIME, a in ASSETS, s in SCENARIO} >=0;
var marketvalue {t in TIME} >=0;

#OBJECTIVE
maximize wealth: sum{s in SCENARIO} prob[s] * (sum{a in ASSETS}
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price [NT,a,s]*hold[NT,a,s]*(1-chold[a]));

#CONSTRAINTS
subject to
assetmarketvalue1: marketvalue[1]  = sum{a in ASSETS, s in SCENARIO}
prob[s] * Initialholdings[a]*price[1,a,s]*(1-chold[a]);
assetmarketvalue2{t in 2..NT}: marketvalue[t]  = sum{a in ASSETS, s in SCENARIO}
prob[s] * hold[t,a,s] * price[t,a,s]*(1-chold[a]);
stockbalance1{t in TIME, a in ASSETS, s in SCENARIO}: hold[1,a,s]
= Initialholdings[a] + buy[1,a,s] - sell[1,a,s];
stockbalance2{t in 2..NT ,a in ASSETS, s in SCENARIO}:
hold[t,a,s] = hold[t-1,a,s] + buy[t,a,s] - sell[t,a,s];


fundbalance1{t in TIME, s in SCENARIO}:
sum{a in ASSETS} sell[t,a,s] * price[t,a,s] * (1-csell) - Liabilities[t] + Income[t]
=  sum{a in ASSETS} buy [t,a,s] * price[t,a,s] * (1+cbuy);


InternationalBalance {t in TIME, s in SCENARIO}:
(sum{a in ASSETS} hold[t,a,s]*price[t,a,s]*.25) >=
(sum{a in ASSETS} International[a] * hold[t,a,s]*price[t,a,s]);


LowerRiskConstraints {t in TIME, s in SCENARIO}:
(sum{a in ASSETS} hold[NT,a,s]*price[NT,a,s] / Target[NT]) >=  LC;

UpperRiskConstraints {t in TIME, s in SCENARIO}:
(sum{a in ASSETS} hold[NT,a,s]*price[NT,a,s] / Target[NT]) <=  UC;


# STAGE 1
na1{a in ASSETS, s in 2..NS}: hold[1,a,1]=hold[1,a,s];
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