Threefolds Isogenous to a Product and Product quotient Threefolds with Canonical Singularities

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Abstract

In this thesis we study varieties isogenous to a product and product quotient varieties with canonical singularities in dimension three.
A *variety X isogenous to a product of curves* is a quotient of a product of compact Riemann surfaces of genus at least two by the free action of a finite group:

$$X = \big(C_1 \times \ldots \times C_n\big)/G.$$

Since Catanese introduced these objects in [Cat00] they turned out to be very useful to find new and interesting examples of varieties of general type. Especially the surface case has been studied extensively (see [BCG08, CP09, Pe10] et al). We are interested in the systematic construction and classification of these varieties in dimension three. Our first main result is the full classification of threefolds isogenous to a product with $\chi(\mathcal{O}_X) = -1$ under the assumption that the induced actions of the maximal subgroups $G_i \leq G$ acting on C_i are faithful. Our approach to achieve this result is algorithmic and relies on techniques from computational group theory, which extend the methods used by the authors above. Part of the classification, namely the unmixed case, where the group G acts diagonally, has been achieved in a joint work with Davide Frapporti [FG15].

The notion of a product quotient variety X generalizes the definition of a variety isogenous to a product by allowing non-free group actions. We study these varieties in dimension three under the assumptions that X has canonical singularities and G acts faithfully on each factor of the product. The first assumption implies that we can consider a crepant terminalisation i.e. a proper birational morphism $\rho\colon \widehat{X} \to X$, where $\widehat{X}$ has only terminal singularities and $\rho^*(K_X) = K_{\widehat{X}}$. Our first aim is to study the geography of these varieties i.e. relations between the Chern invariants

$$\chi(\mathcal{O}_{\widehat{X}}), \qquad e(\widehat{X}) \qquad \text{and} \qquad K_{\widehat{X}}^3.$$

We provide such relations in the form of inequalities and discuss the boundary cases. This leads to a characterization of the examples, where $\widehat{X}$ is smooth i.e. the examples admitting a smooth minimal model. For these varieties, we provide a classification algorithm to determines all examples for a given fixed value of $\chi(\mathcal{O}_{\widehat{X}})$. In the last part

of this thesis, we prove the sharp inequality $K_X^3 \geq 4$ for product quotient threefolds X with canonical singularities and provide the full list of examples realizing the minimum value $K_X^3 = 4$.

Zusammenfassung

Gegenstand dieser Arbeit sind Varietäten isogen zu einem Produkt von Kurven und Produktquotienten mit kanonischen Singularitäten in Dimension drei. Eine Varietät X isogen zu einem Produkt von Kurven ist ein Quotient eines Produktes $C_1 \times \ldots \times C_n$ kompakter Riemannscher Flächen C_i vom Geschlecht größer gleich zwei nach einer endlichen Gruppe G von Automorphismen, die frei auf dem Produkt operiert:

$$X = \big(C_1 \times \ldots \times C_n\big)/G.$$

Diese Objekte wurden von Catanese in [Cat00] eingeführt und haben sich seitdem als sehr nützlich erwiesen, um neue und interessante Beispiele von Varietäten von allgemeinem Typ zu finden. Insbesondere der zweidimensionale Fall wurde intensiv untersucht (siehe [BCG08, CP09, Pe10] et al). In dieser Arbeit sind wir an der systematischen Konstruktion und Klassifikation im dreidimensionalen Fall interessiert. Unser Hauptresultat ist die Klassifikation aller dreidimensionaler Varietäten X isogen zu einem Produkt von Kurven mit $\chi(\mathcal{O}_X) = -1$ unter der Voraussetzung, dass die induzierten Operationen der maximalen Untergruppen $G_i \leq G$, die auf den Kurven C_i wirken, treu sind. Um dieses Ergebnis zu erhalten, verwenden wir Techniken aus der algorithmischen Gruppentheorie, die die Methoden der oben aufgeführten Autoren erweitern und verallgemeinern. Ein Teil unserer Klassifikation, der sogenannte ungemischte Fall, bei dem die Gruppe G diagonal wirkt, ist Gegenstand einer gemeinsamen Arbeit mit Davide Frapporti [FG15].

Der Begriff des Produktquotienten erweitert den Begriff der Varietät isogen zu einem Produkt von Kurven dadurch, dass auch nicht freie Gruppenwirkungen zugelassen werden. Wir untersuchen diese Varietäten im dreidimensionalen Fall unter den Annahmen, dass X kanonische Singularitäten hat und die Gruppe G treu auf jedem Faktor des Produkts operiert. Die erste Annahme impliziert, dass eine krepante Terminalisierung existiert d.h. ein eigentlicher, birationaler Morphismus $\rho\colon \widehat{X} \to X$ mit $\rho^*(K_X) = K_{\widehat{X}}$, so dass $\widehat{X}$ nur terminale Singularitäten besitzt. Unser erstes Ziel ist es die Geographie dieser Varietäten d.h. Relationen zwischen den Chern Invarianten

$$\chi(\mathcal{O}_{\widehat{X}}), \qquad e(\widehat{X}) \qquad \text{und} \qquad K_{\widehat{X}}^3$$

zu untersuchen. Wir leiten Relationen in Form von Ungleichungen her und diskutieren

deren Grenzfälle. Dies führt zu einer Charakterisierung jener Beispiele, bei denen $\widehat{X}$ glatt ist, die also ein glattes minimales Modell besitzen. Wir stellen einen Algorithmus bereit, um diese Varietäten für einen gegebenen, fixierten Wert von $\chi(\mathcal{O}_{\widehat{X}})$ zu klassifizieren. Im letzten Teil dieser Arbeit beweisen wir die scharfe Schranke $K_X^3 \geq 4$ für dreidimensionale Produktquotienten mit kanonischen Singularitäten und berechnen die vollständige Liste aller Beispiele, die den minimalen Wert $K_X^3 = 4$ realisieren.

Contents

Introduction

A complex algebraic variety X is *isogenous to a product of curves* if X is a quotient

$$X = (C_1 \times \ldots \times C_n)/G,$$

where the C_i's are compact Riemann surfaces of genus at least two and G is a finite group acting freely on $C_1 \times \ldots \times C_n$. If the *diagonal group*

$$G^0 := G \cap \big(\operatorname{Aut}(C_1) \times \ldots \times \operatorname{Aut}(C_n)\big)$$

is equal to G, we say that X is of *unmixed type* and otherwise of *mixed type.* This class of smooth minimal projective varieties of general type has been introduced by Catanese [Cat00], and since then a considerable amount of literature appeared, especially in the case of surfaces. In particular, surfaces isogenous to a product with *holomorphic Euler-Poincaré-characteristic* $\chi(\mathcal{O}_X) = 1$, are completely classified (see [BCG08, CP09, Pe10] et al). Also quotients of a product of two curves by a non-free action of a finite group G and their desingularisations have been studied. First under the assumption that the quotient has only canonical singularities (i.e. rational double points) [BCGP12] and later without any restrictions on the singularities (see [BP12], [P09], [Pe10] et al). There are two natural questions regarding these varieties in higher dimension:

I) Is it possible to classify varieties X isogenous to a product for a fixed value of $\chi(\mathcal{O}_X)$ if $\dim(X) \geq 3$?

II) What can be said about quotients of products of curves by a non-free action?

Our aim is to address these questions under the assumption that $\dim(X) = 3$ and, in case of the second question, that the singularities of the quotient are canonical. We want to mention that the holomorphic Euler-Poincaré-characteristic of a smooth projective threefold of general type with ample canonical class is negative, in contrast to the surface case, where $\chi(\mathcal{O}_S)$ is positive if S is of general type (cf. [Mi87]).

To give an answer to the first question we derive an algorithm, i.e. a finite procedure to classify threefolds isogenous to a product for a fixed value of $\chi(\mathcal{O}_X)$. In particular, we determine all Galois groups G and all Hodge numbers $h^{p,q}(X)$. The technical condition we have to impose is that the induced group actions

$$\psi_i \colon G_i \to \mathrm{Aut}(C_i), \quad \text{where} \quad G_i := G \cap \big[\, \mathrm{Aut}(C_1 \times \ldots \times \widehat{C_i} \times \ldots \times C_n) \times \mathrm{Aut}(C_i)\big],$$

have trivial kernels K_i, in which case we say that the action of G is *absolutely faithful*. This assumption allows us to derive an effective bound for the order of G in terms of $\chi(\mathcal{O}_X)$, which makes a complete classification feasible, at least in the boundary case $\chi(\mathcal{O}_X) = -1$. The classification procedure is computationally hard and cannot be carried out by hand. For this reason, we use the computer algebra system MAGMA [Mag]. We run our implementation (see Appendix A), which is based on the code given in [BCGP12], in the boundary case $\chi(\mathcal{O}_X) = -1$. For threefolds of unmixed type, i.e. in the case where the groups G^0, G_i and G coincide, we obtain the following classification theorem, which is also the main theorem in our joint paper with Davide Frapporti [FG15]:

Theorem (A). *Let $X = (C_1 \times C_2 \times C_3)/G$ be a threefold isogenous to a product of curves of unmixed type. Assume that the action of G is absolutely faithful and $\chi(\mathcal{O}_X) = -1$. Then, the tuple*

$$[G, T_1, T_2, T_3, h^{3,0}(X), h^{2,0}(X), h^{1,0}(X), h^{1,1}(X), h^{2,1}(X), d]$$

appears in the table below. Conversely, each row is realized by at least one family of threefolds isogenous to a product of curves of unmixed type with $\chi(\mathcal{O}_X) = -1$, which depends on d parameters and is obtained by an absolutely faithful G-action.

No.	G	Id	T_1	T_2	T_3	$h^{3,0}$	$h^{2,0}$	$h^{1,0}$	$h^{1,1}$	$h^{1,2}$	d
1	$\{1\}$	$\langle 1,1\rangle$	$[2;-]$	$[2;-]$	$[2;-]$	8	12	6	27	36	9
2	$\mathbb{Z}_2$	$\langle 2,1\rangle$	$[0;2^6]$	$[0;2^6]$	$[2;-]$	8	8	2	19	28	9
3	$\mathbb{Z}_2$	$\langle 2,1\rangle$	$[0;2^6]$	$[1;2^2]$	$[2;-]$	6	7	3	17	24	8
4	$\mathbb{Z}_2$	$\langle 2,1\rangle$	$[1;2^2]$	$[1;2^2]$	$[2;-]$	6	8	4	19	26	7
5	$\mathbb{Z}_3$	$\langle 3,1\rangle$	$[0;3^4]$	$[0;3^4]$	$[2;-]$	6	6	2	15	22	5
6	$\mathbb{Z}_2^2$	$\langle 4,2\rangle$	$[0;2^5]$	$[0;2^5]$	$[2;-]$	6	6	2	15	22	7
7	$\mathbb{Z}_2^2$	$\langle 4,2\rangle$	$[0;2^5]$	$[0;2^5]$	$[2;-]$	5	5	2	13	19	7
8	$\mathbb{Z}_2^2$	$\langle 4,2\rangle$	$[0;2^5]$	$[0;2^6]$	$[1;2^2]$	5	4	1	11	17	7
9	$\mathbb{Z}_2^2$	$\langle 4,2\rangle$	$[0;2^5]$	$[0;2^6]$	$[1;2^2]$	6	5	1	13	20	7
10	$\mathbb{Z}_2^2$	$\langle 4,2\rangle$	$[0;2^5]$	$[1;2^2]$	$[1;2^2]$	4	4	2	11	16	6
11	$\mathbb{Z}_2^2$	$\langle 4,2\rangle$	$[0;2^5]$	$[1;2^2]$	$[1;2^2]$	5	5	2	13	19	6
12	$\mathbb{Z}_4$	$\langle 4,1\rangle$	$[0;2^2,4^2]$	$[0;2^2,4^2]$	$[2;-]$	6	6	2	15	22	5
13	$\mathbb{Z}_5$	$\langle 5,1\rangle$	$[0;5^3]$	$[0;5^3]$	$[2;-]$	4	4	2	15	20	3
14	$\mathbb{Z}_5$	$\langle 5,1\rangle$	$[0;5^3]$	$[0;5^3]$	$[2;-]$	5	5	2	13	19	3
15	$\mathbb{Z}_5$	$\langle 5,1\rangle$	$[0;5^3]$	$[0;5^3]$	$[2;-]$	6	6	2	11	18	3
16	$\mathfrak{S}_3$	$\langle 6,1\rangle$	$[0;2^6]$	$[0;2^2,3^2]$	$[1;3]$	4	3	1	9	14	5
17	$\mathfrak{S}_3$	$\langle 6,1\rangle$	$[0;2^2,3^2]$	$[0;2^2,3^2]$	$[2;-]$	5	5	2	13	19	5
18	$\mathbb{Z}_6$	$\langle 6,2\rangle$	$[0;2^2,3^2]$	$[0;2^2,3^2]$	$[2;-]$	6	6	2	15	22	5
19	$\mathbb{Z}_6$	$\langle 6,2\rangle$	$[0;2^2,3^2]$	$[0;3,6^2]$	$[2;-]$	5	5	2	13	19	4
20	$\mathfrak{S}_3$	$\langle 6,1\rangle$	$[0;2^2,3^2]$	$[1;2^2]$	$[1;3]$	4	4	2	11	16	4
21	$\mathbb{Z}_6$	$\langle 6,2\rangle$	$[0;3,6^2]$	$[0;3,6^2]$	$[2;-]$	4	4	2	15	20	3
22	$\mathbb{Z}_6$	$\langle 6,2\rangle$	$[0;3,6^2]$	$[0;3,6^2]$	$[2;-]$	6	6	2	11	18	3

No.	G	Id	T_1	T_2	T_3	$h^{3,0}$	$h^{2,0}$	$h^{1,0}$	$h^{1,1}$	$h^{1,2}$	d
23	$\mathbb{Z}_2^3$	$\langle 8,5\rangle$	$[0;2^5]$	$[0;2^5]$	$[0;2^5]$	4	2	0	7	12	6
24	$\mathbb{Z}_2^3$	$\langle 8,5\rangle$	$[0;2^5]$	$[0;2^5]$	$[0;2^5]$	5	3	0	9	15	6
25	$\mathcal{D}_4$	$\langle 8,3\rangle$	$[0;2^6]$	$[0;2^3,4]$	$[1;2]$	4	3	1	9	14	5
26	$\mathcal{D}_4$	$\langle 8,3\rangle$	$[0;2^3,4]$	$[0;2^3,4]$	$[2;-]$	5	5	2	13	19	5
27	$\mathcal{D}_4$	$\langle 8,3\rangle$	$[0;2^3,4]$	$[0;2^2,4^2]$	$[1;2^2]$	4	3	1	9	14	4
28	$\mathcal{D}_4$	$\langle 8,3\rangle$	$[0;2^3,4]$	$[1;2]$	$[1;2^2]$	4	4	2	11	16	4
29	$\mathbb{Z}_8$	$\langle 8,1\rangle$	$[0;2,8^2]$	$[0;2,8^2]$	$[2;-]$	4	4	2	15	20	3
30	$\mathbb{Z}_8$	$\langle 8,1\rangle$	$[0;2,8^2]$	$[0;2,8^2]$	$[2;-]$	6	6	2	11	18	3
31	Q	$\langle 8,4\rangle$	$[0;4^3]$	$[0;4^3]$	$[2;-]$	5	5	2	13	19	3
32	$\mathbb{Z}_{10}$	$\langle 10,2\rangle$	$[0;2,5,10]$	$[0;2,5,10]$	$[2;-]$	4	4	2	15	20	3
33	$\mathbb{Z}_{10}$	$\langle 10,2\rangle$	$[0;2,5,10]$	$[0;2,5,10]$	$[2;-]$	6	6	2	11	18	3
34	$\mathbb{Z}_{10}$	$\langle 10,2\rangle$	$[0;2,5,10]$	$[0;2,5,10]$	$[2;-]$	5	5	2	13	19	3
35	$\mathcal{D}_6$	$\langle 12,4\rangle$	$[0;2^5]$	$[0;2^3,3]$	$[1;3]$	4	3	1	9	14	4
36	$\mathcal{D}_6$	$\langle 12,4\rangle$	$[0;2^3,3]$	$[0;2^3,3]$	$[2;-]$	5	5	2	13	19	5
37	$\mathcal{D}_6$	$\langle 12,4\rangle$	$[0;2^3,3]$	$[0;2^3,6]$	$[1;2^2]$	4	3	1	9	14	4
38	$\mathbb{Z}_3\times\mathbb{Z}_2^2$	$\langle 12,5\rangle$	$[0;2,6^2]$	$[0;2,6^2]$	$[2;-]$	4	4	2	15	20	3
39	$\mathbb{Z}_3\times\mathbb{Z}_2^2$	$\langle 12,5\rangle$	$[0;2,6^2]$	$[0;2,6^2]$	$[2;-]$	4	4	2	13	18	3
40	$\mathbb{Z}_3\times\mathbb{Z}_2^2$	$\langle 12,5\rangle$	$[0;2,6^2]$	$[0;2,6^2]$	$[2;-]$	5	5	2	11	17	3
41	$\mathbb{Z}_3\times\mathbb{Z}_2^2$	$\langle 12,5\rangle$	$[0;2,6^2]$	$[0;2,6^2]$	$[2;-]$	6	6	2	11	18	3
42	$Dic12$	$\langle 12,1\rangle$	$[0;3,4^2]$	$[0;3,4^2]$	$[2;-]$	5	5	2	13	19	3
43	$\mathcal{D}_4\times\mathbb{Z}_2$	$\langle 16,11\rangle$	$[0;2^5]$	$[0;2^3,4]$	$[0;2^3,4]$	4	2	0	7	12	4
44	$\mathcal{D}_4\times\mathbb{Z}_2$	$\langle 16,11\rangle$	$[0;2^5]$	$[0;2^3,4]$	$[0;2^3,4]$	3	1	0	5	9	4
45	$SD16$	$\langle 16,8\rangle$	$[0;2,4,8]$	$[0;2,4,8]$	$[2;-]$	4	4	2	13	18	3
46	$SD16$	$\langle 16,8\rangle$	$[0;2,4,8]$	$[0;2,4,8]$	$[2;-]$	5	5	2	11	17	3
47	$\mathfrak{S}_4$	$\langle 24,12\rangle$	$[0;2^3,4]$	$[0;2^2,3^2]$	$[0;3,4^2]$	3	1	0	5	9	2
48	$\mathbb{Z}_3\rtimes_\varphi\mathcal{D}_4$	$\langle 24,8\rangle$	$[0;2,4,6]$	$[0;2,4,6]$	$[2;-]$	4	4	2	13	18	3
49	$\mathbb{Z}_3\rtimes_\varphi\mathcal{D}_4$	$\langle 24,8\rangle$	$[0;2,4,6]$	$[0;2,4,6]$	$[2;-]$	5	5	2	11	17	3
50	$\mathrm{SL}(2,\mathbb{F}_3)$	$\langle 24,3\rangle$	$[0;3^2,4]$	$[0;3^2,4]$	$[2;-]$	5	5	2	13	19	3
51	$\mathfrak{S}_4\times\mathbb{Z}_2$	$\langle 48,48\rangle$	$[0;2^5]$	$[0;2,4,6]$	$[0;2,4,6]$	3	1	0	5	9	2
52	$\mathrm{GL}(2,\mathbb{F}_3)$	$\langle 48,29\rangle$	$[0;2,3,8]$	$[0;2,3,8]$	$[2;-]$	4	4	2	13	18	3
53	$\mathrm{GL}(2,\mathbb{F}_3)$	$\langle 48,29\rangle$	$[0;2,3,8]$	$[0;2,3,8]$	$[2;-]$	5	5	2	11	17	3
54	$\mathfrak{A}_5$	$\langle 60,5\rangle$	$[0;2^3,3]$	$[0;2,5^2]$	$[0;3^2,5]$	2	0	0	3	6	1

The table above is organized in the following way:

- the first column gives the number of the example,
- the second column reports the Galois group (see Notation 0.0.1 for the definition of the groups that appear),
- the third column provides the MAGMA identifier of the Galois group: $\langle a,b\rangle$ denotes the b^{th} group of order a in the *Database of Small Groups* [Mag],
- the types

  $$T_i=[g_i';m_{i,1},\dots,m_{i,r_i}]$$

  in column 4-6 yield the branching data of the G-covers $F_i\colon C_i\to C_i'$ induced by the actions

  $$\psi_i\colon G\to\mathrm{Aut}(C_i),$$

  i.e. g_i' is the genus of the quotient curve C_i' and the $m_{i,j}$'s are the branching indices. They are written in a simplified way: for example, $[0;2,2,4,4]$ is abbreviated by $[0;2^2,4^2]$.

- the remaining columns report the Hodge numbers $h^{p,q}(X)$ and the number d of parameters of the families.

The mixed case is algebraically more complicated to handle than the unmixed case. Since G/G^0 can be considered as a subgroup of $\mathfrak{S}_3$, it breaks up in three sub-cases, according to the index of G^0 in G and we need a slightly different strategy for our algorithm depending on the particular case. The full classification of threefolds isogenous to a product of mixed type with $\chi(\mathcal{O}_X) = -1$ is summarized in [Theorem **(B)** p.50] and [Theorem **(C)** p.52].

In principle though, the basic idea of the algorithms is similar in the unmixed and the mixed case. We briefly explain the idea. Let

$$X = (C_1 \times C_2 \times C_3)/G$$

be a threefold isogenous to a product. The associated actions $\psi_i \colon G_i \to \mathrm{Aut}(C_i)$ induce G_i-covers $F_i \colon C_i \to C_i'$, where C_i' is the quotient Riemann surface. Let $\mathcal{B}_i \subset C_i'$ be the branch locus of F_i. Then, the restrictions

$$F_i \colon C_i \setminus F_i^{-1}(\mathcal{B}_i) \to C_i' \setminus \mathcal{B}_i$$

are also Galois covers with the same Galois group G_i. Therefore, they are determined, up to isomorphism, by the monodromy map

$$\eta_i \colon \pi_1\big(C_i' \setminus \mathcal{B}_i\big) \to G_i.$$

Once we choose a *geometric basis* of $\pi_1\big(C_i' \setminus \mathcal{B}_i\big)$ (see Definition 1.1.4), the images of the basis elements under the monodromy map η_i form a tuple V_i, which is called a *generating vector*. It has the property that it's elements generate the group G_i and fulfill certain relations. We can now attach to a threefold X isogenous to a product an *algebraic datum*, i.e. a tuple which basically consists of the group G and the generating vectors V_i. It encodes the geometric information of X that we are interested in. Thanks to Riemann's existence theorem there is also a way back: starting with an abstract datum of a finite group G and generating vectors V_i for G_i, which fulfill certain conditions, we obtain family of threefold isogenous to a product. This allows us to translate the geometric classification problem into a problem of combinatorial group theory. As we already mentioned, the crucial step in this approach is that the freeness assumption for the group action allows us to bound the order of the group G in terms of $\chi(\mathcal{O}_X)$ and derive combinatorial constraints on the genera of the curves and the generating vectors V_i. These constraints are strong enough to imply that the set of algebraic data of all threefolds isogenous to a product, with the same value of $\chi(\mathcal{O}_X)$, is finite. This allows us to develop an algorithm searching systematically through all possible groups in order to find all algebraic data of these threefolds. The algorithm contains a subroutine to determine the Hodge numbers of the threefolds from the corresponding algebraic data. Here, the strategy relies on representation theory: the group actions ψ_i induce representations via pullback of holomorphic 1-forms

$$\varphi_i\colon G_i \to \mathrm{GL}\left(H^{1,0}(C_i)\right).$$

According to the formula of *Chevalley-Weil*, the characters χ_{φ_i} of these representations can be computed from a generating vector V_i of G_i. On the other hand, Künneth's formula allows us to determine the characters $\chi_{p,q}$ of the representations

$$\phi_{p,q}\colon G \to \mathrm{GL}\left(H^{p,q}(C_1 \times C_2 \times C_3)\right), \qquad g \mapsto [\omega \mapsto (g^{-1})^*\omega]$$

in terms of the characters χ_{φ_i}. This provides a way to determine the Hodge numbers $h^{p,q}(X)$, because they are equal to the multiplicity of the trivial character of G in $\chi_{p,q}$.

Now we turn to the second question and study *product quotient threefolds*

$$X = \left(C_1 \times C_2 \times C_3\right)/G,$$

i.e. quotients of a product of curves of genus $g(C_i) \geq 2$ by a non-free action of a finite group G. We assume that G embeds into the automorphism group of each curve C_i and that the quotient X has canonical singularities. The first assumption ensures that the singularities of X are isolated *cyclic quotient singularities* and the second assumption allows us to consider a *crepant terminalisation*

$$\rho\colon \widehat{X} \to X,$$

i.e. a proper birational morphism such that $\widehat{X}$ has terminal singularities and $\rho^*(K_X) = K_{\widehat{X}}$. We show that the Chern invariants

$$\chi(\mathcal{O}_{\widehat{X}}), \qquad e(\widehat{X}) \qquad \text{and} \qquad K_{\widehat{X}}^3,$$

which are independent of the chosen crepant terminalisation (cf. [H89]), fulfill the inequalities

$$i)\ \ 48\chi(\mathcal{O}_{\widehat{X}}) + K_{\widehat{X}}^3 \geq 0 \qquad \text{and} \qquad ii)\ \ 6e(\widehat{X}) + K_{\widehat{X}}^3 \geq 0.$$

Therefore, by dropping the freeness assumption, we obtain a great flexibility to construct threefolds of general type with interesting Chern invariants. The price is that we lose or weaken certain combinatorial constraints which hold in the isogenous case. The inequalities above characterize those product quotient threefolds admitting a smooth minimal model, more precisely we show:

- the inequality *i*) is sharp if and only if $\widehat{X}$ is smooth and
- the inequality *ii*) is sharp if and only if X is a threefold isogenous to a product of

unmixed type, in which case $i)$ is automatically sharp.

Since we already discussed threefolds isogenous to a product, we go on to study the case, where $\widehat{X}$ is smooth but X is singular, i.e. the case where $i)$ is sharp, but $ii)$ is a proper inequality. Here, the crepant terminalisation $\rho\colon \widehat{X} \to X$ is actually a crepant resolution and the singularities of X are Gorenstein. We adapt our algorithm from the isogenous case to classify these varieties for a fixed value of $\chi(\mathcal{O}_{\widehat{X}})$, which must be also negative. Our algorithm allows us to determine the Galois group G and the Hodge numbers $h^{p,q}(\widehat{X})$, which are independent of the chosen crepant resolution, thanks to a theorem of Kontsevich [Kon95]. As it turns out, the Gorenstein condition implies that the Galois groups cannot be 2-groups, which excludes a huge number of groups that should be investigated otherwise. We execute our implementation (see Appendix A) and find that there are no examples with $\chi(\mathcal{O}_{\widehat{X}}) = -1$ and -2 (see Proposition 5.0.7). For $\chi(\mathcal{O}_{\widehat{X}}) = -3$ we obtain examples and provide the full classification (see Theorem 5.0.8).
In the last part of this thesis, we prove the sharp inequality $K_X^3 \geq 4$ for product quotient threefolds X with canonical singularities and provide the full list of examples realizing the minimum value $K_X^3 = 4$ (see Theorem 6.0.1). To determine this value, we give an algorithm to classify for a given positive and sufficiently small number c all product quotient threefolds X with canonical singularities and $K_{\widehat{X}}^3 \leq c$. We refer to Appendix A for our MAGMA implementation. Running the implementation for $c = 4$, we obtain the above result. We point out that $K_{\widehat{X}}$ is, in general, not a Cartier divisor, consequently $K_{\widehat{X}}^3$ does not need to be an integer (see Remark 6.0.3).

The thesis is divided in six chapters. We briefly explain the main content of each chapter.

- In Chapter 1, we present the theoretical prerequisites from Riemann surface theory, group theory and representation theory used in this thesis. The central theorems are *Riemann's existence theorem* and the *Chevalley-Weil formula*.
- In Chapter 2 we introduce varieties isogenous to a product and explain some of their basic properties. Moreover, we study the structure of mixed group actions on a product of three curves and define the algebraic datum of a threefold X isogenous to a product. Based on that, we show how to determine Hodge numbers of a threefold X isogenous to a product from an algebraic datum of X.
- In Chapter 3 we develop an algorithm to classify threefolds isogenous to a product for a fixed value of $\chi(\mathcal{O}_X)$ obtained by an absolutely faithful group action and present our main result: the classification of these varieties in the case $\chi(\mathcal{O}_X) = -1$.
- In Chapter 4 we introduce product quotient threefolds and their singularities. We show how to determine a crepant terminalisation $\widehat{X}$ of a product quotient threefold with canonical singularities, using methods from toric geometry. We go on to relate

the invariants $\chi(\mathcal{O}_{\widehat{X}})$, $e(\widehat{X})$ and $K^3_{\widehat{X}}$ and derive the inequalities $i)$ and $ii)$ that we discussed above.

- In Chapter 5 we specialize to singular product quotient threefolds X with canonical singularities admitting a smooth minimal model $\widehat{X}$. We explain how to determine the Hodge numbers of $\widehat{X}$ and derive an algorithm to classify these varieties for a fixed value of $\chi(\mathcal{O}_{\widehat{X}})$. Running an implementation of this algorithm we show there are no examples for $\chi(\mathcal{O}_{\widehat{X}}) = -1$ and -2 and compute the full list of examples for $\chi(\mathcal{O}_{\widehat{X}}) = -3$.

- In Chapter 6, the last chapter of this thesis, we prove the sharp inequality $K^3_X \geq 4$ for product quotient threefolds X with canonical singularities and provide a classification of these varieties for $K^3_X = 4$.

Notation 0.0.1. *Throughout the thesis all varieties are defined over the field of complex numbers and the standard notation from complex algebraic geometry is used, see for example [GH78]. Moreover, we use the following notations and definitions from group theory.*

- *The cyclic group of order n is denoted by $\mathbb{Z}_n$.*

- $\mathcal{D}_n = \langle s, t \mid s^2 = t^n = 1,\ sts = t^{-1} \rangle$ *is the dihedral group of order $2n$.*

- $\mathfrak{S}_n$ *and* $\mathfrak{A}_n$ *denote the symmetric and alternating group on n letters, respectively.*

- *The quaternion group of order 8 is defined by*

$$Q := \langle -1, i, j, k \mid i^2 = j^2 = k^2 = ijk = -1 \rangle.$$

- *The groups* $\mathrm{GL}(n, \mathbb{F}_q)$ *and* $\mathrm{SL}(n, \mathbb{F}_q)$ *are the general linear and special linear groups of $n \times n$ matrices over the field $\mathbb{F}_q$.*

- *The holomorph* $\mathrm{Hol}(G)$ *of a group G is the semi-direct product* $G \rtimes_{id} \mathrm{Aut}(G)$.

- *Let G_1 and G_2 be groups with isomorphic subgroups $U_i \leq Z(G_i)$ and let $\phi \colon U_1 \to U_2$ be an isomorphism. The central product $G_1 *_\phi G_2$ is defined as the quotient of the direct product $G_1 \times G_2$ by the normal subgroup*

$$N := \{(g_1, g_2) \in U_1 \times U_2 \mid \phi(g_1)g_2 = 1_{G_2}\}.$$

- *The dicyclic group of order $4n$ is $Dic4n := \langle a, b, c \mid a^n = b^2 = c^2 = abc \rangle$.*

- *The semidihedral group of order 2^n is*

$$SD2^n := \langle a, b \mid a^{2^{(n-1)}} = b^2 = 1,\ bab = a^{2^{(n-2)}-1} \rangle.$$

- *The group* M_{16} *of order* 16 *is* $M_{16} := \langle a, b \mid a^8 = b^2 = e,\ bab^{-1} = a^5 \rangle$.
- *The binary octahedral group of order* 48 *is* $2O := \langle a, b, c \mid a^4 = b^3 = c^2 = abc \rangle$.

Chapter 1

Riemann Surfaces

In this chapter we present the theoretical prerequisites from Riemann surface theory, group theory and representation theory used in this thesis. In particular we recall some principles of group actions on compact Riemann surfaces. The central theorems are *Riemann's existence theorem* and the *Chevalley-Weil formula*. Although most of the results in this chapter come without a proof, we decided to give a proof of the Chevalley-Weil formula, because we could not find a suitable modern reference. For an introduction to the theory of Riemann surfaces we refer the reader to the textbooks [L05], [F81], [FK80] and for representation and character theory to textbook [Isa76].

1.1 Group Actions on Riemann Surfaces

Let C be a compact Riemann surface, G be a finite group and

$$\psi\colon G \to \mathrm{Aut}(C)$$

be a faithful group action. If there is no possibility of confusion, we identify G with its image in $\mathrm{Aut}(C)$ and omit writing the map ψ.

We shall introduce some notation and recall the standard constructions: we denote the stabilizer group of a point $p \in C$ by G_p. This is a cyclic group. We say that two points $p_1, p_2 \in C$ are equivalent if $g(p_1) = p_2$ for some $g \in G$ and consider the quotient $C' := C/G$ endowed with the quotient topology. Note that C' is Hausdorff because C is Hausdorff and G is finite. Moreover, there is a unique complex structure on C' such that the projection map $F\colon C \to C'$ is holomorphic. We say that F is a *(ramified) Galois cover*. Let $\mathcal{B} \subset C'$ be the finite set of critical values of F and $\mathcal{R} := F^{-1}(\mathcal{B})$ its preimage, then the restriction

$$f\colon C \setminus \mathcal{R} \to C' \setminus \mathcal{B}$$

of F is an unramifed cover.

We write $\mathrm{Deck}(f)$ for the group of covering transformations of f. It coincides with $\mathrm{Deck}(F)$ and is identified with G via ψ. We choose a point $p_0 \in C \setminus \mathcal{R}$ and define $q_0 := f(p_0)$. The cover f determines the monodromy map

$$\mu\colon \pi_1\big(C' \setminus \mathcal{B}, q_0\big) \to \mathrm{Deck}(f), \quad \gamma \mapsto g_\gamma.$$

Here the element g_γ is the unique covering transformation such that $g_\gamma(p_0)$ is the end point of the unique lift of γ with initial point p_0. Recall that the monodromy map is a surjective group homomorphism with kernel $f_*\big(\pi_1\big(C \setminus \mathcal{R}, p_0\big)\big)$.

Proposition 1.1.1. *Let C' be a compact Riemann surface, $\mathcal{B} \subset C'$ a finite subset and $\eta\colon \pi_1\big(C' \setminus \mathcal{B}, q_0\big) \to G$ be a homomorphism onto a finite group G. Then there exists, up to isomorphism, a unique topological cover*

$$f\colon (\widehat{C}, p_0) \to \big(C' \setminus \mathcal{B}, q_0\big)$$

together with a unique isomorphism of groups $\psi\colon G \to \mathrm{Deck}(f)$ such that the homomorphism

$$(\psi \circ \eta)\colon \pi_1\big(C' \setminus \mathcal{B}, q_0\big) \to \mathrm{Deck}(f)$$

is the monodromy map of f. Moreover, $\widehat{C}$ has a unique complex structure such that f is holomorphic.

Proof. Consider the kernel $\ker(\eta) \trianglelefteq \pi_1\big(C' \setminus \mathcal{B}, q_0\big)$. According to the classification theorem of unramified covering spaces [L05, 3.7.1] there exists, up to isomorphism, an unramified cover $f\colon (\widehat{C}, p_0) \to \big(C' \setminus \mathcal{B}, q_0\big)$ such that $f_*\big(\pi_1\big(\widehat{C}, p_0\big)\big) = \ker(\eta)$. Since the kernel is normal in $\pi_1\big(C' \setminus \mathcal{B}, q_0\big)$, the cover f is Galois. Let

$$\mu\colon \pi_1\big(C' \setminus \mathcal{B}, q_0\big) \to \mathrm{Deck}(f)$$

be the monodromy map of f. Note that the surjective maps μ and η have the same kernel, therefore they differ by a unique isomorphism $\psi\colon G \to \mathrm{Deck}(f)$. For the statement that $\widehat{C}$ admits a unique complex structure such that f is holomorphic we refer to [F81, Theorem 4.6]. □

The crucial point is that the unramified cover f of $C' \setminus \mathcal{B}$ in the proposition above can be uniquely extended to a (ramified) cover of C'.

Theorem 1.1.2 (Riemann's existence theorem, cf. [F81, Theorem 8.4 and 8.5])**.** *Let C' be a compact Riemann surface and $\mathcal{B} \subset C'$ be a finite subset. Let $\widehat{C}$ be a Riemann surface and $f\colon \widehat{C} \to C' \setminus \mathcal{B}$ a finite unramified cover. Then there exists a compact Riemann surface C, a holomorphic map $F\colon C \to C'$ and a biholomorphic map*

$$\Xi\colon C \setminus F^{-1}(\mathcal{B}) \to \widehat{C}$$

such that the following diagram commutes

$$\begin{array}{ccc} \widehat{C} & \xrightarrow{\quad f \quad} & C' \setminus \mathcal{B} \\ & {\scriptstyle \Xi}\nwarrow \qquad \nearrow{\scriptstyle F} & \\ & C \setminus F^{-1}(\mathcal{B}). & \end{array}$$

Moreover, every covering transformation of f can be uniquely extended to a covering transformation of F yielding an isomorphism between $\mathrm{Deck}(f)$ *and* $\mathrm{Deck}(F)$.

We shall summarize the discussion from above.

Theorem 1.1.3. *Given*

i) a compact Riemann surface C',

ii) a finite set $\mathcal{B} \subset C'$, a point $q_0 \in C' \setminus \mathcal{B}$ and

iii) a surjective homomorphism $\eta\colon \pi_1(C' \setminus \mathcal{B}, q_0) \to G$.

