

Published in "International Mathematics Research Notices 2017(3): 715–808, 2017"
which should be cited to refer to this work.

Classification of Flexible Kokotsakis Polyhedra with Quadrangular Base

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A Kokotsakis polyhedron with quadrangular base is a neighborhood of a quadrilateral in a quad surface. Generically, a Kokotsakis polyhedron is rigid. In this article we classify flexible Kokotsakis polyhedra with quadrangular bases. The analysis is based on the fact that any pair of adjacent dihedral angles of a Kokotsakis polyhedron is related by a biquadratic equation. This results in a diagram of branched covers of complex projective lines by elliptic curves. A polyhedron is flexible if and only if all repeated fiber products of coverings meet in the same Riemann surface, which is then the configuration space of the polyhedron.

1 Introduction

1.1 Kokotsakis polyhedra

A *Kokotsakis polyhedron* is a polyhedral surface in \mathbb{R}^3 , which consists of one n -gon (the *base*), n quadrilaterals attached to every side of the n -gon, and n triangles placed between each two consecutive quadrilaterals. See Figure 1 for the case $n = 5$.

If all faces are viewed as rigid plates, and edges as hinges, then the polyhedron is in general rigid. Indeed, every interior vertex has only one degree of freedom, so that

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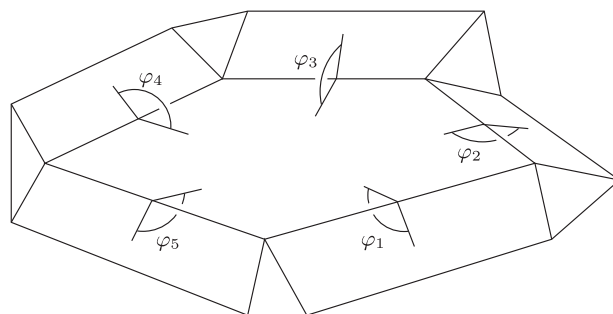


Fig. 1. A Kokotsakis polyhedron with a pentagonal base.

any two dihedral angles adjacent to a vertex are related by an equation:

$$F_{12}(\varphi_1, \varphi_2) = 0, \quad F_{23}(\varphi_2, \varphi_3) = 0, \dots \quad F_{n1}(\varphi_n, \varphi_1) = 0$$

In a generic case, the solutions of this system form a discrete set, and the polyhedron cannot be deformed. A natural question is to find all special shapes that allow isometric deformations.

Antonios Kokotsakis, a student of Carathéodory, studied these polyhedra in his PhD thesis in 1930s. As a result, he found a necessary and sufficient condition for the infinitesimal flexibility and described several classes of flexible polyhedra [10]. At the same time, Sauer and Graf [20] studied the case $n = 4$ and also found several flexible special cases.

1.2 The approach

We provide a classification of flexible Kokotsakis polyhedra with quadrangular base, that is, for $n = 4$. The description is not absolutely explicit, see discussion at the beginning of Section 3.

Our approach is based on a diagram of branched covers between Riemann surfaces constructed as follows.

A substitution $z_i = \tan \frac{\varphi_i}{2}$ transforms $F_{i,i+1}(\varphi_i, \varphi_{i+1}) = 0$ into a polynomial equation $P_{i,i+1}(z_i, z_{i+1}) = 0$ with $z_i \in \mathbb{R}P^1$. By allowing z_i to take complex values we arrive at a Riemann surface

$$C_{i,i+1} = \{(z_i, z_{i+1}) \mid P_{i,i+1}(z_i, z_{i+1}) = 0\} \tag{1}$$

with two coordinate projections

$$z_i \in \mathbb{CP}^1 \leftarrow C_{i,i+1} \rightarrow \mathbb{CP}^1 \ni z_{i+1},$$

which form the following diagram

$$\begin{array}{ccccc} C_{34} & \rightarrow & \mathbb{CP}^1 & \leftarrow & C_{41} \\ \downarrow & & & & \downarrow \\ \mathbb{CP}^1 & & & & \mathbb{CP}^1 \\ \uparrow & & & & \uparrow \\ C_{23} & \rightarrow & \mathbb{CP}^1 & \leftarrow & C_{12} \end{array}$$

Now, if a polyhedron is flexible, then there are paths in $C_{i,i+1}$ whose projections to the common \mathbb{CP}^1 's coincide. These paths can be lifted to a bigger diagram involving fiber products of $C_{i-1,i}$ and $C_{i,i+1}$, see Section 6.2. To find the conditions for this big commutative diagram to exist, we consider holomorphic parametrizations and study the branch points of the surfaces and covers.

1.3 Related work

1.3.1 Branched covers, elliptic curves, and Euler–Chasles correspondences

Another example of an application of complex analysis to an algebraic problem is Ritt's characterization of composite and commutative polynomials [16, 18]. To achieve that, Ritt studied the monodromy of a composition of branched covers. His approach has partly inspired our work.

Instead of the tetrahedral angles at the interior vertices of the Kokotsakis polyhedron, it is more convenient to study the spherical quadrilaterals obtained by intersecting these angles with a unit sphere. It was discovered by Darboux [6] that the configuration space of a Euclidean quadrilateral is an elliptic curve. In particular, he derived from it a very nice but little known porism on quadrilateral folding, see [9]. Darboux's elliptic parametrization does not extend to the spherical case, so we will use a different one.

The equation of the configuration space (1) has the form

$$a_{22}z^2w^2 + a_{20}z^2 + a_{02}w^2 + 2a_{11}zw + a_{00} = 0 \tag{2}$$

as was discovered by Bricard in [4]. On the other hand, this is a special case of the so-called Euler–Chasles correspondence, which in a generic situation describes an elliptic curve. The correspondence is 2–2, that is, to one value of z there usually correspond two values of w , and vice versa.

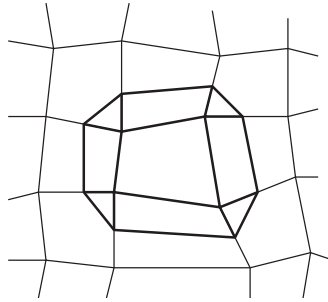


Fig. 2. Kokotsakis polyhedron with a quadrangular base as a part of a quad surface.

Thus, a Kokotsakis polyhedron is flexible if and only if the composition of associated Euler–Chasles correspondences contain an identity component. A similar problem was studied by Krichever [11] in the context of the quantum Yang–Baxter equation. See [7, Section 10] for a survey.

1.3.2 Flexible octahedra and cross-polytopes

A neighborhood of a face of an octahedron is a Kokotsakis polyhedron with a triangular base. It flexes if and only if the octahedron does. Flexible octahedra were studied and classified by Bricard [4]. Bricard derived the equation 2 between the tangents of halves of dihedral angles and used the elimination method to solve the system: the resultant of P_{12} and P_{23} must have a common factor with P_{13} . Nawratil [12] studied flexible Kokotsakis polyhedra with triangular base that cannot be extended to octahedra.

Stachel and Nawratil [13, 14, 22] applied Bricard’s method of resultants to the Kokotsakis polyhedra with quadrangular base and obtained a partial classification of flexible polyhedra. The case of irreducible resultants remained open.

As a generalization of Bricard’s flexible polyhedra, Gaifullin [8] studied flexible cross-polytopes in space-forms of arbitrary dimension. Gaifullin’s approach is similar to ours.

In a different manner, elliptic curves came up in the work of Connelly [5] who studied flexible bipyramids. See also the work of Connelly and Alexandrov [1].

1.3.3 Quad surfaces

A Kokotsakis polyhedron with a quadrangular base can be viewed as a part of a quadrangular mesh, see Figure 2. It is easy to see that a simply connected piece of a quadrangular mesh is flexible if and only if each of its Kokotsakis subpolyhedra is. Quad surfaces are a natural analog of parametrized smooth surfaces and have been studied in this context since about a century, see the book [19].

There was a recent surge of interest to quad surfaces. First, they give rise to various discrete integrable systems [3, 21]. Second, they have obvious applications in the architectural design [17]: curved facades and roofs made of quadrangular glass panels can be found in abundance in the modern cities. Third, flexible Kokotsakis polyhedra (a special case called Miura-ori) found application in industry as solar panels for spacecraft [15].

1.4 Organization of the article

Section 2 introduces the notation and describes parametrizations of the configuration spaces of spherical quadrilaterals. It also explains the terminology later used in Section 3 to list all flexible Kokotsakis polyhedra.

In Section 4 parametrization theorems for configuration spaces of spherical quadrilaterals are proved.

Section 5 studies couplings of two four-bar linkages, that is, the fiber products of configuration spaces of two quadrilaterals. Involutive and reducible couplings (i.e., trivial fiber products) are classified.

Finally, in Section 6 the diagram of branched covers associated with a flexible Kokotsakis polyhedron is introduced and studied based on the results of two previous sections. The most interesting and difficult case of a polyhedron without involutive couplings is dealt with in Section 6.6.

2 Notation and Preliminaries

Vertices of a Kokotsakis polyhedron and the values of its planar angles at the interior vertices are denoted in Figure 3. Clearly, it is only a neighborhood of the base face $A_1A_2A_3A_4$ that matters: replacing, say, the vertex B_1 by any other point on the half-line A_1B_1 does not affect the flexibility or rigidity of the polyhedron. We will assume that all planar angles lie between 0 and π : $0 < \alpha_i, \beta_i, \gamma_i, \delta_i < \pi$.

2.1 Spherical linkage associated with a Kokotsakis polyhedron

The idea of associating with a flexible polyhedron, a movable spherical linkage goes back to Bennett [2, Section 7], who used it in his study of the motion of flexible octahedra.

For each of the four interior vertices A_1, A_2, A_3, A_4 consider its spherical image, that is, the intersection of the cone of adjacent faces with a unit sphere centered at the vertex. This yields four spherical quadrilaterals Q_i with side lengths $\alpha_i, \beta_i, \gamma_i, \delta_i$ in

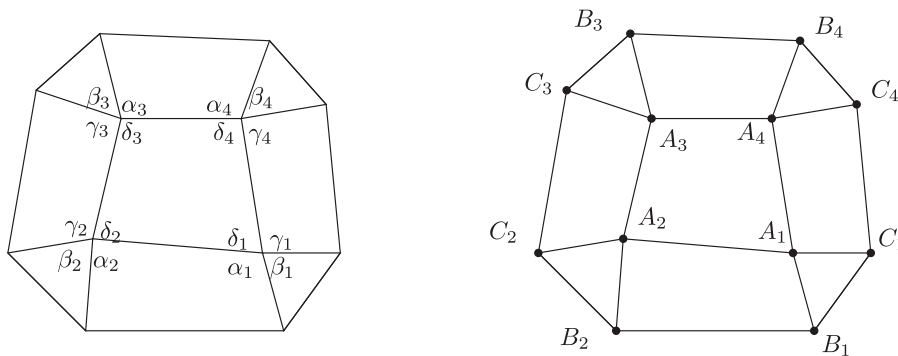


Fig. 3. Notation in a Kokotsakis polyhedron.

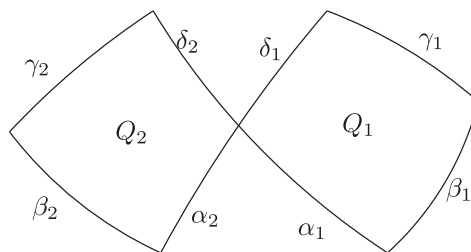


Fig. 4. Two coupled spherical quadrilaterals associated with the edge A_1A_2 .

this cyclic order. Spherical images of two adjacent vertices are *coupled* by means of a common dihedral angle. Equivalently, for every edge A_iA_{i+1} we have a scissors-like coupling of two spherical quadrilaterals, see Figure 4.

On Figure 5 two different ways to couple all four quadrilaterals Q_i are shown. The left one is a closed chain of quadrilaterals on a sphere (due to $\delta_1 + \delta_2 + \delta_3 + \delta_4 = 2\pi$), so that it incorporates all four couplings. The right one fits better with Figure 3 but is in general not closed, so that the coupling between Q_1 and Q_4 must be encoded by requiring the equality of marked angles.

Lemma 2.1. Every Kokotsakis polyhedron gives rise to spherical linkages as shown on Figure 5. Vice versa, every spherical linkage of this form corresponds to some Kokotsakis polyhedron, if we require $\delta_1 + \delta_2 + \delta_3 + \delta_4 = 2\pi$ for the linkage on the right.

A Kokotsakis polyhedron is flexible if and only if either of the corresponding spherical linkages on Figure 5 is flexible. For the linkage on the right, the two marked angles are required to stay equal during the deformation. \square

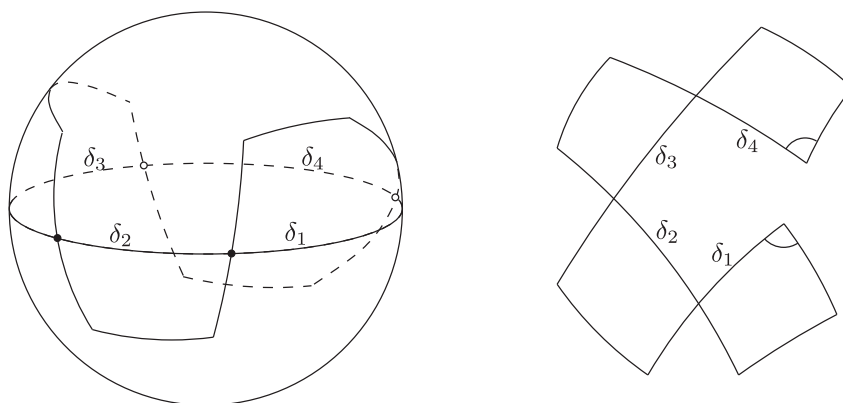


Fig. 5. Spherical linkages with scissors-like joints associated with a Kokotsakis polyhedron.

Proof. How to associate with a Kokotsakis polyhedron a spherical linkage was explained above. To construct a Kokotsakis polyhedron for a given chain of quadrilaterals, start by choosing as the base any Euclidean quadrilateral with the angles $\delta_1, \delta_2, \delta_3, \delta_4$. At the vertices of the base, construct tetrahedral angles that have spherical quadrilaterals Q_i as their spherical images. The fact that the corresponding angles of Q_i are equal in pairs means that these tetrahedral angles fit to form a Kokotsakis polyhedron.

During an isometric deformation of a Kokotsakis polyhedron the planar angles of its faces remain constant. Thus isometric deformations of the polyhedron correspond to motions of the linkages on Figure 5 (with the equal angles condition for the right one). ■

Thus, in order to classify all flexible Kokotsakis polyhedra, it suffices to classify all flexible spherical linkages as on Figure 5 with $\delta_1 + \delta_2 + \delta_3 + \delta_4 = 2\pi$.