Then there exists, up to isomorphism, a unique Galois cover $F\colon C \to C'$ together with a unique inclusion $\psi\colon G \to \mathrm{Aut}(C)$ identifying G with $\mathrm{Deck}(F)$ such that the critical values of F are contained in $\mathcal{B}$ and the restriction $f\colon C \setminus F^{-1}(\mathcal{B}) \to C' \setminus \mathcal{B}$ is an unramified cover with monodromy map

$$(\psi \circ \eta)\colon \pi_1(C' \setminus \mathcal{B}, q_0) \to \mathrm{Deck}(F).$$

In the literature the fundamental group of the non-compact Riemann surface $C' \setminus \mathcal{B}$ is usually described in terms of a *geometric basis*. We recall its definition.

Definition 1.1.4. *Let C' be a compact Riemann surface of genus g' and $\mathcal{B} := \{q_1, \ldots, q_r\}$ a finite subset of C'. A* geometric basis *of $\pi_1(C' \setminus \mathcal{B}, q_0)$ consists of loops*

$$\gamma_1, \ldots, \gamma_r, \alpha_1, \beta_1, \ldots, \alpha_{g'}, \beta_{g'}$$

with base point q_0 and no other intersection points such that γ_i is a simple loop around q_i and

$$\pi_1(C' \setminus \mathcal{B}, q_0) = \Big\langle \gamma_1, \ldots, \gamma_r, \alpha_1, \beta_1, \ldots, \alpha_{g'}, \beta_{g'} \;\Big|\; \gamma_1 \cdots \gamma_r \cdot \prod_{i=1}^{g'} [\alpha_i, \beta_i] \Big\rangle.$$

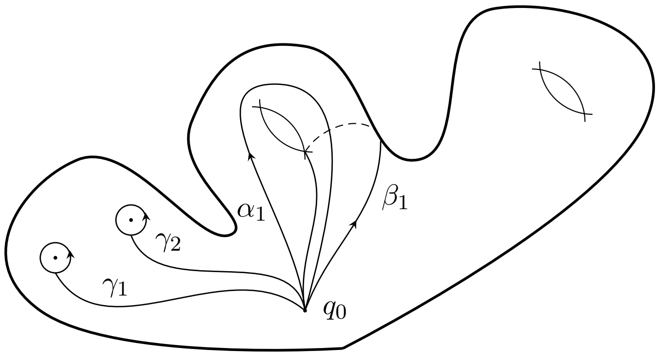

Figure 1.1: geometric basis

By a simple loop γ around a point $q \in \mathcal{B}$ we mean a product of paths of the form $\delta * u * \delta^-$, where δ connects q_0 and the initial point of u, which is a loop inside a centred chart

$$z_q \colon U_q \to \mathbb{E} := \{z \in \mathbb{C} \mid |z| < 1\}, \quad U_q \cap \mathcal{B} = \{q\}$$

such that the composition $z_q \circ u$ is homotopic to

$$\epsilon \colon [0,1] \to \mathbb{E}, \quad t \mapsto (z_q \circ u)(0) \cdot \exp\big(2\pi\sqrt{-1}t\big).$$

Remark 1.1.5. Let $F \colon C \to C'$ be a Galois cover and $\mathcal{B} = \{q_1, \ldots, q_r\}$ be the set of critical values of F. Let $\mu \colon \pi_1\big(C' \setminus \mathcal{B}, q_0\big) \to \mathrm{Deck}(F)$ be the monodromy map and

$$\gamma_1, \ldots, \gamma_r, \alpha_1, \beta_1, \ldots, \alpha_{g'}, \beta_{g'}$$

be a geometric basis of $\pi_1\big(C' \setminus \mathcal{B}, q_0\big)$. Then the elements

$$h_i := \mu(\gamma_i), \quad a_i := \mu(\alpha_i) \quad \text{and} \quad b_i := \mu(\beta_i)$$

generate the group $\mathrm{Deck}(F)$ and the product

$$h_1 \cdots h_r \cdot \prod_{i=1}^{g'} [a_i, b_i]$$

is trivial. Moreover, the *branching index* of F at q_i is equal to $\mathrm{ord}(h_i)$.

This observation motivates the following definition.

Definition 1.1.6. *Let $m_1, \ldots, m_r \geq 2$ and $g' \geq 0$ be integers and G be a finite group. A* generating vector *for G of type $[g'; m_1, \ldots, m_r]$ is a $(2g' + r)$-tuple*

$$(h_1, \ldots, h_r, a_1, b_1, \ldots, a_{g'}, b_{g'})$$

of elements of G *such that:*

i) $G = \langle h_1, \ldots, h_r, a_1, b_1, \ldots, a_{g'}, b_{g'} \rangle$,

ii) $h_1 \cdots h_r \cdot \prod_{i=1}^{g'} [a_i, b_i] = 1_G$ *and*

iii) $\mathrm{ord}(h_i) = m_i$ *for all* $1 \leq i \leq r$.

In the remaining parts of this thesis it is convenient to use the following much weaker version of Riemann's existence theorem.

Theorem 1.1.7. *A finite group* G *acts faithfully and biholomorphically on some compact Riemann surface* C *of genus* $g(C)$ *if and only, if there exists a generating vector*

$$(h_1, \ldots, h_r, a_1, b_1, \ldots, a_{g'}, b_{g'})$$

for G *of type* $[g'; m_1, \ldots, m_r]$ *such that the Hurwitz formula holds:*

$$2g(C) - 2 = |G| \left(2g' - 2 + \sum_{i=1}^{r} \frac{m_i - 1}{m_i} \right).$$

In this case g' *is the genus of the quotient Riemann surface* C/G *and the Galois cover* $F\colon C \to C/G$ *is branched in* r*-points* $\{q_1, \ldots, q_r\}$ *with branching indices* $m_1, \ldots, m_r$*, respectively. Moreover, the cyclic groups* $\langle h_i \rangle$ *and their conjugates provide the non-trivial stabilizers of the action of* G *on* C.

Example 1.1.8. Consider the dihedral group

$$\mathcal{D}_6 = \langle s, t \mid s^2 = t^6 = 1, \ sts = t^5 \rangle$$

of order 12 and the type $T := [1; 2, 2]$. Note that Hurwitz' formula holds for $g(C) = 7$:

$$2g(C) - 2 = 12 \left(2g' - 2 + \frac{1}{2} + \frac{1}{2} \right) = 12.$$

We claim that $V := (st, st, t^5, t^5)$ is a generating vector for $\mathcal{D}_6$ of type T. Clearly the elements in V generate $\mathcal{D}_6$,

$$\mathrm{ord}(st) = 2 \qquad \text{and} \qquad st \cdot st \cdot [t^5, t^5] = 1.$$

We conclude that $\mathcal{D}_6$ acts on a compact Riemann surface C of genus $g(C) = 7$ such that the quotient C/G is an elliptic curve, i.e. $g(C/G) = 1$ and the covering $C \to C/G$ is branched in two points with branching index 2, respectively.

1.2 Representations and Characters

In this section we introduce the fundamental notions of a *representation* and a *character*. In mathematics representation and character theory appears naturally, whenever we deal with symmetries, i.e. groups and vector spaces. The case where we want to apply this theory is when the symmetries are automorphisms of Riemann surfaces, or more generally algebraic varieties, and the vector spaces are cohomology groups. These concepts will be used throughout the thesis.

Definition 1.2.1. *Let G be a finite group and let V be a finite dimensional complex vector space. A group homomorphism $\varrho\colon G \to \mathrm{GL}(V)$ is called a* representation. *The* character *of ϱ is the associated function $\chi_\varrho\colon G \to \mathbb{C}$, where $\chi_\varrho(g) = \mathrm{tr}\big(\varrho(g)\big)$ and the* degree *of ϱ is the dimension of V.*

We want to give two important examples of representations which can be defined for any finite group G.

i) The first one is the trivial representation: it is defined to be the trivial homomorphism

$$\varrho_{triv}\colon G \to \mathbb{C}^*, \qquad g \mapsto 1.$$

We denote its character by χ_{triv}.

ii) The second one is the regular representation: let V be the vector space of maps from G to the complex numbers $\mathbb{C}$. It has a natural basis indexed by the group elements:

$$\{e_h \mid h \in G\}, \quad \text{where} \quad e_h(g) := \begin{cases} 1, & \text{if} \quad g = h \\ 0, & \text{otherwise.} \end{cases}$$

The regular representation is defined as

$$\varrho_{reg}\colon G \to \mathrm{GL}(V), \qquad g \mapsto \left[\sum_{h\in G} \lambda_h e_h \mapsto \sum_{h\in G} \lambda_h e_{gh}\right]$$

and its character, the regular character, is given by

$$\chi_{reg}(g) = \begin{cases} |G|, & \text{if} \quad g = 1_G \\ 0, & \text{otherwise.} \end{cases}$$

Note that a character of a representation belongs to the vector space of class functions:

$$\mathrm{CF}(G) := \{\alpha\colon G \to \mathbb{C} \mid \alpha \text{ is constant on the conjugacy classes of } G\}.$$

This vector space, of dimension equal to the number of conjugacy classes of G, carries a

Hermitian product defined as:

$$\langle \alpha, \beta \rangle := \frac{1}{|G|} \sum_{g \in G} \alpha(g)\overline{\beta(g)}, \quad \text{for} \quad \alpha, \beta \in \mathrm{CF}(G).$$

In representation theory we have the notion of isomorphic representations:

Definition 1.2.2. *Let $\varrho_1 \colon G \to \mathrm{GL}(V_1)$ and $\varrho_2 \colon G \to \mathrm{GL}(V_2)$ be representations. We say that ϱ_1 and ϱ_2 are isomorphic if there exists an isomorphism $f \colon V_1 \to V_2$ of vector-spaces such that for all $g \in G$ the following diagram is commutative:*

$$\begin{array}{ccc} V_1 & \xrightarrow{\ f\ } & V_2 \\ {\scriptstyle \varrho_1(g)} \uparrow & & \uparrow {\scriptstyle \varrho_2(g)} \\ V_1 & \xrightarrow{\ f\ } & V_2 \end{array}$$

Clearly, isomorphic representations have the same character.
The standard operations form linear algebra: the direct sum and the tensor product can be defined for representations as well.

Definition 1.2.3. *Let $\varrho_1 \colon G \to \mathrm{GL}(V_1)$ and $\varrho_2 \colon G \to \mathrm{GL}(V_2)$ be representations. The direct sum of ϱ_1 and ϱ_2 is defined as*

$$\varrho_1 \oplus \varrho_2 \colon G \to \mathrm{GL}(V_1 \oplus V_2), \quad g \mapsto [v_1 \oplus v_2 \mapsto \varrho_1(g)v_1 \oplus \varrho_2(g)v_2].$$

The tensor product of ϱ_1 and ϱ_2 is defined as

$$\varrho_1 \otimes \varrho_2 \colon G \to \mathrm{GL}(V_1 \otimes V_2), \quad g \mapsto [v_1 \otimes v_2 \mapsto \varrho_1(g)v_1 \otimes \varrho_2(g)v_2].$$

The character of the direct sum is the sum of the characters and the character of the tensor product is the product of the characters:

$$\chi_{\varrho_1 \oplus \varrho_2} = \chi_{\varrho_1} + \chi_{\varrho_2} \qquad \text{and} \qquad \chi_{\varrho_1 \otimes \varrho_2} = \chi_{\varrho_1} \cdot \chi_{\varrho_2}.$$

Given a representation ϱ, it is natural to ask if ϱ is isomorphic to a direct sum of representations that are easier to understand. To formalize this question, we give the following definition:

Definition 1.2.4. *Let $\varrho \colon G \to \mathrm{GL}(V)$ be a representation and $U \subset V$ be a linear subspace. We say that*

*i) U is G-*invariant *if $\varrho(g)(U) \subset U$ for all $g \in G$,*

ii) ϱ is irreducible *if the trivial vector space is the unique G-invariant subspace $U \subsetneq V$,*

iii) the character χ_ϱ is irreducible *if ϱ is irreducible.*

Maschke's theorem provides a positive answer to the question stated above:

Theorem 1.2.5. *Let $\varrho\colon G \to \mathrm{GL}(V)$ be a representation, then ϱ is isomorphic to a direct sum of irreducible representations.*

The central theorem in representation theory of finite groups is due to Schur and is usually called *the orthogonality relations.* We we state it in the following form:

Theorem 1.2.6. *Let G be a finite group with d conjugacy classes, then:*

i) There are exactly d isomorphism classes of irreducible representations of G.

ii) The set of characters corresponding to this d irreducible representations form an orthonormal basis of the vector space $\mathrm{CF}(G)$.

The set of irreducible characters of a group G is denoted by $\mathrm{Irr}(G)$. It is customary to give it in the form of a $d \times d$ square matrix, the so called *character table*, where the columns are labelled by the conjugacy classes of G and the rows by the irreducible characters. The entries of this matrix are the values that the characters obtain at the respective classes.

Example 1.2.7. To illustrate Theorem 1.2.6, we consider the regular character

$$\chi_{reg}(g) = \begin{cases} |G|, & \text{if} \quad g = 1_G \\ 0, & \text{otherwise} \end{cases}$$

of a finite group G. Observe that for any other character χ of G it holds

$$\langle \chi, \chi_{reg} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\chi_{reg}(g)} = \frac{1}{|G|}\chi(1_G)|G| = \chi(1_G).$$

Decomposing χ_{reg} in irreducible characters, we obtain the following useful formula:

$$\chi_{reg} = \sum_{\chi \in \mathrm{Irr}(G)} \langle \chi, \chi_{reg} \rangle \cdot \chi = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1_G)\chi.$$

In other words: the regular representation contains every irreducible representation of degree k precisely k-times.

Example 1.2.8. To conclude this section, we determine the character table of the dihedral group

$$\mathcal{D}_6 = \langle s, t \mid s^2 = t^6 = 1,\ sts = t^5 \rangle.$$

It has six conjugacy classes: apart from the trivial one they are given by

$$\{s, st^2, st^4\}, \quad \{t, t^5\}, \quad \{t^2, t^4\}, \quad \{t^3\} \quad \text{and} \quad \{st, st^3, st^5\}.$$

Consequently $\mathcal{D}_6$ admits six (isomorphism classes) of irreducible representations. Four of these representations have degree one, they are obtained from the irreducible representations of the abelian group $\mathbb{Z}_2 \times \mathbb{Z}_2$ via the quotient map

$$\mathcal{D}_6 \to \mathcal{D}_6/\langle t^2 \rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Since they have degree one, we can identify them with their characters

$$\chi_{triv}, \quad \chi_1, \quad \chi_2 \quad \text{and} \quad \chi_3.$$

Clearly the values of these characters can only be ± 1. There is also a natural irreducible representation of degree two $\mathcal{D}_6 \to \mathrm{GL}(\mathbb{R}^2)$ obtained by considering $\mathcal{D}_6$ as the symmetry group of a regular hexagon:

$$t \mapsto \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \qquad \text{and} \qquad s \mapsto \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}.$$

We denote the character of this representation by χ_4. The remaining irreducible representation has degree two and its character is given by

$$\chi_5 = \frac{1}{2}\big(\chi_{reg} - \chi_{triv} - \chi_1 - \chi_2 - \chi_3 - 2\chi_4\big)$$

according to the previous example. Whence, the character table of $\mathcal{D}_6$ is the following:

	1	s	t	t^2	t^3	st
χ_{triv}	1	1	1	1	1	1
χ_1	1	-1	1	1	1	-1
χ_2	1	1	-1	1	-1	-1
χ_3	1	-1	-1	1	-1	1
χ_4	2	0	1	-1	-2	0
χ_5	2	0	-1	-1	2	0

1.3 The Chevalley-Weil Formula

Let G be a finite group and $\psi\colon G \to \mathrm{Aut}(C)$ be a faithful group action on a compact Riemann surface C. The action ψ induces in a natural way a representation of G via pullback of holomorphic 1-forms:

$$\varphi\colon G \to \mathrm{GL}\left(H^{1,0}(C)\right), \quad g \mapsto [\omega \mapsto \psi(g^{-1})^*\omega].$$

The character χ_φ of this representation has, according to Theorem 1.2.6, a decomposition

$$\chi_\varphi = \sum_{\chi \in \mathrm{Irr}(G)} \langle \chi, \chi_\varphi \rangle \cdot \chi$$

in irreducible characters. The Chevalley-Weil formula, which we want to state and prove in this section, provides a way to compute the coefficients $\langle \chi, \chi_\varphi \rangle$ from a generating vector

$$(h_1, \ldots, h_r, a_1, b_1, \ldots, a_{g'}, b_{g'})$$

associated to the covering $F \colon C \to C/G$.

We begin with a definition which is based on the following observation: let $\varrho \colon G \to \mathrm{GL}(V)$ be a representation of a finite group G. Then for each $h \in G$ the endomorphism

$$\varrho(h) \colon V \to V$$

has finite order and is therefore diagonalizable. Its eigenvalues are of the form ξ_m^α, where

$$m = \mathrm{ord}(h), \quad 0 \leq \alpha \leq m-1 \quad \text{and} \quad \xi_m := \exp\left(\frac{2\pi\sqrt{-1}}{m}\right).$$

Definition 1.3.1. *Let G be a finite group and $(h_1, \ldots, h_r, a_1, b_1, \ldots, a_{g'}, b_{g'})$ be a generating vector for G of type $[g'; m_1, \ldots, m_r]$ and*

$$\varrho \colon G \to \mathrm{GL}(V)$$

be a representation. We denote by $N_{i,\alpha}$ the multiplicity of $\xi_{m_i}^\alpha$ as an eigenvalue of $\varrho(h_i)$.

The following lemma describes the local action of the elements h_i from the generating vector around the points that they stabilize. This result is a crucial ingredient in our prove of the Chevalley-Weil formula.

Lemma 1.3.2 (cf. [L05, Satz 4.7.2])**.** *Let $F \colon C \to C/G$ be a Galois cover with branch locus $\mathcal{B} = \{q_1, \ldots, q_r\}$ and*

$$(h_1, \ldots, h_r, a_1, b_1, \ldots, a_{g'}, b_{g'})$$

be an associated generating vector of G. Let q_j be a branch point and $p \in C$ be a point in the fibre of q_j. Let h be an element in G such that the stabilizer of p is generated by hh_jh^{-1}. Then it holds

$$J_p(hh_jh^{-1}) = \exp\left(\frac{2\pi\sqrt{-1}}{m_j}\right),$$

where $J_p(hh_jh^{-1})$ is the Jacobian of hh_jh^{-1} in the point p and $m_j = \mathrm{ord}(h_j)$.

Theorem 1.3.3 (Chevalley-Weil formula, cf. [CW34])**.** *Let G be a finite group acting faithfully on a compact Riemann surface C. Choose a generating vector*

$$(h_1, \dots, h_r, a_1, b_1, \dots, a_{g'}, b_{g'})$$

of type $[g'; m_1, \dots, m_r]$ corresponding to the covering $F\colon C \to C/G$. Let χ be an irreducible character of G and ϱ be a representation with character χ, then it holds

$$\langle \chi, \chi_\varphi \rangle = \chi(1_G)(g'-1) + \sum_{i=1}^{r} \sum_{\alpha=1}^{m_i-1} \frac{\alpha \cdot N_{i,\alpha}}{m_i} + \langle \chi, \chi_{triv} \rangle.$$

Before we give a proof of the formula, we apply it in the special case of a free action. Here we obtain a particularly nice description of the representation

$$\varphi\colon G \to \mathrm{GL}\left(H^{1,0}(C)\right).$$

Example 1.3.4. Under the assumption that the action is free, the Chevalley-Weil formula reads

$$\langle \chi, \chi_\varphi \rangle = \chi(1_G)(g'-1) + \langle \chi, \chi_{triv} \rangle \quad \text{for all} \quad \chi \in \mathrm{Irr}(G).$$

Therefore, the character of φ is given by

$$\chi_\varphi = (g'-1) \sum_{\chi \in \mathrm{Irr}(G)} \chi(1_G)\chi + \chi_{triv} = (g'-1)\chi_{reg} + \chi_{triv}.$$

In other words: the representation φ is isomorphic to the direct sum of $(g'-1)$ copies of the regular representation and one copy of the trivial representation.

Proof of the Chevalley-Weil formula. Let $\mathcal{L}$ be the class function $\chi_{triv} - \chi_\varphi$. By bilinearity of the inner product it holds

$$\langle \chi, \chi_\varphi \rangle = -\langle \chi, \mathcal{L} \rangle + \langle \chi, \chi_{triv} \rangle.$$

Thus, it suffices to determine the inner product $\langle \chi, \mathcal{L} \rangle$, which we expand in the following way

$$\langle \chi, \mathcal{L} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\mathcal{L}(g)} = \frac{1}{|G|}\chi(1_G)\overline{\mathcal{L}(1_G)} + \frac{1}{|G|} \sum_{\substack{g \in G \\ g \neq 1_G}} \chi(g)\overline{\mathcal{L}(g)}.$$

By definition of $\mathcal{L}$ it holds $\mathcal{L}(1_G) = 1 - g(C)$, and for all elements $g \neq 1_G$ we have

$$\mathcal{L}(g) = 1 - \chi_\varphi(g) = \sum_{p \in \mathrm{Fix}(g)} \frac{1}{1 - J_p(g)}$$

according to Eichler's trace formula (see [FK80, Theorem V.2.9]). Since $\overline{\mathcal{L}(g)} = \mathcal{L}(g^{-1})$, it follows

$$\begin{aligned}\frac{1}{|G|}\sum_{\substack{g\in G\\ g\neq 1_G}}\chi(g)\overline{\mathcal{L}(g)} &= \frac{1}{|G|}\sum_{\substack{g\in G\\ g\neq 1_G}}\sum_{p\in \mathrm{Fix}(g)}\frac{\chi(g)}{1-J_p(g^{-1})}\\ &= \frac{1}{|G|}\sum_{p\in \mathrm{Fix}(C)}\sum_{\substack{g\in G_p\\ g\neq 1_G}}\frac{\chi(g)}{1-J_p(g^{-1})},\end{aligned}$$

where

$$\mathrm{Fix}(C) := \big\{p\in C \bigm| G_p \neq \{1_G\}\big\}.$$

Let $\{q_1,\dots,q_r\}$ be the branch locus of F. For all $1\leq i\leq r$ there exists a point $p_i\in F^{-1}(q_i)$ with $G_{p_i} = \langle h_i\rangle$ and for each $h\in G$ it holds

$$G_{h(p_i)} = \langle hh_ih^{-1}\rangle.$$

Moreover, every $p\in \mathrm{Fix}(C)$ maps to a branch point of F. Lemma 1.3.2 implies

$$J_{h(p_i)}(hh_ih^{-1}) = \xi_{m_i} = \exp\left(\frac{2\pi\sqrt{-1}}{m_i}\right)$$

and we conclude that the Jacobian $J_{h(p_i)}(hh_i^lh^{-1})$ is equal to $\xi_{m_i}^l$ for all $l\in\mathbb{Z}$. Since χ is a class function we also have $\chi(h_i^l) = \chi(hh_i^lh^{-1})$. This implies

$$\frac{1}{|G|}\sum_{p\in \mathrm{Fix}(C)}\sum_{\substack{g\in G_p\\ g\neq 1_G}}\frac{\chi(g)}{1-J_p(g^{-1})} = \frac{1}{|G|}\sum_{i=1}^{r}\frac{|G|}{m_i}\sum_{l=1}^{m_i-1}\frac{\chi(h_i^l)}{1-\xi_{m_i}^{-l}},$$

which is equal to

$$\sum_{i=1}^{r}\frac{1}{m_i}\sum_{l=1}^{m_i-1}\frac{\sum_{\alpha=0}^{m_i-1}N_{i,\alpha}\cdot(\xi_{m_i}^{\alpha})^l}{1-\xi_{m_i}^{-l}}.$$

To simplify the last sum, we use the well-known identity

$$\sum_{l=1}^{m_i-1}\frac{(\xi_{m_i}^l)^{\alpha}}{1-\xi_{m_i}^{-l}} = \frac{m_i-1}{2}-\alpha \qquad \text{for all} \qquad 0\leq\alpha\leq m_i-1,$$

(see for example [Reid87, Eq. 8.8]) and get

$$\langle\chi,\mathcal{L}\rangle = \frac{1}{|G|}\chi(1_G)\big(1-g(C)\big) + \sum_{i=1}^{r}\sum_{\alpha=0}^{m_i-1}\frac{N_{i,\alpha}}{m_i}\left(\frac{m_i-1}{2}-\alpha\right).$$

Finally, we use Hurwitz' formula to rewrite the inner product $\langle\chi,\chi_\varphi\rangle$ in the following way

$$\langle \chi, \chi_\varphi \rangle = \langle \chi, \chi_{triv} \rangle + \chi(1_G)(g'-1) + \sum_{i=1}^{r} \left[\chi(1_G)\left(\frac{m_i-1}{2m_i}\right) - \sum_{\alpha=0}^{m_i-1} N_{i,\alpha}\left(\frac{m_i-1}{2m_i} - \frac{\alpha}{m_i}\right)\right].$$

The proof of the Chevalley-Weil formula is finished because $\sum_{\alpha=0}^{m_i-1} N_{i,\alpha}$ is equal to the degree of the representation ϱ which is given by $\chi(1_G)$. □

Remark 1.3.5. If we want to decompose the character χ_φ of the representation

$$\varphi \colon G \to \mathrm{GL}\left(H^{1,0}(C)\right)$$

in irreducible characters with the help of the Chevalley-Weil formula, we face the following computational problem: for each irreducible character χ of G and for each h_i in the generating vector, we need to determine the eigenvalues of the endomorphism $\varrho(h_i)$, where ϱ is an irreducible representation with character χ. While it is relatively easy to compute the irreducible characters of a finite group, it is comparatively hard to determine its irreducible representations. For example: the computer algebra system MAGMA can determine the character table of every finite group but its irreducible representations only over fields of positive characteristic and, if the group is solvable, over cyclotomic fields. The eigenvalues of $\varrho(h_i)$ are, by definition, the roots of the characteristic polynomial P_{h_i} of this endomorphism. Clearly, once we know the polynomial P_{h_i}, we can factorize it easily since its roots are powers of ξ_{m_i}. Thus, we need to determine P_{h_i} from the information encoded in the character χ without using the representation ϱ. This is indeed possible, according to the lemma below.

Lemma 1.3.6. *Let ϱ be a representation of degree n with character χ and $h \in G$. Then, the coefficients c_k of the characteristic polynomial*

$$P_h(x) = x^n + c_1 x^{n-1} + \ldots + c_{n-1}x + c_n$$

of $\varrho(h)$ are given by the following recursive formula:

$$c_k = -\frac{1}{k}\sum_{j=1}^{k} c_{k-j} \cdot \chi(h^j) \quad \textit{for} \quad 1 \le k \le n \quad \textit{and} \qquad c_0 := 1.$$

Proof. The elementary symmetric polynomials and the power sum polynomials in n variables

$$s_k = \sum_{1 \le j_1 < \ldots < j_k \le n} x_{j_1} \cdot \ldots \cdot x_{j_k} \qquad \text{and} \qquad p_k = x_1^k + \ldots + x_n^k$$

are related via the *Newton identities*:

$$k \cdot s_k = \sum_{j=1}^{k} (-1)^{j-1} s_{k-j} \cdot p_j, \qquad \text{for all} \qquad 1 \leq k \leq n.$$

Let $\lambda_1, \dots, \lambda_n$ be the roots of P_h. Since ϱ is a homomorphism and $\varrho(h)$ is diagonalizable, it holds:

$$\chi(h^k) = \operatorname{tr}\left(\varrho(h)^k\right) = \lambda_1^k + \ldots + \lambda_n^k = p_k(\lambda_1, \dots, \lambda_n).$$

To conclude the proof, we combine the equalities

$$s_k(\lambda_1, \dots, \lambda_n) = (-1)^k c_k, \quad \text{and} \quad p_k(\lambda_1, \dots, \lambda_n) = \chi(h^k)$$

with the Newton identities in compliance with $c_0 = s_0 = 1$. □

Example 1.3.7. To illustrate the Chevalley-Weil formula, we consider the dihedral group

$$\mathcal{D}_6 = \langle s, t \mid s^2 = t^6 = 1, \ sts = t^5 \rangle.$$

According to Example 1.1.8 there is a faithful group action of $\mathcal{D}_6$ on a compact Riemann surface C of genus $g(C) = 7$ such that $C/\mathcal{D}_6$ is an elliptic curve and the cover

$$F \colon C \to C/\mathcal{D}_6$$

is branched in two points with index 2, respectively. An associated generating vector is given by (st, st, t^5, t^5). We consider the character χ_5 in the character table of $\mathcal{D}_6$ (see Example 1.2.8) and compute the multiplicity $\langle \chi_5, \chi_\varphi \rangle$: by the lemma above

$$c_1 = -\chi_5(st) = 0, \quad \text{and} \quad c_2 = -\frac{1}{2}\big(c_1 \chi_5(st) + \chi_5((st)^2)\big) = -1$$

are the coefficients of the characteristic polynomial

$$P_{st}(x) = x^2 + c_1 x + c_2 = x^2 - 1$$

of $\varrho(st)$, where ϱ is a representation with character χ_5. The roots of P_{st} are $\xi_2 = -1$ and $\xi_2^0 = 1$. Consequently $N_{i,1} = 1$ for $i = 1, 2$ and the Chevalley-Weil formula yields

$$\langle \chi_5, \chi_\varphi \rangle = 2\big(g(C/\mathcal{D}_6) - 1\big) + \sum_{i=1}^{2} \sum_{\alpha=1}^{m_i - 1} \frac{\alpha \cdot N_{i,\alpha}}{m_i} = \frac{1}{2} + \frac{1}{2} = 1.$$

Performing an analogous computation using the remaining irreducible characters of $\mathcal{D}_6$, we obtain the decomposition

$$\chi_\varphi = \chi_{triv} + \chi_1 + \chi_2 + \chi_4 + \chi_5.$$

Remark 1.3.8. At first glance, our version of the Chevalley-Weil formula differs from the original one (see [CW34]), where the formula reads

$$\langle \chi, \chi_\varphi \rangle = \chi(1_G)(g'-1) + \sum_{i=1}^{r} \sum_{\alpha=1}^{m_i-1} M_{i,\alpha}\left(1 - \frac{\alpha}{m_i}\right) + \langle \chi, \chi_{triv} \rangle.$$

The reason is that the integers $M_{i,\alpha}$ are defined in another way than $N_{i,\alpha}$. The definition of $M_{i,\alpha}$ goes as follows (cf. [Nae05, § 1.4 and § 2.1]): let $p \in C$ be a ramification point which maps to the branch point $q_i \in C/G$ of index m_i. Consider the unique element h_p in the stabilizer group G_p which maps to ξ_{m_i} under the cotangent representation

$$G_p \to \mathrm{GL}\left(\mathfrak{m}_p/\mathfrak{m}_p^2\right) \simeq \mathbb{C}^*, \qquad g \mapsto \left[\omega \mapsto (g^{-1})^*\omega\right].$$

The integer $M_{i,\alpha}$ is defined as the multiplicity of $\xi_{m_i}^{\alpha}$ as an eigenvalue of $\varrho(h_p)$, where ϱ is a representation with character χ. The relation between $M_{i,\alpha}$ and $N_{i,\alpha}$ is easy to describe: recall that the branch point q_i corresponds to the element h_i from the generating vector. We can assume that $G_p = \langle h_i \rangle$. According to Lemma 1.3.2 it holds $J_p(h_i) = \xi_{m_i}$. In other words h_i is the unique element in G_p which maps to ξ_{m_i} under the tangent representation

$$G_p \to \mathrm{GL}\left(T_pC\right) \simeq \mathbb{C}^*.$$

Since the tangent representation is the dual of the cotangent representation, we conclude that $h_p = h_i^{-1}$. Thus, $\xi_{m_i}^{\alpha}$ is an eigenvalue of $\varrho(h_p)$ if and only if $\xi_{m_i}^{m_i-\alpha}$ is an eigenvalue of $\varrho(h_i)$ which implies

$$M_{i,\alpha} = N_{i,m_i-\alpha}$$

and shows that both versions of the Chevalley-Weil formula are equivalent.

Chapter 2

Varieties Isogenous to a Product

In the previous chapter we studied actions of a finite group G on a compact Riemann surface C of genus $g(C) \geq 2$. Here, we want to generalize this concept and consider instead of a single Riemann surface C a product $C_1 \times \ldots \times C_n$ of such surfaces and a group G of automorphisms of the product. Under the assumption that the action of G is free, the quotient space

$$X := (C_1 \times \ldots \times C_n)/G$$

is a projective manifold. It is called a *variety isogenous to a product.* These varieties were introduced by Catanese in [Cat00]. They are the objects we want to study in this chapter, especially in the case $n = 3$, where we call X a *threefold isogenous to a product.*

2.1 Basic Definitions and Properties

In this section we define varieties isogenous to a product and collect some basic properties of these varieties.

Definition 2.1.1. *A complex algebraic variety X is isogenous to a product of curves if there exist compact Riemann surfaces $C_1, \ldots, C_n$ of genus at least two and a finite group $G \leq \mathrm{Aut}(C_1 \times \ldots \times C_n)$ acting freely on the product $C_1 \times \ldots \times C_n$ such that*

$$X = (C_1 \times \ldots \times C_n)/G.$$

As a direct consequence of the definition, a variety X isogenous to a product is smooth, projective, of general type (i.e. $\kappa(X) = \dim(X) = n$) and its canonical class K_X is ample. The *n-fold self-intersection* of the canonical class K_X^n, the *topological Euler number* $e(X)$ and the *holomorphic Euler-Poincaré-characteristic* $\chi(\mathcal{O}_X)$ can be expressed in terms of the genera $g(C_i)$ and the group order $|G|$.

Proposition 2.1.2. *Let $X = (C_1 \times \ldots \times C_n)/G$ be a variety isogenous to a product. Then*

$$\chi(\mathcal{O}_X) = \frac{(-1)^n}{|G|} \prod_{i=1}^{n} \big(g(C_i) - 1\big), \quad K_X^n = (-1)^n n!\ 2^n \chi(\mathcal{O}_X) \quad \textit{and} \quad e(X) = 2^n \chi(\mathcal{O}_X).$$

Proof. We define $Y := C_1 \times \ldots \times C_n$ and denote a fibre of the projection $p_i \colon Y \to C_i$ by F_i. Then, the class of the canonical divisor K_Y it given by

$$K_Y \equiv_{num} \sum_{i=1}^{n} \big(2g(C_i) - 2\big) F_i, \quad \text{and therefore} \quad K_Y^n = n!\, 2^n \prod_{i=1}^{n} \big(g(C_i) - 1\big).$$

Since the G-action on Y is free, the quotient map $\pi \colon Y \to X$ is unramified and it holds $\pi^* c_i(X) = c_i(Y)$ for all i. The equality $c_1(K_Z) = -c_1(Z)$ holds for all projective manifolds Z, in particular for X and Y, and we deduce

$$n!\, 2^n \prod_{i=1}^{n} \big(g(C_i) - 1\big) = K_Y^n = (-1)^n \deg c_1(Y)^n = |G|(-1)^n \deg c_1(X)^n = |G| K_X^n.$$

The topological as well as the holomorphic Euler-Poincaré-characteristic is multiplicative, therefore

$$e(Y) = \prod_{i=1}^{n} e(C_i) = \prod_{i=1}^{n} \big(2 - 2g(C_i)\big) \quad \text{and} \quad \chi(\mathcal{O}_Y) = \prod_{i=1}^{n} \chi(\mathcal{O}_{C_i}) = \prod_{i=1}^{n} \big(1 - g(C_i)\big).$$

To relate these expressions with $e(X)$ and $\chi(\mathcal{O}_X)$, we apply the Gauss-Bonnet formula (see [GH78, p. 416]):

$$\prod_{i=1}^{n} \big(2 - 2g(C_i)\big) = \deg c_n(Y) = |G| \deg c_n(X) = |G| e(X)$$

and the formula of Hirzebruch-Riemann-Roch (see [GH78, p. 437]):

$$\prod_{i=1}^{n} \big(1 - g(C_i)\big) = \deg \mathrm{Td}_n\big(c_1(Y), \ldots, c_n(Y)\big) = |G| \deg \mathrm{Td}_n\big(c_1(X), \ldots, c_n(Y)\big) = |G| \chi(\mathcal{O}_X).$$

□

In order to study group actions on a product $C_1 \times \ldots \times C_n$ of compact Riemann surfaces with $g(C_i) \geq 2$, it is important to understand the structure of the automorphism group of the product. This group has a simple description in terms of the automorphism groups $\mathrm{Aut}(C_i)$ of the factors, thanks to the lemma below:

Lemma 2.1.3 ([Cat00, cf. Rigidity Lemma 3.8])**.** *Let $g \colon C_1 \times \ldots \times C_n \to B_1 \times \ldots \times B_n$ be a surjective holomorphic map between products of compact Riemann surfaces. Assume that $g(B_i) \geq 2$ for all $1 \leq i \leq n$. Then there exists a permutation $\rho \in \mathfrak{S}_n$ and surjective* *holomorphic maps* $h_i\colon C_{\rho(i)} \to B_i$ *such that the following diagram commutes*

$$\begin{array}{ccc} C_1 \times \ldots \times C_n & \xrightarrow{\quad g \quad} & B_1 \times \ldots \times B_n \\ & {}_{f_\rho}\searrow \quad \nearrow_{h} & \\ & C_{\rho(1)} \times \ldots \times C_{\rho(n)}, & \end{array}$$

where $h = (h_1, \ldots, h_n)$ *and* $f_\rho(x_1, \ldots, x_n) = (x_{\rho(1)}, \ldots, x_{\rho(n)})$.

Corollary 2.1.4 ([Cat00, cf. Corollary 3.9])**.** *Let* $D_1, \ldots, D_k$ *be pairwise non-isomorphic compact Riemann surfaces with* $g(D_i) \geq 2$. *Then for all positive integers* $n_1, \ldots, n_k$ *it holds:*

$$\mathrm{Aut}(D_1^{n_1} \times \ldots \times D_k^{n_k}) = \big(\mathrm{Aut}(D_1)^{n_1} \rtimes \mathfrak{S}_{n_1}\big) \times \ldots \times \big(\mathrm{Aut}(D_k)^{n_k} \rtimes \mathfrak{S}_{n_k}\big).$$

The corollary above motivates the next definition.

Definition 2.1.5. *Let* $G \leq \mathrm{Aut}(C_1 \times \ldots \times C_n)$ *be a subgroup. Then we define:*

i) the diagonal subgroup $G^0 := G \cap \big[\mathrm{Aut}(C_1) \times \ldots \times \mathrm{Aut}(C_n)\big]$

ii) the subgroups $G_i := G \cap \big[\mathrm{Aut}(C_1 \times \ldots \times \widehat{C_i} \times \ldots \times C_n) \times \mathrm{Aut}(C_i)\big]$ *and*

iii) the group homomorphisms $\psi_i\colon G_i \to \mathrm{Aut}(C_i)$, *induced by the natural projections*

$$\mathrm{Aut}(C_1 \times \ldots \times \widehat{C_i} \times \ldots \times C_n) \times \mathrm{Aut}(C_i) \to \mathrm{Aut}(C_i).$$

The kernel of ψ_i *is denoted by* K_i. *We say that* G *is of unmixed type if* $G = G^0$ *and otherwise of mixed type. Moreover, we say that the quotient*

$$(C_1 \times \ldots \times C_n)/G$$

is of mixed, respectively unmixed type, if and only if G *is of mixed, respectively unmixed type.*

Remark 2.1.6. Let $G \leq \mathrm{Aut}(C_1 \times \ldots \times C_n)$ be a subgroup, then G^0 is normal in G.

Proof. The diagonal subgroup G^0 is, by definition, the intersection of G with

$$\mathrm{Aut}(C_1) \times \ldots \times \mathrm{Aut}(C_n).$$

The latter is normal in $\mathrm{Aut}(C_1 \times \ldots \times C_n)$, whence G^0 is normal in G, too.

□

2.2 Group Actions on a Product of Curves

Let G be a subgroup of the automorphism group of a product $C_1 \times \ldots \times C_n$ of compact Riemann surfaces, where $g(C_i) \geq 2$. The diagonal subgroup G^0 is normal in G and the quotient G/G^0 can be considered as a permutation group of the coordinates $(x_1, \ldots, x_n)$ of the product in a natural way:

$$G/G^0 \to \mathfrak{S}_n, \qquad \overline{g} \mapsto \rho^{-1},$$

where $g = h \circ f_\rho$ is the factorization of g according to Lemma 2.1.3. If G is of unmixed type, then it is immediate that the action is given in terms of the maps ψ_i:

$$g(x_1, \ldots, x_n) = \big(\psi_1(g)x_1, \ldots, \psi_n(g)x_n\big), \qquad \text{for all} \qquad g \in G.$$

In the mixed case our aim is to show that after conjugation with a suitable automorphism in

$$\mathrm{Aut}(C_1) \times \ldots \times \mathrm{Aut}(C_n),$$

there are analogous formulas describing the G-action on the product in terms of the maps ψ_i. Such a description, i.e. a *normalized form* of the action, is of great importance for the following reasons:

- we can study geometric properties of the quotient $(C_1 \times \ldots \times C_n)/G$ using Riemann surface theory (see Chapter 1).
- the formulas defining the normal form can be used to construct an action of an abstract finite group G on a product of compact Riemann surfaces starting from suitable subgroups $G_i \leq G$ and group actions $\psi \colon G_i \to \mathrm{Aut}(C_i)$.

We assume that $n = 3$, but similar results can be obtained in any dimension. For the case $n = 2$ we refer the reader to [Cat00, Proposition 3.16].

According to the index of G^0 in G, there are three sub-cases of the mixed case:

$$G/G^0 \simeq \mathbb{Z}_2, \qquad G/G^0 \simeq \mathfrak{A}_3 \qquad \text{and} \qquad G/G^0 \simeq \mathfrak{S}_3.$$

We call them index two, index three and index six case, respectively.

Convention: in the index two case we can assume that $C_2 \simeq C_3$. In the index three and six case it holds $C_1 \simeq C_2 \simeq C_3$. If we specialize to one of these cases, then we write $D \times C^2$ or C^3 instead of $C_1 \times C_2 \times C_3$.

Remark 2.2.1. The groups G, G_i and G^0 are related as follows:

i) $G_1 = G$ and $G_2 = G_3 = G^0$ in the index two case,

ii) $G_i = G^0$ for all $1 \leq i \leq 3$ in the index three case and

iii) $|G_i : G^0| = 2$ for all $1 \leq i \leq 3$ and $G_i \cap G_j = G^0$ for all $i \neq j$ in the index six case.

Proposition 2.2.2. *Let G be a subgroup of the automorphism group of a product of three compact Riemann surfaces and $\nu\colon G \to G/G^0 \leq \mathfrak{S}_3$ be the projection map.*

i) In the index two case, fix an element $\delta \in G$ of the form $\delta(x, y, z) = (\delta_1 x, \delta_3 z, \delta_2 y)$, i.e. $\nu(\delta) = (2, 3)$. Then, after conjugating with the automorphism $\xi(x, y, z) := (x, y, \delta_3 z)$, it holds

$$\psi_3(g) = \psi_2(\delta g \delta^{-1}) \quad \textit{for all} \quad g \in G^0$$

and the action is given by the formulas

- $\delta(x, y, z) = \big(\psi_1(\delta)x, z, \psi_2(\delta^2)y\big)$
- $g(x, y, z) = \big(\psi_1(g)x, \psi_2(g)y, \psi_2(\delta g \delta^{-1})z\big)$ *for all* $g \in G^0$.

ii) In the index three case, fix an element $\tau \in G$ of the form $\tau(x, y, z) = (\tau_2 y, \tau_3 z, \tau_1 x)$, i.e. $\nu(\tau) = (1, 3, 2)$. Then, after conjugating with the automorphism $\epsilon(x, y, z) := (x, \tau_2 y, \tau_2\tau_3 z)$, it holds

$$\psi_2(g) = \psi_1(\tau g \tau^{-1}) \quad \textit{and} \quad \psi_3(g) = \psi_1(\tau^2 g \tau^{-2}) \quad \textit{for all} \quad g \in G^0$$

and the action is given by the formulas

- $\tau(x, y, z) = \big(y, z, \psi_1(\tau^3)x\big)$
- $g(x, y, z) = \big(\psi_1(g)x, \psi_1(\tau g \tau^{-1})y, \psi_1(\tau^2 g \tau^{-2})z\big)$ *for all* $g \in G^0$.

iii) In the index six case, fix an element $\tau \in G$ of the form $\tau(x, y, z) = (\tau_2 y, \tau_3 z, \tau_1 x)$, i.e. $\nu(\tau) = (1, 3, 2)$. Then, after conjugating with the automorphism $\epsilon(x, y, z) := (x, \tau_2 y, \tau_2\tau_3 z)$, it holds

$$\psi_2(h) = \psi_1(\tau h \tau^{-1}) \quad \textit{and} \quad \psi_3(k) = \psi_1(\tau^2 k \tau^{-2})$$

for all $h \in G_2$ and $k \in G_3$ and the action is given by the formulas

- $\tau(x, y, z) = \big(y, z, \psi_1(\tau^3)x\big)$
- $g(x, y, z) = \big(\psi_1(g)x, \psi_1(\tau g \tau^{-1})y, \psi_1(\tau^2 g \tau^{-2})z\big)$
- $f(x, y, z) = \big(\psi_1(f)x, \psi_1(\tau f \tau^{-2})z, \psi_1(\tau^2 f \tau^{-1})y\big)$

for all $g \in G^0$ and $f \in G_1 \setminus G^0$.

Proof. We just prove *iii*). The proofs of *i*) and *ii*) are analogous. After conjugation with ϵ, it holds $\tau(x,y,z) = (y,z,\tau' x)$, where $\tau' = \tau_2\tau_3\tau_1$. Since $\tau^3(x,y,z) = (\tau' x, \tau' y, \tau' z)$, it follows that $\tau' = \psi_1(\tau^3)$. The next step is to relate the actions ψ_1, ψ_2 and ψ_3 to each other. The action of $f \in G_1 \setminus G^0$ can be written as

$$f(x,y,z) = \big(\psi_1(f)x, f_3 z, f_2 y\big).$$

The element $f' := \tau f \tau^{-1}$ is contained in $G_3 \setminus G^0$, say $f'(x,y,z) = \big(f_2' y, f_1' x, \psi_3(f')z\big)$. Consequently

$$\big(f_3 z, f_2 y, \psi_1(\tau^3 f)x\big) = \tau f(x,y,z) = f'\tau(x,y,z) = \big(f_2' z, f_1' y, \psi_3(f')\psi_1(\tau^3)x\big)$$

and we conclude that $\psi_3(\tau f \tau^{-1}) = \psi_1(\tau^3 f \tau^{-3})$ for all $f \in G_1 \setminus G^0$. An analogous computation with $f \in G^0$ shows that $\psi_3(\tau f \tau^{-1}) = \psi_1(\tau^3 f \tau^{-3})$ holds for all $f \in G_1$ or equivalently

$$\psi_3(k) = \psi_1(\tau^2 k \tau^{-2}) \quad \text{for all} \quad k \in G_3.$$

The same argument using τ^2 instead of τ yields $\psi_2(h) = \psi_1(\tau h \tau^{-1})$ for all $h \in G_2$. As a by-product

$$g(x,y,z) = \big(\psi_1(g)x, \psi_1(\tau g \tau^{-1})y, \psi_1(\tau^2 g \tau^{-2})z\big) \quad \text{for all} \quad g \in G^0.$$

It remains to show that

$$f(x,y,z) = \big(\psi_1(f)x, \psi_1(\tau f \tau^{-2})z, \psi_1(\tau^2 f \tau^{-1})y\big) \quad \text{for all} \quad f \in G_1 \setminus G^0.$$

Consider arbitrary elements $f \in G_1 \setminus G^0$ and $h \in G_2 \setminus G^0$. They can be written as

$$f(x,y,z) = \big(\psi_1(f)x, f_3 z, f_2 y\big) \quad \text{and} \quad h(x,y,z) = \big(h_3 z, \psi_1(\tau h \tau^{-1})y, h_1 x\big).$$

The product

$$fh(x,y,z) = f\big(h_3 z, \psi_1(\tau h \tau^{-1})y, h_1 x\big) = \big(\psi_1(f)h_3 z, f_3 h_1 x, f_2\psi_1(\tau h \tau^{-1})y\big)$$

is contained in $\nu^{-1}\big((1,2,3)\big)$, whence it is equal to $\tau^2 g' \in \tau^2 G^0$, where $g' := \tau^{-2} fh$. Consequently

$$\tau^2 g'(x,y,z) = \big(\psi_1(fh\tau^{-2})z, \psi_1(\tau fh)x, \psi_1(\tau^2 fh \tau^{-1})y\big) = \big(\psi_1(f)h_3 z, f_3 h_1 x, f_2\psi_1(\tau h \tau^{-1})y\big)$$

and we conclude that $f_2 = \psi_1(\tau^2 f \tau^{-1})$. A similar computation starting with the product hf instead of fh shows that $f_3 = \psi_1(\tau f \tau^{-2})$. □

Remark 2.2.3. In the index six case it holds:

i) $G_3 = \tau G_1 \tau^{-1}$ and $G_2 = \tau^2 G_1 \tau^{-2}$ as we already used in the proof above.

ii) the restriction of the action to the subgroup $H := \nu^{-1}\big(\langle(1,2,3)\rangle\big)$ is in normal form, i.e. in the form given in Proposition 2.2.2.

Convention: from now on we assume that a subgroup $G \leq \mathrm{Aut}(C_1 \times C_2 \times C_3)$ of mixed type is embedded in *normal form* for a fixed choice of δ or τ, respectively.

As already mentioned a very important observation is that the formulas from Proposition 2.2.2 provide a way to define mixed group actions on a product of three compact Riemann surfaces:

Proposition 2.2.4. *Let G be a finite group with a normal subgroup G^0 such that G/G^0 is isomorphic to $\mathbb{Z}_2$, $\mathfrak{A}_3$ or $\mathfrak{S}_3$. Let $\nu\colon G \to G/G^0$ be the quotient map.*

i) In the index two case, let $\psi_1\colon G \to \mathrm{Aut}(D)$ and $\psi_2\colon G^0 \to \mathrm{Aut}(C)$ be group actions on compact Riemann surfaces with kernels K_i such that

$$K_1 \cap K_2 \cap \delta K_2 \delta^{-1} = \{1_G\}$$

for an element $\delta \in G \setminus G^0$. Then the formulas from Proposition 2.2.2 i) define an embedding $i_\delta\colon G \hookrightarrow \mathrm{Aut}(D \times C^2)$.

ii) In the index three case, let $\alpha\colon G/G^0 \to \mathfrak{A}_3$ be an isomorphism and $\psi_1\colon G^0 \to \mathrm{Aut}(C)$ be a group action on a compact Riemann surface with kernel K_1 such that

$$K_1 \cap \tau K_1 \tau^{-1} \cap \tau^2 K_1 \tau^{-2} = \{1_G\}$$

for an element $\tau \in G$ with $(\alpha \circ \nu)(\tau) = (1,3,2)$. Then the formulas from Proposition 2.2.2 ii) define an embedding $i_\tau\colon G \hookrightarrow \mathrm{Aut}(C^3)$.

iii) In the index six case, let $\alpha\colon G/G^0 \to \mathfrak{S}_3$ be an isomorphism. Define

$$G_1 := (\alpha \circ \nu)^{-1}\big(\langle(2,3)\rangle\big)$$

and let $\psi_1\colon G_1 \to \mathrm{Aut}(C)$ be a group action on a compact Riemann surface with kernel K_1 such that

$$K_1 \cap \tau K_1 \tau^{-1} \cap \tau^2 K_1 \tau^{-2} = \{1_G\}$$

for an element $\tau \in G$ with $(\alpha \circ \nu)(\tau) = (1,3,2)$. Then the formulas from Proposition 2.2.2 iii) define an embedding $i_\tau\colon G \hookrightarrow \mathrm{Aut}(C^3)$.

Proof. The verification that the maps i_δ and i_τ are well-defined group homomorphisms, is elementary and will be skipped. To finish the proof, we need to show that the maps i_δ and i_τ are injective. Let $g \in G$ such that $i_\delta(g) = \mathrm{id}$, then $g \in G^0$ and the condition

$i_\delta(g) = \mathrm{id}$ is equivalent to

$$\psi_1(g) = \mathrm{id}, \quad \psi_2(g) = \mathrm{id} \quad \text{and} \quad \psi_2(\delta g \delta^{-1}) = \mathrm{id}\,.$$

This is the same as $g \in K_1 \cap K_2 \cap \delta K_2 \delta^{-1}$. Similarly, $i_\tau(g) = \mathrm{id}$ is equivalent to

$$g \in K_1 \cap \tau K_1 \tau^{-1} \cap \tau^2 K_1 \tau^{-2}.$$

□

Proposition 2.2.5. *Let G be a subgroup of the automorphism group of a product of three compact Riemann surfaces and denote by $K_{i,j}$ the intersection of the kernels K_i and K_j for $i \neq j$. Then it holds:*

i) $K_1 \cap K_2 \cap K_3 = \{1_G\}$.

ii) *a) In the index two case $K_3 = \delta K_2 \delta^{-1}$. In particular $K_{2,3}$ is normal in G.*

b) In the index three and index six case $K_3 = \tau K_1 \tau^{-1}$ and $K_2 = \tau^2 K_1 \tau^{-2}$.

iii) the group $K_{1,2} \cdot K_{1,3} \cdot K_{2,3}$ is a normal subgroup of G.

Proof. *i)* Let $g \in K_1 \cap K_2 \cap K_3 \leq G^0$, then $\psi_i(g) = \mathrm{id}$ for all $1 \leq i \leq 3$ which implies $g = 1_G$.
ii) is a direct consequence of Proposition 2.2.2.
iii) We only consider the index six case. The proof in the other cases is similar. Since $K_{i,j}$ is normal in G^0, it suffices to show that

$$g\big(K_{1,2} \cdot K_{1,3} \cdot K_{2,3}\big)g^{-1} = K_{1,2} \cdot K_{1,3} \cdot K_{2,3} \quad \text{for } g = \tau \text{ and for all } g \in G_i \setminus G^0.$$

For $g = \tau$ this is a consequence of *ii)* part *b)* and the fact that elements $a \in K_{1,2}$, $b \in K_{1,3}$ and $c \in K_{2,3}$ always commute. Consider an element $f \in G_1 \setminus G^0$. Since K_1 is normal in G_1, it holds $fK_1f^{-1} = K_1$. By a direct computation we verify $fK_2f^{-1} = K_3$ and conclude $fK_3f^{-1} = K_2$ using the normality of K_2 in G_2. Putting these informations together we obtain

$$f\big(K_{1,2} \cdot K_{1,3} \cdot K_{2,3}\big)f^{-1} = K_{1,2} \cdot K_{1,3} \cdot K_{2,3}.$$

The proof is finished because every element $g \in G_2 \setminus G^0$ can be written as $g = \tau f$ for a suitable element $f \in G_1 \setminus G^0$ and every $g \in G_3 \setminus G^0$ as $g = \tau^2 f$ for some $f \in G_1 \setminus G^0$. □

Definition 2.2.6. *Let $X := (C_1 \times C_2 \times C_3)/G$ be a threefold isogenous to a product. We say that the action of G is*

i) minimal *if the groups $K_{i,j}$ are trivial.*

ii) absolutely faithful *if the kernels K_i are trivial.*

Remark 2.2.7. Let $X := (C_1 \times C_2 \times C_3)/G$ be a threefold isogenous to a product.

i) Assume that the action of G is minimal, then

$$(G^0 \cap K_i) \times (G^0 \cap K_j) \trianglelefteq G^0 \quad \text{for all} \quad i \neq j.$$

ii) In the index two case the action of G is minimal if and only if

$$K_1 \cap K_2 = \{1_G\} \quad \text{and} \quad K_2 \cap \delta K_2 \delta^{-1} = \{1_G\}.$$

iii) In the index three and index six case the action of G is minimal if and only if

$$K_1 \cap \tau K_1 \tau^{-1} = \{1_G\}.$$

Proof. i) Clearly, the groups $K_i \cap G^0$ are normal in G^0. Moreover, the pairwise intersection of these groups is trivial, which implies that

$$(K_i \cap G^0) \cdot (K_j \cap G^0) \trianglelefteq G^0$$

is a direct product.
ii) and iii) follow immediately from Proposition 2.2.5 ii) a) and b) □

Lemma 2.2.8. *Every threefold isogenous to a product can be obtained by a minimal action.*

Proof. Let $(C_1 \times C_2 \times C_3)/G$ be a realization of a threefold X isogenous to a product. Since $K := K_{1,2} \cdot K_{1,3} \cdot K_{2,3}$ is normal in G and K_i acts trivially on C_i we have the following natural isomorphisms:

$$\frac{C_1 \times C_2 \times C_3}{G} \simeq \frac{(C_1 \times C_2 \times C_3)/K}{G/K} \simeq \frac{C_1/K_{2,3} \times C_2/K_{1,3} \times C_3/K_{1,2}}{G/K}.$$

Note that the Riemann surface $C_k/K_{i,j}$ has genus $g(C_k/K_{i,j}) \geq 2$ because $g(C_k) \geq 2$ and $K_{i,j}$ acts freely on C_k. The kernel of the induced action of G_k/K on $C_k/K_{i,j}$ is given by

$$\overline{K_k} = \ker\left(G_k/K \to \mathrm{Aut}(C_k/K_{i,j})\right) \simeq (K_k \cdot K)/K.$$

We claim that the intersection $\overline{K_i} \cap \overline{K_j}$ is trivial for all $1 \leq i < j \leq 3$. Assume that $i = 1$ and $j = 2$. An element in the intersection $\overline{K_1} \cap \overline{K_2}$ can be represented by an element $k_1 \in K_1$ such that $k_1 = k_2 k$ with $k_2 \in K_2$ and $k \in K$. We want to show that $k_1 \in K$. Write $k = abc$ with $a \in K_{1,2}$, $b \in K_{1,3}$ and $c \in K_{2,3}$. Note that the elements a, b and c commute, because the pairwise intersection of the normal subgroups $K_{i,j} \trianglelefteq G^0$ is trivial.

The product k_2c is contained in $K_{1,2} \leq K$ because

$$k_2c \in K_2 \quad \text{and} \quad k_2c = k_1(ab)^{-1} \in K_1.$$

It follows that $k_1 = (k_2c)ab \in K$. $\square$

Convention: from now on we assume, without loss of generality, that the action of G is minimal.

2.3 The Algebraic Datum

The aim of this section is to give a *group theoretical description* of threefolds isogenous to a product.

We start by describing the freeness of the G-action. Here the following definition is convenient:

Definition 2.3.1. *Let C be a compact Riemann surface and $\psi\colon G \to \mathrm{Aut}(C)$ a group action. The stabilizer set $\Sigma \subset G$ of ψ is defined as the set of elements in G admitting at least one fixed point:*

$$\Sigma = \bigcup_{p \in C} G_p.$$

Let $\psi\colon G \to \mathrm{Aut}(C)$ be a group action and $K := \ker(\psi)$ be the kernel. Since ψ factors through the quotient map

$$\begin{array}{ccc} G/K & \xrightarrow{\overline{\psi}} & \mathrm{Aut}(C)\,, \\ \uparrow & \nearrow_{\psi} & \\ G & & \end{array}$$

it follows that the pre-image of the stabilizer set $\overline{\Sigma}$ of $\overline{\psi}$ is equal to Σ. Let

$$V := (h_1, \ldots, h_r, a_1, b_1, \ldots, a_{g'}, b_{g'})$$

be a generating vector of G/K associated to the cover

$$C \to C/\overline{G}, \qquad \text{where} \qquad \overline{G} := G/K.$$

Then, according to Riemann's existence Theorem, the cyclic groups $\langle h_i \rangle$ and their conjugates provide the non-trivial stabilizer groups of $\overline{\psi}$. Therefore we have

$$\overline{\Sigma} = \bigcup_{g \in \overline{G}} \bigcup_{i \in \mathbb{Z}} \bigcup_{j=1}^{r} \{g h_j^i g^{-1}\}$$

and we also refer to $\overline{\Sigma}$ as the stabilizer set associated to the generating vector V.

Proposition 2.3.2. *Let G be a subgroup of the automorphism group of a product $C_1 \times C_2 \times C_3$ of compact Riemann surfaces and Σ_i be the stabilizer sets of the group actions $\psi_i \colon G_i \to \mathrm{Aut}(C_i)$. Then*

a) In the unmixed case, the action of G is free if and only if:

$$\Sigma_1 \cap \Sigma_2 \cap \Sigma_3 = \{1_G\}.$$

b) In the index two case, the action of G is free if and only if:

i) $\Sigma_1 \cap \Sigma_2 \cap \delta\Sigma_2\delta^{-1} = \{1_G\}$.

ii) for all $g \in G^0$ with $\delta g \in \Sigma_1$, it holds $(\delta g)^2 \notin \Sigma_2$.

c) In the index three case, the action of G is free if and only if:

i) $\Sigma_1 \cap \tau\Sigma_1\tau^{-1} \cap \tau^2\Sigma_1\tau^{-2} = \{1_G\}$.

ii) for all $g \in G^0$ it holds $(\tau g)^3 \notin \Sigma_1$.

d) In the index six case, the action of G is free if and only if:

i) $\Sigma_1 \cap \tau\Sigma_1\tau^{-1} \cap \tau^2\Sigma_1\tau^{-2} = \{1_G\}$.

ii) for all $g \in G^0$ it holds $(\tau g)^3 \notin \Sigma_1$.

iii) for all $f \in G_1 \setminus G^0$ with $f \in \Sigma_1$, it holds $\tau f^2\tau^{-1} \notin \Sigma_1$.

Proof. c) The group G is the disjoint union of G^0, τG^0 and $\tau^2 G^0$. Suppose p is a fixed point of $\tau g \in \tau G^0$, then p is also a fixed point of $(\tau g)^2 \in \tau^2 G^0$. Conversely, let p be a fixed point of $\tau^2 g \in \tau^2 G^0$, then p is also a fixed point of $(\tau^2 g)^2 \in \tau G^0$. Therefore it is sufficient to show:

i) an element $g \in G^0$ has a fixed point if and only if $g \in \Sigma_1 \cap \tau\Sigma_1\tau^{-1} \cap \tau^2\Sigma_1\tau^{-2}$.

ii) an element $\tau g \in \tau G^0$ has a fixed point if and only if $(\tau g)^3 \in \Sigma_1$.

i) According to Proposition 2.2.2, we have

$$g(x, y, z) = \big(\psi_1(g)x, \psi_1(\tau g\tau^{-1})y, \psi_1(\tau^2 g\tau^{-2})z\big),$$

for all $g \in G^0$. Therefore $(x_0, y_0, z_0) \in C^3$ is a fixed point of g if and only if g, $\tau g\tau^{-1}$ and $\tau^2 g\tau^{-2}$ are contained in Σ_1. This is equivalent to

$$g \in \Sigma_1 \cap \tau\Sigma_1\tau^{-1} \cap \tau^2\Sigma_1\tau^{-2}$$

because $\Sigma_1 = \tau^3\Sigma_1\tau^{-3}$.
ii) Let $p \in C^3$ be a fixed point of τg, then p is also a fixed point of $(\tau g)^3 \in G^0$ and it follows from i), that $(\tau g)^3$ is contained in Σ_1. Conversely, assume that $(\tau g)^3 \in \Sigma_1$, then

there exists a point $x_0 \in C$ such that $\psi_1\big((\tau g)^3\big)x_0 = x_0$. We define $y_0 := \psi_1(\tau(\tau g)^2)x_0$ and $z_0 := \psi_1(\tau^3 g)x_0$. Then (x_0, y_0, z_0) is a fixed point of τg.
d) In addition to the elements in G^0, τG^0 and $\tau^2 G^0$, we have to consider the elements in $G_i \setminus G^0$. Note that $G_2 \setminus G^0 = \tau^2(G_1 \setminus G^0)\tau^{-2}$ and $G_3 \setminus G^0 = \tau(G_1 \setminus G^0)\tau^{-1}$. This implies: there are elements in $G_i \setminus G^0$ (for $i = 2, 3$) with fixed points if and only if there are elements in $G_1 \setminus G^0$ with fixed points. It remains to show: an element $f \in G_1 \setminus G^0$ has fixed points if and only if $f \in \Sigma_1$ and $\tau f^2 \tau^{-1} \in \Sigma_1$. According to Proposition 2.2.2:

$$f(x, y, z) = \big(\psi_1(f)x, \psi_1(\tau f \tau^{-2})z, \psi_1(\tau^2 f \tau^{-1})y\big).$$

Let $(x_0, y_0, z_0) \in C^3$ be a fixed point of f, then

$$\psi_1(f)x_0 = x_0, \quad \psi_1(\tau f \tau^{-2})z_0 = y_0, \quad \psi_1(\tau^2 f \tau^{-1})y_0 = z_0.$$

This shows that $f \in \Sigma_1$ and $\psi_1(\tau f^2 \tau^{-1})y_0 = y_0$. The latter implies $\tau f^2 \tau^{-1} \in \Sigma_1$. Conversely, assume $f \in \Sigma_1$ and $\tau f^2 \tau^{-1} \in \Sigma_1$ then there exist points $x_0, y_0 \in C$ such that $\psi_1(f)x_0 = x_0$ and $\psi_1(\tau f^2 \tau^{-1})y_0 = y_0$. Define $z_0 := \psi_1(\tau^2 f \tau^{-1})y_0$, then (x_0, y_0, z_0) is a fixed point of f.
a) is clear and the proof of *b)* is similar, but simpler.

□

Proposition 2.3.3. *Let G be a subgroup of the automorphism group of a product of three curves.*

a) Assume that $G^0 \trianglelefteq G$ is of index six and G is acting freely on the product, then the short exact sequence

$$1 \longrightarrow G^0 \longrightarrow G \longrightarrow \mathfrak{S}_3 \longrightarrow 1$$

does not split.

b) Assume that $G^0 \trianglelefteq G$ is of index three and condition i) in Proposition 2.3.2 c) holds. Then ii) in Proposition 2.3.2 c) is equivalent to the condition, that the short exact sequence

$$1 \longrightarrow G^0 \longrightarrow G \longrightarrow \mathfrak{A}_3 \longrightarrow 1$$

does not split.

Proof. *a)* A short exact sequence $1 \longrightarrow G^0 \longrightarrow G \longrightarrow \mathfrak{S}_3 \longrightarrow 1$ splits, if and only if there exist elements $a, b \in G \setminus G^0$ such that $\mathrm{ord}(a) = 2$, $\mathrm{ord}(b) = 3$ and $aba = b^{-1}$. Assume that the sequence splits, then there exist elements a, b as above. Since $b^2 \notin G^0$ we can assume that $b = \tau g \in \tau G^0$. This leads to the contradiction $(\tau g)^3 = 1 \in \Sigma_1$.
b) A short exact sequence $1 \longrightarrow G^0 \longrightarrow G \longrightarrow \mathfrak{A}_3 \longrightarrow 1$ splits if and only if there exists an element $b \in G \setminus G^0$ of order 3. Suppose that such an element exists. Observe that $\mathrm{ord}(b^2) = 3$ and $b^2 \notin G^0$ so we can assume that $b = \tau g \in \tau G^0$. Since $(\tau g)^3 = b^3 = 1 \in \Sigma_1$

we obtain a contradiction. Assume there is an element $g \in G^0$ such that $(\tau g)^3 \in \Sigma_1$. Then there exists a fixed point $p \in C^3$ of τg. The point p is also a fixed point of $(\tau g)^3 \in G^0$, hence

$$(\tau g)^3 \in \Sigma_1 \cap \tau \Sigma_1 \tau^{-1} \cap \tau^2 \Sigma_1 \tau^{-2} = \{1_G\}.$$

□

Corollary 2.3.4. *Let X be a threefold isogenous to a product of curves. Assume that $G^0 \trianglelefteq G$ is of index three or six, then $|G^0|$ is divisible by* 3.

Proof. Assume that 3 does not divide $|G^0|$, then all elements of order 3 in G are contained in $G \setminus G^0$. Since $|G|$ is divisible by 3, there is least one such element and it follows that the short exact sequence

$$1 \longrightarrow G^0 \longrightarrow G \longrightarrow G/G^0 \longrightarrow 1$$

splits. A contradiction. □

We shall associate to a threefold isogenous to a product certain algebraic data. We have the groups G, G^0, the kernels K_i and the embedding $G/G^0 \leq \mathfrak{S}_3$. In the index three case we choose an element $\tau \in G$ with residue class $(1,3,2)$. In the index six case we choose elements $\tau, h \in G$ with classes $(1,3,2)$ and $(2,3)$. According to Riemann's existence theorem, for each $\overline{\psi_i} \colon G_i/K_i \to \mathrm{Aut}(C_i)$ we can choose a generating vector V_i for G_i/K_i of type T_i. However, this choice is not unique, only the type T_i is uniquely determined.

Definition 2.3.5. *To a threefold X isogenous to a product of we attach the tuple*

- $(G, K_1, K_2, K_3, V_1, V_2, V_3)$ *in the unmixed case,*
- $(G, G^0, K_1, K_2, V_1, V_2)$ *in the index two case,*
- (G, G^0, K_1, τ, V_1) *in the index three case,*
- $(G, G^0, K_1, \tau, h, V_1)$ *in the index six case,*

and call it an algebraic datum *of X.*

Thanks to Riemann's existence theorem and Proposition 2.2.4, we have a way to construct threefolds isogenous to a product starting from group theoretical data. For the two dimensional analogue, we refer to [BCG08, Proposition 2.5].

Proposition 2.3.6. *Let G be a finite group and $G^0 \trianglelefteq G$ be a normal subgroup such that $G/G^0 \leq \mathfrak{S}_3$. Let $\nu \colon G \to G/G^0$ be the quotient map.*

a) Assume that $G^0 = G$ and let $K_i \trianglelefteq G$ be three normal subgroups such that

$$K_i \cap K_j = \{1_G\} \quad \textit{for all} \quad 1 \leq i < j \leq 3.$$

Let V_i be generating vectors for the groups G/K_i for $1 \leq i \leq 3$. Let $\Sigma_i \subset G$ be the pre-images of the stabilizer sets associated to the generating vectors V_i under the quotient maps

$$G \to G/K_i.$$

Assume that the freeness condition from Proposition 2.3.2 a) holds. Then there exists a threefold X isogenous to a product of unmixed type with algebraic datum

$$(G, K_1, K_2, K_3, V_1, V_2, V_3).$$

b) *Assume that $G/G^0 \simeq \mathbb{Z}_2$. Let $\delta \in G \setminus G^0$, $K_1 \trianglelefteq G$ and $K_2 \trianglelefteq G^0$ be normal subgroups such that*

$$K_1 \cap K_2 = \{1_G\} \quad \textit{and} \quad K_2 \cap \delta K_2 \delta^{-1} = \{1_G\}.$$

Let V_1 be a generating vector for G/K_1 and V_2 a generating vector for G^0/K_2. Let $\Sigma_i \subset G$ be the pre-images of the stabilizer sets associated to the generating vectors V_i under the quotient maps

$$G \to G/K_1 \quad \textit{and} \quad G^0 \to G^0/K_2.$$

Assume that the freeness conditions from Proposition 2.3.2 b) hold. Then there exists a threefold X isogenous to a product with algebraic datum

$$(G, G^0, K_1, K_2, V_1, V_2).$$

c) *Assume that $G/G^0 \simeq \mathbb{Z}_3$. Let $\tau \in G \setminus G^0$ and $K_1 \trianglelefteq G^0$ such that*

$$K_1 \cap \tau K_1 \tau^{-1} = \{1_G\}.$$

Let V_1 be a generating vector for G^0/K_1 and $\Sigma_1 \subset G^0$ be the pre-image of the stabilizer set associated to the generating vector V_1 under the quotient map

$$G^0 \to G^0/K_1.$$

Assume that the freeness conditions from Proposition 2.3.2 c) hold. Then there exists a threefold X isogenous to a product with algebraic datum

$$(G, G^0, K_1, \tau, V_1).$$

d) *Assume that $G/G^0 \simeq \mathfrak{S}_3$. Let $\tau, h \in G \backslash G^0$ such that $\tau^2 \notin G^0$ and $h^2 \in G^0$. Define the subgroup*

$$G_1 := \langle h, G^0 \rangle \leq G.$$

Let $K_1 \trianglelefteq G_1$ be a normal subgroup such that

$$K_1 \cap \tau K_1 \tau^{-1} = \{1_G\}.$$

Let V_1 be a generating vector for G_1/K_1 and $\Sigma_1 \subset G_1$ be the pre-image of the stabilizer set associated to the generating vector V_1 under the quotient map

$$G_1 \to G_1/K_1.$$

Assume that the freeness conditions from Proposition 2.3.2 d) hold. Then there exists a threefold X isogenous to a product with algebraic datum

$$(G, G^0, K_1, \tau, h, V_1).$$

Proof. b) By Riemann's existence theorem there are two compact Riemann surfaces C and D and two faithful group actions, $\overline{\psi_1}\colon G/K_1 \to \mathrm{Aut}(D)$ and $\overline{\psi_2}\colon G^0/K_1 \to \mathrm{Aut}(C)$, associated to the generating vectors V_1 and V_2. Composing these homomorphisms with the quotient maps $G \to G/K_1$ and $G^0 \to G^0/K_2$, we obtain two group actions

$$\psi_1\colon G \to \mathrm{Aut}(D) \quad \text{and} \quad \psi_2\colon G^0 \to \mathrm{Aut}(C)$$

with kernels K_1 and K_2. According to Proposition 2.2.4 i) there exists an embedding $i_\delta\colon G \to \mathrm{Aut}(D \times C^2)$. The action of G on $D \times C^2$ is minimal by Remark 2.2.7 and free by Proposition 2.3.2. It follows that the quotient $(D \times C^2)/G$ is a threefold isogenous to a product with algebraic datum $(G, G^0, K_1, K_2, V_1, V_2)$.
d) There is a compact Riemann surface together with a faithful group action

$$\overline{\psi_1}\colon G_1/K_1 \to \mathrm{Aut}(C)$$

associated to the generating vector V_1. Composition with $G_1 \to G_1/K_1$ yields a group action

$$\psi_1\colon G_1 \to \mathrm{Aut}(C) \quad \text{with} \quad \ker(\psi_1) = K_1.$$

Define an isomorphism $\alpha\colon G/G^0 \to \mathfrak{S}_3$ via $\alpha(\overline{\tau}) := (1,3,2)$ and $\alpha(\overline{h}) := (2,3)$. According to Proposition 2.2.4 iii) there is an embedding $i_\tau\colon G \to \mathrm{Aut}(C^3)$. As in part a), the action of G on C^3 is minimal and free, whence the quotient C^3/G is a threefold isogenous to a product with algebraic datum $(G, G^0, \tau, h, K_1, V_1)$. We skip the proof of a) and c) because it is identical. □

In the proposition above, we actually construct families of threefolds. Theorem 1.1.3, which is a stronger version of Riemann's existence theorem, allows us to count the number of parameters of these families. In order to apply it, we need to give additional data namely:

i) a compact Riemann surface C_i', i.e. an oriented compact 2-dimensional topological surface without boundary endowed with a complex structure,

ii) a finite set $\mathcal{B}_i \subset C_i'$ of points and

iii) a surjective homomorphism $\pi_1\big(C_i' \setminus \mathcal{B}_i\big) \to G_i/K_i$

for $i = 1$ in the index three and index six case, for $i = 1$ and 2 in the index two case and for $i = 1$, 2 and 3 in the unmixed case. The choice of C_i' and the set of points $\mathcal{B}_i$ depend on $3g_i' - 3 + r_i$ parameters, where $r_i = |\mathcal{B}_i|$ and $g_i' = g(C_i')$. Once we choose a geometric basis for the fundamental group, the homomorphism

$$\pi_1\big(C_i' \setminus \mathcal{B}_i\big) \to G_i/K_i$$

is determined by a generating vector V_i of type $T_i = [g_i'; m_{i,1}, \ldots, m_{i,r_i}]$. The following remark is an immediate consequence this discussion.