2.2 Algebraic reformulation of the problem

For fixed lengths of the bars, the shape of the spherical linkage introduced in Section 2.1 is uniquely determined by the four angles φ, ψ_1, ψ_2 , and θ on Figure 6. If we introduce the variables

$$z = \tan \frac{\varphi}{2}, \quad w_1 = \tan \frac{\psi_1}{2}, \quad w_2 = \tan \frac{\psi_2}{2}, \quad u = \tan \frac{\theta}{2}, \quad (3)$$

then the relation between each pair of adjacent angles can be expressed as a polynomial equation in the corresponding variables:

$$\begin{aligned} P_1(z, w_1) = 0 & & P_2(z, w_2) = 0 \\ P_3(u, w_1) = 0 & & P_4(u, w_2) = 0 \end{aligned} \quad (4)$$

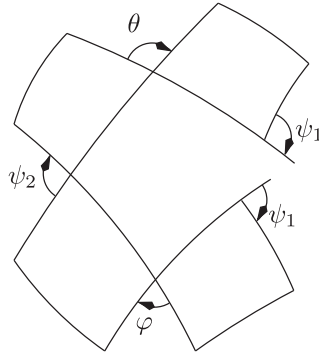


Fig. 6. Angles determining the shape of a spherical linkage.

Thus we have the following lemma.

Lemma 2.2. A Kokotsakis polyhedron is flexible if and only if the system of polynomial equations (4) has a one-parameter family of solutions over the reals. \square

One possible approach to the problem (modifying Bricard's approach to a system of three equations) is to compute the resultant $R_{12}(w_1, w_2)$ of P_1 and P_2 as polynomials in z and similarly the resultant $R_{34}(w_1, w_2)$. The polyhedron is flexible if and only if the algebraic sets $R_{12} = 0$ and $R_{34} = 0$ have a common irreducible component, which, in turn, is equivalent to the vanishing of the resultant of R_{12} and R_{34} . This plan was partially realized by Stachel and Nawratil, but the case of irreducible R_{12} and R_{34} remained out of reach.

2.3 Branched covers between configuration spaces

Put

$$Z_i := \{(z, w_i) \in (\mathbb{C}P^1)^2 \mid P_i(z, w_i) = 0\}, \quad i = 1, 2$$

$$Z_{12} := \{(z, w_1, w_2) \in (\mathbb{C}P^1)^3 \mid P_1(z, w_1) = 0 = P_2(z, w_2)\}$$

Then Z_i is the complexified configuration space of the quadrilateral Q_i , and Z_{12} is the complexified configuration space of the spherical linkage formed by coupling Q_1 and Q_2 . (The projection of Z_{12} to the (w_1, w_2) -plane is the zero set of the resultant R_{12} defined in Section 2.2.)

A component of Z_i is called *trivial*, if it has the form $z = \text{const}$ or $w_i = \text{const}$. For non-trivial components, the restrictions of the maps $Z_{12} \rightarrow Z_i$, $i = 1, 2$ are branched

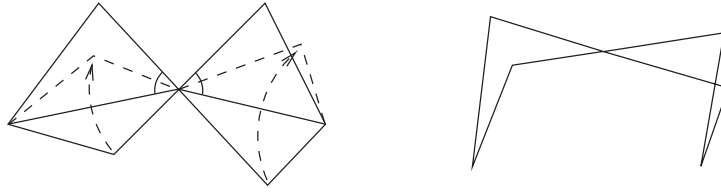


Fig. 7. An involutive coupling and the action of j_{12} on its configuration space.

covers between Riemann surfaces. If the solution set Z_{all} of the system (4) is one dimensional, then the map $Z_{\text{all}} \rightarrow Z_{12}$ is also a branched cover, and we obtain a diagram of branched covers shown on Figure 15, Section 6. All maps in this diagram are at most two-fold, and the configuration spaces Z_i can be classified as in Section 4. However, the analysis of the diagram is still complicated enough, and we bring more structure in it by distinguishing certain sorts of couplings.

First, a coupling (Q_1, Q_2) is called *involutive*, if a component of its configuration space Z_{12} carries an involution that changes the value of z but preserves w_i :

$$j_{12}: (w_1, z, w_2) \mapsto (w_1, z', w_2)$$

This means that the involutions

$$j_1: (w_1, z) \mapsto (w_1, z') \quad \text{and} \quad j_2: (w_2, z) \mapsto (w_2, z'')$$

on Z_1 and Z_2 are compatible: $z' = z''$. Geometrically, j_1 is a “folding” of the quadrilateral Q_1 . Therefore a coupling is involutive if and only if the two marked angles on Figure 7, left, remain equal during the deformation. (The coupling between Q_1 and Q_2 on Figure 7 is chosen in the same way as on Figure 5, right; the involutivity has a more nice geometric meaning if the coupling is chosen as on Figure 5, left.)

A detailed study of the configuration space Z_{12} is the subject of Section 5. Involutive couplings are studied in Section 5.4 and classified in Lemma 5.9.

Second, a coupling (Q_1, Q_2) is called *reducible*, if the algebraic set Z_{12} is reducible, while Z_1 and Z_2 are not. This property is hard to visualize: the real configuration space of a coupling may have several connected components all of which are parts of one complex component.

Reducibility of (Q_1, Q_2) is related to the reducibility of the resultant $R_{12}(w_1, w_2)$ defined in Section 2.2. Moreover, a coupling is involutive if and only if the resultant is a square. For more details see Section 5.1.

2.4 Types of spherical quadrilaterals and associated parameters

Let Q be a spherical quadrilateral with side lengths $\alpha, \beta, \gamma, \delta$, in this cyclic order. The form of the configuration space of Q depends on the number and the type of solutions of the equation

$$\alpha \pm \beta \pm \gamma \pm \delta = 0 \pmod{2\pi} \tag{5}$$

Definition 2.3. A spherical quadrilateral Q is said to be

- of *elliptic* type, if equation (5) has no solutions;
- of *conic* type, if equation (5) has exactly one solution;
- a *deltoid*, if it has two pairs of equal adjacent sides, and an *antideltoid*, if it has two pairs of adjacent sides complementing each other to π ;
- an *isogram*, if pairs of opposite sides have equal lengths, and an *antiisogram*, if lengths of opposite sides complement each other to π . □

Circumscribable spherical quadrilaterals are characterized by $\alpha + \gamma = \beta + \delta$ and thus are of conic type according to our terminology (possibly degenerating to a deltoid). The link of a vertex of a quad surface satisfies this condition if and only if the faces adjacent to this vertex are tangent to a circular cone. Because of this, surfaces with this property are called *conical meshes* by Pottman and Wallner [17] and find an application in the freeform architecture. We call these quadrilaterals conic for a different reason: their configuration spaces are described by quadratic equations.

At the beginning of Section 4 we show that every spherical quadrilateral belongs to one of the types above. The theorems below, also proved in Section 4, describe the configuration spaces of all types of quadrilaterals. They provide a vocabulary for the description of flexible Kokotsakis polyhedra that follows in Section 3.

The classification given in Definition 2.3 and parametrization of the configuration spaces in some cases are due to Bricard [4, Section II].

Theorem 2.4 (Bricard). The configuration space Z of an (anti)isogram with side lengths $\alpha, \beta, \gamma, \delta$ has the following form.

- (1) If $\alpha = \beta = \gamma = \delta = \frac{\pi}{2}$, then all components of Z are trivial:

$$Z = \{z = 0\} \cup \{z = \infty\} \cup \{w = 0\} \cup \{w = \infty\}$$

- (2) If Q is an antiisogram and at the same time (anti)deltoid, then Z has one non-trivial component with the equation

$$z = \kappa w, \text{ where } \kappa = \begin{cases} \frac{1}{\cos \alpha}, & \text{if } \alpha = \beta = \pi - \gamma = \pi - \delta \\ -\cos \alpha, & \text{if } \alpha = \delta = \pi - \beta = \pi - \gamma \end{cases} \quad (6)$$

- (3) If Q is an antiisogram and not an (anti)deltoid, then Z has two non-trivial components of the form

$$z = \kappa w,$$

where

$$\kappa \in \left\{ \frac{\sin \frac{\alpha-\beta}{2}}{\sin \frac{\alpha+\beta}{2}}, \frac{\cos \frac{\alpha-\beta}{2}}{\cos \frac{\alpha+\beta}{2}} \right\} \quad (7)$$

- (4) If Q is an isogram: $\alpha = \gamma, \beta = \delta$ with not all sides equal to $\frac{\pi}{2}$, then the non-trivial components of Z are described by the equation

$$z = \frac{1}{\kappa w}, \quad (8)$$

where κ taking one or two values according to whether Q is a deltoid and given by the same formulas as in the antiisogram case. \square

Since $\kappa \in \mathbb{R}$, we have $z \in \mathbb{R} \Leftrightarrow w \in \mathbb{R}$. This is the real part of the configuration space.

Theorem 2.5. Let Q be a deltoid or an antideltoid that is neither isogram nor antiisogram. Then the affine part of the non-trivial component of its configuration space has the following parametrization.

- 1) If $\alpha = \delta, \beta = \gamma$ or $\alpha + \delta = \pi = \beta + \gamma$, then

$$z^m = p \sin t, \quad w = \varepsilon \sqrt{-\mu} e^{it}, \quad (9)$$

where

$$p = \sqrt{\frac{\sin^2 \delta}{\sin^2 \gamma} - 1}, \quad m = \begin{cases} 1, & \text{if } \alpha = \delta \\ -1, & \text{if } \alpha + \delta = \pi \end{cases}$$

$$\mu = \frac{\tan \delta + \tan \gamma}{\tan \delta - \tan \gamma}, \quad \varepsilon = \begin{cases} m, & \text{if } \gamma + \delta > \pi, \\ -m, & \text{if } \gamma + \delta < \pi \end{cases}$$

2) If $\alpha = \beta, \gamma = \delta$ or $\alpha + \beta = \pi = \gamma + \delta$, then

$$z = \iota \sqrt{-\lambda} e^{it}, \quad w^n = q \sin t, \tag{10}$$

where

$$q = \sqrt{\frac{\sin^2 \delta}{\sin^2 \alpha} - 1}, \quad n = \begin{cases} 1, & \text{if } \gamma = \delta \\ -1, & \text{if } \gamma + \delta = \pi \end{cases}$$

$$\lambda = \frac{\tan \delta + \tan \alpha}{\tan \delta - \tan \alpha}, \quad \iota = \begin{cases} n, & \text{if } \alpha + \delta > \pi, \\ -n, & \text{if } \alpha + \delta < \pi \end{cases}$$

For square roots we adopt the convention $\sqrt{x} \in \mathbb{R}_+ \cup i\mathbb{R}_+$. □

It can be shown that $p^2 > 0 \Leftrightarrow \mu > 0$ and $q^2 > 0 \Leftrightarrow \lambda > 0$. The real part of the configuration space is parametrized by $t \in \pm \frac{\pi}{2} + i\mathbb{R}$ if both parameters are positive and by $t \in \mathbb{R} \cup (\pi + i\mathbb{R})$ if they are negative.

Theorem 2.6. The affine part of the configuration space of a spherical quadrilateral of the conic type has the parametrization

$$z^m = p \sin t, \quad w^n = q \sin(t + t_0), \tag{11}$$

Here the exponents m and n are determined according to the table

	$n = 1$	$n = -1$
$m = 1$	$\alpha + \gamma = \beta + \delta$	$\alpha + \beta = \gamma + \delta$
$m = -1$	$\alpha + \delta = \beta + \gamma$	$\alpha + \beta + \gamma + \delta = 2\pi$

The amplitudes are given by

$$p = \sqrt{\frac{\sin \alpha \sin \delta}{\sin \beta \sin \gamma} - 1} \in \mathbb{R}_{>0} \cup i\mathbb{R}_{>0}, \quad q = \sqrt{\frac{\sin \gamma \sin \delta}{\sin \alpha \sin \beta} - 1} \in \mathbb{R}_{>0} \cup i\mathbb{R}_{>0};$$

The phase shift satisfies

$$\tan t_0 = i \sqrt{\frac{\sin \beta \sin \delta}{\sin \alpha \sin \gamma}}$$

The indeterminacy in t_0 up to the summand π is resolved in the table below: it gives an interval in which t_0 lies, depending on the values of pq and σ .

	$pq \in \mathbb{R}_{>0}$	$pq \in i\mathbb{R}_{>0}$	$pq \in \mathbb{R}_{<0}$
$\sigma < \pi$	$i\mathbb{R}_{>0}$	$\frac{\pi}{2} + i\mathbb{R}_{>0}$	$\pi + i\mathbb{R}_{>0}$
$\sigma > \pi$	$\pi + i\mathbb{R}_{>0}$	$\frac{3\pi}{2} + i\mathbb{R}_{>0}$	$i\mathbb{R}_{>0}$

where

$$\sigma = \begin{cases} \frac{\alpha + \beta + \gamma + \delta}{2}, & \text{if } \alpha + \gamma = \beta + \delta \\ \frac{-\alpha + \beta + \gamma - \delta}{2} + \pi, & \text{if } \alpha + \beta = \gamma + \delta \\ \frac{\alpha + \beta - \gamma - \delta}{2} + \pi, & \text{if } \alpha + \delta = \beta + \gamma \\ \frac{-\alpha + \beta - \gamma + \delta}{2} + \pi, & \text{if } \alpha + \beta + \gamma + \delta = 2\pi \end{cases} \quad \square$$

The real part of the configuration space consists of two lines on a cylinder (plus a singular point at infinity, which corresponds to the folded quadrilateral). The exact parameter range depends on the signs of p^2, q^2 and the relation between σ and π . See [9, Section 3.2.3], which deals with the case of a Euclidean quadrilateral.

In the elliptic case introduce the notation

$$\sigma = \frac{\alpha + \beta + \gamma + \delta}{2}, \quad \bar{\alpha} = \sigma - \alpha = \frac{-\alpha + \beta + \gamma + \delta}{2},$$

and similarly $\bar{\beta} = \sigma - \beta, \bar{\gamma} = \sigma - \gamma, \bar{\delta} = \sigma - \delta$. Applying this transformation twice yields the initial quadruple of numbers. Denote

$$M = \frac{\sin \alpha \sin \beta \sin \gamma \sin \delta}{\sin \bar{\alpha} \sin \bar{\beta} \sin \bar{\gamma} \sin \bar{\delta}},$$

Because of the last equation of Lemma 4.2, if equation (5) has no solution, then we have $M \neq 1$. More exactly, we have

$$\begin{aligned} M < 1 &\Leftrightarrow \alpha_{\min} + \alpha_{\max} < \sigma < \pi \text{ OR } \alpha_{\min} + \alpha_{\max} > \sigma > \pi \\ M > 1 &\Leftrightarrow \alpha_{\min} + \alpha_{\max} > \sigma < \pi \text{ OR } \alpha_{\min} + \alpha_{\max} < \sigma > \pi, \end{aligned}$$

where α_{\min} , respectively, α_{\max} denotes the smallest, respectively, the biggest of the numbers $\alpha, \beta, \gamma, \delta$.