Remark 2.3.7. The families of threefolds constructed in Proposition 2.3.6 depend on

i) $3(g_1' + g_2' + g_3') - 9 + r_1 + r_2 + r_3$ parameters in the unmixed case,

ii) $3(g_1' + g_2') - 6 + r_1 + r_2$ parameters in the index two case and

iii) $3g_1' - 3 + r_1$ parameters in the index three and index six case.

Here, the integers g_i' and r_i are given in terms of the types $T_i = [g_i'; m_{i,1}, \ldots, m_{i,r_i}]$ of the generating vectors V_i.

2.4 The Hodge Diamond

The aim of this section is to explain how to compute the *Hodge diamond* of a threefold X isogenous to a product from an algebraic datum of X. For the readers benefit we briefly recall the terminology used in *Hodge theory* and some basic properties of Hodge numbers (see for example [Hu05, Section 3.2]).

The Hodge numbers $h^{p,q}(X)$ of a projective, or more generally, a compact Kähler manifold X are defined to be the dimensions of the *Dolbeault cohomology groups*

$$H^{p,q}(X) := \frac{\ker\left(\overline{\partial}\colon \Omega^{p,q}(X) \to \Omega^{p,q+1}(X)\right)}{\overline{\partial}\ \Omega^{p,q-1}(X)}, \qquad \text{where} \qquad p, q \leq \dim(X) = n.$$

Here $\Omega^{p,q}$ denotes the sheaf of $\mathcal{C}^\infty$-forms of type (p,q) and $\overline{\partial}$ is the *Dolbeault operator*. According to the theory of *harmonic forms* there is a decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X) \qquad \text{for all} \qquad k \leq 2n.$$

Moreover, we have

$$h^{p,q}(X) = h^{q,p}(X) \qquad \text{and} \qquad h^{p,q}(X) = h^{n-p,n-q}(X),$$

where the equality on the left-hand side is induced by *complex conjugation* (on the level of harmonic forms) and the equality on the right-hand side by *Serre-duality.* The Hodge numbers can be displayed in form of a "diamond" which is called the Hodge diamond of X.

$$\begin{array}{ccccccccc}
 & & & & h^{0,0} & & & & \\
 & & & h^{1,0} & & h^{0,1} & & & \\
 & & h^{2,0} & & h^{1,1} & & h^{0,2} & & \\
 & & & & \vdots & & & & \\
h^{n,0} & h^{n-1,1} & \cdots & & & & \cdots & h^{1,n-1} & h^{0,n} \\
 & & & & \vdots & & & & \\
 & & h^{n,n-2} & & h^{n-1,n-1} & & h^{n-2,n} & & \\
 & & & h^{n,n-1} & & h^{n-1,1} & & & \\
 & & & & h^{n,n} & & & &
\end{array}$$

Let G be a group of automorphisms of X, then for all $g \in G$ there are linear maps

$$g^*\colon H^{p,q}(X) \to H^{p,q}(X), \qquad \omega \mapsto g^*\omega.$$

This fact allows us to define the subspaces

$$H^{p,q}(X)^G := \{\omega \in H^{p,q}(X) \mid g^*\omega = \omega \quad \text{for all} \quad g \in G\} \subseteq H^{p,q}(X)$$

for all $p, q \leq n$. Thanks to the next proposition, these subspaces have a particularly nice interpretation in the case where G is finite group acting freely on X.

Proposition 2.4.1. *Let X be a compact Kähler manifold and G be a finite group of automorphisms acting freely on X. Then, the quotient map $\pi\colon X \to X/G$ induces isomorphisms*

$$H^{p,q}(X/G) \simeq H^{p,q}(X)^G.$$

Proof. According to [Hat02, Proposition 3G.1] there are isomorphisms

$$\pi^*\colon H^k(X/G, \mathbb{Q}) \xrightarrow{\sim} H^k(X, \mathbb{Q})^G \qquad \text{for all} \qquad k \leq 2n.$$

Since a holomorphic map induces a morphism of Hodge structures (cf. [Voi07, Section 7.3.2]) we obtain a graded isomorphism

$$\pi^*\colon \bigoplus_{p+q=k} H^{p,q}(X/G) \xrightarrow{\sim} \Big(\bigoplus_{p+q=k} H^{p,q}(X)\Big)^G = \bigoplus_{p+q=k} H^{p,q}(X)^G$$

and the statement follows. □

In our situation, the proposition motivates the following definition.

Definition 2.4.2. *Let G be a subgroup of $\mathrm{Aut}(C_1 \times C_2 \times C_3)$, where C_i are compact Riemann surfaces of genus $g(C_i) \geq 2$. We define representations of G via pullback*

$$\phi_{p,q}\colon G \to \mathrm{GL}\left(H^{p,q}(C_1 \times C_2 \times C_3)\right), \quad g \mapsto [\omega \mapsto (g^{-1})^*\omega]$$

and denote the characters of $\phi_{p,q}$ by $\chi_{p,q}$.

Remark 2.4.3. Let $X = (C_1 \times C_2 \times C_3)/G$ be a threefold isogenous to a product. As a direct consequence of Proposition 2.4.1 the Hodge numbers of X are given by

$$h^{p,q}(X) = \langle \chi_{p,q}, \chi_{triv} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_{p,q}(g).$$

Note that the group actions $\psi_i\colon G_i \to \mathrm{Aut}(C_i)$ also induce representations

$$\varphi_i\colon G_i \to \mathrm{GL}\left(H^{1,0}(C_i)\right), \quad h \mapsto [\omega \mapsto \psi_i(h^{-1})^*\omega].$$

Our first aim is to show that the characters $\chi_{p,q}$ are completely determined by the characters χ_{φ_i} of the representations φ_i. As usual, we assume that the embedding

$$G \leq \mathrm{Aut}(C_1 \times C_2 \times C_3)$$

is given in normal form (see Proposition 2.2.2). The main tool for our computations is Künneth's formula for Dolbeault cohomology.

Proposition 2.4.4 ([GH78, p.103-104]). *There is an isomorphism*

$$H^{p,q}(C_1 \times C_2 \times C_3) \simeq \bigoplus_{\substack{s_1+s_2+s_3=p \\ t_1+t_2+t_3=q}} H^{s_1,t_1}(C_1) \otimes H^{s_2,t_2}(C_2) \otimes H^{s_3,t_3}(C_3)$$

is induced by the map

$$\omega_1 \otimes \omega_2 \otimes \omega_3 \mapsto p_1^*\omega_1 \wedge p_2^*\omega_2 \wedge p_3^*\omega_3,$$

where $p_i \colon C_1 \times C_2 \times C_3 \to C_i$ *are the natural projections.*

Theorem 2.4.5. *For the characters* $\chi_{p,q}$ *it holds:*

i) $\mathrm{Res}^G_{G^0}\left(\chi_{1,0}\right) = \chi_{\varphi_1} + \chi_{\varphi_2} + \chi_{\varphi_3}$,

ii) $\mathrm{Res}^G_{G^0}\left(\chi_{1,1}\right) = 2\Re e(\chi_{\varphi_1}\overline{\chi_{\varphi_2}} + \chi_{\varphi_1}\overline{\chi_{\varphi_3}} + \chi_{\varphi_2}\overline{\chi_{\varphi_3}}) + 3\chi_{triv}$,

iii) $\mathrm{Res}^G_{G^0}\left(\chi_{2,0}\right) = \chi_{\varphi_1}\chi_{\varphi_2} + \chi_{\varphi_1}\chi_{\varphi_3} + \chi_{\varphi_2}\chi_{\varphi_3}$,

iv) $\mathrm{Res}^G_{G^0}\left(\chi_{2,1}\right) = \overline{\chi_{\varphi_1}}\chi_{\varphi_2}\chi_{\varphi_3} + \chi_{\varphi_1}\overline{\chi_{\varphi_2}}\chi_{\varphi_3} + \chi_{\varphi_1}\chi_{\varphi_2}\overline{\chi_{\varphi_3}} + 2(\chi_{\varphi_1} + \chi_{\varphi_2} + \chi_{\varphi_3})$,

v) $\mathrm{Res}^G_{G^0}\left(\chi_{3,0}\right) = \chi_{\varphi_1}\chi_{\varphi_2}\chi_{\varphi_3}$.

Here, χ_{triv} *is the trivial character and* $\mathrm{Res}^G_{G^0}\left(\chi_{p,q}\right)$ *is the restriction of* $\chi_{p,q}$ *to* G^0.

Proof. According to Künneth's formula

$$H^{p,q}(C_1 \times C_2 \times C_3) = \bigoplus_{\substack{s_1+s_2+s_3=p \\ t_1+t_2+t_3=q}} H^{s_1,t_1}(C_1) \otimes H^{s_2,t_2}(C_2) \otimes H^{s_3,t_3}(C_3).$$

Let $\omega = \omega_1 \otimes \omega_2 \otimes \omega_3$ be a pure tensor contained in some direct summand of this decomposition. Since the action of G^0 is diagonal the tensors ω and $(g^{-1})^*\omega$ are in the same summand for all $g \in G^0$. This implies that each summand is a subrepresentation, whence the character $\chi_{p,q}$ is the sum of the characters of these subrepresentations. By definition of χ_{φ_i} and the fact that the character of a tensor product is the product of the characters, the statement follows. □

Now we turn to the mixed case, where we need to determine the values of the characters $\chi_{p,q}$ for the elements outside of G^0. Here, according to Proposition 2.2.2, the actions ψ_i

are related to each other. Since the representations χ_{φ_i} are defined in terms of ψ_i, they must be related in the same way:

Remark 2.4.6.

i) $\chi_{\varphi_3}(g) = \chi_{\varphi_2}(\delta g \delta^{-1})$ for all $g \in G^0$ in the index two case,

ii) $\chi_{\varphi_2}(g) = \chi_{\varphi_1}(\tau g \tau^{-1})$ and $\chi_{\varphi_3}(g) = \chi_{\varphi_1}(\tau^2 g \tau^{-2})$ for all $g \in G^0$ in the index three case and

iii) $\chi_{\varphi_2}(h) = \chi_{\varphi_1}(\tau h \tau^{-1})$ and $\chi_{\varphi_3}(k) = \chi_{\varphi_1}(\tau^2 k \tau^{-2})$ for all $h \in G_2$ and $k \in G_3$ in the index six case.

In the proof of the next theorem, we need a lemma from linear algebra.

Lemma 2.4.7. *Let A, B and C be endomorphisms of a finite dimensional vector space V over the field $\mathbb{K}$. Then*

i) the trace of the unique linear map

$$V \otimes V \to V \otimes V \quad \textit{with} \quad u \otimes v \mapsto Av \otimes Bu$$

is equal to the trace of $A \circ B$,

ii) the trace of the unique linear map

$$V \otimes V \otimes V \to V \otimes V \otimes V \quad \textit{with} \quad u \otimes v \otimes w \mapsto Av \otimes Bw \otimes Cu$$

is equal to the trace of $A \circ B \circ C$.

Proof. We just prove *i*), the proof of *ii*) is similar. Let $\{v_1, \dots, v_n\}$ be a basis of V. We write

$$A(v_j) = \sum_{i=1}^{n} a_{ij} v_i, \quad \text{and} \quad B(v_k) = \sum_{l=1}^{n} b_{lk} v_l,$$

where $a_{ij} \in \mathbb{K}$ and $b_{lk} \in \mathbb{K}$ are the coefficients of the matrices of A and B with respect to the basis above. From the formula

$$v_k \otimes v_j \mapsto Av_j \otimes Bv_k = \sum_{i,l} a_{ij} b_{lk} v_i \otimes v_l$$

it follows that the trace of the map $u \otimes v \mapsto Av \otimes Bu$ is the sum of the elements $a_{kj} b_{jk}$:

$$\sum_{k,l} a_{kj} b_{jk} = \sum_{k=1}^{n} \sum_{j=1}^{n} a_{kj} b_{jk}.$$

The proof is finished, because the elements $c_{kk} := \sum_{j=1}^{n} a_{kj} b_{jk}$ are precisely the diagonal entries of the matrix of $A \circ B$. □

Theorem 2.4.8. *The values of the characters $\chi_{p,q}$ for the elements outside of G^0 are displayed in the table below:*

	$(1,0)$	$(1,1)$	$(2,0)$	$(2,1)$	$(3,0)$
$\chi_{p,q}(\delta g)$	$\chi_{\varphi_1}(\delta g)$	1	$-\chi_{\varphi_2}\big((\delta g)^2\big)$	$-\overline{\chi_{\varphi_1}(\delta g)}\chi_{\varphi_2}\big((\delta g)^2\big)$	$-\chi_{\varphi_1}(\delta g)\chi_{\varphi_2}\big((\delta g)^2\big)$
$\chi_{p,q}(\tau g)$	0	0	0	0	$\chi_{\varphi_1}\big((\tau g)^3\big)$
$\chi_{p,q}(\tau^2 g)$	0	0	0	0	$\chi_{\varphi_1}\big((\tau^2 g)^3\big)$
$\chi_{p,q}(f)$	$\chi_{\varphi_1}(f)$	1	$-\chi_{\varphi_2}(f^2)$	$-\overline{\chi_{\varphi_1}(f)}\chi_{\varphi_2}(f^2)$	$-\chi_{\varphi_1}(f)\chi_{\varphi_2}(f^2)$

Here,

- *the first row holds for all $\delta g \in \delta G^0$ in the index two case,*
- *the second and third row holds for all $\tau g \in \tau G^0$ and $\tau^2 g \in \tau^2 G^0$ in the index three as well as the index six case and*
- *the last row holds for all $f \in G_1 \setminus G^0$ in the index six case.*

Remark 2.4.9. The table above gives the values of the characters $\chi_{p,q}$ for all $G \setminus G^0$. In the index two and index three case this is clear. In the index six case we use the identities

$$G_1 \setminus G^0 = \tau(G_2 \setminus G^0)\tau^{-1} = \tau^2(G_3 \setminus G^0)\tau^{-2}$$

and the fact that a character is a class function.

Proof. Via $\nu\colon G \to G/G^0 \leq \mathfrak{S}_3$, the elements in $G \setminus G^0$ can either map to a three cycle or a transposition. For this reason we will compute the values of the characters $\chi_{p,q}$ just in two cases:

a) for the elements $\tau g \in \tau G^0$, i.e. $\nu(\tau g) = (1,3,2)$ and

b) for the elements $\delta g \in \delta G^0$, i.e. $\nu(\delta g) = (2,3)$.

For the elements contained in $\tau^2 G^0$, the computation is identical to *a*) and for the elements $f \in G_1 \setminus G^0$ it is identical to *b*).

a) The inverse of an element $\tau g \in \tau G^0$ acts on C^3 via

$$(\tau g)^{-1}(x,y,z) = \big(\psi_1(g^{-1}\tau^{-3})z, \psi_1(\tau g^{-1}\tau^{-1})x, \psi_1(\tau^2 g^{-1}\tau^{-2})y\big).$$

Let $\omega = \omega_1 \otimes \omega_2 \otimes \omega_3$ be a pure tensor in $H^{s_1,t_1}(C) \otimes H^{s_2,t_2}(C) \otimes H^{s_3,t_3}(C)$, where

$$s_1 + s_2 + s_3 = p \qquad \text{and} \qquad t_1 + t_2 + t_3 = q.$$

Under Künneth's isomorphism ω maps to $p_1^*\omega_1 \wedge p_2^*\omega_2 \wedge p_3^*\omega_3$. The pullback of this element via $(\tau g)^{-1}$ is:

$$\pm\, p_1^*\psi_1(\tau g^{-1}\tau^{-1})^*\omega_2 \wedge p_2^*\psi_1(\tau^2 g^{-1}\tau^{-2})^*\omega_3 \wedge p_3^*\psi_1(g^{-1}\tau^{-3})^*\omega_1,$$

where the sign depends on the degrees of the classes ω_i. The corresponding tensor

$$\pm\, \psi_1(\tau g^{-1}\tau^{-1})^*\omega_2 \otimes \psi_1(\tau^2 g^{-1}\tau^{-2})^*\omega_3 \otimes \psi_1(g^{-1}\tau^{-3})^*\omega_1$$

is an element in

$$H^{s_2,t_2}(C) \otimes H^{s_3,t_3}(C) \otimes H^{s_1,t_1}(C).$$

Hence ω and $\big((\tau g)^{-1}\big)^*\omega$ are contained in different direct summands for all pairs

$$(p,q) \in \big\{(1,0),(1,1),(2,0),(2,1)\big\}.$$

This implies that the traces of the linear maps

$$\big((\tau g)^{-1}\big)^* \colon H^{p,q}(C^3) \to H^{p,q}(C^3)$$

are equal to zero for these pairs. In other words $\chi_{p,q}(\tau g) = 0$. In the case $(p,q) = (3,0)$, the forms ω_i are all of type $(1,0)$, consequently the sign in the formula for the pullback of ω is $+1$ and there is only one summand in the decomposition of $H^{3,0}(C^3)$. According to the definition of the representations φ_i it holds

$$\big((\tau g)^{-1}\big)^*\omega = \varphi_1(\tau g\tau^{-1})\omega_2 \otimes \varphi_1(\tau^2 g\tau^{-2})^*\omega_3 \otimes \varphi_1(\tau^3 g)^*\omega_1.$$

We apply Lemma 2.4.7 with $A := \varphi_1(\tau g\tau^{-1})$, $B := \varphi_1(\tau^2 g\tau^{-2})$ and $C := \varphi_1(\tau^3 g)$ and conclude

$$\chi_{3,0}(\tau g) = \operatorname{tr}(ABC) = \operatorname{tr}\big(\varphi_1(\tau g))^3\big) = \chi_{\varphi_1}\big((\tau g)^3\big).$$

b) Consider an element $\delta g \in \delta G^0$ and a pure tensor $\omega = \omega_1 \otimes \omega_2 \otimes \omega_3$ in

$$H^{s_1,t_1}(D) \otimes H^{s_2,t_2}(C) \otimes H^{s_3,t_3}(C) \subset H^{p,q}(D \times C^2).$$

The pullback of ω via $(\delta g)^{-1}$ is

$$\big((\delta g)^{-1}\big)^*\omega \quad = \quad \pm\, \psi_1(g^{-1}\delta^{-1})^*\omega_1 \otimes \psi_2(\delta g^{-1}\delta^{-1})^*\omega_3 \otimes \psi_2(g^{-1}\delta^{-2})^*\omega_2.$$

It is a tensor in

$$H^{s_1,t_1}(D) \otimes H^{s_3,t_3}(C) \otimes H^{s_2,t_2}(C).$$

For all pairs (p,q) in the table below, there is exactly one direct summand of $H^{p,q}(D\times C^2)$ containing both ω and $\big((\delta g)^{-1}\big)^*\omega$. Hence, the trace of $\big((\delta g)^{-1}\big)^*$ is equal to the trace of the restriction of $\big((\delta g)^{-1}\big)^*$ to this invariant direct summand. Using Lemma 2.4.7 in the same way as above, we obtain the following table:

(p,q)	invariant summand	$\chi_{p,q}(\delta g)$
$(1,0)$	$H^{1,0}(D)\otimes H^{0,0}(C)\otimes H^{0,0}(C)$	$\chi_{\varphi_1}(\delta g)$
$(1,1)$	$H^{1,1}(D)\otimes H^{0,0}(C)\otimes H^{0,0}(C)$	1
$(2,0)$	$H^{0,0}(D)\otimes H^{1,0}(C)\otimes H^{1,0}(C)$	$-\chi_{\varphi_2}\big((\delta g)^2\big)$
$(2,1)$	$H^{0,1}(D)\otimes H^{1,0}(C)\otimes H^{1,0}(C)$	$-\overline{\chi_{\varphi_1}(\delta g)}\chi_{\varphi_2}\big((\delta g)^2\big)$
$(3,0)$	$H^{1,0}(D)\otimes H^{1,0}(C)\otimes H^{1,0}(C)$	$-\chi_{\varphi_1}(\delta g)\chi_{\varphi_2}\big((\delta g)^2\big)$

□

Remark 2.4.10. Let X be a threefold isogenous to a product. The group actions

$$\overline{\psi_i}\colon G_i/K_i \to \mathrm{Aut}(C_i)$$

induce representations

$$\overline{\varphi_i}\colon G_i/K_i \to \mathrm{GL}\left(H^{1,0}(C_i)\right),$$

whose characters $\chi_{\overline{\varphi_i}}$ can be determined from an algebraic datum of X with the help of the Chevalley-Weil formula (Theorem 1.3.3). Clearly, the composition of the quotient map $G_i \to G_i/K_i$ and the character $\chi_{\overline{\varphi_i}}$ is equal to χ_{φ_i}. According to Remark 2.4.3 the Hodge numbers of X are given by

$$h^{p,q}(X) = \frac{1}{|G|}\sum_{g\in G}\chi_{p,q}(g),$$

where the characters $\chi_{p,q}$ are determined by the characters χ_{φ_i} according to Theorem 2.4.5 and Theorem 2.4.8.

Chapter 3

Combinatorics, Bounds and Algorithms

Given a threefold isogenous to a product $X = (C_1 \times C_2 \times C_3)/G$, we consider the following numerical information:

- the group order $n := |G|$,
- the orders k_i of the kernels K_i of $\psi_i \colon G_i \to \operatorname{Aut}(C_i)$ and
- the types $T_i = [g_i'; m_{i,1}, \ldots, m_{i,r_i}]$ (see Theorem 1.1.7) of the corresponding Galois covers

$$C_i \to C_i/\overline{G_i}, \quad \text{where} \quad \overline{G_i} := G_i/K_i.$$

Note that the collection above determines the genera $g_i := g(C_i)$ via Hurwitz' formula

$$g_i = \frac{|G_i|}{2k_i}\left(2g_i' - 2 + \sum_{j=1}^{r_i} \frac{m_{i,j} - 1}{m_{i,j}}\right) + 1,$$

and therefore also the invariants $\chi(\mathcal{O}_X)$, $e(X)$ and K_X^3 of the threefold X (cf. Proposition 2.1.2). However, for threefolds of mixed type, some of the information above is redundant: in the index two case $k_2 = k_3$ and $T_2 = T_3$, whereas in the index three and six case $k_1 = k_2 = k_3$ and $T_1 = T_2 = T_3$. Therefore, the following definition is convenient.

Definition 3.0.1. *The* numerical datum *of a threefold X isogenous to a product is the tuple*

- $\mathcal{D} := (n, k_1, k_2, k_3, T_1, T_2, T_3)$ *in the unmixed case,*
- $\mathcal{D} := (n, k_1, k_2, T_1, T_2)$ *in the index two case,*
- $\mathcal{D} := (n, k_1, T_1)$ *in the index three and index six case.*

In the absolutely faithful case $k_i = 1$ for all $1 \leq i \leq 3$. Here, as a convention, we omit writing the $k_i's$.

Note that an algebraic datum $\mathcal{A}$ of X (see Definition 2.3.5) determines the numerical datum $\mathcal{D}$ of X. In this case we say that the numerical datum $\mathcal{D}$ is *realized by the algebraic datum* $\mathcal{A}$.

In this section we derive *combinatorial constraints* on the numerical data. These constraints will imply that there are only finitely many possibilities for the numerical data, once the value of $\chi(\mathcal{O}_X)$ is fixed. Consequently, there can be only finitely many algebraic data, realizing these numerical data. This fact can be turned into an algorithm searching systematically through all possibilities and thereby classifying all threefolds isogenous to a product with a fixed value of $\chi(\mathcal{O}_X)$.

To have a compact notation of Hurwitz' formula, we give the following definition.

Definition 3.0.2. *To a type $T := [g'; m_1, \ldots, m_r]$, we associate the rational number*

$$\Theta(T) := 2g' - 2 + \sum_{j=1}^{r} \frac{m_j - 1}{m_j}.$$

According Hurwitz' formula $\Theta(T)$ is positive, when T is the type associated to a Galois cover of a compact Riemann surface C of genus $g(C) \geq 2$. The next remark is a well-known estimation for $\Theta(T)$.

Remark 3.0.3 (cf. [Mir95, Lemma III.3.8]). If $\Theta(T) > 0$, then $\Theta(T) \geq \Theta_{min}(T)$, where

$$\Theta_{min}(T) := \begin{cases} 1/42, & \text{if} \quad g' = 0 \\ 1/2, & \text{if} \quad g' = 1 \\ 2g' - 2, & \text{if} \quad g' \geq 2 \end{cases}$$

Moreover, $\Theta(T) = \Theta_{min}(T)$ if and only if

$$T \in \Big\{ [0; 2, 3, 7],\ [1; 2],\ [g'; -] \Big\}.$$

Definition 3.0.4. *We define the function $N_{\max} \colon \mathbb{N}_{\geq 2} \to \mathbb{N}$, where*

$$N_{\max}(g) := \max \big\{ |\operatorname{Aut}(C)| \ \big|\ C \text{ is a compact Riemann surface with } g(C) = g \big\}.$$

Remark 3.0.5. According to Hurwitz $N_{\max}(g)$ is bounded by $84(g - 1)$. In [Con14] the author provides a table listing the values of $N_{\max}(g)$ for $2 \leq g \leq 301$. According to our knowledge this paper is the most comprehensive reference.

Proposition 3.0.6. *Let* $X = \big(C_1 \times C_2 \times C_3\big)/G$ *be a threefold isogenous to a product with numerical datum* $\mathcal{D}$*. Then*

$$n \leq \left\lfloor \sqrt{-d \cdot \chi(\mathcal{O}_X) \prod_{i=1}^{3} \frac{k_i}{\Theta_{min}(T_i)}} \right\rfloor$$

where $d = 8$ *in the unmixed case,* $d = 32$ *in the index two case and* $d = 216$ *in the index three and index six case.*

Proof. Since $g_i \geq 2$, Hurwitz' formula

$$g_i - 1 = \frac{1}{2}\frac{|G_i|}{k_i}\Theta(T_i)$$

implies $\Theta(T_i) > 0$. In combination with Proposition 2.1.2 and Remark 3.0.3 we can estimate

$$-\chi(\mathcal{O}_X) = \frac{1}{n}\prod_{i=1}^{3}\big(g_i - 1\big) = \frac{1}{8n}\prod_{i=1}^{3}\frac{|G_i|}{k_i}\Theta(T_i) \geq \frac{1}{8n}\prod_{i=1}^{3}\frac{|G_i|}{k_i}\Theta_{min}(T_i).$$

Now the claim follows from Remark 2.2.1. □

In the absolutely faithful case Proposition 3.0.6 immediately yields a bound of the group order in terms of $\chi(\mathcal{O}_X)$.

Corollary 3.0.7. *Let* X *be a threefold isogenous to a product. Assume that the action is absolutely faithful. Then*

$$n \leq \lfloor 42\sqrt{-d \cdot 42\chi(\mathcal{O}_X)} \rfloor,$$

where $d = 8$ *in the unmixed case,* $d = 32$ *in the index two case and* $d = 216$ *in the index three and index six case.*

Also in the general case it is possible to show that n is bounded in terms of $\chi(\mathcal{O}_X)$: let $X = (C_1 \times C_2 \times C_3)/G$ be a threefold isogenous to a product. We consider the quotient

$$X^0 := (C_1 \times C_2 \times C_3)/G^0.$$

It is a threefold isogenous to a product of unmixed type with

$$\chi(\mathcal{O}_{X^0}) = \big|G/G^0\big|\chi(\mathcal{O}_X).$$

Therefore it suffices to give a bound in the unmixed case.

Proposition 3.0.8. *Let X be a threefold isogenous to a product of unmixed type, then*

$$n \leq 84^6 \cdot \chi(\mathcal{O}_X)^2.$$

Proof. It holds

$$-\chi(\mathcal{O}_X) = \frac{(g_1-1)}{k_3}\frac{(g_2-1)}{k_3}\frac{(g_3-1)k_3}{n}k_3 \geq \left(\frac{1}{84}\right)^3 k_3 .$$

The inequality follows from Hurwitz' bound which can be applied since K_3 acts faithfully on C_1 and C_2, by the minimality assumption, and G/K_3 acts faithfully on C_3. By symmetry, the inequality

$$-84^3\chi(\mathcal{O}_X) \geq k_i$$

also holds for $i = 1, 2$. The claim follows combining these inequalities with Proposition 3.0.6 and using $\Theta(T_i) \geq 1/42$. □

Unfortunately, even in the simplest case $\chi(\mathcal{O}_X) = -1$, the bound for the group order n is very large.

Remark 3.0.9. It would be interesting to understand if there exists a significantly better bound for n in terms of $\chi(\mathcal{O}_X)$.

Proposition 3.0.10. *Let X be a threefold isogenous to a product, with numerical datum $\mathcal{D}$. Then*

i) $k_i \mid (g_{[i+1]}-1)(g_{[i+2]}-1)$,

ii) $m_{i,j} \mid (g_{[i+1]}-1)(g_{[i+2]}-1)$,

iii) $(g_i-1) \mid \chi(\mathcal{O}_X)\dfrac{n}{k_i}$,

iv) $r_i \leq \dfrac{4d_ik_i(g_i-1)}{n} - 4g_i' + 4$,

v) $m_{i,j} \leq 4g_i + 2$,

vi) $g_i' \leq 1 - \dfrac{d_ik_i\chi(\mathcal{O}_X)}{(g_{[i+1]}-1)(g_{[i+2]}-1)} \leq 1 - d_i\chi(\mathcal{O}_X)$.

vii) $n/(k_id_i) \leq N_{\max}(g_i)$

Here, $[\cdot]$ denotes the residue mod 3 *and*

- $d_i = 1$ *for all i in the unmixed case,*
- $d_1 = 1$ *and $d_2 = d_3 = 2$ in the index two case,*
- $d_i = 3$ *for all i in the index three and index six case.*

Proof. $i-ii$) We assume that $i=1$. Let $V_1 := (h_1,\dots,h_r,a_1,b_1,\dots,a_{g_1'},b_{g_1'})$ be a generating vector of type T_1 associated to the cover

$$C_1 \to C_1/\overline{G_1}, \quad \text{where} \quad \overline{G_1} := G_1/K_1.$$

Let $\gamma_j \in G_1 \leq G$ be a representative of $h_j \in \overline{G_1}$ (if $r=0$, then we set $\gamma_j := 1_G$). By the minimality of the G-action, the subgroup $\langle\gamma_j\rangle \cdot K_1 \leq G_1$ acts faithfully on $C_2 \times C_3$. Furthermore, $\langle\gamma_j\rangle \cdot K_1$ is contained in Σ_1, therefore, it acts freely on $C_2 \times C_3$. In other words

$$S := \frac{C_2 \times C_3}{\langle\gamma_j\rangle \cdot K_1}$$

is a surface isogenous to a product with holomorphic Euler-Poincaré characteristic

$$\chi(\mathcal{O}_S) = \frac{\big(g_2-1\big)\big(g_3-1\big)}{|\langle\gamma_j\rangle \cdot K_1|}.$$

We conclude that

$$k_1 \mid \big(g_2-1\big)\big(g_3-1\big) \quad \text{and} \quad m_{1,j} = \operatorname{ord}(h_j) \mid \big(g_2-1\big)\big(g_3-1\big).$$

iii) The statement follows from part i) and Proposition 2.1.2:

$$-\chi(\mathcal{O}_X)\frac{n}{k_i} = \big(g_i-1\big)\frac{\big(g_{[i+1]}-1\big)\big(g_{[i+2]}-1\big)}{k_i}.$$

iv) is a straightforward consequence of Hurwitz' formula, using the fact $m_{i,j} \geq 2$.

v) For a cyclic group H acting faithfully on a compact Riemann surface C of genus $g(C) \geq 2$, Wiman's bound (see [Wim95]) holds:

$$|H| \leq 4g(C) + 2.$$

In particular $m_{i,j} \leq 4g_i + 2$.

vi) According to Proposition 2.1.2 and Hurwitz' formula

$$g_i' - 1 \leq \frac{\Theta(T_i)}{2} = \frac{k_i}{|G_i|}\big(g_i-1\big) = \frac{-k_i}{\big(g_{[i+1]}-1\big)\big(g_{[i+2]}-1\big)}|G/G_i|\chi(\mathcal{O}_X).$$

Now, the second inequality follows from part i).

vii) The group G_i/K_i of order $n/(d_ik_i)$ acts faithfully on C_i, hence $n/(k_id_i) \leq N_{\max}(g_i)$.

□

An immediate consequence of Proposition 3.0.8 and Proposition 3.0.10 is the following Corollary.

Corollary 3.0.11. *Let χ be an integer.*

a) If $\chi \geq 0$ there are no threefolds X isogenous to a product with $\chi(\mathcal{O}_X) = \chi$.

b) If $\chi \leq -1$ there are only finitely many algebraic data of threefolds X isogenous to a product with $\chi(\mathcal{O}_X) = \chi$.

A trivial but useful consequence of Proposition 2.1.2 is:

Remark 3.0.12.

a) In the index two case $g_2 = \sqrt{\dfrac{-n \cdot \chi(\mathcal{O}_X)}{g_1 - 1}} + 1$.

b) In the index three and index six case $g_1 = \sqrt[3]{-n \cdot \chi(\mathcal{O}_X)} + 1$.

The combinatorial constraints that we found enable us to give an algorithm to classify threefolds isogenous to a product with a fixed value of $\chi(\mathcal{O}_X)$. Since the bound for the group order is very large (cf. Proposition 3.0.8), a complete classification, even with the help of a computer and just for small values of $\chi(\mathcal{O}_X)$, seems to be out of reach. On the other hand, if the group action is assumed to be absolutely faithful, then the bound drops significantly and a full classification, at least in the boundary case $\chi(\mathcal{O}_X) = -1$, is possible. For this reason, we restrict ourselves to the absolutely faithful case. The exact strategy, that we follow in our algorithm, differs slightly according to the index of G^0 in G. Our MAGMA implementation is based on the code given in [BCGP12, Appendix]. We point out that the program relies heavily on *MAGMA's Database of Small Groups*, which contains:

- all groups of order up to 2000, excluding the groups of order 1024,
- the groups whose order is a product of at most 3 primes,
- the groups of order dividing p^6 for p prime,
- the groups of order $p^k q$, where p^k is a prime-power dividing 2^8, 3^6, 5^5 or 7^4 and q is a prime different from p.

Since the full code is long, we give it in Appendix A. Here we just explain the strategy:

Input: A value χ for the holomorphic Euler-Poincaré-characteristic.

Part 1: In the first part we determine the set of *admissible numerical data*. This is the finite set of tuples of the form

- (n, T_1, T_2, T_3) in the unmixed case,
- (n, T_1, T_2) in the index two case and
- (n, T_1) in the index three and index six case,

such that the combinatorial constraints form Proposition 3.0.10 and Remark 3.0.12, the inequality from Proposition 3.0.6 and Hurwitz' formula are satisfied.

Note that the set of numerical data of threefolds isogenous to a product with $\chi(\mathcal{O}_X) = \chi$ is a subset of the set of admissible numerical data.

In our implementation, this computation is performed by the functions `AdNDunmixed`, `AdNDindexTwo`, `AdNDindexThree` and `AdNDindexSix` in the respective cases. The functions just return the set of admissible numerical data such that the groups of order n in the unmixed case, $n/2$ in the index two case and $n/3$ in the index three and index six case are contained in the Database of Small Groups. The exceptions are stored in the files `ExcepUnmixed`χ`.txt`, `ExcepIndexTwo`χ`.txt`, `ExcepIndexThree`χ`.txt` and `ExcepIndexSix`χ`.txt`.

Part 2: In the second part of the algorithm, we search for algebraic data.

Unmixed case:

For each 4-tuple (n, T_1, T_2, T_3) contained in the set `AdNDunmixed(`χ`)` [1] search through the groups G of order n for groups admitting at least one generating vector of type T_1, one of type T_2 and one of type T_3. For such groups compute all generating vectors V_i of type T_i and consider all possible triples (V_1, V_2, V_3). If the associated stabilizer sets Σ_i fulfill the condition

$$\Sigma_1 \ \cap \ \Sigma_2 \ \cap \ \Sigma_3 = \{1_G\}$$

(cf. Proposition 2.3.2 a)), then there exists a threefold X isogenous to a product with algebraic datum (G, V_1, V_2, V_3) and $\chi(\mathcal{O}_X) = \chi$ (see Proposition 2.3.6). Compute the Hodge diamond of X with the method described in Section 2.4 and save the occurrence

$$[G, T_1, T_2, T_3, h^{3,0}, h^{2,0}, h^{1,0}, h^{1,1}, h^{1,2}]$$

in the file `unmixed`χ`.txt`. The classification is performed calling `ClassifyUnmixed(`χ`)`.

Index two case:

Step 1: Starting from the triples (n, T_1, T_2) contained in the set `AdNDindexTwo(`χ`)`, compute the set of 4-tuples (n, T_1, T_2, H), where H is a group of order $n/2$ admitting at least one generating vector of type T_2.

In our implementation, this computation is performed by the function `NDHIndexTwo`. The set of 4-tuples (n, T_1, T_2, H) such that the groups of order n are contained in the Database of Small Groups is returned. The remaining tuples are stored in the file `ExcepIndexTwo`χ`.txt`.

[1] Here we mean the set of admissible numerical data which is returned after applying the function `AdNDunmixed` to the value χ.

Step 2: For each integer n belonging to some 4-tuple in the set `NDHIndexTwo(`χ`)` consider the groups of order n. For each group G of order n construct the list of subgroups of index two. For each G^0 in this list consider the 4-tuples (n, T_1, T_2, H) from Step 1 such that $H \simeq G^0$. For each of this 4-tuples compute the set of generating vectors V_1 for G of type T_1 and the set of generating vectors V_2 for G^0 of type T_2. Check the freeness conditions *i)* and *ii)* of Proposition 2.3.2 *b)*. If they are fulfilled, then there exists a threefold X isogenous to a product with algebraic datum (G, G^0, V_1, V_2) and $\chi(\mathcal{O}_X) = \chi$ (see Proposition 2.3.6). Compute the Hodge diamond of X and save the occurrence

$$[G, T_1, T_2, h^{3,0}, h^{2,0}, h^{1,0}, h^{1,1}, h^{1,2}]$$

in the file `IndexTwo`χ`.txt`. Step 2 is performed calling `ClassifyIndexTwo(`χ`)`.

Index three case:

Step 1: Starting from the pairs (n, T_1) contained in the set `AdNDindexThree(`χ`)`, compute the set of triples (n, T_1, H), where H is a group of order $n/3$ admitting three generating vectors V_1,V_1' and V_1'' of type T_1 such that the associated stabilizer sets Σ_1, Σ_1' and Σ_1'' fulfill the condition

$$\Sigma_1 \ \cap \ \Sigma_1' \ \cap \ \Sigma_1'' = \{1_H\}.$$

Here we use the fact that a threefold isogenous to a product of mixed type with numerical datum (n, T_1) is covered by a threefold of unmixed type with numerical datum $(n/3, T_1, T_1, T_1)$.

In our implementation, this computation is performed by the function `NDHIndexThree`. The set of triples (n, T_1, H) such that the groups of order n are contained in the Database of Small Groups is returned. The remaining triples are stored in the file `ExcepIndexThree`χ`.txt`.

Step 2: For each integer n belonging to a triple from Step 1 consider the groups of order n. For each group G of order n construct the list of normal subgroups G^0 of index three such that the short exact sequence

$$1 \to G^0 \to G \to \mathfrak{A}_3 \to 1$$

does not split. For each G^0 in this list consider the triples (n, T_1, H) from Step 1 such that $H \simeq G^0$. For each of these 4-tuples choose an element $\tau \in G \setminus G^0$ and compute all generating vectors V_1 for G^0 of type T_1. Check the freeness condition *i)* of Proposition 2.3.2 *c)*. If it holds, then the second condition of the proposition is also fulfilled, since the sequence

$$1 \to G^0 \to G \to \mathfrak{A}_3 \to 1$$

is non-split which is an equivalent condition according to Proposition 2.3.3. Therefore, there exists a threefold X isogenous to a product with algebraic datum (G, G^0, τ, V_1) and $\chi(\mathcal{O}_X) = \chi$ (see Proposition 2.3.6). Compute the Hodge diamond of X and save the occurrence

$$[G, T_1, h^{3,0}, h^{2,0}, h^{1,0}, h^{1,1}, h^{1,2}]$$

in the file `IndexThree`χ`.txt`. Step 2 is performed calling `ClassifyIndexThree(`χ`)`.

<u>**Index six case:**</u>

Step 1: Starting from the pairs (n, T_1) contained in the set `AdNDindexSix(`χ`)`, compute the set of triples (n, T_1, H), where H is a group of order $n/3$ admitting a generating vector V_1 of type T_1.

In our implementation, this computation is performed by the function `NDHIndexSix`. The set of triples (n, T_1, H) such that the groups of order n are contained in the Database of Small Groups is returned. The remaining triples are stored in the file `ExcepIndexSix`χ`.txt`.

Step 2: For each integer n belonging to a triple from Step 1 consider the list of groups of order n. For each group G of order n, consider the triples of the form (n, T_1, H) such that G admits a subgroup of index three isomorphic to H. Compute the set of normal subgroups G^0 of G of index six such that the short exact sequence

$$1 \to G^0 \to G \to \mathfrak{S}_3 \to 1$$

does not split. Choose elements $\tau, h \in G \setminus G^0$ such that $\tau^2 \notin G^0$ and $h^2 \in G^0$. If the group $G_1 := G^0 \cup h \cdot G^0$ is isomorphic to H, then compute all generating vectors V_1 of type T_1 for this group. For each of these vectors compute the associated stabilizer set Σ_1 and check the freeness conditions *i*), *ii*) and *iii*) of Proposition 2.3.2 *d*). If they are fulfilled, then there exists a threefold X isogenous to a product with algebraic datum (G, G^0, τ, h, V_1) and $\chi(\mathcal{O}_X) = \chi$ (see Proposition 2.3.6). Compute the Hodge diamond of X and save the occurrence

$$[G, T_1, h^{3,0}, h^{2,0}, h^{1,0}, h^{1,1}, h^{1,2}]$$

in the file `IndexSix`χ`.txt`. Step 2 is performed calling `ClassifyIndexSix(`χ`)`.

Computational Remark 3.0.13.

- In Part 2 of the algorithm we search for generating vectors. We point out that different generating vectors may determine threefolds with the same invariants. For example, this happens if (but not only if) they differ by some *Hurwitz moves*. These moves are described in [CLP15], [Zim87] and [Pe10] and we refer to these sources for further details.

- Note that for a generating vector of type $[g'; -]$ the corresponding stabilizer set is trivial and the corresponding character χ_φ is the sum of the trivial character and $(g'-1)$ copies of the regular character (cf. Example 1.3.4). Consequently, in this case it is sufficient for us to know the existence of a generating vector, but there is no need to compute all of them.

Main Computation

We execute the implementation for the input value $\chi = -1$. Note that the combinatorial constraints in Part 1 of the program are very strong, so relatively few admissible numerical data are returned. The total number of admissible group orders turns out to be relatively small and the maximum possible group order drops significantly compared to the theoretical bound from Corollary 3.0.7.

The table below summarizes the occurrences:

	unmixed	index two	index three	index six
No. AdNumData	672	253	8	5
No. G-Orders	38	39	2	1
n_{max}	504	576	216	216
n_{theo}	769	1539	4000	4000

In the first row we report the total number of admissible numerical data, in the second row the total number of group orders, in the third row the maximum possible group order after performing Part 1 of the algorithm and in the last row the theoretical bound for the group order according to Corollary 3.0.7. There are no exceptional numerical data to be considered, i.e. the files `ExcepUnmixed`χ`.txt`, `ExcepIndexTwo`χ`.txt`, `ExcepIndexThree`χ`.txt` and `ExcepIndexSix`χ`.txt` remain empty. The table below reports the computation time to run the complete program (Part 1 and Part 2) on a 8×2.5GHz Intel Xenon L5420 workstation with 16GB RAM in the respective cases:

	unmixed	index two	index three	index six
time	11h 35min	10h 28min	24 sec	30 sec

This computation yields our main result: the classification of threefolds isogenous to a product with $\chi(\mathcal{O}_X) = -1$ and absolutely faithful G-action. For the definition of the groups occurring in the tables below, we refer to Notation 0.0.1.

In the unmixed case we have the following theorem.

Theorem (A). *Let $X = (C_1 \times C_2 \times C_3)/G$ be a threefold isogenous to a product of curves of unmixed type. Assume that the action of G is absolutely faithful and $\chi(\mathcal{O}_X) = -1$. Then, the tuple*

$$[G, T_1, T_2, T_3, h^{3,0}(X), h^{2,0}(X), h^{1,0}(X), h^{1,1}(X), h^{1,2}(X), d]$$

appears in the table below. Conversely, each row is realized by at least one family of threefolds isogenous to a product of curves of unmixed type with $\chi(\mathcal{O}_X) = -1$, which depends on d parameters and is obtained by an absolutely faithful G-action.

No.	G	Id	T_1	T_2	T_3	$h^{3,0}$	$h^{2,0}$	$h^{1,0}$	$h^{1,1}$	$h^{1,2}$	d
1	$\{1\}$	$\langle 1,1\rangle$	$[2;-]$	$[2;-]$	$[2;-]$	8	12	6	27	36	9
2	$\mathbb{Z}_2$	$\langle 2,1\rangle$	$[0;2^6]$	$[0;2^6]$	$[2;-]$	8	8	2	19	28	9
3	$\mathbb{Z}_2$	$\langle 2,1\rangle$	$[0;2^6]$	$[1;2^2]$	$[2;-]$	6	7	3	17	24	8
4	$\mathbb{Z}_2$	$\langle 2,1\rangle$	$[1;2^2]$	$[1;2^2]$	$[2;-]$	6	8	4	19	26	7
5	$\mathbb{Z}_3$	$\langle 3,1\rangle$	$[0;3^4]$	$[0;3^4]$	$[2;-]$	6	6	2	15	22	5
6	$\mathbb{Z}_2^2$	$\langle 4,2\rangle$	$[0;2^5]$	$[0;2^5]$	$[2;-]$	6	6	2	15	22	7
7	$\mathbb{Z}_2^2$	$\langle 4,2\rangle$	$[0;2^5]$	$[0;2^5]$	$[2;-]$	5	5	2	13	19	7
8	$\mathbb{Z}_2^2$	$\langle 4,2\rangle$	$[0;2^5]$	$[0;2^6]$	$[1;2^2]$	5	4	1	11	17	7
9	$\mathbb{Z}_2^2$	$\langle 4,2\rangle$	$[0;2^5]$	$[0;2^6]$	$[1;2^2]$	6	5	1	13	20	7
10	$\mathbb{Z}_2^2$	$\langle 4,2\rangle$	$[0;2^5]$	$[1;2^2]$	$[1;2^2]$	4	4	2	11	16	6
11	$\mathbb{Z}_2^2$	$\langle 4,2\rangle$	$[0;2^5]$	$[1;2^2]$	$[1;2^2]$	5	5	2	13	19	6
12	$\mathbb{Z}_4$	$\langle 4,1\rangle$	$[0;2^2,4^2]$	$[0;2^2,4^2]$	$[2;-]$	6	6	2	15	22	5
13	$\mathbb{Z}_5$	$\langle 5,1\rangle$	$[0;5^3]$	$[0;5^3]$	$[2;-]$	4	4	2	15	20	3
14	$\mathbb{Z}_5$	$\langle 5,1\rangle$	$[0;5^3]$	$[0;5^3]$	$[2;-]$	5	5	2	13	19	3
15	$\mathbb{Z}_5$	$\langle 5,1\rangle$	$[0;5^3]$	$[0;5^3]$	$[2;-]$	6	6	2	11	18	3
16	$\mathfrak{S}_3$	$\langle 6,1\rangle$	$[0;2^6]$	$[0;2^2,3^2]$	$[1;3]$	4	3	1	9	14	5
17	$\mathfrak{S}_3$	$\langle 6,1\rangle$	$[0;2^2,3^2]$	$[0;2^2,3^2]$	$[2;-]$	5	5	2	13	19	5
18	$\mathbb{Z}_6$	$\langle 6,2\rangle$	$[0;2^2,3^2]$	$[0;2^2,3^2]$	$[2;-]$	6	6	2	15	22	5
19	$\mathbb{Z}_6$	$\langle 6,2\rangle$	$[0;2^2,3^2]$	$[0;3,6^2]$	$[2;-]$	5	5	2	13	19	4
20	$\mathfrak{S}_3$	$\langle 6,1\rangle$	$[0;2^2,3^2]$	$[1;2^2]$	$[1;3]$	4	4	2	11	16	4
21	$\mathbb{Z}_6$	$\langle 6,2\rangle$	$[0;3,6^2]$	$[0;3,6^2]$	$[2;-]$	4	4	2	15	20	3
22	$\mathbb{Z}_6$	$\langle 6,2\rangle$	$[0;3,6^2]$	$[0;3,6^2]$	$[2;-]$	6	6	2	11	18	3
23	$\mathbb{Z}_2^3$	$\langle 8,5\rangle$	$[0;2^5]$	$[0;2^5]$	$[0;2^5]$	4	2	0	7	12	6
24	$\mathbb{Z}_2^3$	$\langle 8,5\rangle$	$[0;2^5]$	$[0;2^5]$	$[0;2^5]$	5	3	0	9	15	6
25	$\mathcal{D}_4$	$\langle 8,3\rangle$	$[0;2^6]$	$[0;2^3,4]$	$[1;2]$	4	3	1	9	14	5
26	$\mathcal{D}_4$	$\langle 8,3\rangle$	$[0;2^3,4]$	$[0;2^3,4]$	$[2;-]$	5	5	2	13	19	5
27	$\mathcal{D}_4$	$\langle 8,3\rangle$	$[0;2^3,4]$	$[0;2^2,4^2]$	$[1;2^2]$	4	3	1	9	14	4
28	$\mathcal{D}_4$	$\langle 8,3\rangle$	$[0;2^3,4]$	$[1;2]$	$[1;2^2]$	4	4	2	11	16	4
29	$\mathbb{Z}_8$	$\langle 8,1\rangle$	$[0;2,8^2]$	$[0;2,8^2]$	$[2;-]$	4	4	2	15	20	3
30	$\mathbb{Z}_8$	$\langle 8,1\rangle$	$[0;2,8^2]$	$[0;2,8^2]$	$[2;-]$	6	6	2	11	18	3
31	Q	$\langle 8,4\rangle$	$[0;4^3]$	$[0;4^3]$	$[2;-]$	5	5	2	13	19	3
32	$\mathbb{Z}_{10}$	$\langle 10,2\rangle$	$[0;2,5,10]$	$[0;2,5,10]$	$[2;-]$	4	4	2	15	20	3
33	$\mathbb{Z}_{10}$	$\langle 10,2\rangle$	$[0;2,5,10]$	$[0;2,5,10]$	$[2;-]$	6	6	2	11	18	3
34	$\mathbb{Z}_{10}$	$\langle 10,2\rangle$	$[0;2,5,10]$	$[0;2,5,10]$	$[2;-]$	5	5	2	13	19	3
35	$\mathcal{D}_6$	$\langle 12,4\rangle$	$[0;2^5]$	$[0;2^3,3]$	$[1;3]$	4	3	1	9	14	4
36	$\mathcal{D}_6$	$\langle 12,4\rangle$	$[0;2^3,3]$	$[0;2^3,3]$	$[2;-]$	5	5	2	13	19	5
37	$\mathcal{D}_6$	$\langle 12,4\rangle$	$[0;2^3,3]$	$[0;2^3,6]$	$[1;2^2]$	4	3	1	9	14	4
38	$\mathbb{Z}_3\times\mathbb{Z}_2^2$	$\langle 12,5\rangle$	$[0;2,6^2]$	$[0;2,6^2]$	$[2;-]$	4	4	2	15	20	3
39	$\mathbb{Z}_3\times\mathbb{Z}_2^2$	$\langle 12,5\rangle$	$[0;2,6^2]$	$[0;2,6^2]$	$[2;-]$	4	4	2	13	18	3
40	$\mathbb{Z}_3\times\mathbb{Z}_2^2$	$\langle 12,5\rangle$	$[0;2,6^2]$	$[0;2,6^2]$	$[2;-]$	5	5	2	11	17	3
41	$\mathbb{Z}_3\times\mathbb{Z}_2^2$	$\langle 12,5\rangle$	$[0;2,6^2]$	$[0;2,6^2]$	$[2;-]$	6	6	2	11	18	3
42	$Dic12$	$\langle 12,1\rangle$	$[0;3,4^2]$	$[0;3,4^2]$	$[2;-]$	5	5	2	13	19	3
43	$\mathcal{D}_4\times\mathbb{Z}_2$	$\langle 16,11\rangle$	$[0;2^5]$	$[0;2^3,4]$	$[0;2^3,4]$	4	2	0	7	12	4
44	$\mathcal{D}_4\times\mathbb{Z}_2$	$\langle 16,11\rangle$	$[0;2^5]$	$[0;2^3,4]$	$[0;2^3,4]$	3	1	0	5	9	4
45	$SD16$	$\langle 16,8\rangle$	$[0;2,4,8]$	$[0;2,4,8]$	$[2;-]$	4	4	2	13	18	3
46	$SD16$	$\langle 16,8\rangle$	$[0;2,4,8]$	$[0;2,4,8]$	$[2;-]$	5	5	2	11	17	3
47	$\mathfrak{S}_4$	$\langle 24,12\rangle$	$[0;2^3,4]$	$[0;2^2,3^2]$	$[0;3,4^2]$	3	1	0	5	9	2
48	$\mathbb{Z}_3\rtimes_\varphi\mathcal{D}_4$	$\langle 24,8\rangle$	$[0;2,4,6]$	$[0;2,4,6]$	$[2;-]$	4	4	2	13	18	3
49	$\mathbb{Z}_3\rtimes_\varphi\mathcal{D}_4$	$\langle 24,8\rangle$	$[0;2,4,6]$	$[0;2,4,6]$	$[2;-]$	5	5	2	11	17	3
50	$\mathrm{SL}(2,\mathbb{F}_3)$	$\langle 24,3\rangle$	$[0;3^2,4]$	$[0;3^2,4]$	$[2;-]$	5	5	2	13	19	3

No.	G	Id	T_1	T_2	T_3	$h^{3,0}$	$h^{2,0}$	$h^{1,0}$	$h^{1,1}$	$h^{1,2}$	d
51	$\mathfrak{S}_4 \times \mathbb{Z}_2$	$\langle 48, 48\rangle$	$[0; 2^5]$	$[0; 2, 4, 6]$	$[0; 2, 4, 6]$	3	1	0	5	9	2
52	$\mathrm{GL}(2, \mathbb{F}_3)$	$\langle 48, 29\rangle$	$[0; 2, 3, 8]$	$[0; 2, 3, 8]$	$[2; -]$	4	4	2	13	18	3
53	$\mathrm{GL}(2, \mathbb{F}_3)$	$\langle 48, 29\rangle$	$[0; 2, 3, 8]$	$[0; 2, 3, 8]$	$[2; -]$	5	5	2	11	17	3
54	$\mathfrak{A}_5$	$\langle 60, 5\rangle$	$[0; 2^3, 3]$	$[0; 2, 5^2]$	$[0; 3^2, 5]$	2	0	0	3	6	1

In the index two case we have the following theorem.

Theorem (B). *Let X be a threefold isogenous to a product of curves of mixed type. Assume that the action of G is absolutely faithful, $\chi(\mathcal{O}_X) = -1$ and the index of G^0 in G is two. Then, the tuple*

$$[G, T_1, T_2, h^{3,0}(X), h^{2,0}(X), h^{1,0}(X), h^{1,1}(X), h^{1,2}(X), d]$$

appears in the table below. Conversely, each row in the table is realized by at least one family of threefolds, which depends on d parameters and is obtained by an absolutely faithful G-action.

No.	G	Id	T_1	T_2	$h^{3,0}$	$h^{2,0}$	$h^{1,0}$	$h^{1,1}$	$h^{1,2}$	d
1	$\mathbb{Z}_2$	$\langle 2, 1\rangle$	$[2; -]$	$[2; -]$	5	7	4	18	24	6
2	$\mathbb{Z}_4$	$\langle 4, 1\rangle$	$[0; 2^2, 4^2]$	$[2; -]$	4	4	2	11	16	4
3	$\mathbb{Z}_4$	$\langle 4, 1\rangle$	$[2; -]$	$[0; 2^6]$	7	7	2	14	22	6
4	$\mathbb{Z}_4$	$\langle 4, 1\rangle$	$[2; -]$	$[1; 2^2]$	5	6	3	14	20	5
5	$\mathbb{Z}_2^2$	$\langle 4, 2\rangle$	$[2; -]$	$[0; 2^6]$	5	5	2	14	20	6
6	$\mathbb{Z}_2^2$	$\langle 4, 2\rangle$	$[2; -]$	$[1; 2^2]$	4	5	3	14	19	5
7	$\mathfrak{S}_3$	$\langle 6, 1\rangle$	$[2; -]$	$[0; 3^4]$	4	4	2	12	17	4
8	$\mathbb{Z}_6$	$\langle 6, 2\rangle$	$[2; -]$	$[0; 3^4]$	5	5	2	12	18	4
9	$\mathbb{Z}_8$	$\langle 8, 1\rangle$	$[2; -]$	$[0; 2^2, 4^2]$	5	5	2	12	18	4
10	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\langle 8, 2\rangle$	$[2; -]$	$[0; 2^2, 4^2]$	5	5	2	12	18	4
11	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\langle 8, 2\rangle$	$[0; 2^2, 4^2]$	$[1; 2^2]$	3	2	1	7	11	3
12	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\langle 8, 2\rangle$	$[2; -]$	$[0; 2^5]$	5	5	2	12	18	5
13	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\langle 8, 2\rangle$	$[2; -]$	$[0; 2^5]$	6	6	2	12	19	5
14	$\mathcal{D}_4$	$\langle 8, 3\rangle$	$[2; -]$	$[0; 2^2, 4^2]$	4	4	2	12	17	4
15	$\mathcal{D}_4$	$\langle 8, 3\rangle$	$[0; 2^2, 4^2]$	$[1; 2^2]$	3	2	1	8	12	3
16	$\mathcal{D}_4$	$\langle 8, 3\rangle$	$[1; 2]$	$[0; 2^6]$	3	2	1	8	12	4
17	$\mathcal{D}_4$	$\langle 8, 3\rangle$	$[1; 2]$	$[1; 2^2]$	2	2	2	8	11	3
18	$\mathcal{D}_4$	$\langle 8, 3\rangle$	$[2; -]$	$[0; 2^5]$	4	4	2	11	16	5
19	$\mathcal{D}_4$	$\langle 8, 3\rangle$	$[2; -]$	$[0; 2^5]$	5	5	2	12	18	5
20	Q	$\langle 8, 4\rangle$	$[2; -]$	$[0; 2^2, 4^2]$	6	6	2	12	19	4
21	$\mathbb{Z}_2^3$	$\langle 8, 5\rangle$	$[2; -]$	$[0; 2^5]$	4	4	2	12	17	5
22	$\mathcal{D}_5$	$\langle 10, 1\rangle$	$[2; -]$	$[0; 5^3]$	4	4	2	10	15	3
23	$\mathbb{Z}_{10}$	$\langle 10, 2\rangle$	$[2; -]$	$[0; 5^3]$	4	4	2	12	17	3
24	$Dic12$	$\langle 12, 1\rangle$	$[2; -]$	$[0; 2^2, 3^2]$	6	6	2	12	19	4
25	$Dic12$	$\langle 12, 1\rangle$	$[2; -]$	$[0; 3, 6^2]$	5	5	2	10	16	3
26	$\mathbb{Z}_{12}$	$\langle 12, 2\rangle$	$[2; -]$	$[0; 2^2, 3^2]$	5	5	2	12	18	4
27	$\mathbb{Z}_{12}$	$\langle 12, 2\rangle$	$[2; -]$	$[0; 3, 6^2]$	4	4	2	12	17	3
28	$\mathcal{D}_6$	$\langle 12, 4\rangle$	$[2; -]$	$[0; 2^2, 3^2]$	4	4	2	12	17	4
29	$\mathcal{D}_6$	$\langle 12, 4\rangle$	$[2; -]$	$[0; 3, 6^2]$	4	4	2	10	15	3
30	$\mathcal{D}_6$	$\langle 12, 4\rangle$	$[2; -]$	$[0; 2^2, 3^2]$	4	4	2	11	16	4
31	$\mathbb{Z}_3 \times \mathbb{Z}_2^2$	$\langle 12, 5\rangle$	$[2; -]$	$[0; 2^2, 3^2]$	5	5	2	12	18	4
32	$\mathbb{Z}_3 \times \mathbb{Z}_2^2$	$\langle 12, 5\rangle$	$[2; -]$	$[0; 3, 6^2]$	4	4	2	12	17	3

No.	G	Id	T_1	T_2	$h^{3,0}$	$h^{2,0}$	$h^{1,0}$	$h^{1,1}$	$h^{1,2}$	d
33	$\mathbb{Z}_{16}$	$\langle 16,1\rangle$	$[2;-]$	$[0;2,8^2]$	4	4	2	12	17	3
34	$\mathbb{Z}_2^2 \rtimes_\varphi \mathbb{Z}_4$	$\langle 16,3\rangle$	$[1;2]$	$[0;2^2,4^2]$	3	2	1	6	10	2
35	$\mathbb{Z}_2^2 \rtimes_\varphi \mathbb{Z}_4$	$\langle 16,3\rangle$	$[0;2^2,4^2]$	$[0;2^5]$	4	2	0	6	11	3
36	$\mathbb{Z}_2^2 \rtimes_\varphi \mathbb{Z}_4$	$\langle 16,3\rangle$	$[0;2^2,4^2]$	$[0;2^5]$	3	1	0	5	9	3
37	$\mathbb{Z}_2^2 \rtimes_\varphi \mathbb{Z}_4$	$\langle 16,3\rangle$	$[0;2^2,4^2]$	$[0;2^5]$	4	2	0	5	10	3
38	$\mathbb{Z}_2^2 \rtimes_\varphi \mathbb{Z}_4$	$\langle 16,3\rangle$	$[0;2^2,4^2]$	$[0;2^5]$	4	2	0	7	12	3
39	$\mathbb{Z}_2^2 \rtimes_\varphi \mathbb{Z}_4$	$\langle 16,3\rangle$	$[1;2]$	$[0;2^5]$	3	2	1	6	10	3
40	$\mathbb{Z}_2^2 \rtimes_\varphi \mathbb{Z}_4$	$\langle 16,3\rangle$	$[1;2]$	$[0;2^5]$	4	3	1	6	11	3
41	$\mathbb{Z}_4 \rtimes_\varphi \mathbb{Z}_4$	$\langle 16,4\rangle$	$[1;2]$	$[0;2^2,4^2]$	3	2	1	6	10	2
42	$\mathbb{Z}_4 \rtimes_\varphi \mathbb{Z}_4$	$\langle 16,4\rangle$	$[1;2]$	$[0;2^2,4^2]$	4	3	1	6	11	2
43	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\langle 16,5\rangle$	$[2;-]$	$[0;2,8^2]$	4	4	2	12	17	3
44	M_{16}	$\langle 16,6\rangle$	$[2;-]$	$[0;2,8^2]$	5	5	2	10	16	3
45	$\mathcal{D}_8$	$\langle 16,7\rangle$	$[2;-]$	$[0;2,8^2]$	4	4	2	10	15	3
46	$\mathcal{D}_8$	$\langle 16,7\rangle$	$[2;-]$	$[0;2^3,4]$	4	4	2	11	16	4
47	$SD16$	$\langle 16,8\rangle$	$[2;-]$	$[0;2,8^2]$	4	4	2	12	17	3
48	$SD16$	$\langle 16,8\rangle$	$[2;-]$	$[0;4^3]$	4	4	2	11	16	3
49	$SD16$	$\langle 16,8\rangle$	$[0;2,4,8]$	$[1;2^2]$	3	2	1	7	11	2
50	$SD16$	$\langle 16,8\rangle$	$[2;-]$	$[0;2^3,4]$	5	5	2	11	17	4
51	$Dic16$	$\langle 16,9\rangle$	$[2;-]$	$[0;2,8^2]$	6	6	2	10	17	3
52	$Dic16$	$\langle 16,9\rangle$	$[2;-]$	$[0;4^3]$	5	5	2	11	17	3
53	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	$\langle 16,10\rangle$	$[0;2^2,4^2]$	$[0;2^5]$	3	1	0	5	9	3
54	$\mathcal{D}_4 \times \mathbb{Z}_2$	$\langle 16,11\rangle$	$[2;-]$	$[0;2^3,4]$	4	4	2	11	16	4
55	$\mathcal{D}_4 \times \mathbb{Z}_2$	$\langle 16,11\rangle$	$[0;2^2,4^2]$	$[0;2^5]$	3	1	0	6	10	3
56	$Q \times \mathbb{Z}_2$	$\langle 16,12\rangle$	$[2;-]$	$[0;4^3]$	5	5	2	11	17	3
57	$\mathcal{D}_4 *_\phi \mathbb{Z}_4$	$\langle 16,13\rangle$	$[2;-]$	$[0;4^3]$	4	4	2	11	16	3
58	$\mathcal{D}_4 *_\phi \mathbb{Z}_4$	$\langle 16,13\rangle$	$[2;-]$	$[0;2^3,4]$	5	5	2	11	17	4
59	$Dic20$	$\langle 20,1\rangle$	$[2;-]$	$[0;2,5,10]$	6	6	2	10	17	3
60	$\mathbb{Z}_{20}$	$\langle 20,2\rangle$	$[2;-]$	$[0;2,5,10]$	4	4	2	12	17	3
61	$\mathcal{D}_{10}$	$\langle 20,4\rangle$	$[2;-]$	$[0;2,5,10]$	4	4	2	10	15	3
62	$\mathbb{Z}_2^2 \times \mathbb{Z}_5$	$\langle 20,5\rangle$	$[2;-]$	$[0;2,5,10]$	4	4	2	12	17	3
63	$Dic24$	$\langle 24,4\rangle$	$[2;-]$	$[0;3,4^2]$	5	5	2	11	17	3
64	$\mathfrak{S}_3 \times \mathbb{Z}_4$	$\langle 24,5\rangle$	$[2;-]$	$[0;3,4^2]$	4	4	2	11	16	3
65	$\mathfrak{S}_3 \times \mathbb{Z}_4$	$\langle 24,5\rangle$	$[2;-]$	$[0;2^3,3]$	5	5	2	11	17	4
66	$\mathcal{D}_{12}$	$\langle 24,6\rangle$	$[2;-]$	$[0;2^3,3]$	4	4	2	11	16	4
67	$Dic12 \times \mathbb{Z}_2$	$\langle 24,7\rangle$	$[2;-]$	$[0;2,6^2]$	5	5	2	10	16	3
68	$Dic12 \times \mathbb{Z}_2$	$\langle 24,7\rangle$	$[2;-]$	$[0;2,6^2]$	6	6	2	10	17	3
69	$Dic12 \times \mathbb{Z}_2$	$\langle 24,7\rangle$	$[2;-]$	$[0;3,4^2]$	5	5	2	11	17	3
70	$\mathbb{Z}_3 \rtimes_\varphi \mathcal{D}_4$	$\langle 24,8\rangle$	$[2;-]$	$[0;2^3,3]$	5	5	2	11	17	4
71	$\mathbb{Z}_3 \rtimes_\varphi \mathcal{D}_4$	$\langle 24,8\rangle$	$[2;-]$	$[0;3,4^2]$	4	4	2	11	16	3
72	$\mathbb{Z}_3 \rtimes_\varphi \mathcal{D}_4$	$\langle 24,8\rangle$	$[2;-]$	$[0;2,6^2]$	4	4	2	10	15	3
73	$\mathbb{Z}_3 \rtimes_\varphi \mathcal{D}_4$	$\langle 24,8\rangle$	$[2;-]$	$[0;2,6^2]$	4	4	2	12	17	3
74	$\mathbb{Z}_6 \times \mathbb{Z}_4$	$\langle 24,9\rangle$	$[2;-]$	$[0;2,6^2]$	4	4	2	12	17	3
75	$\mathcal{D}_4 \times \mathbb{Z}_3$	$\langle 24,10\rangle$	$[2;-]$	$[0;2,6^2]$	4	4	2	11	16	3
76	$\mathcal{D}_4 \times \mathbb{Z}_3$	$\langle 24,10\rangle$	$[2;-]$	$[0;2,6^2]$	5	5	2	10	16	3
77	$\mathcal{D}_6 \times \mathbb{Z}_2$	$\langle 24,14\rangle$	$[2;-]$	$[0;2^3,3]$	4	4	2	11	16	4
78	$\mathcal{D}_6 \times \mathbb{Z}_2$	$\langle 24,14\rangle$	$[2;-]$	$[0;2,6^2]$	4	4	2	10	15	3
79	$\mathbb{Z}_2^3 \times \mathbb{Z}_3$	$\langle 24,15\rangle$	$[2;-]$	$[0;2,6^2]$	4	4	2	12	17	3
80	$\mathbb{Z}_2^2 \rtimes_\varphi \mathbb{Z}_8$	$\langle 32,5\rangle$	$[1;2]$	$[0;2,8^2]$	2	1	1	6	9	1
81	$\mathbb{Z}_2^3 \rtimes_\varphi \mathbb{Z}_4$	$\langle 32,6\rangle$	$[0;2^2,4^2]$	$[0;2^3,4]$	3	1	0	4	8	2
82	$\mathbb{Z}_2^3 \rtimes_\varphi \mathbb{Z}_4$	$\langle 32,6\rangle$	$[1;2]$	$[0;2^3,4]$	3	2	1	6	10	2
83	$M16 \rtimes_\varphi \mathbb{Z}_2$	$\langle 32,7\rangle$	$[1;2]$	$[0;2^3,4]$	3	2	1	6	10	2
84	$\mathcal{D}_4 \rtimes_\varphi \mathbb{Z}_4$	$\langle 32,9\rangle$	$[0;2^2,4^2]$	$[0;2^3,4]$	3	1	0	5	9	2
85	$\mathcal{D}_4 \rtimes_\varphi \mathbb{Z}_4$	$\langle 32,9\rangle$	$[0;2^2,4^2]$	$[0;2^3,4]$	3	1	0	4	8	2
86	$\mathcal{D}_4 \rtimes_\varphi \mathbb{Z}_4$	$\langle 32,9\rangle$	$[0;2,4,8]$	$[0;2^5]$	3	1	0	5	9	2
87	$\mathbb{Z}_4 \rtimes_\varphi \mathbb{Z}_8$	$\langle 32,12\rangle$	$[1;2]$	$[0;2,8^2]$	3	2	1	6	10	1
88	$\mathcal{D}_4 \times \mathbb{Z}_4$	$\langle 32,25\rangle$	$[0;2^2,4^2]$	$[0;2^3,4]$	3	1	0	4	8	2
89	$\mathbb{Z}_4 \rtimes_\varphi \mathcal{D}_4$	$\langle 32,28\rangle$	$[0;2^2,4^2]$	$[0;2^3,4]$	3	1	0	5	9	2

No.	G	Id	T_1	T_2	$h^{3,0}$	$h^{2,0}$	$h^{1,0}$	$h^{1,1}$	$h^{1,2}$	d
90	$SD16 \times \mathbb{Z}_2$	$\langle 32, 40\rangle$	$[2; -]$	$[0; 2, 4, 8]$	4	4	2	11	16	3
91	$\mathcal{D}_8 *_\phi \mathbb{Z}_4$	$\langle 32, 42\rangle$	$[2; -]$	$[0; 2, 4, 8]$	4	4	2	11	16	3
92	$\mathrm{Hol}(\mathbb{Z}_8)$	$\langle 32, 43\rangle$	$[2; -]$	$[0; 2, 4, 8]$	4	4	2	10	15	3
93	$SD16 \rtimes_\varphi \mathbb{Z}_2$	$\langle 32, 44\rangle$	$[2; -]$	$[0; 2, 4, 8]$	5	5	2	10	16	3
94	$2O$	$\langle 48, 28\rangle$	$[2; -]$	$[0; 3^2, 4]$	5	5	2	11	17	3
95	$\mathrm{GL}(2, \mathbb{F}_3)$	$\langle 48, 29\rangle$	$[2; -]$	$[0; 3^2, 4]$	4	4	2	11	16	3
96	$\mathrm{SL}(2, 3) \times \mathbb{Z}_2$	$\langle 48, 32\rangle$	$[2; -]$	$[0; 3^2, 4]$	5	5	2	11	17	3
97	$\mathrm{SL}(2, 3) \rtimes_\varphi \mathbb{Z}_2$	$\langle 48, 33\rangle$	$[2; -]$	$[0; 3^2, 4]$	4	4	2	11	16	3
98	$Dic24 \rtimes_\varphi \mathbb{Z}_2$	$\langle 48, 37\rangle$	$[2; -]$	$[0; 2, 4, 6]$	4	4	2	11	16	3
99	$\mathcal{D}_4 \times \mathfrak{S}_3$	$\langle 48, 38\rangle$	$[2; -]$	$[0; 2, 4, 6]$	4	4	2	10	15	3
100	$\mathcal{D}_4 \rtimes_\varphi \mathfrak{S}_3$	$\langle 48, 39\rangle$	$[2; -]$	$[0; 2, 4, 6]$	5	5	2	10	16	3
101	$\mathbb{Z}_6 \rtimes_\varphi \mathcal{D}_4$	$\langle 48, 43\rangle$	$[2; -]$	$[0; 2, 4, 6]$	4	4	2	11	16	3
102	$\mathfrak{S}_4 \times \mathbb{Z}_4$	$\langle 96, 186\rangle$	$[0; 2^2, 4^2]$	$[0; 2, 4, 6]$	3	1	0	3	7	1
103	$\mathrm{GL}(2, \mathbb{F}_3) \times \mathbb{Z}_2$	$\langle 96, 189\rangle$	$[2; -]$	$[0; 2, 3, 8]$	4	4	2	11	16	3
104	$(Q \times \mathbb{Z}_2) \rtimes_\varphi \mathfrak{S}_3$	$\langle 96, 190\rangle$	$[2; -]$	$[0; 2, 3, 8]$	5	5	2	10	16	3
105	$2O \rtimes_\varphi \mathbb{Z}_2$	$\langle 96, 192\rangle$	$[2; -]$	$[0; 2, 3, 8]$	4	4	2	11	16	3
106	$\mathrm{GL}(2, \mathbb{F}_3) \rtimes_\varphi \mathbb{Z}_2$	$\langle 96, 193\rangle$	$[2; -]$	$[0; 2, 3, 8]$	4	4	2	10	15	3
107	$\mathrm{GL}(2, \mathbb{Z}_4)$	$\langle 96, 195\rangle$	$[0; 2^2, 4^2]$	$[0; 2, 4, 6]$	3	1	0	4	8	1

In the index three and index six case we have the following theorem.