Theorem 2.7. The configuration space of a spherical quadrilateral of the elliptic type has the following parametrization.

(1) If $M < 1$, then

$$z = p \operatorname{sn} t, \quad w = q \operatorname{sn}(t + t_0), \quad (12)$$

where sn is the elliptic sine function with modulus $k = \sqrt{1 - M}$, and the phase shift t_0 satisfies

$$\operatorname{dn} t_0 = \sqrt{\frac{\sin \alpha \sin \gamma}{\sin \bar{\alpha} \sin \bar{\gamma}}}$$

(2) If $M > 1$, then

$$z = p \operatorname{cn} t, \quad w = q \operatorname{cn}(t + t_0), \quad (13)$$

where cn is the elliptic cosine function with modulus $k = \sqrt{1 - M^{-1}}$, and the phase shift t_0 satisfies

$$\operatorname{dn} t_0 = \sqrt{\frac{\sin \bar{\alpha} \sin \bar{\gamma}}{\sin \alpha \sin \gamma}}$$

In both cases the amplitudes p and q are given by

$$p = \sqrt{\frac{\sin \alpha \sin \delta}{\sin \bar{\alpha} \sin \bar{\delta}} - 1} \in \mathbb{R}_{>0} \cup i\mathbb{R}_{>0}, \quad q = \sqrt{\frac{\sin \gamma \sin \delta}{\sin \bar{\gamma} \sin \bar{\delta}} - 1} \in \mathbb{R}_{>0} \cup i\mathbb{R}_{>0}$$

The value of $\operatorname{dn} t_0$ determines the phase shift up to a real half-period $2K$. This indeterminacy is resolved in the table below: it gives an interval in which t_0 lies, depending on

the values of pq and σ .

	$pq \in \mathbb{R}_{>0}$	$pq \in i\mathbb{R}_{>0}$	$pq \in \mathbb{R}_{<0}$	
$\sigma < \pi$	$(0, iK')$	$(K, K + iK')$	$(2K, 2K + iK')$	□
$\sigma > \pi$	$(2K, 2K + iK')$	$(3K, 3K + iK')$	$(0, iK')$	

The real part of the configuration space has two components in the sn-case and one component in the cn-case. See [9, Section 3.3.5] for a discussion of the Euclidean case.

To finish setting up the notation, consider orthodiagonal quadrilaterals, that is, those whose side lengths satisfy the relation

$$\cos \alpha \cos \gamma = \cos \beta \cos \delta$$

An orthodiagonal quadrilateral is either (anti)deltoid or of elliptic type, see Section 4.5. We exclude the case when $\alpha = \beta = \gamma = \delta = \frac{\pi}{2}$, as it leads only to trivial deformations, see Theorem 2.4 and Section 6.1.

We refer to a vertex of a quadrilateral by naming the two sides incident to it. We say that an (anti)deltoid has apices $\alpha\delta$ and $\beta\gamma$, if $\alpha = \delta$, $\beta = \gamma$, or $\alpha + \delta = \pi = \beta + \gamma$.

Definition 2.8. Let Q be an orthodiagonal quadrilateral. We define the *involution factors* at each of its vertices, excluding the apices if Q is an (anti)deltoid, as follows.

The involution factor at the vertex $\alpha\delta$ is

$$\lambda := \begin{cases} \frac{\tan \delta + \tan \alpha}{\tan \delta - \tan \alpha}, & \text{if } \alpha \neq \frac{\pi}{2} \text{ or } \delta \neq \frac{\pi}{2} \\ \frac{\cos \beta + \cos \gamma}{\cos \beta - \cos \gamma}, & \text{if } \alpha = \delta = \frac{\pi}{2} \end{cases}$$

Similarly, the involution factor at the vertex $\gamma\delta$ is

$$\mu := \begin{cases} \frac{\tan \delta + \tan \gamma}{\tan \delta - \tan \gamma}, & \text{if } \gamma \neq \frac{\pi}{2} \text{ or } \delta \neq \frac{\pi}{2} \\ \frac{\cos \beta + \cos \alpha}{\cos \beta - \cos \alpha}, & \text{if } \gamma = \delta = \frac{\pi}{2} \end{cases}$$

Besides, for an orthodiagonal quadrilateral of elliptic type we put

$$v := \begin{cases} \frac{(\lambda-1)(\mu-1)}{\cos \delta}, & \text{if } \delta \neq \frac{\pi}{2} \\ 2(\mu-1) \tan \alpha, & \text{if } \delta = \gamma = \frac{\pi}{2} \\ 2(\lambda-1) \tan \gamma, & \text{if } \delta = \alpha = \frac{\pi}{2} \end{cases}$$

for an (anti)deltoid with apex $\alpha\delta$ we put

$$\zeta := \begin{cases} \frac{\mu-1}{\cos \delta}, & \text{if } \delta \neq \frac{\pi}{2} \\ 2 \tan \gamma, & \text{if } \delta = \alpha = \frac{\pi}{2} \end{cases}$$

and for an (anti)deltoid with apex $\gamma\delta$ we put

$$\xi := \begin{cases} \frac{\lambda-1}{\cos \delta}, & \text{if } \delta \neq \frac{\pi}{2} \\ 2 \tan \alpha, & \text{if } \delta = \gamma = \frac{\pi}{2} \end{cases} \quad \square$$

In the (anti)deltoid case the values of λ and μ coincide with those given in Theorem 2.5. Also there are the identities

$$p^2 \zeta^2 = 4\mu, \quad q^2 \xi^2 = 4\lambda$$

The involution factors are well-defined real numbers different from 0. For example, if $\alpha = \frac{\pi}{2}$ and $\delta \neq \frac{\pi}{2}$, then $\lambda = \frac{\infty}{-\infty} = -1$. If $\alpha = \delta$ or $\alpha + \delta = \pi$, so that the denominator or numerator in the first formula for λ vanish, then either Q is an (anti)deltoid and has no involution factor at the vertex $\delta\alpha$, or $\alpha = \delta = \frac{\pi}{2}$, and λ is computed by the second formula. The second formula makes sense since $\beta \neq \gamma$ by assumption that $\delta\alpha$ is not an apex.

2.5 Switching a boundary strip

Here we describe an operation that transforms one flexible Kokotsakis polyhedron to another flexible one. *Switching the right boundary strip* consists in replacing on Figure 3 the vertex C_1 by its mirror image with respect to A_1 and C_4 by its mirror image with respect to A_4 . Switching the left, lower, and upper boundary strips is defined similarly.

In the intrinsic terms, switching the right boundary strip consists in replacing $\beta_1, \beta_4, \gamma_1, \gamma_4$ by their complements to π . It can transform the spherical quadrilateral $Q_i, i = 1, 4$ from an isogram to an antiisogram, or from deltoid to antideltoid, or change n_i to $-n_i$ if Q_i was of conic type. We will use this repeatedly to simplify the case distinction. The action of switching on the equation of the configuration space is described in Lemma 4.5.

3 The List of Flexible Kokotsakis Polyhedra

Main theorem. Every flexible Kokotsakis polyhedron with quadrangular base belongs to one of the following classes, possibly after switching some of the boundary strips (as described in Section 2.5).

- (1) Orthodiagonal (or T-surfaces, Graf-Sauer surfaces).
- (2) Isogonal (or V-surfaces, discrete Voss surfaces).
- (3) Equimodular.
- (4) Conjugate-modular.
- (5) Linear compounds.
- (6) Linearly conjugate.
- (7) Chimeras.
- (8) Trivial. □

Each of these classes is further subdivided into subclasses. A subclass is described through a system of equations on the angles $\alpha_i, \beta_i, \gamma_i, \delta_i$ of the faces of the polyhedron or on the parameters $p_i, q_i, t_i, \kappa_i, \lambda_i$, etc. defined in terms of these angles in Section 2.4. Usually the number of equations is less than the number of parameters, and often there is a way to solve the system by the elimination method. We did not do it for all the cases, therefore we do not guarantee that there is a flexible Kokotsakis polyhedron in each subclass.

Another (and more serious) problem is that solving the corresponding system of equations produces angle values $\alpha_i, \beta_i, \gamma_i, \delta_i$ for which the *complexified* configuration space is not empty. This does not imply the non-emptiness of its real part (i.e., the configuration space of two coupled spherical quadrilaterals is always non-empty, but its real part may be empty).

Also, the real part of the configuration space may (and usually does) contain self-intersecting polyhedra. Thus, still another question is whether every subclass contains a non-self-intersecting flexible polyhedron. Sometimes the self-intersections can be removed by switching the boundary strips, see Section 2.5.

Certainly, it would be useful to give an example of a flexible polyhedron in each of the cases. Also one would like to know the extrinsic geometric properties of the polyhedra in each class, as it is the case for Bricard octahedra, or for the orthodiagonal involutive type, the first one in the list below.

3.1 Orthodiagonal types

3.1.1 Orthodiagonal involutive type

A Kokotsakis polyhedron belongs to the *orthodiagonal type*, if its planar angles satisfy the following conditions.

- (1) All spherical quadrilaterals Q_i are orthodiagonal:

$$\cos \alpha_i \cos \gamma_i = \cos \beta_i \cos \delta_i$$

Geometrically this means that the plane $C_1A_1A_2$ is orthogonal to the plane $B_1A_1A_4$ (this property is preserved during a deformation of a tetrahedral angle), and the same holds for the corresponding pairs of planes through the vertices A_2, A_3, A_4 . We exclude the case $\alpha_i = \beta_i = \gamma_i = \delta_i = \frac{\pi}{2}$, as it leads to trivial deformations only, which are described in Section 3.8.

- (2) The couplings of adjacent quadrilaterals are compatible, see Definition 4.18. Geometrically this means that each of the polygonal lines $C_1A_1A_2C_2$, $B_2A_2A_3B_3$, $C_3A_3A_4C_4$, $B_4A_4A_1B_1$ on Figure 3 remains planar during the deformation. Because of condition (1) each of the AB -planes is orthogonal to each of the AC -planes.
- (3) The angles of the base quadrilateral satisfy the condition

$$\cos \delta_1 \cos \delta_3 = \cos \delta_2 \cos \delta_4$$

Together with $\delta_1 + \delta_2 + \delta_3 + \delta_4$ this implies that the base quadrilateral is a trapezoid.

The compatibility condition 2) can be made more explicit. For example, if $\delta_1 + \delta_4 = \delta_2 + \delta_3 = \pi$ and $\alpha_i, \delta_i \neq \frac{\pi}{2}$ for all i , then the compatibility is equivalent to

$$\gamma_1 + \gamma_4 = \pi, \quad \gamma_2 + \gamma_3 = \pi, \quad \frac{\tan \alpha_1}{\tan \alpha_2} = \frac{\tan \alpha_4}{\tan \alpha_3} = \frac{\tan \delta_1}{\tan \delta_2}$$

so that, in particular, the AC -planes are parallel.

This class of flexible Kokotsakis polyhedra was described by Sauer and Graf in [20] and called "*T-Flache*" (from "*Trapezflache*"). Sauer and Graf proved their flexibility by a geometric argument.

3.1.2 Orthodiagonal antiinvolutive type

- (1) All quadrilaterals Q_i are orthodiagonal and elliptic:

$$\cos \alpha_i \cos \gamma_i = \cos \beta_i \cos \delta_i, \quad \alpha_i \pm \beta_i \pm \gamma_i \pm \delta_i \neq 0 \pmod{2\pi}$$

- (2) The involution factors at common vertices are opposite:

$$\lambda_1 = -\lambda_2, \quad \mu_1 = -\mu_4, \quad \mu_2 = -\mu_3, \quad \lambda_3 = -\lambda_4$$

(3) The following relations hold:

$$\frac{v_1^2}{\lambda_1\mu_1} = \frac{v_3^2}{\lambda_3\mu_3}, \quad \frac{v_2^2}{\lambda_2\mu_2} = \frac{v_4^2}{\lambda_4\mu_4}, \quad \frac{v_1^2}{\lambda_1\mu_1} + \frac{v_2^2}{\lambda_2\mu_2} = 1$$

3.2 Isogonal type

A polyhedron of *basic* isogonal type is characterized by the following conditions.

(1) All quadrilaterals Q_i are antiisograms:

$$\alpha_i + \gamma_i = \pi = \beta_i + \delta_i, \quad i = 1, 2, 3, 4$$

(2) One of the following equalities hold:

$$\kappa_1\kappa_3 = \kappa_2\kappa_4,$$

where κ_i are as in Theorem 2.4. Note that κ_i may take two values if Q_i is not an (anti)deltoid.

A general polyhedron of isogonal type can be obtained from a basic one by switching some of the boundary strips. For example, switching the left and the right or the upper and the lower boundary strips transforms all antiisograms to isograms.

Flexible polyhedra of isogonal type were described in [20] and named *discrete Voss surfaces*. They have two flat realizations corresponding to $\alpha_i + \beta_i + \gamma_i + \delta_i = 2\pi$ and to $\alpha_i - \beta_i + \gamma_i - \delta_i = 0$. Miura-ori [15] is of this type.

3.3 Equimodular type

Here we have two subcases:

3.3.1 Elliptic case

Assume that

$$\alpha_i \pm \beta_i \pm \gamma_i \pm \delta_i \neq 0 \pmod{2\pi} \tag{14}$$

for all choices of \pm . Introduce the notation

$$\sigma_i := \frac{\alpha_i + \beta_i + \gamma_i + \delta_i}{2}, \quad \bar{\alpha}_i := \sigma_i - \alpha_i = \frac{-\alpha_i + \beta_i + \gamma_i + \delta_i}{2}$$

Similarly, $\bar{\beta}_i := \sigma_i - \beta_i$, $\bar{\gamma}_i := \sigma_i - \gamma_i$, $\bar{\delta}_i := \sigma_i - \delta_i$. Denote

$$a_i := \frac{\sin \alpha_i}{\sin \bar{\alpha}_i}, \quad b_i := \frac{\sin \beta_i}{\sin \bar{\beta}_i}, \quad c_i := \frac{\sin \gamma_i}{\sin \bar{\gamma}_i}, \quad d_i := \frac{\sin \delta_i}{\sin \bar{\delta}_i}, \quad M_i := a_i b_i c_i d_i$$

The polyhedron is of equimodular elliptic type if the following three conditions are satisfied.

- (1) Quadrilaterals have equal moduli:

$$M_1 = M_2 = M_3 = M_4 =: M$$

- (2) Amplitudes at common vertices are equal:

$$a_1 d_1 = a_2 d_2, \quad b_2 c_2 = b_3 c_3, \quad a_3 d_3 = a_4 d_4, \quad b_4 c_4 = b_1 c_1$$

- (3) The sum of shifts is a period:

$$t_1 \pm t_2 \pm t_3 \pm t_4 \in \Lambda, \quad \Lambda := \begin{cases} 4K\mathbb{Z} + 2iK'\mathbb{Z}, & \text{if } M < 1 \\ 4K\mathbb{Z} + (2K + 2iK')\mathbb{Z}, & \text{if } M > 1, \end{cases}$$

where t_i is determined up to the real half-period $2K$ by

$$\operatorname{dn} t_i = \begin{cases} \sqrt{a_i c_i}, & \text{if } M < 1 \\ \frac{1}{\sqrt{a_i c_i}}, & \text{if } M > 1 \end{cases}$$

and the indeterminacy is resolved at the end of Theorem 2.7.

Together with the restriction $\delta_1 + \delta_2 + \delta_3 + \delta_4 = 2\pi$ this gives nine conditions on sixteen angles of a Kokotsakis polyhedron. However, as follows from [10, Sections 9, 10], these conditions are not independent, so that a polyhedron of elliptic equimodular type depends on eight parameters instead of seven.

3.3.2 Conic case

The polyhedron is of *basic* equimodular conic type if the following conditions are satisfied.

- (1) All spherical quadrilaterals Q_i are circumscribed: $\alpha_i + \gamma_i = \beta_i + \delta_i$.

(2) Their amplitudes at common vertices are equal:

$$\frac{\sin \alpha_1 \sin \delta_1}{\sin \beta_1 \sin \gamma_1} = \frac{\sin \alpha_2 \sin \delta_2}{\sin \beta_2 \sin \gamma_2} = \frac{\sin \alpha_2 \sin \beta_2}{\sin \gamma_2 \sin \delta_2} = \frac{\sin \alpha_3 \sin \beta_3}{\sin \gamma_3 \sin \delta_3}$$

$$\frac{\sin \alpha_3 \sin \delta_3}{\sin \beta_3 \sin \gamma_3} = \frac{\sin \alpha_4 \sin \delta_4}{\sin \beta_4 \sin \gamma_4} = \frac{\sin \alpha_4 \sin \beta_4}{\sin \gamma_4 \sin \delta_4} = \frac{\sin \alpha_1 \sin \beta_1}{\sin \gamma_1 \sin \delta_1}$$

(3) The sum of shifts is a multiple of 2π :

$$t_1 \pm t_2 \pm t_3 \pm t_4 \in 2\pi\mathbb{Z}$$

Here $t_i \in \mathbb{C}$ is determined up to π by

$$\tan t_i = i \sqrt{\frac{\sin \beta_i \sin \delta_i}{\sin \alpha_i \sin \gamma_i}},$$

(one of the i 's on the right-hand side is the imaginary unit), with the indeterminacy resolved as in Theorem 2.6.

Switching all four boundary strips results in replacing condition (1) by

(1)' All spherical quadrilaterals Q_i have perimeter 2π : $\alpha_i + \beta_i + \gamma_i + \delta_i$.

This is an origami.

3.4 Conjugate-modular type

3.4.1 First elliptic conjugate-modular type

- (1) All quadrilaterals Q_i are elliptic of cn-type: $M_i > 1$ for all i .
- (2) The moduli of Q_1 and Q_3 are equal and conjugate to those of Q_2 and Q_4 :

$$M_1 = M_3, \quad M_2 = M_4, \quad \frac{1}{M_1} + \frac{1}{M_2} = 1$$

- (3) The amplitudes satisfy the following relations:

$$\frac{p_1}{p_2} = \pm i \frac{k}{k'}, \quad \frac{q_3}{q_2} = \pm i \frac{k}{k'}, \quad \frac{p_3}{p_4} = \pm i \frac{k}{k'}, \quad \frac{q_1}{q_4} = \pm i \frac{k}{k'},$$

where $k = \sqrt{1 - M_1^{-1}}$ is the Jacobi modulus of Q_1 .

- (4) For some of the eight choices of the \pm signs on the left-hand side, the following equation hold:

$$t_1 \pm it_2 \pm t_3 \pm it_4 = \begin{cases} 0(\text{mod } \Lambda), & \text{if } \frac{p_1}{p_2} = \frac{q_3}{q_2}, \frac{q_1}{q_4} = \frac{p_3}{p_4} \text{ or } \frac{p_1}{p_2} = -\frac{q_3}{q_2}, \frac{q_1}{q_4} = -\frac{p_3}{p_4} \\ 2K(\text{mod } \Lambda), & \text{if } \frac{p_1}{p_2} = \frac{q_3}{q_2}, \frac{q_1}{q_4} = -\frac{p_3}{p_4} \text{ or } \frac{p_1}{p_2} = -\frac{q_3}{q_2}, \frac{q_1}{q_4} = \frac{p_3}{p_4} \end{cases}$$

The last condition implies that $t_1 \pm t_3 \in K\mathbb{Z} + iK'\mathbb{Z}$ and $t_2 \pm t_4 \in K'\mathbb{Z} + iK\mathbb{Z}$.

3.4.2 Second elliptic conjugate-modular type

- (1) All quadrilaterals Q_i are elliptic of cn-type: $M_i > 1$ for all i .
- (2) The moduli of Q_1 and Q_4 are equal and conjugate to those of Q_2 and Q_3 :

$$M_1 = M_4, \quad M_2 = M_3, \quad \frac{1}{M_1} + \frac{1}{M_2} = 1$$

- (3) The amplitudes satisfy the following relations:

$$\frac{p_1}{p_2} = \pm i \frac{k}{k'}, \quad q_2 = q_3, \quad \frac{p_3}{p_4} = \pm i \frac{k'}{k}, \quad q_4 = q_1,$$

where $k = \sqrt{1 - M_1^{-1}}$ is the Jacobi modulus of Q_1 .

- (4) For some of the eight choices of the \pm signs on the left-hand side, the following equation hold:

$$t_1 \pm it_2 \pm it_3 \pm t_4 = \begin{cases} 0(\bmod \Lambda), & \text{if } \frac{p_1}{p_2} = \frac{p_4}{p_3} \\ 2K(\bmod \Lambda), & \text{if } \frac{p_1}{p_2} = -\frac{p_4}{p_3} \end{cases}$$

The last condition is rather restrictive. It implies that $t_1 \pm t_4 \in K\mathbb{Z} + iK'\mathbb{Z}$ and $t_2 \pm t_3 \in K'\mathbb{Z} + iK\mathbb{Z}$. If $t_1 \pm t_4$ is a half-period of Λ , then the polyhedron is of linear compound type described in Section 3.5.

3.5 Linear compound type

A coupling (Q_1, Q_2) is called *linear* if it results in a linear dependence between $\tan \frac{\psi_1}{2}$ and $\tan \frac{\psi_2}{2}$:

$$w_1 = cw_2 \tag{15}$$

(Strictly speaking, if one of the components of the configuration space of the coupling has this equation.) If the coupling (Q_3, Q_4) is also linear with the same value of c , then the polyhedron is flexible, and we say that it belongs to the *linear compound type*. Switching the right or the left boundary strip transforms the linear dependence to $w_1 w_2 = c'$.

Planar-symmetric and translational types from [22] are special cases of linear compounds.

Below we list all linear couplings (Q_1, Q_2) together with the corresponding values of c .

3.5.1 Coupling of (anti)isograms

Both Q_1 and Q_2 are isograms or both are antiisograms. The coefficient c in (15) equals

$$c = \frac{\kappa_2}{\kappa_1}$$

with κ_i as in Theorem 2.4. Switching the lower boundary strip transforms isograms to antiisograms.

3.5.2 Linear lateral (anti)deltoid coupling

- (1) Q_1 and Q_2 are both deltoids or both antideltoids, coupled laterally:

$$\alpha_i = \beta_i, \quad \gamma_i = \delta_i, \quad \text{or}$$

$$\alpha_i + \beta_i = \pi = \gamma_i + \delta_i$$

- (2) The coupling is involutive, that is, $\lambda_1 = \lambda_2$. If $\alpha_i, \delta_i \neq \frac{\pi}{2}$, then this is equivalent to

$$\frac{\tan \alpha_1}{\tan \delta_1} = \frac{\tan \alpha_2}{\tan \delta_2}$$

The coefficient c equals

$$c = \begin{cases} \frac{\xi_2}{\xi_1}, & \text{if } Q_1 \text{ and } Q_2 \text{ are deltoids,} \\ \frac{\xi_1}{\xi_2}, & \text{if } Q_1 \text{ and } Q_2 \text{ are antideltoids} \end{cases}$$

with ξ_i as in Definition 2.8. In particular, if $\delta_i \neq \frac{\pi}{2}$, then $\frac{\xi_2}{\xi_1} = \frac{\cos \delta_1}{\cos \delta_2}$.

3.5.3 Linear frontal (anti)deltoid coupling

- (1) Q_1 and Q_2 are both deltoids or both antideltoids, coupled frontally:

$$\alpha_i = \delta_i, \quad \beta_i = \gamma_i, \quad i = 1, 2, \text{ or}$$

$$\alpha_i + \delta_i = \pi = \beta_i + \gamma_i, \quad i = 1, 2$$

- (2) The coupling is reducible, which means that

$$\frac{\sin \alpha_1}{\sin \beta_1} = \frac{\sin \alpha_2}{\sin \beta_2}$$

The coefficient c equals

$$c = \frac{\varepsilon_1 \sqrt{-\mu_1}}{\varepsilon_2 \sqrt{-\mu_2}} = \pm \sqrt{\frac{\sin(\alpha_1 + \beta_1) \sin(\alpha_2 - \beta_2)}{\sin(\alpha_2 + \beta_2) \sin(\alpha_1 - \beta_1)'}}$$

see Theorem 2.5. Switching the lower boundary strip transforms deltoids to antideltoids while preserving the coefficient c .

3.5.4 Linear elliptic coupling

Configuration spaces of Q_1 and Q_2 are elliptic curves. Besides, in the notation of Section 3.3.1 we have

- (1) Q_1 and Q_2 have equal moduli $M_1 = M_2$, as well as equal amplitudes at the common vertex:

$$a_1 d_1 = a_2 d_2, \quad b_1 c_1 = b_2 c_2$$

- (2) The shift difference is a (real) half-period: $t_1 - t_2 \in \{0, 2K\}$. Or, equivalently,

$$a_1 c_1 = a_2 c_2$$

Then we have

$$c = \begin{cases} \sqrt{\frac{c_1 d_1 - 1}{c_2 d_2 - 1}}, & \text{if } \sin \sigma_1 \sin \sigma_2 > 0, \\ -\sqrt{\frac{c_1 d_1 - 1}{c_2 d_2 - 1}}, & \text{if } \sin \sigma_1 \sin \sigma_2 < 0 \end{cases}$$

3.5.5 Linear conic coupling

- (1) Quadrilaterals Q_1 and Q_2 are either both circumscribed or both have perimeter 2π :

$$\alpha_i + \gamma_i = \beta_i + \delta_i, \quad i = 1, 2 \tag{16a}$$

$$\alpha_i + \beta_i + \gamma_i + \delta_i = 2\pi, \quad i = 1, 2 \tag{16b}$$

- (2) Their amplitudes at the common vertex are equal, and the shifts difference is a half-period:

$$\frac{\sin \alpha_1}{\sin \beta_1} = \frac{\sin \alpha_2}{\sin \beta_2}, \quad \frac{\sin \gamma_1}{\sin \delta_1} = \frac{\sin \gamma_2}{\sin \delta_2}$$

The value of c is given by the following table

	(16a)	(16b)
$\sin \sigma_1 \sin \sigma_2 > 0$	\bar{c}	\bar{c}^{-1}
$\sin \sigma_1 \sin \sigma_2 < 0$	$-\bar{c}$	$-\bar{c}^{-1}$

where $\bar{c} = \sqrt{\frac{\sin \gamma_1 \sin \delta_1 - 1}{\sin \alpha_1 \sin \beta_1} \frac{\sin \gamma_2 \sin \delta_2 - 1}{\sin \alpha_2 \sin \beta_2}}$

3.6 Linearly conjugate types

These are polyhedra, where Q_2 and Q_4 are antiisograms, so that they result in linear dependencies $z = \kappa_2 w_2$ and $u = \kappa_4 w_1$. Polyhedra Q_1 and Q_3 must be of the same type, and their equations are related by the above linear substitutions.

3.6.1 Linearly conjugate antideltooids

- (1) Q_1 and Q_3 are "parallel" antideltooids:

$$\alpha_1 + \delta_1 = \pi = \beta_1 + \gamma_1, \quad \alpha_3 + \beta_3 = \pi = \gamma_3 + \delta_3$$

- (2) Q_2 and Q_4 are antiisograms:

$$\alpha_2 + \gamma_2 = \pi = \beta_2 + \gamma_2, \quad \alpha_4 + \gamma_4 = \pi = \beta_4 + \delta_4$$

- (3) The following conditions are satisfied:

$$\kappa_4^2 \mu_1 = \lambda_3, \quad \kappa_4 \zeta_1 = \kappa_2 \xi_3,$$

where $\mu_1, \zeta_1, \lambda_3, \xi_3$ are as in Definition 2.8, and κ_2, κ_4 as in Theorem 2.4.

3.6.2 Linearly conjugate conics

- (1) Q_1 and Q_3 are conic quadrilaterals of perimeter 2π :

$$\alpha_1 + \beta_1 + \gamma_1 + \delta_1 = 2\pi = \alpha_3 + \beta_3 + \gamma_3 + \delta_3,$$

and equations $\alpha_i \pm \beta_i \pm \gamma_i \pm \delta_i \equiv 0 \pmod{2\pi}$ have for $i = 1, 3$ no other solutions.

- (2) Q_2 and Q_4 are antiisograms:

$$\alpha_2 + \gamma_2 = \pi = \beta_2 + \gamma_2, \quad \alpha_4 + \gamma_4 = \pi = \beta_4 + \delta_4$$

(3) The following relations hold:

$$q_3 = |\kappa_2|p_1, \quad q_1 = |\kappa_4|p_3, \quad t_1 = \begin{cases} t_3, & \text{if } c_2c_4 > 0 \\ t_3 + \pi, & \text{if } c_2c_4 < 0 \end{cases}$$

3.6.3 Linearly conjugate elliptics

(1) Quadrilaterals Q_1 and Q_3 are elliptic and have the same modulus:

$$M_1 = M_3$$

(2) Q_2 and Q_4 are antiisograms:

$$\alpha_2 + \gamma_2 = \pi = \beta_2 + \gamma_2, \quad \alpha_4 + \gamma_4 = \pi = \beta_4 + \delta_4$$

(3) The following relations hold:

$$p_1 = |\kappa_2|q_3, \quad p_3 = |\kappa_4|q_1, \quad t_1 = \begin{cases} t_3, & \text{if } \kappa_2\kappa_4 > 0 \\ t_3 + 2K, & \text{if } \kappa_2\kappa_4 < 0 \end{cases}$$

3.7 Chimeras

This is the largest class; we do not split it into smaller classes, in order not to imitate Borges' taxonomy of animals.