Theorem (C).

a) *There is a unique group G acting absolutely faithful and freely on a product of curves such that the quotient X has $\chi(\mathcal{O}_X) = -1$ and the index of G^0 in G is three. The group G has MAGMA id $\langle 27, 4\rangle$ and is a semidirect product*

$$G \simeq \mathbb{Z}_9 \rtimes_\varphi \mathbb{Z}_3.$$

The numerical datum of X is $\big(27, [0; 3^4]\big)$ and the Hodge numbers are:

$$h^{3,0}(X) = 4, \quad h^{2,0}(X) = 2, \quad h^{1,0}(X) = 0, \quad h^{1,1}(X) = 5 \quad \text{and} \quad h^{1,2}(X) = 10$$

b) *There is no group acting absolutely faithful and freely on a product of curves such that the quotient X has $\chi(\mathcal{O}_X) = -1$ and the index of G^0 in G is six.*

Computational Remark 3.0.14. For values of χ different from -1 there might occur exceptional numerical data, when we run Part 1 of the program. We executed Part 1 in the unmixed, index two, index three and index six case for all values of χ in the range

$$-40 \leq \chi \leq -1$$

and found no exceptional numerical data. Albeit it is not of great importance in our context, we we shall mention that there are methods to deal with the exceptional numerical data, if they should occur for $\chi \leq -41$. We refer the reader to the paper [BCG08], where the authors classify surfaces isogenous to a product with $p_g = q = 0$ and the analogous problem appears. Their strategy can be easily adapted to the threefold case.

Nevertheless, running Part 2 of the program for χ different from -1 is very time and memory consuming, in particular in the unmixed and the index two case: when we decrease χ, then the maximal possible value for g_i' increases, according to Proposition 3.0.10 vi). Similarly, the maximal length r_i of the types

$$T_i = [g_i'; m_{i,1}, \ldots, m_{i,r_i}]$$

that we obtain increases as well. This leads to a large number of generating vectors that need to be determined and analysed, which slows down the computation and requires a lot of memory.

To conclude this chapter we give two further examples of threefolds X isogenous to a product with $\chi(\mathcal{O}_X) = -1$. The first one is of mixed type and obtained by an index six action, the second one is of unmixed type without parameters, i.e. a rigid example. Note that there are no such examples in the absolutely faithful case with $\chi(\mathcal{O}_X) = -1$. The files `ExampleIndexSix.magma` and `ExampleRigid.magma` in Appendix A contain the source codes to perform the required MAGMA computations.

Example 3.0.15.

a) We begin with the index six example. Consider the group $G := \mathit{SmallGroup}(216, 90)$, it admits a unique normal subgroup G^0 such that $G/G^0 \simeq \mathfrak{S}_3$. Moreover, the extension

$$1 \to G^0 \to G \to \mathfrak{S}_3 \to 1$$

is non-split. For the elements $h := G.1 * G.2 * G.4^2$ and $\tau := G.3 * G.4^2$ in $G \setminus G^0$ it holds

$$\tau^2 \notin G^0 \qquad \text{and} \qquad h^2 \in G^0,$$

i.e. h and τ define an isomorphism $G/G^0 \to \mathfrak{S}_3$. The cyclic group K_1 generated by $G1.3 * G1.4$ is the unique normal subgroup in $G_1 := \langle h, G^0 \rangle$ of order six such that

$$K_1 \cap \tau K_1 \tau^{-1} = \{1_G\}.$$

The quotient G_1/K_1 is isomorphic to the dihedral group $\mathcal{D}_6$ via the map $G_1/K_1 \to \mathcal{D}_6$ defined by

$$\overline{G1.1} \mapsto s \qquad \text{and} \qquad \overline{G1.2 * G1.5} \mapsto t.$$

According to Example 1.1.8 there is a faithful group action $\mathcal{D}_6 \to \mathrm{Aut}(C)$, where C is a compact Riemann surface of genus $g(C) = 7$. A corresponding generating vector is given by $V_1 := (st, st, t^5, t^5)$. The stabilizer set Σ_1 of the action

$$\psi_1 \colon G_1 \to G_1/K_1 \simeq \mathcal{D}_6 \to \mathrm{Aut}(C)$$

fulfills the freeness conditions:

i) $\Sigma_1 \cap \tau\Sigma_1\tau^{-1} \cap \tau^2\Sigma_1\tau^{-2} = \{1_G\}$.

ii) $(\tau g)^3 \notin \Sigma_1$ for all $g \in G^0$ and

iii) $\tau f^2\tau^{-1} \notin \Sigma_1$ for all $f \in G_1 \setminus G^0 \cap \Sigma_1$.

According to Proposition 2.3.6 c), the tuple $(G, G^0, K_1, \tau, h, V_1)$ is an algebraic datum of a threefold $X = C^3/G$ isogenous to a product. Since $g(C) = 7$, it holds

$$\chi(\mathcal{O}_X) = -\frac{(g(C)-1)^3}{216} = -1.$$

For completeness, we also determine the Hodge numbers $h^{p,q}$ of the threefold X: the character χ_{φ_1} of the representation

$$\varphi_1 \colon G_1 \to G_1/K_1 \simeq \mathcal{D}_6 \to \mathrm{GL}\left(H^{1,0}(C)\right)$$

is equal to the composition of the homomorphism $G_1 \to G_1/K_1 \simeq \mathcal{D}_6$ with the character of the representation $\mathcal{D}_6 \to \mathrm{GL}\left(H^{1,0}(C)\right)$. The latter is, according to Example 1.3.7, given by

$$\chi_{triv} + \chi_1 + \chi_2 + \chi_4 + \chi_5,$$

where χ_i are the irreducible characters of $\mathcal{D}_6$ determined in Example 1.2.8. Using the explicit description of the characters $\chi_{p,q}$ in terms of χ_{φ_1} (see Theorem 2.4.5 and Theorem 2.4.8), we compute the Hodge numbers of X via the formulae

$$h^{p,q}(X) = \langle \chi_{p,q}, \chi_{triv} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_{p,q}(g)$$

and obtain

$$h^{3,0}(X) = 2, \quad h^{2,0}(X) = 1, \quad h^{1,0}(X) = 1, \quad h^{1,1}(X) = 5 \quad \text{and} \quad h^{1,2}(X) = 8.$$

b) Let $S = \big(C_1 \times C_2\big)/G$ be a rigid surface isogenous to a product of unmixed type, then $C_i/G \simeq \mathbb{P}^1$ and the G-covers $C_i \to \mathbb{P}^1$ are branched over $0, 1$ and ∞. These surfaces are called Beauville surfaces, since Beauville provided the first example of such a surface (cf. [Be83]). In his example $G = \mathbb{Z}_5^2$ and $g(C_i) = 6$, which yields $\chi(\mathcal{O}_S) = 1$, according to Proposition 2.1.2. Appropriate generating vectors V_i for $\mathbb{Z}_5^2$ are given by

$$V_1 = \big[(0,3),(3,3),(2,4)\big] \qquad \text{and} \qquad V_2 = \big[(2,0),(2,1),(1,4)\big].$$

We can easily modify this example to obtain a rigid threefold isogenous to a product with $\chi(\mathcal{O}_X) = -1$. Consider the generating vector $V_3 = (1, 1, 3)$ of $\mathbb{Z}_5$. It corresponds to an action

$$\psi_3 \colon \mathbb{Z}_5 \to \mathrm{Aut}(C_3),$$

where C_3 is a curve of genus two, $C_3/\mathbb{Z}_5 \simeq \mathbb{P}^1$ and the $\mathbb{Z}_5$-cover $C_3 \to \mathbb{P}^1$ is branched over $0, 1$ and ∞. We obtain a diagonal, free action of $\mathbb{Z}_5^2$ on $C_1 \times C_2 \times C_3$, where $\mathbb{Z}_5^2$ acts on C_3 via ψ_3 composed with the projection to the first factor. The quotient

$$X = \big(C_1 \times C_2 \times C_3\big)/\mathbb{Z}_5^2$$

is a rigid threefold isogenous to a product with $\chi(\mathcal{O}_X) = -1$ and the Hodge numbers of X are the following:

$$h^{3,0}(X) = 3, \quad h^{2,0}(X) = 1, \quad h^{1,0}(X) = 0, \quad h^{1,1}(X) = 5 \quad \text{and} \quad h^{1,2}(X) = 9.$$

Chapter 4

Product Quotient Threefolds

4.1 Generalities

Until now we considered varieties isogenous to a product, i.e. varieties obtained as a quotient of a product of compact Riemann surfaces by the free action of a finite group G. Now we drop the freeness assumption. For simplicity, we also assume that the action is unmixed and absolutely faithful.

Definition 4.1.1. *A variety $X = \big(C_1 \times \ldots \times C_n\big)/G$, where G is a finite group with an unmixed action on the product of compact Riemann surfaces C_i of respective genera at least two such that the maps $\psi_i \colon G \to \operatorname{Aut}(C_i)$ are injective is called a* product quotient variety*.*

Since we do not assume that G acts freely, a product quotient variety X is in general singular. The singularities of X must be quotient singularities, i.e. locally analytically isomorphic to a quotient of $\mathbb{C}^n$ modulo a finite subgroup of $\mathrm{GL}(n, \mathbb{C})$. More precisely we have:

Proposition 4.1.2. *The singular locus* $\operatorname{Sing}(X)$ *of a product quotient variety*

$$X = \big(C_1 \times \ldots \times C_n\big)/G$$

is the image of the finite set $\big\{q \in C_1 \times \ldots \times C_n \;\big|\; \operatorname{Stab}(q) \neq \{1_G\}\big\}$ under the quotient map. Each singularity is an isolated cyclic quotient singularity

$$\frac{1}{m}(a_1, \ldots, a_n),$$

i.e. locally around the singular point, the variety X is isomorphic to a quotient $\mathbb{C}^n/H$, *where H is a cyclic group generated by a diagonal matrix*

$$\operatorname{diag}\left(\xi^{a_1}, \ldots, \xi^{a_n}\right), \quad \textit{where} \quad \xi := \exp\left(\frac{2\pi\sqrt{-1}}{m}\right) \quad \textit{and} \quad \gcd(a_i, m) = 1.$$

Proof. Clearly, a point $q \in C_1 \times \ldots \times C_n$ with trivial stabilizer group can not decent to a singular point of X. From the assumption that the maps ψ_i are injective, it follows that there are at most finitely many points on the product admitting a non-trivial stabilizer group. Let $q = (q_1, \ldots, q_n)$ be such a point and x its class in X. According to Cartan [Car57], we know that locally around x the quotient X is isomorphic to $\mathbb{C}^n/H$ where $H < \mathrm{GL}(n, \mathbb{C})$ is identified, via the tangent representation, with the stabilizer group $\mathrm{Stab}(q)$. Since the action of G on $C_1 \times \ldots \times C_n$ is assumed to be diagonal also H acts on $\mathbb{C}^n$ diagonally, i.e. via diagonal matrices. Note that the stabilizer group $\mathrm{Stab}(q)$ is equal to

$$G_{q_1} \cap \ldots \cap G_{q_n}$$

which is, as an intersection of cyclic groups, cyclic. We conclude that $H \simeq \mathrm{Stab}(q)$ is generated by a diagonal matrix of order $m := |H|$:

$$\operatorname{diag}\left(\xi^{a_1}, \ldots, \xi^{a_n}\right), \quad \text{where} \quad \xi := \exp\left(\frac{2\pi\sqrt{-1}}{m}\right) \quad \text{and} \quad \gcd(a_i, m) = 1.$$

The condition $\gcd(a_i, m) = 1$ reflects the fact that in a sufficiently small neighbourhood of q there are no other points with non-trivial stabilizer group. The Chevalley–Shephard–Todd theorem (cf. [Che55] and [S-T54]) implies that x is singular. □

Definition 4.1.3 ([Cat07, Def 1.1]). *Let $\pi\colon Y \to X$ be a regular map between varieties of the same dimension. The map π is called* quasi-étale *if there is a subvariety $Z \subset Y$ of codimension at least two such that*

$$\pi_{|Y \setminus Z}\colon Y \setminus Z \to \pi(Y \setminus Z)$$

is étale.

Let X be a product quotient variety. Since quotient singularities are $\mathbb{Q}$-factorial (see [KM98, Proposition 5.15]) we have a well-defined intersection product $K_X^n \in \mathbb{Q}$, which is in general not an integer. It can be determined in the same way as for varieties isogenous to a product (cf. Proposition 2.1.2):

Proposition 4.1.4. *Let $X = \big(C_1 \times \ldots \times C_n\big)/G$ be a product quotient variety, then*

$$K_X^n = \frac{n!\,2^n}{|G|} \prod_{i=1}^{n} \big(g(C_i) - 1\big).$$

Proof. Choose an integer I such that IK_X is Cartier and define $Y := C_1 \times \ldots \times C_n$.

Since the quotient map $\pi\colon Y \to X$ is quasi-étale it holds $\pi^*(IK_X) = IK_Y$. Moreover, the degree of π is the order of G, which implies $(IK_Y)^n = |G|(IK_X)^n$. The claim follows because

$$K_Y^n = n!\, 2^n \prod_{i=1}^{n} \big(g(C_i) - 1\big).$$

□

In recent years, there has been a considerable interest in product quotient varieties of dimension two, see for example [Pe11], [BCGP12] and [BP12]. A lot of new examples of varieties of general type have been constructed in this way. Our aim is to generalize the methods of the above authors in order to study higher dimensional product quotient varieties admitting canonical singularities, in particular the three dimensional case. For the readers benefit we recall the definition of *canonical singularities*:

Definition 4.1.5. *A normal variety X has* canonical singularities *if the canonical divisor K_X is $\mathbb{Q}$-Cartier, and if for a resolution $\rho\colon Z \to X$ with exceptional prime divisors E_i the rational numbers a_i defined by the formula*

$$K_Z = \rho^*(K_X) + \sum a_i E_i$$

are non-negative. If $a_i > 0$ for all exceptional prime divisors E_i then the variety X has terminal singularities*.*

An important property of quasi-étale morphisms between varieties with canonical singularities is the following:

Proposition 4.1.6 ([Cat07, Section 3])**.** *Let $\pi\colon Y \to X$ be a quasi-étale morphism between varieties with canonical singularities. Then X and Y have the same Kodaira dimension, i.e. $\kappa(X) = \kappa(Y)$.*

Corollary 4.1.7. *A product quotient variety X with canonical singularities is of general type.*

Proof. The claim follows immediately from the fact that the quotient map $\pi\colon C_1 \times \ldots \times C_n \to X$ is quasi-étale and

$$\kappa(C_1 \times \ldots \times C_n) = \kappa(C_1) + \ldots + \kappa(C_n) = n$$

because $g(C_i) \geq 2$. □

Theorem 4.1.8 ([Reid87, 3.2 b])**.** *Let X be a threefold with canonical singularities then there exists a* crepant terminalisation*, i.e. a crepant partial resolution $\rho\colon \widehat{X} \to X$ such that $\widehat{X}$ has only terminal singularities.*

Remark 4.1.9.

i) Recall that a crepant partial resolution is a proper birational morphism $\rho\colon \widehat{X} \to X$ such that $\rho^*(K_X) = K_{\widehat{X}}$.

ii) Let X be a threefold with canonical singularities and $\rho\colon \widehat{X} \to X$ be a crepant terminalisation. Assume that $\widehat{X}$ is minimal but singular. Let $Z \to \widehat{X}$ be a resolution of singularities. Then, by the Negativity Lemma [KM98, Lemma 3.39], there exists a curve C contained in the exceptional locus of the resolution such that $C.K_Z < 0$, i.e. Z is not minimal.

The object we want to study in the remaining part of this thesis is a crepant terminalisation $\widehat{X}$ of a product quotient threefold

$$X := \big(C_1 \times C_2 \times C_3\big)/G$$

with canonical singularities. In particular we want to understand which values of the invariants

$$e(\widehat{X}), \qquad K_{\widehat{X}}^3 \qquad \text{and} \qquad \chi(\mathcal{O}_{\widehat{X}})$$

may be realized and how to classify these varieties. These questions are well-posed, because the invariants mentioned above are independent of the chosen crepant terminalisation (cf. [H89]).

Similar to the case of threefolds isogenous to a product it is convenient to introduce and work with an algebraic datum:

Definition 4.1.10. *Let $X = \big(C_1 \times C_2 \times C_3\big)/G$ be a product quotient threefold and V_i be a generating vector for G associated to the cover*

$$C_i \to C_i/G, \qquad \textit{for all} \qquad 1 \leq i \leq 3.$$

The tuple (G, V_1, V_2, V_3) is called an algebraic datum *of X.*

4.2 The Singularities

In the previous section we saw that the singular locus of a product quotient variety consists of finitely many isolated cyclic quotient singularities:

$$\frac{1}{m}(a_1, \ldots, a_n), \quad \text{where} \quad \gcd(a_i, m) = 1 \quad \text{for all} \quad 1 \leq i \leq n.$$

The case of interest for us is when the singularities are canonical and the dimension is three. These singularities were classified by Morrison (see. [Mor85]). In this section we present Morrison's theorem and determine for each possible type of singularity a crepant terminalisation (cf. Theorem 4.1.8).

Before we start, we explain when two cyclic quotient singularities are isomorphic. This is important because we are only interested in the isomorphism class of a singularity. The following proposition gives a criterion.

Proposition 4.2.1 ([Fuj74, Lemma 2.]). *The singularities*

$$\frac{1}{m}(a_1,\dots,a_n) \qquad \textit{and} \qquad \frac{1}{m}(b_1,\dots,b_n)$$

are isomorphic if and only if there exists a permutation $\delta \in \mathfrak{S}_n$ *and an integer* $0 < k < m$ *with* $\gcd(m,k) = 1$ *such that*

$$a_i k \equiv b_{\delta(i)} \mod m \qquad \textit{for all} \qquad 1 \leq i \leq n.$$

Note that the proposition allows us to assume that an isolated cyclic quotient singularity is of type $\frac{1}{m}(1, a_2, \dots, a_n)$.

Example 4.2.2. The isolated cyclic quotient singularities

$$\frac{1}{m}(1,a,b) \qquad \text{and} \qquad \frac{1}{m}(1,c,d)$$

are isomorphic if and only if (a,b) or (b,a) is one of the following tuples

$$(c,d), \quad (c^{-1} \mod m, dc^{-1} \mod m) \quad \text{or} \quad (d^{-1} \mod m, d^{-1}c \mod m).$$

Next, we introduce affine toric varieties and show that cyclic quotient singularities are examples of such varieties. Toric geometry provides a convenient way to work with this class of singularities. For details about toric varieties we refer the reader to the textbook [Ful93].

Let $N \simeq \mathbb{Z}^n$ be a lattice and $\sigma \subset N \otimes \mathbb{R}$ be a *strongly convex rational polyhedral cone*, i.e.

$$\sigma = \operatorname{cone}(v_1,\dots,v_l) \qquad \text{and} \qquad \sigma \cap (-\sigma) = \{0\},$$

where the vectors v_i belong to the lattice. The *dual lattice* of N is defined as $M := \operatorname{Hom}(N,\mathbb{Z})$ and the dual cone of σ as

$$\sigma^\vee := \left\{ u \in M \otimes \mathbb{R} \;\middle|\; \langle u,v \rangle \geq 0 \text{ for all } v \in \sigma \right\},$$

where $\langle\ ,\ \rangle$ denotes the dual pairing. According to Gordan's lemma [Ful93, Section 1.2] the *semigroup* $M \cap \sigma^\vee$ is finitely generated. Therefore, the associated $\mathbb{C}$-algebra $\mathbb{C}[\sigma^\vee \cap M]$ is also finitely generated which allows us to define the *affine toric variety* U_σ

as

$$\operatorname{Spec}\big(\mathbb{C}[\sigma^\vee \cap M]\big).$$

We mention that U_σ is normal (see [Ful93, Section 2.1]) and of dimension n.

Proposition 4.2.3 ([Ful93, Section 2.1])**.** *The affine toric variety U_σ is smooth if and only if σ is generated by a part of a $\mathbb{Z}$-basis of the lattice N, in which case*

$$U_\sigma \simeq \mathbb{C}^k \times \big(\mathbb{C}^*\big)^{n-k}, \qquad \textit{where} \qquad k := \dim\big(\operatorname{Span}(\sigma)\big).$$

Example 4.2.4. Let U be an isolated cyclic quotient singularity of type

$$\frac{1}{m}(a_1, \ldots, a_n)$$

To give it the structure of an affine toric variety, we consider the lattice

$$N = \mathbb{Z}^n + \frac{\mathbb{Z}}{m}(a_1, \ldots, a_n)$$

and the cone $\sigma := \operatorname{cone}(e_1, \ldots, e_n)$. The dual lattice of N is given by

$$M = \Big\{(u_1, \ldots, u_n) \;\Big|\; m \text{ divides } \sum_{i=1}^{n} u_i a_i\Big\}$$

and we find that the coordinate ring $\mathbb{C}[\sigma^\vee \cap M]$ is the ring of invariants for the action of the cyclic group generated by the matrix

$$\operatorname{diag}\big(\xi^{a_1}, \ldots, \xi^{a_n}\big), \quad \text{where} \quad \xi := \exp\left(\frac{2\pi\sqrt{-1}}{m}\right),$$

i.e. $U = \operatorname{Spec}\big(\mathbb{C}[\sigma^\vee \cap M]\big)$.

Up to now, we saw the definition of an affine toric variety and our main example: a cyclic quotient singularity. A general *toric variety* is obtained by gluing affine toric varieties U_σ, more precisely:

Definition 4.2.5. *Let N be a lattice.*

i) A fan *$\mathcal{F}$ in $N \otimes \mathbb{R}$ is a finite collection of strongly convex rational polyhedral cones $\sigma \subset N \otimes \mathbb{R}$ such that*

- *each face of a cone in $\mathcal{F}$ is a cone in $\mathcal{F}$ and*
- *the intersection of two cones in $\mathcal{F}$ is a face of each of them.*

ii) The toric variety $U_{\mathcal{F}}$ associated to the fan $\mathcal{F}$ is the variety obtained by gluing the affine varieties

$$U_\sigma \qquad \textit{for all} \qquad \sigma \in \mathcal{F}$$

along the open sets $U_{\sigma\cap\sigma'}$ of U_σ and $U_{\sigma'}$ for all cones σ and σ' contained in $\mathcal{F}$.

Example 4.2.6. Consider the lattice $N = \mathbb{Z}^2$ and the fan $\mathcal{F}$ in $\mathbb{Z}^2 \otimes \mathbb{R} \simeq \mathbb{R}^2$ which consists of the cones

$$\sigma_1 := \text{cone}(e_1, e_2), \quad \sigma_2 := \text{cone}(e_2, -e_1 - e_2) \quad \text{and} \quad \sigma_3 := \text{cone}(e_1, -e_1 - e_2).$$

The affine toric varieties U_{σ_i} are smooth, because the generators of σ_i form a $\mathbb{Z}$-basis of $\mathbb{Z}^2$ for all $1 \leq i \leq 3$. By Proposition 4.2.3 they are isomorphic to $\mathbb{C}^2$ and it is not hard to verify that the toric variety associated to $\mathcal{F}$ is $\mathbb{P}^2_{\mathbb{C}}$.

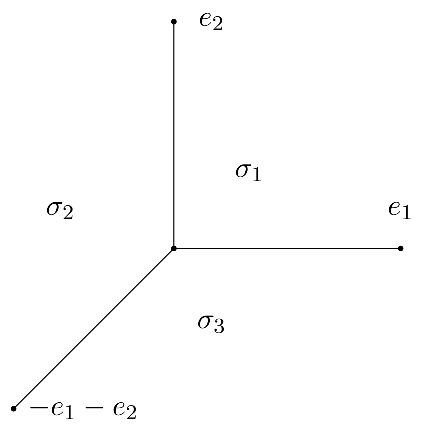

Figure 4.1: toric fan of $\mathbb{P}^2$

Proposition 4.2.7 ([Ful93, Section 2.4])**.** *Let N be a lattice and $\mathcal{F}$ be a fan in $N \otimes \mathbb{R}$. Then, the toric variety $U_{\mathcal{F}}$ is complete, i.e. compact in the Euclidean topology if and only if*

$$N \otimes \mathbb{R} = \bigcup_{\sigma \in \mathcal{F}} \sigma.$$

Now that we have the notion of a toric variety, we shall also introduce morphisms between them. They are induced by certain homomorphisms of lattices.

Remark-Construction 4.2.8.

i) Let N and N' be lattices together with fans $\mathcal{F}$ in $N \otimes \mathbb{R}$ and $\mathcal{F}'$ in $N' \otimes \mathbb{R}$. A homomorphism of lattices $\phi\colon N' \to N$ with the property that for each cone $\sigma' \in \mathcal{F}'$ there exists cone $\sigma \in \mathcal{F}$ with $\phi(\sigma') \subset \sigma$ induces in a natural way a morphism $U_{\mathcal{F}'} \to U_{\mathcal{F}}$ of toric varieties.

ii) Let N be a lattice together with a fan $\mathcal{F}$ and let v be a lattice point and $\tau := \mathbb{R}^{\geq 0} v$ be the ray generated by v. We construct a new fan $\mathcal{F}'$ out of $\mathcal{F}$ via subdivision along τ according to the following recipe: replace any cone σ containing v by the collection of convex hulls of τ with the faces of σ that do not contain τ. By i) the identity map of N induces a morphism

$$\rho\colon U_{\mathcal{F}'} \to U_{\mathcal{F}}.$$

Theorem 4.2.9 ([Ful93, Section 2.4 and Section 2.6]).

i) Let $\mathcal{F}$ be a fan in $N \otimes \mathbb{R}$ and $\mathcal{F}'$ be the fan obtained from $\mathcal{F}$ via subdivision along the ray τ generated by $v \in N$. Then the induced morphism

$$\rho\colon U_{\mathcal{F}'} \to U_{\mathcal{F}}.$$

is proper and birational.

ii) Iterating the procedure in i) for appropriate lattice points leads to a resolution of singularities of $U_{\mathcal{F}}$.

Remark 4.2.10. The exceptional locus of the birational map $\rho\colon U_{\mathcal{F}'} \to U_{\mathcal{F}}$ from Theorem 4.2.9 i) has the following description: define the quotient lattice

$$N(\tau) := \frac{N}{N \cap \mathbb{R}v}$$

and consider all cones σ which contain τ as a face. The collection of the quotient cones

$$\overline{\sigma} := \frac{\sigma + \mathbb{R}v}{\mathbb{R}v} \subset N(\tau) \otimes \mathbb{R}$$

form a fan in the vector space $N(\tau) \otimes \mathbb{R}$ which is denoted by $\mathrm{Star}(\tau)$. The exceptional locus of ρ is the associated toric variety $E(\tau)$. According to Proposition 4.2.7 the variety $E(\tau)$ is compact if and only if

$$N(\tau) \otimes \mathbb{R} = \bigcup_{\overline{\sigma} \in \mathrm{Star}(\tau)} \overline{\sigma}.$$

Now we apply the above to our example of an isolated cyclic quotient singularity.

Proposition 4.2.11 (Reid-Shepherd-Barron-Tai [Reid80] and [Tai82]). *Let U be an isolated cyclic quotient singularity of type*

$$\frac{1}{m}(a_1, \ldots, a_n).$$

i) Let $v = \frac{1}{m}(v_1, \ldots, v_n) \in \sigma \cap N \setminus \{0\}$ be a primitive vector and τ be the ray generated by v, where

$$N = \mathbb{Z}^n + \frac{\mathbb{Z}}{m}(a_1, \ldots, a_n).$$

The subdivision of the fan consisting of the single cone σ yields a proper birational morphism $\rho\colon U_{\mathcal{F}} \to U$, i.e. a partial resolution of U with exceptional divisor $E(\tau)$. It holds

$$K_{U_{\mathcal{F}}} = \rho^*(K_U) + \left[\frac{1}{m}\left(\sum_{i=1}^{n} v_i\right) - 1\right] E(\tau).$$

ii) *The singularity U is canonical (or terminal) if and only if*

$$\sum_{j=1}^{n} [k \cdot a_j] \geq m, \qquad \text{for all} \qquad k = 1, \dots, m-1$$

(respectively $> m$). Here, $[\,\cdot\,]$ denotes the residue mod m.

Remark 4.2.12. Note that the partial resolution in i) is crepant if and only if v is contained in the hyperplane

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 1\}.$$

In dimension three isolated canonical singularities are classified by Morrison. They are either terminal or Gorenstein, except for two examples.

Theorem 4.2.13 ([Mor85]). *Let U be an isolated, canonical, cyclic quotient singularity. Then precisely one of the following holds:*

i) *U is a terminal singularity,*

ii) *U is a Gorenstein singularity,*

iii) *U is isomorphic to a singularity of type $\frac{1}{9}(1,4,7)$ or of type $\frac{1}{14}(1,9,11)$.*

We call the singularities $\frac{1}{9}(1,4,7)$ and $\frac{1}{14}(1,9,11)$ singularities of type III.

Remark 4.2.14.

i) Recall that a normal variety is Gorenstein if it is Cohen-Macauly and its canonical Weil divisor is a Cartier divisor.

ii) According to Watanabe [Wat74] a quotient singularity $\mathbb{C}^n/G$ is Gorenstein if and only if $G \leq \mathrm{SL}(n, \mathbb{C})$. Therefore, a cyclic quotient singularity, as in the theorem above, is Gorenstein if and only if it is isomorphic to a singularity of type $\frac{1}{m}(1, a, m-a-1)$. In which case m is odd, because

$$\gcd(m, a) = \gcd(m, a+1) = 1.$$

Theorem 4.2.15 ([MS84]). *A three dimensional cyclic quotient singularity is terminal if and only if it is isomorphic to a singularity of type $\frac{1}{m}(1, a, m-a)$ for some $a \in \mathbb{N}$ such that* $\gcd(a, m) = 1$.

Next, we want to determine a crepant terminalisation

$$\rho\colon \widehat{U} \to U$$

for each type of singularity U occurring in the theorem above. In the terminal case, we simply set $\widehat{U} := U$ and $\rho := \mathrm{id}$. In the other (non-trivial) cases, we apply Proposition 4.2.11 i), iteratively if needed. Remark 4.2.12 provides a necessary and sufficient condition for ρ to be crepant. We begin with the singularities of type III.

Lemma 4.2.16. *A cyclic quotient singularity of type* $\frac{1}{9}(1,4,7)$ *admits a crepant partial resolution with exactly three terminal singularities of type* $\frac{1}{3}(1,1,2)$.

Proof. As above we consider the singularity $\frac{1}{9}(1,4,7)$ as an affine toric variety. The lattice point $v = \frac{1}{3}(1,1,1)$ is primitive and contained in the plane

$$\{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 1\}.$$

We subdivide the cone $\sigma = \mathrm{cone}(e_1,e_2,e_3)$ along the ray generated by v in three subcones:

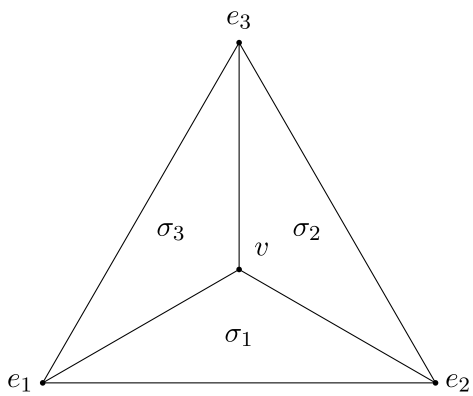

Figure 4.2: crepant terminalisation of $\frac{1}{9}(1,4,7)$

$$\sigma_1 := \mathrm{cone}(e_1,e_2,v), \quad \sigma_2 := \mathrm{cone}(e_2,e_3,v) \quad \text{and} \quad \sigma_3 := \mathrm{cone}(e_3,e_1,v).$$

The matrices

$$A_1 := \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix}, \quad A_2 := \begin{pmatrix} 3 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A_3 := \begin{pmatrix} 0 & 3 & 0 \\ 0 & -1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

induce $\mathbb{Z}$-module isomorphisms $\phi_i\colon N \to M$, where

$$M = \mathbb{Z}e_2 + \mathbb{Z}e_3 + \frac{\mathbb{Z}}{3}(1,1,2)$$

such that $\phi_i(\sigma_i) = \sigma$. It follows that the affine toric varieties corresponding to N and σ_i are cyclic quotient singularities of type $\frac{1}{3}(1,1,2)$. They are terminal according to Theorem 4.2.15. □

Lemma 4.2.17. *A cyclic quotient singularity of type* $\frac{1}{14}(1,9,11)$ *admits a crepant partial resolution with exactly seven terminal singularities of type* $\frac{1}{2}(1,1,1)$.

Proof. We consider the lattice points

$$v_1 := \frac{1}{7}(1,2,4), \quad v_2 := \frac{1}{7}(4,1,2), \quad \text{and} \quad v_3 := \frac{1}{7}(2,4,1).$$

Clearly, these points are primitive and contained in the plane

$$\{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 1\}.$$

We subdivide the cone $\sigma = \operatorname{cone}(e_1,e_2,e_3)$ in the following way:

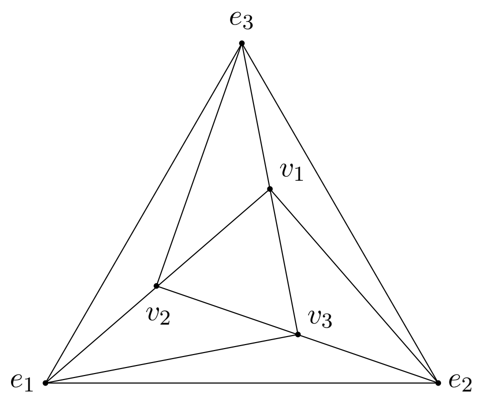

Figure 4.3: crepant terminalisation of $\frac{1}{14}(1,9,11)$

Like in the proof of the previous lemma one can show that the affine toric varieties corresponding to N and the seven cones σ_i are cyclic quotient singularities of type $\frac{1}{2}(1,1,1)$, whence terminal. □

The remaining case to consider is the case of a Gorenstein singularity $\frac{1}{m}(1,a,m-a-1)$. Here, it is possible to determine a crepant terminalisation $\rho\colon \widehat{U} \to U$, where $\widehat{U}$ is smooth, i.e. a crepant resolution. [1]

[1] For the moment, we drop the assumption that the singularity is isolated.

A crepant resolution of $\frac{1}{m}(1, a, m-a-1)$

We subdivide $\sigma = \operatorname{cone}(e_1, e_2, e_3)$ along the ray generated by $v_0 := \frac{1}{m}(1, a, m-a-1)$ into the subcones

$$\operatorname{cone}(e_i, e_j, v_0), \quad \text{where} \quad i \neq j \quad \text{and} \quad v_0 \notin \operatorname{cone}(e_i, e_j).$$

The corresponding partial resolution is crepant because v_0 is contained in the plane

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 1\}.$$

Note that the cone σ_2 generated by $\{e_2, e_3, v_0\}$ is always smooth, since these vectors form a $\mathbb{Z}$-basis of N. The other cones are in general singular. If $m = 2$, the singularity is of type $\frac{1}{2}(1, 1, 0)$. Here, the cones

$$\sigma_3 = \operatorname{cone}(e_1, v_0, e_3) \qquad \text{and} \qquad \sigma_2 = \operatorname{cone}(v_0, e_2, e_3)$$

are both smooth, which yields the desired resolution. Otherwise, we proceed by induction on m. We claim that the affine toric variety U_{σ_3} given by (N, σ_3) is a Gorenstein cyclic quotient singularity of type $\frac{1}{a}(1, -m, m-a-1)$. Indeed, the matrix

$$A = \frac{1}{a}\begin{pmatrix} a & -1 & 0 \\ 0 & m & 0 \\ 0 & a+1-m & a \end{pmatrix}$$

defines a $\mathbb{Z}$-module isomorphism $\phi\colon (N, \sigma_3) \to (N_3, \sigma)$, where

$$N_3 := \mathbb{Z}e_2 + \mathbb{Z}e_3 + \frac{\mathbb{Z}}{a}(1, -m, m-a-1).$$

Similarly, if $v_0 \notin \operatorname{cone}(e_1, e_2)$ then U_{σ_1} given by (N, σ_1) with $\sigma_1 := \operatorname{cone}(e_1, e_2, v_0)$ is also a Gorenstein cyclic quotient singularity. The type is

$$\frac{1}{m-a-1}(1, a, -a-1).$$

By induction hypothesis these singularities admit crepant toric resolutions.

Remark 4.2.18.

i) If the singularity $\frac{1}{m}(1, a, m-a-1)$ is isolated i.e.

$$\gcd(m, a) = \gcd(m, a+1) = 1,$$

we shall perform $(m-1)/2$ subdivisions to resolve it. The resulting fan contains m maximal cones, each of dimension three and the exceptional locus Δ of the resolution consists of $(m-1)/2$ prime divisors with compact support.

ii) More generally, any quotient singularity $\mathbb{C}^3/G$, where G is a finite abelian subgroup of $\mathrm{SL}(3,\mathbb{C})$ admits a crepant resolution (cf. [Nak01]).