A common feature of the polyhedra listed here is that each of them contains tetrahedral angles of different types. Potentially they could be used to join pieces of flexible quad surfaces of equimodular, orthodiagonal, or isogonal types.

3.7.1 Conic-deltoid

(1) Quadrilaterals Q_2 and Q_3 have perimeter 2π :

$$\alpha_2 + \beta_2 + \gamma_2 + \delta_2 = 2\pi, \quad \alpha_3 + \beta_3 + \gamma_3 + \delta_3 = 2\pi$$

and form a reducible coupling: $q_2 = q_3$.

(2) Quadrilaterals Q_1 and Q_4 are antideltoids reducibly coupled to Q_2 and Q_3 :

$$\alpha_1 + \delta_1 = \pi = \beta_1 + \gamma_1, \quad p_1 = p_2$$

$$\alpha_4 + \delta_4 = \pi = \beta_4 + \gamma_4, \quad p_4 = p_3$$

- (3) The number t_0 determined by

$$\varepsilon_1 \sqrt{-\mu_1} = \varepsilon_4 \sqrt{-\mu_4} e^{it_0}$$

is related to the shifts of Q_2 and Q_3 through

$$t_0 = \pm t_1 \pm t_2,$$

where any of the four combinations of signs are allowed.

3.7.2 First orthodiagonal-isogram

- (1) The quadrilateral Q_1 is orthodiagonal:

$$\cos \alpha_1 \cos \gamma_1 = \cos \beta_1 \cos \delta_1$$

- (2) Q_2 and Q_4 are antideltooids that form involutive couplings with Q_1 :

$$\alpha_2 + \beta_2 = \pi = \gamma_2 + \delta_2, \quad \lambda_1 = \lambda_2$$

$$\alpha_4 + \delta_4 = \pi = \beta_4 + \gamma_4, \quad \mu_1 = \mu_4$$

(In particular, this implies that Q_1 is not an (anti)deltoid.)

- (3) Q_3 is an isogram: $\alpha_3 = \gamma_3, \beta_3 = \delta_3$.
 (4) the parameters of Q_i must satisfy the relation

$$\nu_1 = \kappa_3 \xi_2 \zeta_4$$

with κ, ν, ξ , and ζ as in Theorem 2.4 and Definition 2.8. In particular, if $\delta_i \neq \frac{\pi}{2}$ for $i = 1, 2, 4$, then this condition becomes

$$\kappa_3 \cos \delta_1 = \cos \delta_2 \cos \delta_4$$

3.7.3 Second orthodiagonal-isogram

- (1) The quadrilateral Q_1 is orthodiagonal:

$$\cos \alpha_1 \cos \gamma_1 = \cos \beta_1 \cos \delta_1$$

Besides, Q_1 is neither deltoid nor antideltooid.

- (2) Q_2 is an antideltooid forming an involutive coupling with Q_1 :

$$\alpha_2 + \beta_2 = \pi = \gamma_2 + \delta_2, \quad \lambda_1 = \lambda_2$$

- (3) Q_3 is a deltoid coupled with Q_2 frontally, and Q_4 is an antiisogram:

$$\alpha_3 = \beta_3, \quad \gamma_3 = \delta_3, \quad \alpha_4 + \gamma_4 = \beta_4 + \delta_4$$

- (4) The following two equations hold:

$$\kappa_4^2 \mu_1 = \lambda_3, \quad \kappa_4 \nu_1 = \xi_2 \xi_3,$$

with $\kappa, \lambda, \mu, \nu, \xi$ as in Theorem 2.4 and Definition 2.8.

3.7.4 Conic-isogram

- (1) The quadrilateral Q_2 is a conic quadrilateral of perimeter 2π :

$$\alpha_2 + \beta_2 + \gamma_2 + \delta_2 = 2\pi,$$

equation $\alpha_2 \pm \beta_2 \pm \gamma_2 \pm \delta_2 \equiv 0 \pmod{2\pi}$ has no other solutions.

- (2) Quadrilaterals Q_1 and Q_3 are antideltoids reducibly coupled with Q_2 :

$$p_1 = p_2, \quad q_2 = q_3$$

- (3) Q_4 is an isogram with

$$2\varepsilon_1 i \sqrt{-\mu_1} e^{\pm it_2} \kappa_4 = q_3 \xi_3$$

3.7.5 Conic-antiisogram

- (1) The quadrilateral Q_2 is a conic quadrilateral of perimeter 2π :

$$\alpha_2 + \beta_2 + \gamma_2 + \delta_2 = 2\pi,$$

equation $\alpha_2 \pm \beta_2 \pm \gamma_2 \pm \delta_2 \equiv 0 \pmod{2\pi}$ has no other solutions.

- (2) Quadrilaterals Q_1 and Q_3 are antideltoids reducibly coupled with Q_2 :

$$p_1 = p_2, \quad q_2 = q_3$$

- (3) Q_4 is an antiisogram with

$$\varepsilon_1 i \sqrt{-\mu_1} e^{\pm it_2} q_3 \xi_3 \kappa_4 = 2$$

3.7.6 Frontally coupled deltoid and antideltoid versus reducibly coupled elliptics

- (1) (Q_1, Q_2) is a frontal coupling of a deltoid with an antideltoid:

$$\alpha_1 = \delta_1, \quad \beta_1 = \gamma_1, \quad \alpha_2 + \delta_2 = \pi = \beta_2 + \gamma_2,$$

which is irreducible: $\frac{\sin \alpha_1}{\sin \beta_1} \neq \frac{\sin \alpha_2}{\sin \beta_2}$.

- (2) Quadrilaterals Q_3 and Q_4 are elliptic with configuration spaces parametrized by sn and form a reducible coupling:

$$M_3 = M_4 < 1, \quad p_3 = p_4$$

- (3) either the sum or the difference of shifts equals a quarter-period with the imaginary part $\frac{K'}{2}$:

$$\pm t_3 \pm t_4 = lK + i\frac{K'}{2}$$

- (4) The following relations hold:

$$\begin{aligned} \mu_1 &= \frac{q_1^2}{k}, \quad \mu_2 = \frac{q_2^2}{k}, \quad \zeta_1 \zeta_2 = \frac{2(1+k)}{k\sqrt{k}} q_1 q_2, & \text{if } l = 0 \\ \mu_1 &= -\frac{q_1^2}{k}, \quad \mu_2 = -\frac{q_2^2}{k}, \quad \zeta_1 \zeta_2 = \frac{2i(1-k)}{k\sqrt{k}} q_1 q_2, & \text{if } l = 1 \\ \mu_1 &= \frac{q_1^2}{k}, \quad \mu_2 = \frac{q_2^2}{k}, \quad \zeta_1 \zeta_2 = -\frac{2(1+k)}{k\sqrt{k}} q_1 q_2, & \text{if } l = 2 \\ \mu_1 &= -\frac{q_1^2}{k}, \quad \mu_2 = -\frac{q_2^2}{k}, \quad \zeta_1 \zeta_2 = -\frac{2i(1-k)}{k\sqrt{k}} q_1 q_2, & \text{if } l = 3 \end{aligned}$$

3.7.7 Reducible conic-deltoid coupling versus isogram-deltoid coupling

- (1) The quadrilateral Q_3 is a conic quadrilateral of perimeter 2π :

$$\alpha_3 + \beta_3 + \gamma_3 + \delta_3 = 2\pi,$$

equation $\alpha_3 \pm \beta_3 \pm \gamma_3 \pm \delta_3 \equiv 0 \pmod{2\pi}$ has no other solutions.

- (2) The quadrilateral Q_4 is an antideltoid:

$$\alpha_4 + \delta_4 = \pi = \beta_4 + \gamma_4$$

that forms a reducible coupling with Q_3 :

$$p_3 = p_4$$

- (3) The quadrilateral Q_1 is an antideltoid:

$$\alpha_1 + \delta_1 = \pi = \beta_1 + \gamma_1$$

- (4) The quadrilateral Q_2 is an antiisogram:

$$\alpha_2 + \gamma_2 = \pi = \beta_2 + \delta_2$$

- (5) The following relations hold:

$$\mu_1 = \mu_4 e^{\pm 2it_3}, \quad \zeta_1 = c_2 \frac{2i\varepsilon_4 \sqrt{-\mu_4} e^{\pm it_3}}{q_2},$$

where \pm in the first equation must match the \pm in the second equation, and $z = c_2 w_2$ is the equation of an irreducible component of Z_2 .

3.7.8 Three reducibly coupled conics versus an isogram

- (1) Quadrilaterals Q_3, Q_4, Q_1 are conic with perimeter 2π :

$$\alpha_i + \beta_i + \gamma_i + \delta_i = 2\pi, \quad \text{for } i = 3, 4, 1,$$

and equations $\alpha_i \pm \beta_i \pm \gamma_i \pm \delta_i \equiv 0 \pmod{2\pi}$ have no other solutions.

- (2) Couplings (Q_3, Q_4) and (Q_4, Q_1) are reducible:

$$p_3 = p_4, \quad q_4 = q_1$$

- (3) Quadrilateral Q_2 is an antiisogram:

$$\alpha_2 + \gamma_2 = \pi = \beta_2 + \delta_2$$

- (4) The following relations hold:

$$q_3 = |\kappa_2| p_1, \quad t_1 \pm t_3 \pm t_4 = \begin{cases} 0, & \text{if } \kappa_2 > 0 \\ \pi, & \text{if } \kappa_2 < 0 \end{cases}$$

3.7.9 Three reducibly coupled elliptics versus an isogram

- (1) Quadrilaterals Q_3, Q_4, Q_1 are elliptic and have equal moduli:

$$M_1 = M_3 = M_4$$

- (2) Couplings (Q_3, Q_4) and (Q_4, Q_1) are reducible:

$$p_3 = p_4, \quad q_4 = q_1$$

- (3) Quadrilateral Q_2 is an antiisogram:

$$\alpha_2 + \gamma_2 = \pi = \beta_2 + \delta_2$$

- (4) The following relations hold:

$$p_1 = |\kappa_2|q_3, \quad t_1 \pm t_3 \pm t_4 = \begin{cases} 0, & \text{if } \kappa_2 > 0 \\ 2K, & \text{if } \kappa_2 < 0 \end{cases}$$

3.7.10 Involutively coupled orthodiagonal and antideltoid versus reducibly coupled conic and deltoid

- (1) The quadrilateral Q_1 is orthodiagonal:

$$\cos \alpha_1 \cos \gamma_1 = \cos \beta_1 \cos \delta_1,$$

but is neither deltoid nor antideltoid.

- (2) Q_2 is an antideltoid forming an involutive coupling with Q_1 :

$$\alpha_2 + \beta_2 = \pi = \gamma_2 + \delta_2, \quad \lambda_1 = \lambda_2$$

- (3) Q_3 is circumscribed:

$$\alpha_3 = \gamma_3, \quad \beta_3 = \delta_3$$

- (4) Q_4 is a deltoid forming with Q_3 a reducible coupling:

$$\alpha_4 = \delta_4, \quad \beta_4 = \gamma_4, \quad \frac{\sin^2 \alpha_4}{\sin^2 \beta_4} = \frac{\sin \alpha_3 \sin \delta_3}{\sin \beta_3 \sin \gamma_3}$$

- (5) The following relations hold:

$$\mu_1 = \mu_4 e^{\pm 2it_3}, \quad \frac{v_1}{\xi_2} = \frac{2i\varepsilon_4 \sqrt{-\mu_4} e^{\pm it_3}}{q_3}$$

(the \pm in the first equation must match the \pm in the second equation).

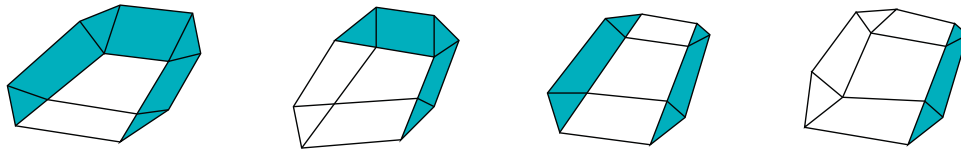


Fig. 8. Trivially flexible Kokotsakis polyhedra.



Fig. 9. The only degenerate configurations that don't count as quadrilaterals.

3.8 Trivial types

We call an isometric deformation of a Kokotsakis polyhedron *trivial*, if it preserves one of the dihedral angles at the central face. In Section 6.1 trivial deformations are classified, which leads to the four types shown on Figure 8. During an isometric deformation, shaded faces move, while white faces stay fixed in \mathbb{R}^3 . Any other trivially flexible polyhedron is obtained from one of these by switching some of the immobile boundary strips.

4 Configuration Space of a Spherical Four-Bar Linkage

4.1 Side lengths of a spherical quadrilateral

A spherical quadrilateral is a collection of four points (vertices) on the unit sphere, together with a cyclic order such that no two consecutive vertices form a pair of antipodes. Thus for any two consecutive vertices there is a unique great circle passing through them; the shortest of its arcs joining the vertices is called a side of the quadrilateral. We allow the edges of a quadrilateral to intersect and overlap, except in one of the ways shown on Figure 9.

Lemma 4.1. There exists a spherical quadrilateral with side lengths $(\alpha, \beta, \gamma, \delta)$ if and only if the inequalities

$$0 < \alpha < \pi, \quad \alpha < \beta + \gamma + \delta < \alpha + 2\pi \quad (17)$$

are fulfilled, as are all those obtained by exchanging α with β , γ , or δ . \square

Proof. The necessity of the first three inequalities in (17) follows from our definition of a spherical quadrilateral. To prove the necessity of the fourth one, replace a vertex by its antipode and apply the third inequality. The equality cases in the third and the fourth inequalities correspond to degenerate quadrilaterals shown on Figure 9.

To prove the sufficiency, note that a spherical triangle with side lengths $\alpha, \beta, \varepsilon$ exists for all $\varepsilon \in [|\alpha - \beta|, \min\{\alpha + \beta, 2\pi - \alpha - \beta\}]$. The inequalities (17) together with those obtained by permutations imply that

$$[|\alpha - \beta|, \min\{\alpha + \beta, 2\pi - \alpha - \beta\}] \cap [|\gamma - \delta|, \min\{\gamma + \delta, 2\pi - \gamma - \delta\}] \neq \emptyset,$$

therefore we can construct a quadrilateral by putting two triangles together. ■

Recall the notation

$$\sigma = \frac{\alpha + \beta + \gamma + \delta}{2}, \quad \bar{\alpha} = \sigma - \alpha = \frac{-\alpha + \beta + \gamma + \delta}{2},$$

$\bar{\beta} = \sigma - \beta, \bar{\gamma} = \sigma - \gamma, \bar{\delta} = \sigma - \delta$. Inequalities (17) imply similar inequalities for $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$, so that the latter are also side lengths of a quadrilateral.

The following lemma is proved by using standard trigonometric identities.

Lemma 4.2. For any $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and for $\sigma, \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$ as defined above the following identities (and all obtained from them by permutations) hold.

$$\begin{aligned} \bar{\alpha} + \bar{\beta} &= \gamma + \delta \\ \sin \bar{\alpha} \sin \bar{\beta} - \sin \alpha \sin \beta &= \sin \sigma \sin(\sigma - \alpha - \beta) = \sin \gamma \sin \delta - \sin \bar{\gamma} \sin \bar{\delta} \\ \sin \alpha \sin \beta - \sin \bar{\gamma} \sin \bar{\delta} &= \sin(\sigma - \alpha - \gamma) \sin(\sigma - \beta - \gamma) \\ &= \sin \bar{\alpha} \sin \bar{\beta} - \sin \gamma \sin \delta \\ \sin \sigma \sin(\sigma - \alpha - \beta) \sin(\sigma - \beta - \gamma) \sin(\sigma - \alpha - \gamma) \\ &= \sin \alpha \sin \beta \sin \gamma \sin \delta - \sin \bar{\alpha} \sin \bar{\beta} \sin \bar{\gamma} \sin \bar{\delta} \end{aligned} \quad \square$$

4.2 The configuration space as an algebraic curve

For any numbers $\alpha, \beta, \gamma, \delta$ that satisfy inequalities (17), the associated *configuration space* is the set of all spherical quadrilaterals with side lengths $\alpha, \beta, \gamma, \delta$ in this cyclic order, up to an orientation-preserving isometry. A quadrilateral with given side lengths

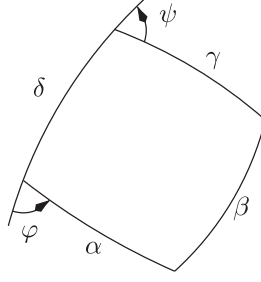


Fig. 10. Angles φ and ψ determine the shape of the quadrilateral.

is uniquely determined by the values of two adjacent angles; on the other hand, these angles satisfy a certain relation. By performing the substitution

$$z = \tan \frac{\varphi}{2}, \quad w = \tan \frac{\psi}{2}, \quad (18)$$

where φ and ψ are as on Figure 10, we arrive at a polynomial equation in z and w .

Lemma 4.3 (Bricard [4]). The configuration space of quadrilaterals with side lengths $\alpha, \beta, \gamma, \delta$ in this cyclic order is the solution set of the equation

$$c_{22}z^2w^2 + c_{20}z^2 + c_{02}w^2 + 2c_{11}zw + c_{00} = 0, \quad \text{where} \quad (19)$$

$$c_{22} = \sin \frac{\alpha + \beta + \gamma - \delta}{2} \sin \frac{\alpha - \beta + \gamma - \delta}{2} = \sin \bar{\delta} \sin(\sigma - \beta - \delta)$$

$$c_{20} = \sin \frac{\alpha - \beta - \gamma - \delta}{2} \sin \frac{\alpha + \beta - \gamma - \delta}{2} = \sin \bar{\alpha} \sin(\sigma - \beta - \alpha)$$

$$c_{02} = \sin \frac{\alpha + \beta - \gamma + \delta}{2} \sin \frac{\alpha - \beta - \gamma + \delta}{2} = \sin \bar{\gamma} \sin(\sigma - \beta - \gamma)$$

$$c_{11} = -\sin \alpha \sin \gamma$$

$$c_{00} = \sin \frac{\alpha - \beta + \gamma + \delta}{2} \sin \frac{\alpha + \beta + \gamma + \delta}{2} = \sin \bar{\beta} \sin \sigma \quad \square$$

Proof can also be found in [9, 22].

The substitution (18) identifies $\mathbb{R}/2\pi\mathbb{Z}$ with $\mathbb{R} \cup \{\infty\} = \mathbb{RP}^1$. Therefore equation (19) must be viewed as an equation in two projective variables. This is achieved by *bihomogenization*

$$c_{22}z_1^2w_1^2 + c_{20}z_1^2w_0^2 + c_{02}z_0^2w_1^2 + 2c_{11}z_1w_1z_0w_0 + c_{00}z_0^2w_0^2 = 0, \quad (20)$$

where $z = \frac{z_1}{z_0}$, $w = \frac{w_1}{w_0}$, see [9] for more details.

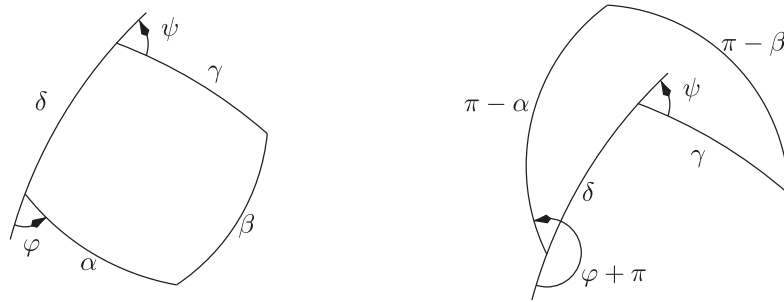


Fig. 11. Isomorphism between the spaces $Z(\alpha, \beta, \gamma, \delta)$ and $Z(\alpha, \pi - \beta, \pi - \gamma, \delta)$.

Definition 4.4. The solution set of equation (20) in $(\mathbb{CP}^1)^2$ is called the *complexified configuration space* of quadrilaterals with side lengths $\alpha, \beta, \gamma, \delta$ and is denoted by $Z(\alpha, \beta, \gamma, \delta)$ or just briefly by Z . □

The following lemma will be useful in the next section.

Lemma 4.5. Let $\alpha, \beta, \gamma, \delta$ be a quadruple of numbers satisfying inequalities (17). Then the map

$$\begin{aligned} (\mathbb{CP}^1)^2 &\rightarrow (\mathbb{CP}^1)^2 \\ (z, w) &\mapsto (-z^{-1}, w) \end{aligned}$$

restricts to a bijection $Z(\alpha, \beta, \gamma, \delta) \rightarrow Z(\pi - \alpha, \pi - \beta, \gamma, \delta)$, and the map

$$\begin{aligned} (\mathbb{CP}^1)^2 &\rightarrow (\mathbb{CP}^1)^2 \\ (z, w) &\mapsto (z, -w^{-1}) \end{aligned}$$

restricts to a bijection $Z(\alpha, \beta, \gamma, \delta) \rightarrow Z(\alpha, \pi - \beta, \pi - \gamma, \delta)$. □

Proof. It suffices to prove only the first part. This can be done by making the substitutions $\alpha \rightarrow \pi - \alpha, \beta \rightarrow \pi - \beta$ in (19). The bijection between the real parts of the configuration spaces is established by replacing the vertex $\alpha\beta$ by its antipode and using $\tan \frac{\varphi + \pi}{2} = -(\tan \frac{\varphi}{2})^{-1}$, see Figure 11. ■

4.3 Classifying configuration spaces

Here we prove Theorems 2.4–2.7.

Let Q be a spherical quadrilateral with side lengths $\alpha, \beta, \gamma, \delta$ in this cyclic order. The shape of the configuration space will depend on the number of solutions of the equation

$$\alpha \pm \beta \pm \gamma \pm \delta \equiv 0 \pmod{2\pi} \tag{21}$$

Because of (17), there is no solution with one or three minus signs. Thus, every solution of (21) corresponds either to the sum of two sides being equal to the sum of two others or to the sum of all sides being equal to 2π .

If equation (21) has at least two solutions, then it is easy to show that Q has either two pairs of sides of equal lengths or two pairs of sides, lengths in each pair complementing each other to π . If these are pairs of opposite sides, then Q is an isogram or antiisogram; if these are pairs of adjacent sides, then Q is a deltoid or antideltoid. It follows that every quadrilateral belongs to one of the types described in Definition 2.3.

Proof of Theorem 2.4. If Q is an antiisogram, then equation (19) becomes

$$\sin(\alpha - \delta)z^2 + 2 \sin \alpha z w + \sin(\alpha + \delta)w^2 = 0 \tag{22}$$

If $\alpha = \delta = \frac{\pi}{2}$, then the bihomogenization (20) yields $z_1 z_0 w_1 w_0 = 0$. That is, the configuration space consists of four trivial components: $z = 0$, $z = \infty$, $w = 0$, and $w = \infty$. Similarly, if $\alpha = \beta$ or $\alpha + \beta = \pi$, then there are two trivial components and one non-trivial of the form $z = \kappa w$ with κ given by (6). Finally, if $\alpha \neq \beta$ and $\alpha + \beta \neq \pi$, then by solving the quadratic equation (22) (for which the identity $\sin(\alpha - \beta) \sin(\alpha + \beta) = \sin^2 \alpha - \sin^2 \beta$ might be useful) we find two non-trivial components with κ as given in (7).

If Q is an isogram, then equation (19) becomes

$$\sin(\alpha - \beta)z^2 w^2 - 2 \sin \alpha z w + \sin(\alpha + \beta) = 0,$$

and the argument is similar. Alternatively, one can use the first switching isomorphism of Lemma 4.5. ■

To deal with the (anti)deltoid case, we need the following lemma.

Lemma 4.6. The affine algebraic curve $aw^2 + 2bzw + c = 0$ with $a, b, c \neq 0$ has the parametrization $z = p \sin t$, $w = re^{it}$, where

$$p = \sqrt{\frac{ac}{b^2}}, \quad r = \begin{cases} \sqrt{\frac{-c}{a}}, & \text{if } bc > 0 \\ -\sqrt{\frac{-c}{a}}, & \text{if } bc < 0 \end{cases} \quad \square$$

The proof is straightforward. Recall our convention $\sqrt{x} \in i\mathbb{R}_+$ for $x < 0$.

The (1,2)-bihomogenization of the curve $aw^2 + 2bzw + c = 0$ contains two additional points $(\infty, 0)$ and (∞, ∞) .

Proof of Theorem 2.5. If Q satisfies $\alpha = \delta, \beta = \gamma$, then equation (19) becomes

$$\sin(\delta - \gamma)w^2 - 2 \sin \gamma zw + \sin(\delta + \gamma) = 0 \tag{23}$$

The (2,2)-bihomogenization (20) contains a trivial component $z = \infty$. The affine part of the non-trivial component can be parametrized according to Lemma 4.6, which yields parametrization (9) for the case $m = 1$. The case $m = 1$ follows by the first switching isomorphism of Lemma 4.5. Finally, parametrizations (10) are obtained by exchanging z with w and α with γ . ■

Let now Q be a quadrilateral of conic type. Consider first the case when the unique solution of equation (21) is $\alpha + \gamma = \beta + \delta$. Then we have

$$\begin{aligned} \sigma &= \alpha + \gamma = \beta + \delta, \\ \bar{\alpha} &= \gamma, \quad \bar{\beta} = \delta, \quad \bar{\gamma} = \alpha, \quad \bar{\delta} = \beta, \\ \sigma - \beta - \alpha &= \delta - \alpha, \quad \sigma - \beta - \gamma = \delta - \gamma \end{aligned}$$

It follows that $c_{22} = 0$ in (19) and that the other coefficients are

$$\begin{aligned} c_{20} &= \sin \gamma \sin(\delta - \alpha), \quad c_{02} = \sin \alpha \sin(\delta - \gamma), \\ c_{11} &= -\sin \alpha \sin \gamma, \quad c_{00} = \sin \sigma \sin \delta \end{aligned} \tag{24}$$

Lemma 4.7. The affine algebraic curve

$$c_{20}z^2 + c_{02}w^2 + 2c_{11}zw + c_{00} = 0 \tag{25}$$

with $c_{20} \neq 0, c_{02} \neq 0, c_{00} \neq 0, c_{11}^2 - c_{20}c_{02} \neq 0$ has the parametrization

$$z = p \sin t, \quad w = q \sin(t + t_0),$$

where the amplitudes and the phase shift are given by

$$p = \sqrt{\frac{c_{02}c_{00}}{c_{11}^2 - c_{20}c_{02}}}, \quad q = \sqrt{\frac{c_{02}c_{00}}{c_{11}^2 - c_{20}c_{02}}}, \quad \cos t_0 = -\frac{c_{11}c_{00}}{(c_{11}^2 - c_{20}c_{02})pq}$$

(the phase shift is determined only up to the sign). □

Proof. It is easy to show that the functions $x = \sin t$, $y = \sin(t + t_0)$ parametrize the curve

$$x^2 + y^2 - 2 \cos t_0 xy - \sin^2 t_0 = 0 \tag{26}$$

Substitution $x = \frac{z}{p}$, $y = \frac{w}{q}$ transforms this to

$$q^2 z^2 + p^2 w^2 - 2pq \cos t_0 zw - p^2 q^2 \sin^2 t_0 = 0$$

and the formulas for p , q , $\cos t_0$ are found by solving the proportion

$$q^2 : p^2 : -pq \cos t_0 : -p^2 q^2 \sin^2 t_0 = c_{20} : c_{02} : c_{11} : c_{00} \quad \blacksquare$$

In order to apply Lemma 4.7 to the equation (19), we need the following specialization of Lemma 4.2.

Lemma 4.8. If $\alpha + \gamma = \beta + \delta =: \sigma$, then the following identities hold.

$$\begin{aligned} \sin \alpha \sin \gamma - \sin \beta \sin \delta &= \sin(\delta - \alpha) \sin(\delta - \gamma) \\ \sin \gamma \sin \delta - \sin \alpha \sin \beta &= \sin \sigma \sin(\delta - \alpha) \quad \square \\ \sin \alpha \sin \delta - \sin \beta \sin \gamma &= \sin \sigma \sin(\delta - \gamma) \end{aligned}$$

Proof of Theorem 2.6. If $\alpha + \gamma = \beta + \delta$, then equation (19) takes the form (25) with c_{ij} as in (24). If equation (21) has no other solutions, then $c_{20}, c_{02}, c_{00} \neq 0$. Besides, because of Lemma 4.8 we have

$$c_{11}^2 - c_{20}c_{02} = \sin \alpha \sin \beta \sin \gamma \sin \delta \neq 0$$

Thus we are in a position to apply Lemma 4.7. It yields

$$p = \sqrt{\frac{\sin \sigma \sin(\delta - \gamma)}{\sin \beta \sin \gamma}}, \quad q = \sqrt{\frac{\sin \sigma \sin(\delta - \alpha)}{\sin \alpha \sin \beta}},$$

which coincides with the formulas given in Theorem 2.6 due to Lemma 4.8. Furthermore we have

$$\cos t_0 = \frac{\sin \sigma}{pq \sin \beta}, \quad \cos^2 t_0 = \frac{\sin \alpha \sin \gamma}{\sin \alpha \sin \gamma - \sin \beta \sin \delta} \in \mathbb{R} \setminus [0, 1] \tag{27}$$

It follows that $\tan^2 t_0 = -\frac{\sin \beta \sin \delta}{\sin \alpha \sin \gamma}$. Since the phase shift is determined only up to the sign, we choose t_0 with $\text{Im } t_0 > 0$, that is, $\text{Im } \tan t_0 > 0$, which leads to the formula in Theorem 2.6. Finally, the range of cosine

$$i\mathbb{R}_+ \mapsto [0, +\infty), \quad \frac{\pi}{2} + i\mathbb{R}_+ \mapsto (-\infty, 1]$$

and equations (27) lead to the table determining $\text{Re } t_0$.