The above resolution allows us to compute the Betti numbers of the exceptional locus of the resolution.

Proposition 4.2.19 (cf. Example 5.5 [BD96])**.** *Let U be an isolated cyclic quotient singularity of type*

$$\frac{1}{m}(1, a, m-a-1).$$

Let $\rho\colon U_{\mathcal{F}} \to U$ be the crepant resolution constructed above and $\Delta := \rho^{-1}(\{0\})$ be the exceptional locus. Then the Betti numbers of Δ are:

- $b^i(\Delta) = \dim\big(H^i(\Delta,\mathbb{C})\big) = 0$ *for* $i = 1, 3$ *and*

- $b^i(\Delta) = \dim\big(H^i(\Delta,\mathbb{C})\big) = \dfrac{m-1}{2}$ *for* $i = 2, 4.$

In particular $e(\Delta) = m$.

Proof. First we recall that canonical singularities are rational (see [El81] and [Fl81]), i.e. $R^i\rho_*(\mathcal{O}_{U_{\mathcal{F}}}) = 0$ for all $i \geq 1$. Leray's spectral sequence implies

$$H^i(U, \mathcal{O}_U) \simeq H^i(U_{\mathcal{F}}, \mathcal{O}_{U_{\mathcal{F}}})$$

for all $i \geq 0$. Since U is affine $H^i(U, \mathcal{O}_U) = 0$ for all $i \geq 1$ according to Serre [Se57]. The exponential sequence yields an isomorphism $\mathrm{Pic}(U_{\mathcal{F}}) \simeq H^2(U_{\mathcal{F}}, \mathbb{Z})$. Let d be the number of edges in the fan $\mathcal{F}$. By construction $d = (m-1)/2 + 3$ and all maximal cones in $\mathcal{F}$ are smooth and have dimension 3. According to [Ful93, Section 3.4] we have $\mathrm{rank}\big(\mathrm{Pic}(U_{\mathcal{F}})\big) = d - 3$ and it follows that the classes of the $(m-1)/2$ exceptional divisors form a basis of $H^2(U_{\mathcal{F}}, \mathbb{C})$. All these divisors have compact support. This implies that their classes are contained in $H^2_c(U_{\mathcal{F}}, \mathbb{C})$. Since $H^4(U_{\mathcal{F}}, \mathbb{C})$ is the Poincaré dual of $H^2_c(U_{\mathcal{F}}, \mathbb{C})$ we conclude that $h^4(U_{\mathcal{F}}, \mathbb{C}) = (m-1)/2$. Moreover, $H^i(\Delta, \mathbb{C}) \simeq H^i(U_{\mathcal{F}}, \mathbb{C})$ because Δ is a deformation retract of $U_{\mathcal{F}}$. To finish the proof it is enough to mention that the Euler number of a toric variety is equal to the number of cones of maximal dimension in its fan [Ful93, Section 3.2]. This implies $e(\Delta) = e(U_{\mathcal{F}}) = m$. □

Remark 4.2.20. Proposition 4.2.19 and its proof are just a special case of the so called *McKay correspondence*, which is a tool to compute the cohomology of a crepant resolution $\widehat{U}$ of a three dimensional Gorenstein quotient singularity $\mathbb{C}^3/G$. We refer the reader to [IR96] for this generalization.

4.3 The Invariants of Product Quotient Threefolds

In Chapter 2 we saw that the invariants $\chi(\mathcal{O}_X)$, $e(X)$ and K_X^3 of a threefold isogenous to a product

$$X = (C_1 \times C_2 \times C_3)/G$$

are determined by the order of the group G and the genera of the curves C_i. Explicitly

$$\chi(\mathcal{O}_X) = -\frac{1}{|G|}\prod_{i=1}^{3}\big(g(C_i)-1\big), \quad K_X^3 = -48\chi(\mathcal{O}_X) \quad \text{and} \quad 6e(X) = -K_X^3.$$

In this section we derive analogous formulas relating the invariants $\chi(\mathcal{O}_{\widehat{X}})$, $e(\widehat{X})$ and $K_{\widehat{X}}^3$ of a crepant terminalisation $\widehat{X}$ of a product quotient threefold X with canonical singularities (cf. Theorem 4.1.8). The difference to the formulas displayed above is a correction term depending on the singularities of X.

Definition 4.3.1. *Let X be an algebraic variety with isolated singularities. The basket of singularities $\mathcal{B}(X)$ is by definition the collection of the analytic germs $\big(X,x\big)$, where $x \in \operatorname{Sing}(X)$, modulo equivalence of isomorphic germs.*

In the case of a product quotient threefold X, the singularities are isolated cyclic quotient singularities (see Proposition 4.1.2) and we can think of $\mathcal{B}(X)$ as a multi-set

$$\left\{\lambda \times \frac{1}{m}(1,a,b) \;\middle|\; X \text{ has exactly } \lambda \text{ singularities of type } \tfrac{1}{m}(1,a,b)\right\}$$

modulo equivalence of isomorphic singularities (cf. Proposition 4.2.1). If X is canonical then, according to Theorem 4.2.13, each singularity is one of the following:

- a terminal singularity $\frac{1}{m}(1,a,m-a)$,
- a Gorenstein singularity $\frac{1}{m}(1,a,m-a-1)$ or
- a singularity of type III: $\frac{1}{9}(1,4,7)$ or $\frac{1}{14}(1,9,11)$.

In the previous section we explained how to construct a (smooth) crepant resolution of a Gorenstein singularity $\frac{1}{m}(1,a,m-a-1)$ and crepant terminalisations of the singularities $\frac{1}{9}(1,4,7)$ and $\frac{1}{14}(1,9,11)$. Recall that the latter admit three terminal singularities of type $\frac{1}{3}(1,1,2)$ and seven terminal singularities of type $\frac{1}{2}(1,1,1)$, respectively. Performing these resolutions and crepant terminalisations for all Gorenstein and for all singularities of type III of X we obtain a crepant terminalisation

$$\rho\colon \widehat{X} \to X$$

which we shall fix in the following. Note that the terminal singularities of X are not modified.

Convention: let X be a product quotient threefold with canonical singularities. Throughout this section we use the following notation: for a singularity $x \in \operatorname{Sing}(X)$, or more precisely a singular germ (X,x), we denote by m_x the order of the corresponding cyclic group. The integers N_1 and N_2 are the number of singularities of type

$$\frac{1}{9}(1,4,7) \qquad \text{and of type} \qquad \frac{1}{14}(1,9,11),$$

respectively.

Remark 4.3.2. By construction, the basket $\mathcal{B}(\widehat{X})$ consists of all terminal singularities of X together with $3N_1$ singularities of type $\frac{1}{3}(1,1,2)$ and $7N_2$ singularities of type $\frac{1}{2}(1,1,1)$.

Proposition 4.3.3. *Let $X = (C_1 \times C_2 \times C_3)/G$ be a product quotient threefold with canonical singularities. Then*

i) $K_{\widehat{X}}^3 = \dfrac{48}{|G|}\displaystyle\prod_{i=1}^{3}\big(g(C_i)-1\big)$ *and*

ii) $e(\widehat{X}) = -\dfrac{8}{|G|}\displaystyle\prod_{i=1}^{3}\big(g(C_i)-1\big) + \sum_{x\ gor}\frac{m_x^2-1}{m_x} + \sum_{x\ ter}\frac{m_x-1}{m_x} + \frac{26}{9}N_1 + \frac{97}{14}N_2.$

Proof. i) Since $\rho\colon \widehat{X} \to X$ is crepant, it holds $K_{\widehat{X}}^3 = K_X^3$ and the claim follows from Proposition 4.1.4:

$$K_{\widehat{X}}^3 = K_X^3 = \frac{48}{|G|}\prod_{i=1}^{3}\big(g(C_i)-1\big).$$

ii) Let $\pi\colon Y \to X$ be the quotient map, where $Y := C_1 \times C_2 \times C_3$ and define

$$X^0 := X \setminus \operatorname{Sing}(X) \qquad \text{and} \qquad Y^0 := Y \setminus \pi^{-1}\big(\operatorname{Sing}(X)\big).$$

The restriction $\pi|_{Y^0} : Y^0 \to X^0$ is unramified and has degree $|G|$, therefore

$$e(X^0) = \frac{e(Y^0)}{|G|}$$

which implies

$$e(X) - |\operatorname{Sing}(X)| = \frac{e(Y)}{|G|} - \sum_{x\in\operatorname{Sing}(X)}\frac{|\pi^{-1}(x)|}{|G|}$$

by the additivity of the Euler number. Using the equalities

$$|G| = m_x\ |\pi^{-1}(x)| \qquad \text{and} \qquad e(Y) = -8\prod_{i=1}^{3}\big(g(C_i)-1\big)$$

we obtain the formula

$$e(X) = -\frac{8}{|G|}\prod_{i=1}^{3}\big(g(C_i)-1\big) + \sum_{x\in \operatorname{Sing}(X)}\left(1-\frac{1}{m_x}\right).$$

It remains to relate $e(X)$ and $e(\widehat{X})$. The additivity of the Euler number yields

$$e(\widehat{X}) = e(X) + \sum_{x\ gor}\big(e(\Delta_x)-1\big) + \sum_{x\ type\ \text{III}}\big(e(\Delta_x)-1\big), \qquad \text{where} \qquad \Delta_x := \rho^{-1}(\{x\}).$$

According to Proposition 4.2.19 we have $e(\Delta_x) = m_x$ for a Gorenstein singularity of type

$$\frac{1}{m_x}(1, a_x, m_x - a_x - 1).$$

To derive this identity, we used that $e(\Delta_x)$ is equal to the number of three-dimensional cones in the fan defining the toric resolution of the singularity. When we consider the fans of the toric partial resolutions of

$$\frac{1}{9}(1,4,7) \qquad \text{and} \qquad \frac{1}{14}(1,9,11)$$

(see Lemma 4.2.16 and Lemma 4.2.17) and count the number of three-dimensional cones, we obtain $e(\Delta_x) = 3$ for $\frac{1}{9}(1,4,7)$ and $e(\Delta_x) = 7$ for $\frac{1}{14}(1,9,11)$ in the same way. □

We proceed to derive a similar formula for $\chi(\mathcal{O}_{\widehat{X}})$. The method we want to use is representation theoretic. As a first step, we show how to determine the invariants $h^p(\widehat{X}, \mathcal{O}_{\widehat{X}})$ form the characters $\chi_{p,0}$ of the representations

$$\phi_{p,0} \colon G \to \operatorname{GL}\big(H^{p,0}(C_1\times C_2\times C_3)\big), \quad g \mapsto [\omega \mapsto (g^{-1})^*\omega]$$

(cf. Section 2.4 for the case of threefolds isogenous to a product). Before we start, we introduce the *sheaf of reflexive differentials* on X, which is defined as

$$\Omega_X^{[p]} := j_*\Omega^p_{X\setminus \operatorname{Sing}(X)},$$

where $j\colon X\setminus \operatorname{Sing}(X) \to X$ is the inclusion map.

Proposition 4.3.4. *Let $X = (C_1\times C_2\times C_3)/G$ be a product quotient threefold and $h\colon Z\to X$ be a resolution of singularities. Then*

$$\Omega_X^{[p]} \simeq h_*\Omega_Z^p \qquad \textit{and} \qquad H^0(Z,\Omega_Z^p) \simeq H^0(X,\Omega_X^{[p]}) \simeq H^0(Y,\Omega_Y^p)^G,$$

where $Y := C_1\times C_2\times C_3$ and $0\le p\le 3$.

Proof. For a proof of the isomorphism $\Omega_X^{[p]} \simeq h_*\Omega_Z^p$ we refer the reader to [Ste77, Lemma

1.11]. The isomorphism yields

$$H^0(Z, \Omega^p_Z) \simeq H^0(X, \Omega^{[p]}_X),$$

by taking global sections, and it remains to show

$$H^0(X, \Omega^{[p]}_X) \simeq H^0(Y, \Omega^p_Y)^G.$$

We follow the argument presented in the proof of [Ste77, Lemma 1.8]. According to Proposition 4.1.2 the singular locus $\mathrm{Sing}(X)$ of X is the image of the finite set

$$\mathcal{R} := \big\{ q \in C_1 \times \ldots \times C_n \;\big|\; \mathrm{Stab}(q) \neq \{1_G\} \big\}$$

under the quotient map π. Clearly, the restriction $\pi_{|Y \setminus \mathcal{R}} \colon Y \setminus \mathcal{R} \to X \setminus \mathrm{Sing}(X)$ is an unramified Galois cover with group G. Therefore, pull-back along $\pi_{|Y\setminus\mathcal{R}}$ induces an isomorphism

$$H^0(X \setminus \mathrm{Sing}(X), \Omega^p_{X\setminus \mathrm{Sing}(X)}) \to H^0(Y \setminus \mathcal{R}, \Omega^p_{Y\setminus\mathcal{R}})^G$$

cf. [Be83, Lemma VI.11 and Example VI.12 1)]. By definition, the cohomology group on the left-hand side coincides with $H^0(X, \Omega^{[p]}_X)$. Finally, by Hartogs theorem, every holomorphic p-form on $Y \setminus \mathcal{R}$ can be extended holomorphically to a p-form on Y yielding an isomorphism $H^0(Y \setminus \mathcal{R}, \Omega^p_{Y\setminus\mathcal{R}}) \simeq H^0(Y, \Omega^p_Y)$ which restricts to the G-invariant forms. □

Proposition 4.3.5. *Let $X = (C_1 \times C_2 \times C_3)/G$ be a product quotient threefold with canonical singularities. Then*

$$h^p(\widehat{X}, \mathcal{O}_{\widehat{X}}) = \langle \chi_{p,0}, \chi_{triv} \rangle, \qquad \textit{for all} \qquad 0 \leq p \leq 3.$$

Proof. The crepant terminalisation ρ fits into a commutative diagram

$$\begin{array}{ccc} Z & \longrightarrow & X \\ & {\scriptstyle h}\searrow \qquad \nearrow{\scriptstyle \rho} & \\ & \widehat{X}, & \end{array}$$

where $h \colon Z \to \widehat{X}$ is a resolution of singularities of $\widehat{X}$. Since $\widehat{X}$ has only terminal singularities, terminal singularities are canonical and canonical singularities are rational (see [El81] and [Fl81]), we have

$$H^p(\widehat{X}, \mathcal{O}_{\widehat{X}}) \simeq H^p(Z, \mathcal{O}_Z).$$

According to Proposition 4.3.4 it holds $H^0(Z, \Omega^p_Z) \simeq H^0(Y, \Omega^p_Y)^G$, where Y denotes the product $C_1 \times C_2 \times C_3$. Therefore,

$$h^p(\widehat{X}, \mathcal{O}_{\widehat{X}}) = h^p(Z, \mathcal{O}_Z) = h^0(Z, \Omega_Z^p) = \langle \chi_{p,0}, \chi_{triv} \rangle.$$

Note that the equality in the middle is a consequence of Hodge symmetry on the projective manifold Z. □

Remark 4.3.6.

i) The proposition above allows us to compute the invariants

$$p_g(\widehat{X}) := h^3(\widehat{X}, \mathcal{O}_{\widehat{X}}), \quad q_2(\widehat{X}) := h^2(\widehat{X}, \mathcal{O}_{\widehat{X}}) \quad \text{and} \quad q_1(\widehat{X}) := h^1(\widehat{X}, \mathcal{O}_{\widehat{X}})$$

using an algebraic datum $\mathcal{A}$ of X, because $\mathcal{A}$ determines the characters $\chi_{p,0}$ (cf. Remark 2.4.10).

ii) From the proof of the proposition it follows that

$$p_g(X') = p_g(\widehat{X}), \quad q_2(X') = q_2(\widehat{X}) \quad \text{and} \quad q_1(X') = q_1(\widehat{X})$$

for any other crepant terminalisation $\rho\colon X' \to X$.

Lemma 4.3.7. *Let X be a product quotient threefold with canonical singularities. Then*

$$\chi(\mathcal{O}_{\widehat{X}}) = -\frac{1}{|G|}\prod_{i=1}^{3}\big(g(C_i) - 1\big) + \frac{1}{|G|}\sum_{\substack{g \in G \\ g \neq 1_G}} \chi_{hol}(g),$$

where $\chi_{hol} := \chi_{triv} - \chi_{1,0} + \chi_{2,0} - \chi_{3,0}$.

Proof. According to Proposition 4.3.5 it holds $\chi(\mathcal{O}_{\widehat{X}}) = \langle \chi_{hol}, \chi_{triv} \rangle$. Expanding the inner product yields:

$$\chi(\mathcal{O}_{\widehat{X}}) = \frac{1}{|G|}\sum_{g \in G} \chi_{hol}(g) = \frac{\chi_{hol}(1_G)}{|G|} + \frac{1}{|G|}\sum_{\substack{g \in G \\ g \neq 1_G}} \chi_{hol}(g).$$

To finish the proof, it suffices to mention that

$$\chi_{hol}(1_G) = \chi(\mathcal{O}_{C_1 \times C_2 \times C_3}) = \prod_{i=1}^{3}\chi(\mathcal{O}_{C_i}) = -\prod_{i=1}^{3}\big(g(C_i) - 1\big),$$

which follows from the definition of χ_{hol} and the product property of the holomorphic Euler-Poincaré-characteristic. □

Our remaining task is to rewrite the correction term

$$\frac{1}{|G|}\sum_{\substack{g\in G\\ g\neq 1_G}} \chi_{hol}(g)$$

in terms of the singularities of X. The tool for this purpose is the *holomorphic Lefschetz fixed point formula.*

Theorem 4.3.8 ([GH78, p. 426])**.** *Let Y be a compact complex manifold and $g\colon Y\to Y$ be an automorphism of finite order with isolated fixed points. Then*

$$\sum_{q=0}^{\dim(Y)} (-1)^q \operatorname{tr}(g^*_{|H^{0,q}(Y)}) = \sum_{p\in \operatorname{Fix}(g)} \frac{1}{\det\big(\operatorname{id} - J_p(g)\big)},$$

where $J_p(g)$ is the Jacobian of g in the point p.

To relate the left-hand side of the holomorphic Lefschetz fixed fixed point formula and χ_{hol} we use the following lemma from Kähler geometry.

Lemma 4.3.9. *Let Y be a compact Kähler manifold and $g\colon Y\to Y$ be a holomorphic map then*

$$Tr(g^*|_{H^{p,q}(Y)}) = \overline{Tr(g^*|_{H^{q,p}(Y)})}\,.$$

Proof. Let $\{[\omega_1],...,[\omega_k]\}$ be a basis of $H^{p,q}(Y)$ with ω_i harmonic, then $\{[\overline{\omega_1}],...,[\overline{\omega_k}]\}$ is a basis of $H^{q,p}(Y)$ and the forms $\overline{\omega_i}$ are harmonic. Now we write

$$[g^*\omega_j] = \sum_{i=1}^{k} a_{ij}\cdot[\omega_i] \qquad \text{or equivalently} \qquad g^*\omega_j = \sum_{i=1}^{k} a_{ij}\cdot\omega_i + \overline{\partial}\eta_j,$$

where $a_{ij}\in\mathbb{C}$ and η_j is a form of type $(p,q-1)$. Complex conjugation yields

$$g^*\overline{\omega_j} = \sum_{i=1}^{k} \overline{a_{ij}}\cdot\overline{\omega_i} + \partial\overline{\eta_j}\,.$$

Note that both $g^*\overline{\omega_j}$ and $\sum_{i=1}^{k}\overline{a_{ij}}\cdot\overline{\omega_i}$ are d-closed forms, therefore $\partial\overline{\eta_j}$ is a d-closed form which is ∂-exact. By the $\partial\overline{\partial}$-lemma [Hu05, Corollary 3.2.10] there exists a $(q,p-1)$ form ξ_j such that $\partial\overline{\eta_j} = \overline{\partial}\xi_j$ and we obtain

$$[g^*\overline{\omega_j}] = \sum_{i=1}^{k} \overline{a_{ij}}\cdot[\overline{\omega_i}]\,.$$

□

Theorem 4.3.10. *Let X be a product quotient threefold with canonical singularities. Then*

$$\chi(\mathcal{O}_{\widehat{X}}) = -\frac{1}{|G|}\prod_{i=1}^{3}\big(g(C_i)-1\big) + \frac{1}{24}\sum_{x\ ter}\frac{m_x^2-1}{m_x} + \frac{1}{3}N_1 + \frac{7}{16}N_2.$$

Proof. By definition of $\chi_{p,0}$ and Lemma 4.3.9 it holds

$$\begin{aligned}\chi_{hol}(g) = \sum_{p=0}^{3}(-1)^p \cdot \overline{\chi_{p,0}(g^{-1})} \quad &= \quad \sum_{p=0}^{3}(-1)^p \cdot \overline{\operatorname{tr}\left(g^*_{|H^{p,0}(Y)}\right)} \\ &= \quad \sum_{p=0}^{3}(-1)^p \cdot \operatorname{tr}\left(g^*_{|H^{0,p}(Y)}\right),\end{aligned}$$

for all $g \in G$. As usual, Y denotes the product $C_1 \times C_2 \times C_3$. Next, we apply the holomorphic Lefschetz fixed point formula and rewrite the correction term from Lemma 4.3.7:

$$\frac{1}{|G|}\sum_{\substack{g\in G\\ g\neq 1_G}}\chi_{hol}(g) = \frac{1}{|G|}\sum_{\substack{g\in G\\ g\neq 1_G}}\sum_{p\in \operatorname{Fix}(g)}\frac{1}{\det\big(\operatorname{id}-J_p(g)\big)}.$$

The double sum on the right-hand side is equal to

$$\frac{1}{|G|}\sum_{p\in\operatorname{Fix}(Y)}\sum_{\substack{g\in G_p\\ g\neq 1_G}}\frac{1}{\det\big(\operatorname{id}-J_p(g)\big)},$$

where $\operatorname{Fix}(Y)$ is the set of points of Y with non-trivial stabilizer group, which is precisely the pre-image of the singular locus of X under the quotient map

$$\pi\colon Y \to X.$$

For all $x \in \operatorname{Sing}(X)$, we choose a point $y_x \in \pi^{-1}(x)$. Then $m_x = |\operatorname{Stab}(y_x)|$ and, according to the orbit-stabilizer correspondence, the points in the fibre $\pi^{-1}(x)$ are in bijection with the quotient $G/\operatorname{Stab}(y_x)$. Moreover,

$$\det\big(\operatorname{id}-J_{h(y_x)}(hgh^{-1})\big) = \det\big(\operatorname{id}-J_{y_x}(g)\big)$$

for all $g \in \operatorname{Stab}(y_x)$ and $h \in G$. This enables us to rewrite the double sum above as

$$\frac{1}{|G|}\sum_{x\in\operatorname{Sing}(X)}\frac{|G|}{m_x}\sum_{\substack{g\in G_{y_x}\\ g\neq 1_G}}\frac{1}{\det\big(\operatorname{id}-J_{y_x}(g)\big)} = \sum_{x\in\operatorname{Sing}(X)}\frac{1}{m_x}\sum_{\substack{g\in G_{y_x}\\ g\neq 1_G}}\frac{1}{\det\big(\operatorname{id}-J_{y_x}(g)\big)}.$$

The stabilizer group G_{y_x} acts, in suitable local coordinates, linearly. Let $\frac{1}{m_x}(1,a_x,b_x)$ be the type of the analytic germ $\big(X,x\big)$, then G_{y_x} is generated by the diagonal matrix

$$\operatorname{diag}\left(\exp\left(\frac{2\pi\sqrt{-1}}{m_x}\right),\exp\left(\frac{2\pi a_x\sqrt{-1}}{m_x)}\right),\exp\left(\frac{2\pi b_x\sqrt{-1}}{m_x}\right)\right).$$

Therefore

$$\sum_{\substack{g\in G_{y_x}\\ g\neq 1_G}} \frac{1}{\det\big(\mathrm{id}-J_{y_x}(g)\big)} = \sum_{\substack{\xi^{m_x}=1\\ \xi\neq 1}} \frac{1}{(1-\xi)(1-\xi^{a_x})(1-\xi^{b_x})},$$

where the sum on the right-hand side runs over all $\xi \in \mathbb{C}\setminus\{1\}$ with $\xi^{m_x}=1$. To finish the proof, it suffices to give a closed expression for the *Dedekind sums*

$$\sigma\big(\frac{1}{m_x}(1,a_x,b_x)\big) := \sum_{\substack{\xi^{m_x}=1\\ \xi\neq 1}} \frac{1}{(1-\xi)(1-\xi^{a_x})(1-\xi^{b_x})}$$

for all types of singularities that can occur. We have

- $\sigma\big(\frac{1}{m_x}(1,a_x,m_x-a_x)\big) = \frac{m_x^2-1}{24}$ according to [Reid87, 8.10] and

- $\sigma\big(\frac{1}{9}(1,4,7)\big) = 3$ and $\sigma\big(\frac{1}{14}(1,9,11)\big) = \frac{49}{8}$ via an explicit computation.

The remaining case is the case of a Gorenstein singularity, here we claim that

$$\sigma\big(\frac{1}{m_x}(1,a_x,m_x-a_x-1)\big) = 0,$$

i.e. these singularities do not contribute. First we recall that m_x-1 is even because m_x is odd (see Remark 4.2.14 ii)). With the help of the identity

$$\sum_{i=0}^{m_x-1} \xi^i = 0,$$

which holds for all $\xi\in\mathbb{C}\setminus\{1\}$ with $\xi^{m_x}=1$, we get

$$\begin{aligned}\sigma\big(\frac{1}{m_x}(1,a_x,-a_x-1)\big) \quad = \quad & -\sum_{i=1}^{(m_x-1)/2}\sum_{\substack{\xi^{m_x}=1\\ \xi\neq 1}} \frac{\xi^i}{(1-\xi)(1-\xi^{a_x})(1-\xi^{-a_x-1})}\\ & -\sum_{i=1}^{(m_x-1)/2}\sum_{\substack{\xi^{m_x}=1\\ \xi\neq 1}} \frac{\xi^{m_x-i}}{(1-\xi)(1-\xi^{a_x})(1-\xi^{-a_x-1})}.\end{aligned}$$

Next we look at the denominator of the summands. It holds

$$(1-\xi)(1-\xi^{a_x})(1-\xi^{-a_x-1}) = (\xi^{-1}-1)(\xi^{-a_x}-1)(\xi^{a_x+1}-1)$$

and therefore

$$\sigma\big(\frac{1}{m_x}(1,a_x,-a_x-1)\big) = -\sum_{i=1}^{(m_x-1)/2}\sum_{\substack{\xi^{m_x}=1\\ \xi\neq 1}}\frac{\xi^i}{(1-\xi)(1-\xi^{a_x})(1-\xi^{-a_x-1})} + \sum_{i=1}^{(m_x-1)/2}\sum_{\substack{\xi^{m_x}=1\\ \xi\neq 1}}\frac{\xi^{-i}}{(1-\xi^{-1})(1-\xi^{-a_x})(1-\xi^{a_x+1})}.$$

Now we exchange ξ with ξ^{-1} in the second double sum and the claim

$$\sigma\big(\frac{1}{m_x}(1,a_x,-a_x-1)\big) = 0$$

follows. We are allowed to exchange ξ with ξ^{-1}, because the sum runs over all $\xi \in \mathbb{C}\setminus\{1\}$ with $\xi^{m_x}=1$. □

Remark 4.3.11. We should mention that the theorem above can also be deduced from Miles Reid's *orbifold Riemann Roch formula* ([Reid87, 10.3]). However, we decided to give a proof which relies on methods from character theory that we developed and used in the previous parts of the thesis. Note the similarity to the proof of the Chevalley-Weil formula (see Theorem 1.3.3).

Proposition 4.3.3 and Theorem 4.3.10 imply the following result:

Proposition 4.3.12. *Let X be a product quotient threefold with canonical singularities. Then*

i) $48\chi(\mathcal{O}_{\widehat{X}}) + K_{\widehat{X}}^3 = 2\sum_{x\ ter}\frac{m_x^2-1}{m_x} + 16N_1 + 21N_2$ *and*

ii) $6e(\widehat{X}) + K_{\widehat{X}}^3 = 6\sum_{x\ gor}\frac{m_x^2-1}{m_x} + 6\sum_{x\ ter}\frac{m_x-1}{m_x} + \frac{52}{3}N_1 + \frac{291}{7}N_2.$

In particular, it holds

$$48\chi(\mathcal{O}_{\widehat{X}}) + K_{\widehat{X}}^3 \geq 0$$

with equality if and only if $\widehat{X}$ is smooth or equivalently X has Gorenstein singularities and

$$6e(\widehat{X}) + K_{\widehat{X}}^3 \geq 0$$

with equality if and only if X is a threefold isogenous to a product.

Remark 4.3.13. Let $\rho\colon X' \to X$ be another crepant terminalisation of X then $\mathcal{B}(X') = \mathcal{B}(\widehat{X})$ according to [Ko89, Corollary 4.11]. In particular $\widehat{X}$ is smooth if and only if X' is smooth.

4.4 How to determine the Basket

In this section we explain how to compute the basket of singularities of a product quotient threefold

$$X = \big(C_1 \times C_2 \times C_3\big)/G$$

from an algebraic datum of X. We follow the method presented in [BP12, § 1.2].

Note that the natural projections $p_i \colon C_1 \times C_2 \times C_3 \to C_i$ induce holomorphic maps

$$\mathfrak{p}_i \colon X \to C_i/G.$$

Clearly, for each $x \in \operatorname{Sing}(X)$ the image $q_i := \mathfrak{p}_i(x)$ is contained in the branch locus $\mathcal{B}_i$ of $F_i \colon C_i \to C_i/G$. Since the action of G on the product $C_1 \times C_2 \times C_3$ is diagonal, we can restrict it to the product of the fibres $F_1^{-1}(q_1) \times F_2^{-1}(q_2) \times F_3^{-1}(q_3)$ and consider the quotient

$$\frac{F_1^{-1}(q_1) \times F_2^{-1}(q_2) \times F_3^{-1}(q_3)}{G}$$

as a subset of X. The important observation is that the singular points of X belonging to the intersection of the fibres $\mathfrak{p}_i^{-1}(q_i)$ are contained in the quotient above. We need to describe this quotient in terms of an algebraic datum of X, detect the points which are singular and determine the analytic type of these singularities.

Definition 4.4.1. *Let X be a product quotient threefold with algebraic datum (G, V_1, V_2, V_3). We define the cyclic subgroups*

$$H_{i,j} := \langle h_{i,j} \rangle \leq G \quad \textit{for all} \quad 1 \leq i \leq 3 \quad \textit{and} \quad 1 \leq j \leq r_i,$$

where $h_{i,j}$ are the elements in the generating vectors

$$V_i = (h_{i,1}, \ldots, h_{i,r_i}, a_{i,1}, b_{i,1}, \ldots, a_{i,g_i'}, b_{i,g_i'}).$$

Recall that the branch points $q_{i,1}, \ldots, q_{i,r_i}$ of the covers F_i are in one-to-one correspondence with the elements $h_{i,1}, \ldots, h_{i,r_i}$ (see Section 1.1) and the choice of a point $x_{i,j} \in F_i^{-1}(q_{i,j})$ with stabilizer group $H_{i,j} = \langle h_{i,j} \rangle$ determines a G-equivariant bijection

$$G/H_{i,j} \to F_i^{-1}(q_{i,j}), \qquad gH_{i,j} \mapsto g(x_{i,j}).$$

Under this bijection the stabilizer group of the "point" $gH_{i,j}$ is $gH_{i,j}g^{-1}$.

Remark 4.4.2. Let H be a non-trivial subgroup of a finite cyclic group $\langle g \rangle$ and

$$\delta := \min\{0 < k < \operatorname{ord}(g) \mid g^k \in H\}.$$

Then

$$H = \langle g^\delta \rangle \qquad \text{and} \qquad |H| = \frac{\mathrm{ord}(g)}{\delta}.$$

Now it is straightforward to determine the basket of singularities $\mathcal{B}(X)$ of a product quotient threefold X from an algebraic datum.

Proposition 4.4.3. *Let X be a product quotient threefold with algebraic datum (G, V_1, V_2, V_3). Then:*

i) there are (non-canonical) bijections

$$\frac{G/H_{2,k} \times G/H_{3,l}}{H_{1,j}} \longrightarrow \frac{F_1^{-1}(q_{1,j}) \times F_2^{-1}(q_{2,k}) \times F_3^{-1}(q_{3,l})}{G}.$$

ii) a class $[gH_{2,k}, g'H_{3,l}]$ corresponds to a singular point in the intersection of the fibres

$$\mathfrak{p}_1^{-1}(q_{1,j}), \quad \mathfrak{p}_2^{-1}(q_{2,k}) \quad \textit{and} \qquad \mathfrak{p}_3^{-1}(q_{3,l})$$

if and only if

$$H := H_{1,j} \cap gH_{2,k}g^{-1} \cap g'H_{3,l}g'^{-1}$$

is a non-trivial subgroup of $H_{1,j}$. In this case, we write $H = \langle h_{1,j}^\delta \rangle$ according to Remark 4.4.2. Then, the type of the corresponding singular point is $\frac{1}{m}(1,a,b)$, where

$$m := |H|, \quad a := \frac{m\gamma}{\mathrm{ord}(h_{2,k})} \quad \textit{and} \quad b := \frac{m\mu}{\mathrm{ord}(h_{3,l})}.$$

Here $1 \leq \gamma < \mathrm{ord}(h_{2,k})$ and $1 \leq \mu < \mathrm{ord}(h_{3,l})$ are the unique integers such that

$$gh_{2,k}^\gamma g^{-1} = h_{1,j}^\delta \qquad \textit{and} \qquad g'h_{3,l}^\mu g'^{-1} = h_{1,j}^\delta.$$

Proof. *i)* We choose a point

$$(x_{1,j}, x_{2,k}, x_{3,l}) \in F_1^{-1}(q_{1,j}) \times F_2^{-1}(q_{2,k}) \times F_3^{-1}(q_{3,l})$$

such that

$$G_{x_{1,j}} = H_{1,j}, \quad G_{x_{2,k}} = H_{2,k} \quad \text{and} \quad G_{x_{3,l}} = H_{3,l}.$$

It is easy to see that the map sending the class $[gH_{2,k}, g'H_{3,l}]$ to the class $[x_{1,j}, g(x_{2,k}), g'(x_{3,l})]$ is a bijection.
ii) The point $[x_{1,j}, g(x_{2,k}), g'(x_{3,l})]$ corresponding to the class $[gH_{2,k}, g'H_{3,l}]$ is singular if and only if the intersection of the stabilizer groups

$$G_{x_{1,j}} \cap G_{g(x_{2,k})} \cap G_{g'(x_{3,l})}$$

is non-trivial. The intersection above is equal to H.

iii) The cyclic group H is generated by the element

$$h := h_{1,j}^{\delta} = g h_{2,k}^{\gamma} g^{-1} = g' h_{3,l}^{\mu} g'^{-1}.$$

It acts in suitable local coordinates around the point $\big(x_{1,j}, g(x_{2,k}), g'(x_{3,l})\big)$ via the diagonal matrix

$$\operatorname{diag}\left(\exp\left(\frac{2\pi\delta\sqrt{-1}}{\operatorname{ord}(h_{1,j})}\right), \exp\left(\frac{2\pi\gamma\sqrt{-1}}{\operatorname{ord}(h_{2,k})}\right), \exp\left(\frac{2\pi\mu\sqrt{-1}}{\operatorname{ord}(h_{3,l})}\right)\right)$$

(see Lemma 1.3.2). Since h has order m, there are unique integers $1 \leq a, b < m$ such that

$$\frac{a}{m} = \frac{\gamma}{\operatorname{ord}(h_{2,k})} \qquad \text{and} \qquad \frac{b}{m} = \frac{\mu}{\operatorname{ord}(h_{3,l})}.$$

We conclude that the type of the corresponding singularity is $\frac{1}{m}(1, a, b)$. □

4.5 The numerical Datum of a Product Quotient Threefold

In analogy to the case of threefolds isogenous to a product we attach to a product quotient threefold $X = \big(C_1 \times C_2 \times C_3\big)/G$ a numerical datum

$$(n, T_1, T_2, T_3),$$

where $n = |G|$ and $T_i = [g_i'; m_{i,1}, \ldots, m_{i,r_i}]$ are the types of the generating vectors V_i from an algebraic datum (G, V_1, V_2, V_3) of X (cf. Definition 4.1.10). In this section we provide combinatorial constraints on the numerical datum. Some of the constraints holding in the case of threefolds isogenous to a product (see Chapter 3) are still valid for product quotient threefolds, but in general we have only weaker versions. Clearly, if X is not isogenous to a product, then $r_i \geq 1$ for all $1 \leq i \leq 3$.

Proposition 4.5.1. *Let X be a product quotient threefold. Then*

$$n \leq \left\lfloor \sqrt{\frac{K_X^3}{6} \prod_{i=1}^{3} \frac{1}{\Theta_{min}(T_i)}} \right\rfloor, \qquad \textit{where} \qquad \Theta_{min}(T_i) := \begin{cases} 1/42, & \textit{if} \;\; g_i' = 0 \\ 1/2, & \textit{if} \;\; g_i' = 1 \\ 2g_i' - 2, & \textit{if} \;\; g_i' \geq 2 \end{cases}$$

We skip the proof of the proposition because it is similar to the proof of the corresponding bound in the case of threefolds isogenous to a product (cf. Proposition 3.0.6).