The other cases can be reduced to $\alpha + \gamma = \beta + \delta$ with the help of switching isomorphisms of Lemma 4.5. For example, if $\alpha + \beta = \gamma + \delta$, then we denote

$$\alpha' := \alpha, \beta' := \pi - \beta, \gamma' := \pi - \gamma, \delta' := \delta,$$

so that $\alpha' + \gamma' = \beta' + \delta'$. Hence the configuration space $Z(\alpha', \beta', \gamma', \delta')$ has the parametrization $z = p' \sin t$, $w = q' \sin(t \pm t'_0)$, where p' , q' , and t'_0 are computed by applying the formulas of Theorem 2.6 to the angles α' , β' , γ' , δ' . By composing this with the second switching isomorphism of Lemma 4.5, we obtain the parametrization

$$z = p' \sin t, \quad w^{-1} = -q' \sin(t + t'_0) = q' \sin(t + (t'_0 + \pi))$$

of the space $Z(\alpha, \beta, \gamma, \delta)$. It remains to note that $p' = p$, $q' = q$, $\tan t_0 = \tan t'_0$. On the other hand, we have

$$\sigma' = \frac{\alpha' + \beta' + \gamma' + \delta'}{2} = \frac{\alpha - \beta - \gamma + \delta}{2} + \pi,$$

which is $< \pi$ if and only if $\sigma > \pi$ with σ given by the table in Theorem 2.6. This accounts for $t_0 = t'_0 + \pi$. ■

In the elliptic case we use the following lemma.

Lemma 4.9. For any given $k \in (0, 1)$ and $t_0 \in \mathbb{C}$, the functions

$$x(t) = \text{sn}(t; k), \quad y(t) = \text{sn}(t + t_0; k)$$

parametrize the affine algebraic curve

$$x^2 + y^2 - k^2 \text{sn}^2 t_0 x^2 y^2 - 2 \text{cn } t_0 \text{dn } t_0 xy - \text{sn}^2 t_0 = 0 \tag{28}$$

(all elliptic functions in the formula have Jacobi modulus k).

Similarly, the functions

$$x(t) = \operatorname{cn}(t; k), \quad y(t) = \operatorname{cn}(t + t_0; k)$$

parametrize the affine algebraic curve

$$x^2 + y^2 + k^2 \frac{\operatorname{sn}^2 t_0}{\operatorname{dn}^2 t_0} x^2 y^2 - 2 \frac{\operatorname{cn} t_0}{\operatorname{dn}^2 t_0} xy - (k')^2 \frac{\operatorname{sn}^2 t_0}{\operatorname{dn}^2 t_0} = 0, \quad (29)$$

where $k' = \sqrt{1 - k^2}$ is the conjugate modulus. □

See [9] for a proof and historical references.

Proof of Theorem 2.7. If Q is of elliptic type, then neither of the coefficients in equation (19) vanishes. We need to show that the substitution $z = px$, $w = qy$ transforms (19) into one of the equations (28) or (29), where p , q , k , and t_0 are as described in Theorem 2.7.

We don't give an analog of Lemma 4.7, because the formulas expressing p , q , k , and t_0 through the coefficients c_{ij} are rather complicated. On the other hand, in [9] it is explained how the values of p and q can be guessed. So let's just substitute

$$z = \sqrt{\frac{\sin \sigma \sin(\sigma - \beta - \gamma)}{\sin \bar{\alpha} \sin \bar{\delta}}} x, \quad w = \sqrt{\frac{\sin \sigma \sin(\sigma - \beta - \alpha)}{\sin \bar{\gamma} \sin \bar{\delta}}}$$

(due to Lemma 4.2, these coefficients are p and q of Theorem 2.7) into

$$\begin{aligned} & \sin \bar{\alpha} \sin(\sigma - \beta - \alpha) z^2 + \sin \bar{\gamma} \sin(\sigma - \beta - \gamma) w^2 \\ & + \sin \bar{\delta} \sin(\sigma - \beta - \delta) z^2 w^2 - 2 \sin \alpha \sin \gamma zw + \sin \bar{\beta} \sin \sigma = 0 \end{aligned}$$

We obtain

$$\begin{aligned} x^2 + y^2 - \frac{\sin \sigma \sin(\sigma - \alpha - \gamma)}{\sin \bar{\alpha} \sin \bar{\gamma}} x^2 y^2 \\ - \frac{2pq \sin \alpha \sin \gamma \sin \bar{\delta}}{\sin \sigma \sin(\sigma - \alpha - \beta) \sin(\sigma - \beta - \gamma)} xy + \frac{\sin \bar{\beta} \sin \bar{\delta}}{\sin(\sigma - \alpha - \beta) \sin(\sigma - \beta - \gamma)} = 0 \end{aligned} \quad (30)$$

Two cases must be distinguished: $M := \frac{\sin \alpha \sin \beta \sin \gamma \sin \bar{\delta}}{\sin \bar{\alpha} \sin \bar{\beta} \sin \bar{\gamma} \sin \bar{\delta}} < 1$ and $M > 1$.

If $M < 1$, then we have to show that equation (30) arises from (28) by substituting

$$k = \sqrt{1 - M} = \sqrt{\frac{\sin \sigma \sin(\sigma - \alpha - \beta) \sin(\sigma - \beta - \gamma) \sin(\sigma - \beta - \delta)}{\sin \bar{\alpha} \sin \bar{\beta} \sin \bar{\gamma} \sin \bar{\delta}}}$$

and t_0 as described in Theorem 2.7. From

$$\operatorname{dn} t_0 = \sqrt{\frac{\sin \alpha \sin \gamma}{\sin \bar{\alpha} \sin \bar{\gamma}}}$$

and $\operatorname{sn}^2 t_0 + k^2 \operatorname{dn}^2 t_0 = 1$ we compute

$$\operatorname{sn}^2 t_0 = \frac{-\sin \bar{\beta} \sin \bar{\delta}}{\sin(\sigma - \alpha - \beta) \sin(\sigma - \beta - \gamma)} = \frac{-\sin \bar{\beta} \sin \bar{\delta}}{\sin \alpha \sin \gamma - \sin \bar{\beta} \sin \bar{\delta}}$$

It follows that the constant term and the coefficient at $x^2 y^2$ in (30) are equal to $-\operatorname{sn}^2 t_0$ and $-k^2 \operatorname{sn}^2 t_0$, respectively. Further, $\operatorname{sn}^2 t_0 + \operatorname{cn}^2 t_0 = 1$ implies that

$$\operatorname{cn}^2 t_0 = \frac{\sin \alpha \sin \gamma}{\sin(\sigma - \alpha - \beta) \sin(\sigma - \beta - \gamma)} \in \left(-\frac{(k')^2}{k^2}, 0 \right) \cup (1, +\infty)$$

It follows that the square of the coefficient at xy in (30) equals $4 \operatorname{cn}^2 t_0 \operatorname{dn}^2 t_0$. The sign of the coefficient resolves the $+2K$ -indeterminacy of t_0 . Indeed, equating the coefficient with $-2 \operatorname{cn} t_0 \operatorname{dn} t_0$ leads to

$$\operatorname{cn} t_0 = \frac{\sin \sigma}{pq \sin \bar{\delta}} \quad (31)$$

Together with the following information about the range of elliptic cosine:

$$\begin{aligned} (0, iK') &\mapsto (1, +\infty), & (K, K + iK') &\mapsto \left(0, -i\frac{k'}{k}\right), \\ (2K, 2K + iK') &\mapsto (-\infty, -1), & (3K, 3K + iK') &\mapsto \left(0, i\frac{k'}{k}\right) \end{aligned}$$

this yields the table in Theorem 2.7.

If $M > 1$, then a similar argument shows that equation (30) arises from (29) by substituting

$$k = \sqrt{\frac{\sin \sigma \sin(\sigma - \alpha - \beta) \sin(\sigma - \beta - \gamma) \sin(\sigma - \alpha - \gamma)}{\sin \alpha \sin \beta \sin \gamma \sin \delta}}$$

and t_0 as described in Theorem 2.7. The value of $\operatorname{dn} t_0$ implies this time that

$$\operatorname{sn}^2 t_0 = \frac{-\sin \beta \sin \delta}{\sin \bar{\alpha} \sin \bar{\gamma} - \sin \beta \sin \delta}, \quad \operatorname{cn}^2 t_0 = \frac{\sin \bar{\alpha} \sin \bar{\gamma}}{\sin \bar{\alpha} \sin \bar{\gamma} - \sin \beta \sin \delta}$$

The coefficient at xy yields the same formula (31) for $\operatorname{cn} t_0$. ■

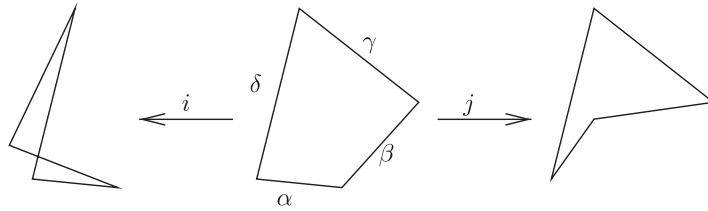


Fig. 12. Involutions i and j on the configuration space of a quadrilateral.

4.4 Branch points and involutions

Let $Z = Z(\alpha, \beta, \gamma, \delta) \subset (\mathbb{CP}^1)^2$ be the complexified configuration space of a quadrilateral with side lengths $\alpha, \beta, \gamma, \delta$, see Section 4.2. An irreducible component of Z is called *trivial*, if it is described by an equation of the form $z = \text{const}$ or $w = \text{const}$.

If Z^0 is a non-trivial component of Z , then both projections

$$\begin{aligned} f: Z^0 &\rightarrow \mathbb{CP}^1 & g: Z^0 &\rightarrow \mathbb{CP}^1 \\ (z, w) &\mapsto z & (z, w) &\mapsto w \end{aligned}$$

are branched covers. Since equation (19) has degree 2 both in z and w , the degrees of f and g are at most 2. Let $A \subset \mathbb{CP}^1$ and $B \subset \mathbb{CP}^1$ be the branch sets of the maps f and g , respectively. Denote by

$$\begin{aligned} i: Z^0 &\rightarrow Z^0 & j: Z^0 &\rightarrow Z^0 \\ (z, w) &\mapsto (z, w') & (z, w) &\mapsto (z', w) \end{aligned}$$

the deck transformations of f and g (defined only if the corresponding branched cover is two-fold). Geometrically, involutions i and j act by folding the quadrilateral along one of its diagonals, see Figure 12.

Lemma 4.10. Branch sets and involutions of the configuration space Z of a spherical quadrilateral Q have the following form.

- (1) If Q is an (anti)isogram, then for each of the non-trivial components of Z (there can be 0, 1, or 2 of them), the maps f and g are homeomorphisms.
- (2) If Q is an (anti)deltoid with apex $\alpha\delta$ (i.e., if either $\alpha = \delta$ and $\beta = \gamma$ or $\alpha + \delta = \pi = \beta + \gamma$), then the map g is a homeomorphism, and the map f is two-fold with the branch set and involution

$$A = \{\pm p^m\}, \quad i: t \mapsto \pi - t$$

Similarly, if Q is an (anti)deltoid with apex $\alpha\beta$, then f is a homeomorphism, and g is a two-fold branched cover with

$$B = \{\pm q^n\}, \quad j: t \mapsto \pi - t$$

- (3) If Q is a conic quadrilateral, then the curve $Z \subset (\mathbb{CP}^1)^2$ is irreducible and has a unique singular point $(0^{-m}, 0^{-n})$ (where $0^{-1} := \infty$). On the regular part of Z , both maps f and g are two-fold branched covers with branch sets and involutions

$$\begin{aligned} A &= \{\pm p^m\}, & i: t &\mapsto \pi - t \\ B &= \{\pm q^n\}, & j: t &\mapsto (\pi - 2t_0) - t \end{aligned}$$

- (4) If Q is an elliptic quadrilateral with $M < 1$ (i.e., Z is parametrized by sn), then both maps f and g are two-fold branched covers with branch sets and involutions

$$\begin{aligned} A &= \left\{ \pm p, \pm \frac{p}{k} \right\}, & i: t &\mapsto 2K - t \\ B &= \left\{ \pm q, \pm \frac{q}{k} \right\}, & j: t &\mapsto (2K - 2t_0) - t \end{aligned}$$

If Q has $M > 1$ (i.e., Z parametrized by cn), then the branch sets and involutions are given by

$$\begin{aligned} A &= \left\{ \pm p, \pm ip \frac{k'}{k} \right\}, & i: t &\mapsto -t \\ B &= \left\{ \pm q, \pm iq \frac{k'}{k} \right\}, & j: t &\mapsto -2t_0 - t \end{aligned}$$

□

Proof. In the (anti)isogram case, the assertion follows from the fact that each non-trivial component has the form $z = cw$ or $zw = cw^{-1}$, see Theorem 2.4. The affine curve $z = cw$ contains in its biprojective completion the point (∞, ∞) , and the curve $z = cw^{-1}$ contains $(0, \infty)$ and $(\infty, 0)$, which ensures that both maps f and g are homeomorphisms.

If Q is a deltoid with apex $\alpha\delta$, then Z^0 is the $(1, 2)$ -biprojective completion of the affine curve (23). It is readily seen that g is a homeomorphism (we have $g^{-1}(0) = (\infty, 0)$ and $g^{-1}(\infty) = (\infty, \infty)$). Formulas for the branch points of f and for the involution i follow from the parametrization (9) of the affine part of Z^0 .

If Q is a conic quadrilateral with $\alpha + \gamma = \beta + \delta$, then the affine part of Z has the form (25), which is a nonsingular affine curve. The biprojective completion adds a single

point (∞, ∞) , whose neighborhood is equivalent to $z_0^2 w_0^2 = 0$. Formulas for branch sets A and B and involutions i and j follow from the parametrization (11). For other types of conic quadrilaterals the situation is similar.

Finally, if Q is elliptic, then the parametrization (12), respectively (13), defines an analytic diffeomorphism $\mathbb{C}/\Lambda \rightarrow Z$. The branch sets A and B can be computed as the sets of critical values of the functions $z = z(t)$ and $w = w(t)$, respectively. Formulas for the involutions follow from the properties of the functions sn and cn . ■

4.5 Orthodiagonal quadrilaterals

A quadrilateral is called *orthodiagonal*, if its diagonals are orthogonal to each other.