Proposition 4.5.2. *Let X be a product quotient threefold with numerical datum (n, T_1, T_2, T_3). Then*

$$\frac{48}{K_X^3} \leq n \leq \lfloor 42\sqrt{7 \cdot K_X^3} \rfloor.$$

Proof. The inequality on the left-hand side follows from

$$1 \leq \prod_{i=1}^{3} \big(g(C_i) - 1\big) = n\frac{K_X^3}{48}.$$

The inequality on the right-hand side is a direct consequence of Proposition 4.5.1 using $\Theta_{min}(T_i) \geq 1/42$ (cf. Corollary 3.0.7). □

Next we show that product quotient threefolds with $K_X^3 < 1$ do not exist.

Corollary 4.5.3. *For a product quotient threefold X it holds $K_X^3 \geq 1$.*

Proof. Let X be a product quotient threefold with $K_X^3 \leq 1$. Then, by Proposition 4.5.2, the group order n is in the range $48 \leq n \leq 111$. It follows that

$$\prod_{i=1}^{3} \big(g(C_i) - 1\big) = n\frac{K_X^3}{48} \leq \frac{111}{48}.$$

W.l.o.g. we assume $g(C_1) \leq g(C_2) \leq g(C_3)$. Then, there are two possibilities for the triple

$$\big(g(C_1), g(C_2), g(C_3)\big),$$

namely $(2, 2, 2)$ and $(2, 2, 3)$. According to Conders table [Con14], the order of a group G acting faithfully on a compact Riemann surface of genus 2 is bounded by 48, a fact which was already known to Bolza [Bol87] in 1887. Consequently $n = 48$ and we can exclude the possibility $(2, 2, 3)$. It follows that $K_X^3 = 1$. □

The bound from Corollary 4.5.3 is actually sharp:

Example 4.5.4. The group $\mathrm{GL}(2, \mathbb{F}_3)$ of order 48 admits the generating vector

$$V = \left(\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \right)$$

of type $T = [0; 2, 3, 8]$. By Riemann's existence theorem, there is a Galois cover $C \to \mathbb{P}^1$ with group $\mathrm{GL}(2, \mathbb{F}_3)$ branched over $0, 1$ and ∞ with branching indices $2, 3$ and 8, respectively. The compact Riemann surface C has genus 2. Consider the diagonal action of $\mathrm{GL}(2, \mathbb{F}_3)$ on C^3 given by three copies of the action $\psi\colon \mathrm{GL}(2, \mathbb{F}_3) \to \mathrm{Aut}(C)$ from above, then

$$X := C^3 / \mathrm{GL}(2, \mathbb{F}_3)$$

is a product quotient threefold with $K_X^3 = 1$.

Remark 4.5.5. Unfortunately our example admits non-canonical singularities: indeed, there exists a point $p \in C$ in the fibre over ∞ with stabilizer group generated by the matrix

$$h := \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

such that the Jacobian of h at p is given by

$$J_p(h) = \exp\left(\frac{\pi\sqrt{-1}}{4}\right).$$

Since the action of $\mathrm{GL}(2, \mathbb{F}_3)$ on C^3 is the same on each factor, the point $(p, p, p) \in C^3$ descends to a singularity of type $\frac{1}{8}(1, 1, 1)$ which is, according to Theorem 4.2.13, non-canonical. However, this is not a coincidence: we refer the reader to Chapter 6, where we prove that $K_X^3 \geq 4$ is the sharp bound for product quotient threefolds with canonical singularities.

Proposition 4.5.6. *Let $X = (C_1 \times C_2 \times C_3)/G$ be a product quotient threefold with numerical datum (n, T_1, T_2, T_3), where $T_i := [g_i'; m_{i,1}, \dots, m_{i,r_i}]$. Then*

i) $\big(g(C_i) - 1\big)$ *divides the integer* $K_X^3 \dfrac{n}{48}$,

ii) $r_i \leq \dfrac{4\big(g(C_i) - 1\big)}{n} - 4g_i' + 4$,

iii) $m_{i,j} \leq 4g(C_i) + 2$,

iv) $g_i' \leq 1 + \dfrac{K_X^3}{48\big(g(C_{[i+1]}) - 1\big)\big(g(C_{[i+2]}) - 1\big)} \leq 1 + \dfrac{K_X^3}{48}$,

for all $1 \leq i \leq 3$ and $1 \leq j \leq r_i$. The brackets $[\cdot]$ denote the residue mod 3;

Proof. Albeit the proof works in the same way as the proof of the corresponding result for threefolds isogenous to a product (cf. Proposition 3.0.10), we shall sketch it:

i) follows directly from the formula

$$K_X^3 = \frac{48}{n} \prod_{i=1}^{3} \big(g(C_i) - 1\big).$$

ii) is a direct consequence of Hurwitz' formula and iii) follows from Wiman's bound. To prove iv), we estimate

$$g_i' - 1 \leq \frac{\Theta(T_i)}{2} = \frac{g(C_i) - 1}{n} = \frac{K_X^3}{48\big(g(C_{[i+1]}) - 1\big)\big(g(C_{[i+2]}) - 1\big)} \leq \frac{K_X^3}{48}.$$

The inequality on the right-hand side holds, because $\big(g(C_{[i+1]}) - 1\big)\big(g(C_{[i+2]}) - 1\big) \geq 1$.

□

Chapter 5

Smooth Minimal Models

In this chapter we study product quotient threefolds

$$X = (C_1 \times C_2 \times C_3)/G$$

with canonical singularities admitting a crepant resolution $\rho\colon \widehat{X} \to X$ of singularities. This is equivalent to the condition that X is Gorenstein (see Proposition 4.3.12). The invariants of $\widehat{X}$ are related as follows:

$$48\chi(\mathcal{O}_{\widehat{X}}) + K_{\widehat{X}}^3 = 0 \quad \text{and} \quad 6e(\widehat{X}) + K_{\widehat{X}}^3 \geq 0,$$

where the inequality on the right-hand side is strict if and only if X is singular (see Proposition 4.3.12). From the equation on the left-hand side we observe that $\chi(\mathcal{O}_{\widehat{X}})$ is negative because $K_{\widehat{X}}^3$ is a positive integer, just like in the case of threefolds isogenous to a product. Our goal in this section is to give an algorithm to classify these varieties for a fixed value of $\chi(\mathcal{O}_{\widehat{X}}) \leq -1$.

For threefolds isogenous to a product we have a method to compute the Hodge numbers (see Section 2.4). Here we are confronted with the following questions:

- how can we compute the Hodge numbers of $\widehat{X}$?
- are the Hodge numbers independent of the chosen crepant resolution?

The answer to the second question is provided by a celebrated theorem of Kontsevich (see [Kon95]).

Theorem 5.0.1 (cf. [Craw04, Theorem 3.6])**.** *Let X be a complex projective variety with at worst Gorenstein canonical singularities. If X admits a crepant resolution $\rho\colon \widehat{X} \to X$ then the Hodge numbers of $\widehat{X}$ are independent of the choice of the crepant resolution.*

Convention: let X be a product quotient threefold with Gorenstein singularities. In the following we shall fix the crepant resolution

$$\rho\colon \widehat{X} \to X$$

obtained by resolving the Gorenstein singularities

$$\frac{1}{m_x}(1, a_x, m_x - a_x - 1), \qquad x \in S := \mathrm{Sing}(X)$$

as described in Section 4.2. The exceptional locus of ρ is denoted by

$$E := \bigsqcup_{x \in S} \Delta_x, \qquad \text{where} \qquad \Delta_x = \rho^{-1}(x).$$

Now we address the first question from above. According to Proposition 4.3.5 it holds

$$h^{p,0}(\widehat{X}) = \langle \chi_{p,0}, \chi_{triv} \rangle, \quad \text{for all} \quad 0 \le p \le 3.$$

where $\chi_{p,0}$ are the characters of the representations

$$\phi_{p,0}\colon G \to \mathrm{GL}\left(H^{p,0}(C_1 \times C_2 \times C_3)\right), \quad g \mapsto [\omega \mapsto (g^{-1})^* \omega].$$

Hodge decomposition and symmetry yields

$$h^{1,1}(\widehat{X}) = b^2(\widehat{X}) - 2h^{2,0}(\widehat{X}) \quad \text{and} \quad h^{2,1}(\widehat{X}) = \frac{b^3(\widehat{X})}{2} - h^{3,0}(\widehat{X}),$$

where

$$b^i(\widehat{X}) := \dim\left(H^i(\widehat{X}, \mathbb{C})\right)$$

are the Betti numbers. Hence, it suffices to compute $b^2(\widehat{X})$ and $b^3(\widehat{X})$. As a first step, we need to understand the relation between $H^*(X, \mathbb{C})$ and $H^*(\widehat{X}, \mathbb{C})$.

Proposition 5.0.2 ([BK82, Proposition 3.A.7])**.** *Let $\rho\colon \widehat{X} \to X$ be a morphism between projective varieties. Let $S \subset X$ be a closed subvariety and $E := \rho^{-1}(S)$. Assume that ρ induces an isomorphism between $\widehat{X} \setminus E$ and $X \setminus S$, then there exists a long exact cohomology sequence*

$$\ldots \to H^k(X, \mathbb{C}) \to H^k(\widehat{X}, \mathbb{C}) \oplus H^k(S, \mathbb{C}) \to H^k(E, \mathbb{C}) \to H^{k+1}(X, \mathbb{C}) \to \ldots .$$

In particular $e(\widehat{X}) = e(X) - e(S) + e(E)$.

Proof. To avoid a bulky notation, we omit the coefficients of the cohomology groups. By assumption S is closed in X and therefore $E = \rho^{-1}(S)$ is closed in $\widehat{X}$. The long exact sequence for cohomology with compact support yields:

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & H^k_c(X \setminus S) & \xrightarrow{i_k} & H^k_c(X) & \xrightarrow{j_k} & H^k_c(S) & \xrightarrow{\delta_k} & H^{k+1}_c(X \setminus S) & \longrightarrow \cdots \\
& \downarrow{a_k} & & \downarrow{b_k} & & \downarrow{c_k} & & \downarrow{a_{k+1}} & \\
\cdots \longrightarrow & H^k_c(\widehat{X} \setminus E) & \xrightarrow{\widehat{i_k}} & H^k_c(\widehat{X}) & \xrightarrow{\widehat{j_k}} & H^k_c(E) & \xrightarrow{\widehat{\delta_k}} & H^{k+1}_c(\widehat{X} \setminus E) & \longrightarrow \cdots
\end{array}$$

The homomorphisms a_k, b_k and c_k are induced by (the proper map) ρ and a_k is an isomorphism for all k. Note that the cohomology groups $H^k_c(X)$, $H^k_c(S)$, $H^k_c(\widehat{X})$ and $H^k_c(E)$ coincide with the singular cohomology groups because the underlying spaces are compact. A standard diagram chase shows that the sequence

$$\cdots \longrightarrow H^k(X) \overset{b_k \oplus j_k}{\longrightarrow} H^k(\widehat{X}) \oplus H^k(S) \overset{\widehat{j_k} - c_k}{\longrightarrow} H^k(E) \overset{i_{k+1} \circ a^{-1}_{k+1} \circ \widehat{\delta_k}}{\longrightarrow} H^{k+1}(X) \longrightarrow \cdots$$

is exact. □

Proposition 5.0.3. *Let $X = \big(C_1 \times C_2 \times C_3\big)/G$ be a product quotient threefold with Gorenstein singularities, then:*

i) the resolution map $\rho \colon \widehat{X} \to X$ induces isomorphisms

$$\rho^* \colon H^k(X, \mathbb{C}) \to H^k(\widehat{X}, \mathbb{C}) \quad \textit{for} \quad k = 1, 3 \textit{ and } 5$$

ii) the cohomology sequence from above breaks up into short exact sequences

$$0 \to H^k(X, \mathbb{C}) \to H^k(\widehat{X}, \mathbb{C}) \to H^k(E, \mathbb{C}) \to 0 \quad \textit{for} \quad k = 2, 4.$$

Proof. Since

$$E = \bigsqcup_{x \in \mathrm{Sing}(X)} \Delta_x$$

is a disjoint union of the exceptional loci Δ_x it holds

$$H^k(E) = \bigoplus_{x \in S} H^k(\Delta_x).$$

Note that $H^k(\Delta_x) = 0$ for k odd (see Proposition 4.2.19) and $H^k(S) = 0$ for $k \geq 1$ for dimension reasons. The vanishing of $H^1(E)$ and Proposition 5.0.2 give the following exact sequence

$$0 \to H^0(X) \to H^0(\widehat{X}) \oplus H^0(S) \to H^0(E) \to H^1(X) \to H^1(\widehat{X}) \to 0.$$

We conclude that the map

$$\rho^* \colon H^1(X) \to H^1(\widehat{X})$$

is an isomorphism, because $\dim\big(H^0(S)\big) = \dim\big(H^0(E)\big) = |\,\mathrm{Sing}(X)|$ and the vector spaces $H^0(X)$ and $H^0(\widehat{X})$ are both isomorphic to $\mathbb{C}$. The map

$$\rho^*\colon H^5(X) \to H^5(\widehat{X})$$

is surjective by the long exact cohomology sequence and the vanishing of $H^5(E)$. According to [Mac62, § 1.2] the quotient map $\pi\colon C_1 \times C_2 \times C_3 \to X$ induces an isomorphism

$$\pi^*\colon H^k(X) \to H^k(C_1 \times C_2 \times C_3)^G$$

for all k, which implies, in combination with the usual Poincaré duality for compact manifolds, that Poincaré duality is valid for X. Therefore

$$b^5(X) = b^1(X) \quad \text{and} \quad b^5(\widehat{X}) = b^1(\widehat{X})$$

and it follows that $\rho^*\colon H^5(X) \to H^5(\widehat{X})$ is an isomorphism, too. Using the above information, the long exact cohomology sequence and the vanishing of $H^3(E)$, we obtain the following exact sequence

$$\begin{array}{l} 0 \longrightarrow H^2(X) \longrightarrow H^2(\widehat{X}) \longrightarrow H^2(E) \longrightarrow H^3(X) \longrightarrow H^3(\widehat{X}) \\ \longrightarrow 0 \longrightarrow H^4(X) \longrightarrow H^4(\widehat{X}) \longrightarrow H^4(E) \longrightarrow 0. \end{array}$$

The bottom row is claim *ii)* in the case $k = 4$. Next we show *ii)* in the case $k = 2$ which immediately implies *i)* in the case $k = 3$ using the exactness of the first row in the sequence above. By Poincaré duality we have

$$b^2(X) = b^4(X) \quad \text{and} \quad b^2(\widehat{X}) = b^4(\widehat{X}),$$

Proposition 4.2.19 implies

$$b^2(E) = b^4(E) = \sum_{x \in \operatorname{Sing}(X)} \frac{m_x - 1}{2}.$$

Thus, the exactness of the sequence

$$0 \to H^k(X) \to H^k(\widehat{X}) \to H^k(E) \to 0$$

for $k = 2$ follows from the exactness for $k = 4$ which was already shown. $\square$

Corollary 5.0.4. *Let $X = \big(C_1 \times C_2 \times C_3\big)/G$ be a product quotient threefold with Gorenstein singularities, then:*

i) $h^{1,1}(\widehat{X}) = \langle \chi_{1,1}, \chi_{triv} \rangle + \sum_{x \in \operatorname{Sing}(X)} \frac{m_x - 1}{2}$ *and* *ii)* $h^{2,1}(\widehat{X}) = \langle \chi_{2,1}, \chi_{triv} \rangle$.

Proof. According to Proposition 5.0.3 i) and ii) it holds:

$$b^3(\widehat{X}) = b^3(X) \quad \text{and} \quad b^2(\widehat{X}) = b^2(X) + b^2(E).$$

We obtain

$$h^{2,1}(\widehat{X}) = \frac{b^3(X)}{2} - h^{3,0}(\widehat{X}) = \dim\big(H^{2,1}(Y)^G\big)$$

and

$$h^{1,1}(\widehat{X}) = b^2(X) + b^2(E) - 2h^{2,0}(\widehat{X}) = \dim\big(H^{1,1}(Y)^G\big) + b^2(E)$$

using Hodge decomposition, Hodge symmetry, the isomorphisms

$$\rho^* \colon H^k(X) \to H^k(C_1 \times C_2 \times C_3)^G \qquad \text{for} \quad k = 2, 3$$

and Proposition 4.3.5. We are done with ii). To finish the proof of i), we substitute the Betti number $b^2(E)$ in the equation $h^{1,1}(\widehat{X}) = \dim\big(H^{1,1}(Y)^G\big) + b^2(E)$ with the expression

$$\sum_{x \in \mathrm{Sing}(X)} \frac{m_x - 1}{2}$$

according to Proposition 4.2.19. □

To give an effective classification algorithm, we need to derive additional combinatorial constraints on the numerical data.

Proposition 5.0.5. *Let X be a singular Gorenstein product quotient threefold with numerical datum (n, T_1, T_2, T_3), where $T_i := [g_i'; m_{i,1}, \ldots, m_{i,r_i}]$. Then for all $1 \leq i \leq 3$ at least one $m_{i,j}$ is not a power of 2. In particular G can not be a 2-group.*

Proof. Recall that m_x is odd for all singularities

$$\frac{1}{m_x}(1, a_x, m_x - a_x - 1)$$

of X. The claim follows immediately because m_x must divide at least one $m_{i,j}$ for all $1 \leq i \leq 3$ and $x \in \mathrm{Sing}(X)$. □

Proposition 5.0.6 (cf. [BCGP12, Lemma 5.8])**.** *Let X be a Gorenstein product quotient threefold with numerical datum (n, T_1, T_2, T_3), where $T_i = [g_i'; m_{i,1}, \ldots, m_{i,r_i}]$. Then*

$$m_{i,j} \mid 8\big(g(C_{[i+1]}) - 1\big)\big(g(C_{[i+2]}) - 1\big)$$

for all i and j. As usual, the brackets $[\cdot]$ denote the residue mod 3.

Proof. Each $m_{i,j}$ is the branching index of a branch point $q_{i,j}$ of the Galois cover $F_i\colon C_i \to C_i/G$. Let $W_{i,j}$ be the fibre over $q_{i,j}$ of the natural map

$$\mathfrak{p}_i : X \to C_i/G.$$

Then $W_{i,j} = m_{i,j}D_{i,j}$, where $D_{i,j}$ is an irreducible Weil divisor. A general fibre E_i of $\mathfrak{p}_i$ is isomorphic to $C_{[i+1]} \times C_{[i+2]}$ and contained in the smooth locus of X. Via the adjunction formula

$$K_{E_i} = \big(K_X + E_i\big)_{|E_i},$$

we can compute the intersection product

$$8\big(g(C_{[i+1]}) - 1\big)\big(g(C_{[i+2]}) - 1\big) = K_{E_i}^2 = K_X^2.E_i + 2K_X.E_i^2 + E_i^3 = K_X^2.E_i.$$

Since fibres are numerical equivalent, it holds

$$K_X^2.E_i = K_X^2.W_{i,j} = m_{i,j}K_X^2.D_{i,j}$$

and the claim follows from the observation that the intersection product $K_X^2.W_{i,j}$ is an integer, because K_X is Cartier. □

Now we can give our Algorithm, which is a modified version of the algorithm presented in Chapter 3. For the full code we refer to Appendix A. Our strategy is the following.

Input: A value for the holomorphic Euler-Poincaré-characteristic χ.

Part 1: Determine the set of admissible numerical data, i.e. the set of tuples

$$(n, T_1, T_2, T_3)$$

such that the combinatorial constraints form Proposition 4.5.6, 5.0.5 and 5.0.6, the inequality from Proposition 4.5.1 and Hurwitz' formula are satisfied.

In our implementation, this computation is performed by the function `AdNumGorenstein`. The function just returns the admissible numerical data (n, T_1, T_2, T_3) such that the groups of order n are contained in the Database of Small Groups. The exceptions are stored in the file `ExcepGorenstein`χ`.txt`.

Part 2: In the second part of the algorithm, we search for algebraic data.

For each 4-tuple (n, T_1, T_2, T_3) contained in the set `AdNumGorenstein(`χ`)` search through the groups of order n for groups admitting at least one generating vector of type T_1, one of T_2 and of type T_3. For such groups G compute all generating vectors V_i of type T_i and consider all possible 4-tuples (G, V_1, V_2, V_3). To each of these tuples there corresponds a family of product quotient threefolds X with holomorphic Euler-Poincaré-characteristic equal to the input value χ and algebraic datum (G, V_1, V_2, V_3). Next, compute the basket of singularities of X from the algebraic datum. If all singularities are Gorenstein, compute the Hodge diamond of a crepant resolution $\widehat{X}$ according to Proposition 4.3.5 and Corollary 5.0.4 and save the occurrence

$$[G, T_1, T_2, T_3, h^{3,0}(\widehat{X}), h^{2,0}(\widehat{X}), h^{1,0}(\widehat{X}), h^{1,1}(\widehat{X}), h^{2,1}(\widehat{X}), \mathcal{B}(X)]\,.$$

in the file `Gorenstein`χ`.txt`. Part 2 is performed by calling the function `ClassifyGorenstein(`χ`)`.

Main Computation

We execute the implementation for the input values $\chi = -1, -2$ and -3. There are no exceptional numerical data, i.e. the files `ExcepGorenstein`χ`.txt` remain empty. The computation time on a 8×2.5GHz Intel Xenon L5420 workstation with 16GB RAM is given in the table below:

χ	-1	-2	-3
time	22 min	43 min	9 h 55 min

We obtain the following results:

Proposition 5.0.7. *There are no singular product quotient threefolds X with Gorenstein singularities and $\chi(\mathcal{O}_{\widehat{X}}) = -1$ or $\chi(\mathcal{O}_{\widehat{X}}) = -2$.*

Theorem 5.0.8. *Let $X = (C_1 \times C_2 \times C_3)/G$ be a singular product quotient threefold with Gorenstein singularities and $\chi(\mathcal{O}_{\widehat{X}}) = -3$. Then the data*

$$[G, T_1, T_2, T_3, h^{3,0}(\widehat{X}), h^{2,0}(\widehat{X}), h^{1,0}(\widehat{X}), h^{1,1}(\widehat{X}), h^{2,1}(\widehat{X}), \mathcal{B}(X)]$$

appears in the table below. Conversely, each row in the table is realized by at least one family of product quotient threefolds.

No.	G	Id	T_1	T_2	T_3	$h^{3,0}$	$h^{2,0}$	$h^{1,0}$	$h^{1,1}$	$h^{2,1}$	$\mathcal{B}$
1	$\mathbb{Z}_3^2$	$\langle 9,2\rangle$	$[0;3^4]$	$[0;3^4]$	$[0;3^4]$	10	6	0	117	12	$1/3^{90}$
2	$\mathbb{Z}_3^2$	$\langle 9,2\rangle$	$[0;3^4]$	$[0;3^4]$	$[0;3^4]$	8	4	0	64	17	$1/3^{45}$
3	$\mathbb{Z}_3^2$	$\langle 9,2\rangle$	$[0;3^4]$	$[0;3^4]$	$[0;3^4]$	8	4	0	46	23	$1/3^{27}$
4	$\mathbb{Z}_3^2$	$\langle 9,2\rangle$	$[0;3^4]$	$[0;3^4]$	$[0;3^4]$	9	5	0	53	19	$1/3^{36}$
5	$\mathbb{Z}_3^2$	$\langle 9,2\rangle$	$[0;3^4]$	$[0;3^4]$	$[0;3^4]$	7	3	0	24	24	$1/3^{9}$
6	$\mathbb{Z}_3^2$	$\langle 9,2\rangle$	$[0;3^4]$	$[0;3^4]$	$[0;3^4]$	8	4	0	53	18	$1/3^{36}$
7	$\mathbb{Z}_3^2$	$\langle 9,2\rangle$	$[0;3^4]$	$[0;3^4]$	$[0;3^4]$	9	5	0	33	23	$1/3^{18}$
8	$\mathbb{Z}_3^2$	$\langle 9,2\rangle$	$[0;3^4]$	$[0;3^4]$	$[0;3^4]$	8	4	0	33	22	$1/3^{18}$
9	$\mathbb{Z}_3^2$	$\langle 9,2\rangle$	$[0;3^4]$	$[0;3^4]$	$[0;3^4]$	9	5	0	22	24	$1/3^{9}$
10	$\mathbb{Z}_3^2$	$\langle 9,2\rangle$	$[0;3^4]$	$[0;3^4]$	$[0;3^4]$	9	5	0	31	21	$1/3^{18}$
11	$\mathbb{Z}_3^2$	$\langle 9,2\rangle$	$[0;3^4]$	$[0;3^4]$	$[0;3^4]$	10	6	0	93	12	$1/3^{72}$

No.	G	Id	T_1	T_2	T_3	$h^{3,0}$	$h^{2,0}$	$h^{1,0}$	$h^{1,1}$	$h^{2,1}$	$\mathcal{B}$
12	$\mathfrak{A}_4 \times \mathbb{Z}_3$	$\langle 36, 11\rangle$	$[0; 3^4]$	$[0; 3^2, 6]$	$[0; 3^2, 6]$	6	2	0	42	17	$1/3^{27}$
13	$\mathfrak{A}_4 \times \mathbb{Z}_3$	$\langle 36, 11\rangle$	$[0; 3^4]$	$[0; 3^2, 6]$	$[0; 3^2, 6]$	7	3	0	29	17	$1/3^{18}$
14	$\mathfrak{A}_4 \times \mathbb{Z}_3$	$\langle 36, 11\rangle$	$[0; 3^4]$	$[0; 3^2, 6]$	$[0; 3^2, 6]$	7	3	0	18	18	$1/3^{9}$
15	$\mathfrak{A}_4 \times \mathbb{Z}_3$	$\langle 36, 11\rangle$	$[0; 3^4]$	$[0; 3^2, 6]$	$[0; 3^2, 6]$	6	2	0	60	11	$1/3^{45}$
16	$\mathfrak{A}_4 \times \mathbb{Z}_3$	$\langle 36, 11\rangle$	$[0; 3^4]$	$[0; 3^2, 6]$	$[0; 3^2, 6]$	7	3	0	49	13	$1/3^{36}$

Remark 5.0.9. The symbol a/m^λ in the last row of the table denotes λ singularities of type $\frac{1}{m}(1, a, m-a-1)$ in the basket $\mathcal{B}(X)$.

Chapter 6

Product Quotient Threefolds with minimal Volume

Let X be a product quotient threefold with canonical singularities and $\rho\colon \widehat{X} \to X$ be a crepant terminalisation. Then, according to Proposition 4.3.12, the invariants $\chi(\mathcal{O}_{\widehat{X}})$, $e(\widehat{X})$ and $K_{\widehat{X}}^3$ satisfy the following inequalities

$$\text{i)}\ \ 48\chi(\mathcal{O}_{\widehat{X}}) + K_{\widehat{X}}^3 \geq 0 \qquad \text{and} \qquad \text{ii)}\ \ 6e(\widehat{X}) + K_{\widehat{X}}^3 \geq 0.$$

Until now, we considered the boundary cases, i.e. the cases where the inequalities are actually equalities. Recall that i) is sharp if and only if $\widehat{X}$ is smooth and ii) is sharp if and only if X is a threefold isogenous to a product of unmixed type which implies that i) must be also sharp. In both cases $\chi(\mathcal{O}_{\widehat{X}})$ is a negative integer. Away from the boundary cases this is far from being true: the holomorphic Euler-Poincaré-characteristic $\chi(\mathcal{O}_{\widehat{X}})$ can be zero or even positive. However, the self-intersection $K_{\widehat{X}}^3$, which is the same as K_X^3, is always positive. In Section 4.5 we derived the inequality $K_X^3 \geq 1$, which is sharp once we drop the assumption that X is canonical. The main purpose of this chapter is to determine the smallest value of K_X^3 that can be can be realised under the assumption that X has canonical singularities. To solve this problem, we provide an algorithm to classify product quotient threefolds X with canonical singularities and $K_X^3 \leq c$, where c is a fixed positive number serving as the input value of the algorithm. Running a MAGMA implementation for $c = 4$ we find that $K_X^3 = 4$ is the minimum value.

The algorithm is the following:

Input: An upper bound c for the self-intersection K_X^3.

Step 1: In the first part we determine the set of *admissible numerical data*

$$(n, T_1, T_2, T_3)$$

of product quotient threefolds with $K_X^3 \leq c$:

- Construct the set of triples $(g_1, g_2, g_3) \in \mathbb{N}^3$ such that $2 \leq g_1 \leq g_2 \leq g_3$ and
$$\prod_{i=1}^{3}(g_i - 1) \leq \frac{7}{8}\sqrt{7c^3}.$$
- For every triple in the output of Step 1, construct the set of 4-tuples (n, g_1, g_2, g_3) such that
$$48/c \leq n \leq 42 \cdot \sqrt{7c}, \qquad n \leq N_{\max}(g_i) \quad \text{and} \quad \frac{48}{n}\prod_{i=1}^{3}(g_i - 1) \leq c.$$
- For every 4-tuple in the output of Step 2 construct the set of 4-tuples
$$(n, T_1, T_2, T_3),$$
where $T_i = [g_i'; m_{i,1}, \ldots, m_{i,r_i}]$ are the types which satisfy the conditions of Proposition 4.5.6 and Proposition 4.5.1.

In our implementation Step 1 is performed by the function `AdNumSmallVol`.

Step 2: In the second part of the algorithm, we search for algebraic data. For each 4-tuple (n, T_1, T_2, T_3) contained in the set `AdNumSmallVol(c)` search through the groups G of order n for groups admitting at least one generating vector of type T_1, one of type T_2 and one of type T_3. For such groups compute all generating vectors V_i of type T_i for G and form all possible combinations (G, V_1, V_2, V_3). To each of these 4-tuples corresponds a product quotient threefold X with algebraic datum (G, V_1, V_2, V_3). Compute the basket of singularities of X. If all singularities are canonical, compute the basket $\mathcal{B}(\widehat{X})$ of singularities of a crepant terminalisation $\widehat{X}$ and the invariants
$$p_g := p_g(\widehat{X}), \qquad q_2 := q_2(\widehat{X}) \qquad \text{and} \qquad q_1 := q_1(\widehat{X}).$$
Save the occurrence
$$[G, T_1, T_2, T_3, p_g, q_2, q_1, e(\widehat{X}), K_X^3, \mathcal{B}(\widehat{X})]$$
in the file `SmallVol`χ`.txt`. Step 2 is performed calling `ClassifySmallVol(c)`.

For the full code we refer to Appendix A. Running the implementation for the value $c = 4$, we obtain the following theorem.

Theorem 6.0.1. *Let $X := (C_1 \times C_2 \times C_3)/G$ be a product quotient threefold with canonical singularities, $K_X^3 \leq 4$ and let $\rho\colon \widehat{X} \to X$ be a crepant partial resolution with* *terminal singularities. Then* $K_X^3 = 4$ *and the data*

$$[G, T_1, T_2, T_3, p_g, q_2, q_1, e, K_X^3, \mathcal{B}(\widehat{X})]$$

appears in the table below. Conversely, each row in the table is realized by at least one family of product quotient threefolds.

No.	G	*Id*	T_1	T_2	T_3	p_g	q_2	q_1	e	K^3	$\mathcal{B}$
1	$\mathcal{D}_6$	$\langle 12,4\rangle$	$[0;2^3,3]$,	$[0;2^3,3]$	$[0;2^3,3]$	0	3	0	40	4	$1/3^{12}, 1/2^{44}$
2	$\mathbb{Z}_3\times\mathbb{Z}_2^2$	$\langle 12,5\rangle$	$[0;2,6^2]$	$[0;2,6^2]$	$[0;2,6^2]$	0	2	0	28	4	$1/6^4, 1/3^{10}, 1/2^{16}$
3	$\mathbb{Z}_3\times\mathbb{Z}_2^2$	$\langle 12,5\rangle$	$[0;2,6^2]$	$[0;2,6^2]$	$[0;2,6^2]$	0	4	0	40	4	$1/6^8, 1/3^8, 1/2^{36}$
4	$\mathbb{Z}_3\times\mathbb{Z}_2^2$	$\langle 12,5\rangle$	$[0;2,6^2]$	$[0;2,6^2]$	$[0;2,6^2]$	0	1	0	24	4	$1/3^{12}, 1/2^{12}$
5	$\mathbb{Z}_3\times\mathbb{Z}_2^2$	$\langle 12,5\rangle$	$[0;2,6^2]$	$[0;2,6^2]$	$[0;2,6^2]$	1	3	0	28	4	$1/6^4, 1/3^{10}, 1/2^{16}$
6	$\mathbb{Z}_3\times\mathbb{Z}_2^2$	$\langle 12,5\rangle$	$[0;2,6^2]$	$[0;2,6^2]$	$[0;2,6^2]$	2	3	0	24	4	$1/3^{12}, 1/2^{12}$
7	$\mathrm{SL}(2,\mathbb{F}_3)$	$\langle 24,3\rangle$	$[0;3^2,4]$	$[0;3^2,4]$	$[0;3^2,6]$	0	3	0	52	4	$1/3^{30}, 1/2^{12}$

Remark 6.0.2. The symbol a/m^λ in the last row of the table denotes λ singularities of type $\frac{1}{m}(1, a, m-a)$ in the basket $\mathcal{B}(\widehat{X})$.

Remark 6.0.3. When we run the algorithm for larger values of c, we also obtain threefolds, such that K_X^3 is not an integer. The smallest value of K_X^3, where this phenomena happens, is 72/5.

Appendix A

MAGMA Codes

In this appendix, we provide the MAGMA implementations of our algorithms. The code is contained in the accompanying CD-ROM [1] which has the following directory tree

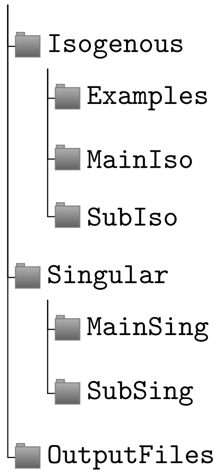

Figure A.1: directory tree CD-ROM

We briefly describe the content and structure of the main-folders `Isogenous`, `Singular` and `OutputFiles`.

1) The folder `Isogenous` consists of three sub-folders. The sub-folder `MainIso` contains the implementation of the algorithm from Chapter 3 in the unmixed, index two, index three and index six case. The corresponding MAGMA files are:

   – `MainUnmixed.magma`,

   – `MainIndexTwo.magma`,

[1]See also `http://www.staff.uni-bayreuth.de/~bt300503`.

– `MainIndexThree.magma`,

– `MainIndexSix.magma`.

In the files above, we combine several subroutines, which perform a specific task within the algorithm: the computation of the types and the generating vectors, the verification of the freeness conditions and the computation of the Hodge numbers using the Chevalley-Weil formula. These subroutines are stored in the second sub-folder `SubIso`:

– `ChevalleyWeil.magma`,

– `FreenessCond.magma`,

– `GenVectors.magma`,

– `HodgeDiamond.magma`,

– `Types.magma`.

The reason why we split up the program is to avoid redundancy and to achieve a better readability of the code.
The third sub-folder `Examples` contains two files:

– `ExampleIndexSix.magma`,

– `ExampleRigid.magma`.

They are used to perform the computations in Example 3.0.15 a) and b).

2) The folder `Singular` consists of two sub-folders. The first one: `MainSing` contains the implementation of the algorithms from Chapter 5 and Chapter 6. The corresponding MAGMA files are:

– `MainGorenstein.magma`,

– `MainSmallVol.magma`.

As above, we use specific subroutines. They are stored in the folder `SubSing`:

– `BasketOfSings.magma`,

– `CohInvariants.magma`.

The first file is used to compute the basket of singularities of a product quotient threefold and the second one to determine the cohomological invariants. It should be mentioned that the routines in the folder `Singular` also use subroutines contained in `SubIso`.

3) The folder `OutputFiles` contains the following `txt` files:

   – `Unmixed-1.txt`,

   – `IndexTwo-1.txt`,

   – `IndexThree-1.txt`,

   – `Gorenstein-3.txt`,

   – `SmallVol4.txt`.

   The first three files provide the full list of examples of threefolds X isogenous to a product of curves with $\chi(\mathcal{O}_X) = -1$ obtained by an absolutely faithful G-action, see [Theorem (**A**) p.48], [Theorem (**B**) p.50] and [Theorem (**C**) p.52]. To produce these files, we shall load the MAGMA programs `MainUnmixed.magma`, `MainIndexTwo.magma` and `MainIndexThree.magma` and call the functions

   – `ClassifyUnmixed(`χ`)`,

   – `ClassifyIndexTwo(`χ`)`,

   – `ClassifyIndexThree(`χ`)`

   for $\chi = -1$, respectively. Recall, there are no examples in the index six case with $\chi(\mathcal{O}_X) = -1$ which are obtained by an absolutely faithful G-action (cf. [Theorem (**C**) p.52]). To verify this claim we load the program `MainIndexSix.magma` and call the function `ClassifyIndexSix(`χ`)` for $\chi = -1$.

   The file `Gorenstein-3.txt` contains the classification of crepant resolutions $\widehat{X}$ of singular Gorenstein product quotient threefolds X with $\chi(\mathcal{O}_{\widehat{X}}) = -3$ (cf. Theorem 5.0.8). To produce this file, we load `MainGorenstein.magma` and execute the function `ClassifyGorenstein(`χ`)` for $\chi = -3$.

   The last file `SmallVol4.txt` contains the full list of examples of product quotient threefolds X with $K_X^3 \leq 4$ (cf. Theorem 6.0.1). To produce it, we load the program `MainSmallVol.magma` and execute the function `ClassifySmallVol(`c`)` for $c = 4$.

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