Lemma 4.11. A spherical quadrilateral with side lengths $\alpha, \beta, \gamma, \delta$ in this cyclic order is orthodiagonal if and only if any of the following equivalent conditions is fulfilled.

$$\cos \alpha \cos \gamma = \cos \beta \cos \delta \tag{32a}$$

$$\sin \bar{\alpha} \sin \bar{\gamma} = \sin \bar{\beta} \sin \bar{\delta} \tag{32b}$$

In particular, isometric deformations preserve the orthodiagonality. □

Proof. Condition (32a) is equivalent to the orthodiagonality due to the spherical Pythagorean theorem. Equivalence between (32a) and (32b) follows by simple trigonometry. ■

Clearly, deltoids and antideltoids are orthodiagonal. Also, an (anti)isogram is orthodiagonal only if it is an (anti)deltoid at the same time.

Lemma 4.12. If an orthodiagonal quadrilateral is not an (anti)deltoid, then it is of elliptic type, its configuration space has the sn -parametrization (Case 1 of Theorem 2.7), and the phase shift t_0 is a quarter-period: $\text{Im } t_0 = \frac{K'}{2}$. Conversely, every quadrilateral with these properties is orthodiagonal. □

Proof. By the remark preceding the theorem, if Q is not an (anti)deltoid, then it is either conic or elliptic. Thus its configuration space has one of the parametrizations from Theorems 2.6 and 2.7, and hence by Lemma 4.10 carries the involutions i and j .

A simple geometric argument shows that Q is orthodiagonal if and only if i and j commute or, equivalently, if and only if $(i \circ j)^2 = \text{id}$. On the other hand, since

$$i \circ j(t) = t + 2t_0,$$

the involutions i and j commute if and only if t_0 is a quarter-period. In the conic case, t_0 cannot be a quarter-period since $\text{Im } t_0 > 0$. Neither can it in the cn case, since there we have $\frac{1}{4}\Lambda = K\mathbb{Z} + \frac{K+iK'}{2}\mathbb{Z}$, which has an empty intersection with $K\mathbb{Z} + (0, iK') \ni t_0$. Thus the configuration space is parametrized by sn , and the lemma is proved. ■

If in an orthodiagonal quadrilateral we have $\delta = \frac{\pi}{2}$, then due to (32a) we must have either $\alpha = \frac{\pi}{2}$ or $\gamma = \frac{\pi}{2}$.

Lemma 4.13. The configuration space of an elliptic orthodiagonal quadrilateral is described by one of the following equations.

(1) If $\delta \neq \frac{\pi}{2}$, then

$$\left(\sin(\delta - \alpha)z + \frac{\sin(\delta + \alpha)}{z} \right) \left(\sin(\delta - \gamma)w + \frac{\sin(\delta + \gamma)}{w} \right) = 4 \sin \alpha \sin \gamma \cos \delta,$$

(2) If $\alpha = \delta = \frac{\pi}{2}$, then

$$\left((\cos \beta - \cos \gamma)z + \frac{\cos \beta + \cos \gamma}{z} \right) \left(w + \frac{1}{w} \right) = 4 \sin \gamma$$

(3) If $\gamma = \delta = \frac{\pi}{2}$, then

$$\left(z + \frac{1}{z} \right) \left((\cos \beta - \cos \alpha)w + \frac{\cos \beta + \cos \alpha}{w} \right) = 4 \sin \alpha \quad \square$$

Proof. Equation $\cos \alpha \cos \gamma = \cos \beta \cos \delta$ can be shown to imply

$$\sin(\delta - \alpha) \sin(\delta - \gamma) = 2 \cos \delta \sin \bar{\delta} \sin(\sigma - \beta - \delta)$$

$$\sin(\delta - \alpha) \sin(\delta + \gamma) = 2 \cos \delta \sin \bar{\alpha} \sin(\sigma - \beta - \alpha)$$

$$\sin(\delta + \alpha) \sin(\delta - \gamma) = 2 \cos \delta \sin \bar{\gamma} \sin(\sigma - \beta - \gamma)$$

$$\sin(\delta + \alpha) \sin(\delta + \gamma) = 2 \cos \delta \sin \bar{\beta} \sin \sigma$$

(the proof can start with $\sin \bar{\delta} \sin(\sigma - \beta - \delta) = \frac{1}{2}(\cos \beta - \cos(\alpha + \gamma - \delta))$). This shows that the equation in the first part of the lemma is equivalent to (19). The second and the third part are proved by similar transformations. ■

Remark 4.14. With the help of the identities from Lemma 4.2 one can show that

$$\det \begin{pmatrix} c_{22} & c_{20} \\ c_{02} & c_{00} \end{pmatrix} = \sin \alpha \sin \gamma (\sin \bar{\beta} \sin \bar{\delta} - \sin \bar{\alpha} \sin \bar{\gamma}),$$

where c_{ij} are the coefficients from (19). By Lemma 4.11, the right-hand side vanishes if and only if the quadrilateral is orthodiagonal. It follows that equation (19) takes the form $(az^2 + b)(cw^2 + d) = zw$ if and only if the quadrilateral is orthodiagonal. This fact is also expressed and proved in a different way in Lemma 4.16 below. \square

The involution factors and other parameters introduced in Definition 2.8 allow to abbreviate the equation of the configuration space of an orthodiagonal quadrilateral in the following way.

Corollary 4.15. The configuration space of an orthodiagonal quadrilateral has the equation

$$(z + \lambda z^{-1})(w + \mu w^{-1}) = \nu, \quad \text{if } Q \text{ is not an (anti)deltoid} \quad (33a)$$

$$z + \lambda z^{-1} = \xi w^n, \quad \text{if } Q \text{ is an (anti)deltoid with apex } \alpha\beta \quad (33b)$$

$$w + \mu w^{-1} = \zeta z^m, \quad \text{if } Q \text{ is an (anti)deltoid with apex } \alpha\delta \quad (33c)$$

Here $m = 1$, respectively, $n = 1$, if Q is a deltoid, and $m = -1$, respectively, $n = -1$, if Q is antideltoid, $\lambda, \mu, \nu, \xi, \zeta$ are as in Definition 2.8. \square

The involution factor λ is defined if and only if the projection $g: (z, w) \mapsto w$ restricted to the non-trivial component Z^0 is a two-fold branched cover of \mathbb{CP}^1 . Recall that $j: (z, w) \mapsto (z', w)$ denotes the deck transformation of g . In general, z' depends both on z and w . But equations (33) show that in the orthodiagonal case z' depends only on z . In other words, the involution j descends to an involution $f_*(j): \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$.

The next lemma explains the term “involution factor” and shows that the pushout $f_*(j)$ exists only in the orthodiagonal case.

Lemma 4.16. The involution $j: (z, w) \mapsto (z', w)$ on the configuration space of an orthodiagonal quadrilateral acts by $z' = \lambda z^{-1}$, where λ is as in Definition 2.8.

Conversely, if the involution j descends to an involution on \mathbb{CP}^1 , then Q is an orthodiagonal quadrilateral. \square

Proof. The first part is immediate from equations (33).

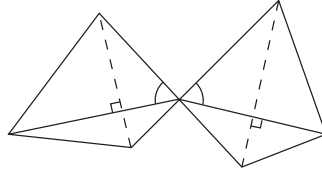


Fig. 13. A compatible coupling of orthodiagonal quadrilaterals.

If f is two-fold, then the map $f_*(j) : z \mapsto z'$ is well defined if and only if $f \circ j = f \circ j \circ i$. The last equation implies that either $j = j \circ i$ or $i \circ j = j \circ i$ holds. As $j \neq j \circ i$, we conclude that i and j must commute. This implies that t_0 is a quarter-period, and thus by Lemma 4.12 the quadrilateral is orthodiagonal. ■

Lemma 4.17. Coefficients λ, μ, ν in the equation (33a) are expressed in terms of the modulus and amplitudes as follows:

$$(\lambda, \mu, \nu) = \begin{cases} \left(\frac{p^2}{k}, \frac{q^2}{k}, \frac{2(1+k)}{k\sqrt{k}} pq \right), & \text{if } t_0 = \frac{iK'}{2} \\ \left(\frac{p^2}{k}, \frac{q^2}{k}, -\frac{2(1+k)}{k\sqrt{k}} pq \right), & \text{if } t_0 = 2K + \frac{iK'}{2} \\ \left(-\frac{p^2}{k}, -\frac{q^2}{k}, \frac{2i(1-k)}{k\sqrt{k}} pq \right), & \text{if } t_0 = K + \frac{iK'}{2} \\ \left(-\frac{p^2}{k}, -\frac{q^2}{k}, -\frac{2i(1-k)}{k\sqrt{k}} pq \right), & \text{if } t_0 = 3K + \frac{iK'}{2} \end{cases} \quad \square$$

Proof. The formulas are obtained by substituting $z = px$ and $w = qy$ in the equations

$$\left(x + \frac{1}{kx} \right) \left(y + \frac{1}{ky} \right) = \pm \frac{2(1+k)}{k\sqrt{k}}, \quad \left(x - \frac{1}{kx} \right) \left(y - \frac{1}{ky} \right) = \pm \frac{2i(1-k)}{k\sqrt{k}}$$

describing the curves $x = \operatorname{sn} t, y = \operatorname{sn}(t + t_0)$ for the values of t_0 given above. ■

Definition 4.18. Orthodiagonal quadrilaterals Q_1 and Q_2 are called *compatible* if one of the following holds:

- the involution factors of Q_1 and Q_2 at their common vertex are equal: $\lambda_1 = \lambda_2$;
- Q_1 and Q_2 are frontally coupled deltoids;
- Q_1 and Q_2 are frontally coupled antideltoids. ■

Geometrically, a coupling of compatible orthodiagonal quadrilaterals is characterized by the property that during a deformation the angles marked on Figure 13 remain equal.

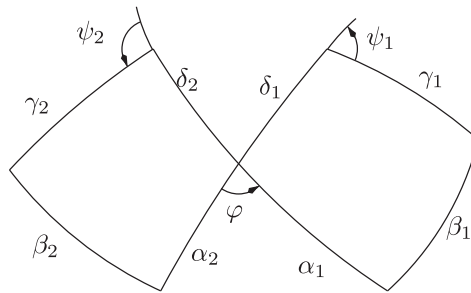


Fig. 14. Coupled four-bar linkages.

The coupling (Q_1, Q_2) on Figure 4 is compatible orthodiagonal if and only if during a deformation of the polyhedron on Figure 3 the points A_1, A_2, C_1, C_2 remain in one plane, and this plane is always orthogonal to the planes $B_1A_1A_4$ and $B_2A_2A_3$.

5 Configuration Space of Two Coupled Four-Bar Linkages

In Section 4 we studied the complexified configuration space Z of a spherical four-bar linkage on Figure 10, together with the projections $f: Z \rightarrow \mathbb{CP}^1$ and $g: Z \rightarrow \mathbb{CP}^1$ that output the tangents of half the angles φ and ψ . In this section we study the configuration space of two four-bar linkages coupled as shown on Figure 14.

5.1 Coupled four-bar linkages and fiber products of branched coverings

The configuration space Z_{12} of the coupling on Figure 14 is the solution set of a system of equations:

$$Z_{12} := \{(w_1, z, w_2) \in (\mathbb{CP}^1)^3 \mid P_1(z, w_1) = 0, P_2(z, w_2) = 0\}$$

Here P_1 and P_2 are polynomials whose zero sets are the configuration spaces Z_1 and Z_2 . Thus we have a commutative diagram

$$\begin{array}{ccccc}
 & & \mathbb{CP}^1 & & \\
 & & \swarrow & & \searrow \\
 \mathbb{CP}^1 & & Z_{12} & & \mathbb{CP}^1 \\
 & \swarrow \tilde{f}_1 & & \searrow \tilde{f}_2 & \\
 & Z_2 & & Z_1 & \\
 & \swarrow f_2 & & \searrow f_1 & \\
 & & \mathbb{CP}^1 & &
 \end{array} \tag{34}$$

where each of the maps is the restriction of a projection $(\mathbb{CP}^1)^3 \rightarrow (\mathbb{CP}^1)^2$ or $(\mathbb{CP}^1)^2 \rightarrow \mathbb{CP}^1$. It is easy to see that the commutative square in (34) is a fiber product diagram:

$$Z_{12} = Z_1 \times_{\mathbb{CP}^1} Z_2 := \{(x_1, x_2) \in Z_1 \times Z_2 \mid f_1(x_1) = f_2(x_2)\}$$

The set Z_{12} is an algebraic set and can have several irreducible components. In particular, this is the case when Z_1 or Z_2 is reducible. We are not interested in trivial components of Z_i given by $z_i = \text{const}$ or $w = \text{const}$, which will be dealt with in Section 6.1. Therefore let us restrict the diagram (34) to some non-trivial components Z_1^0 and Z_2^0 of Z_1 and Z_2 .

Each of the maps $f_i: Z_i^0 \rightarrow \mathbb{CP}^1$ is either a homeomorphism or a two-fold branched cover. If f_1 is a homeomorphism, then so is \tilde{f}_1 , and the branched cover \tilde{f}_2 is equivalent to $f_2: Z_2^0 \rightarrow \mathbb{CP}^1$.

The following lemma describes the fiber product $Z_1^0 \times_{\mathbb{CP}^1} Z_2^0$ in the case when both maps $f_1|_{Z_1^0}$ and $f_2|_{Z_2^0}$ are two-fold branched covers. This is a simple special case of the general description of the components of a fiber product over \mathbb{CP}^1 , see [16, Section 2].

Lemma 5.1. Let $f_1: Z_1^0 \rightarrow \mathbb{CP}^1$ and $f_2: Z_2^0 \rightarrow \mathbb{CP}^1$ be two-fold branched covers with branch sets $A_1, A_2 \subset \mathbb{CP}^1$, respectively. Then their fiber product Z_{12} has one of the following forms.

- (1) If $A_1 \neq A_2$, then both \tilde{f}_1 and \tilde{f}_2 are two-fold branched covers with branch sets $f_1^{-1}(A_2)$ and $f_2^{-1}(A_1)$, respectively. The composition $f_1 \circ \tilde{f}_2 = f_2 \circ \tilde{f}_1$ is four-fold, and its monodromy has the form (12)(34) around $A_1 \setminus A_2$, (13)(24) around $A_1 \cap A_2$, (14)(23) around $A_2 \setminus A_1$. Here 1, 2, 3, 4 is an appropriate labeling of the preimage of some point.
- (2) If $A_1 = A_2$ (so that the covers f_1 and f_2 are equivalent), then Z_{12} is equivalent to the union of two copies of Z_1^0 :

$$Z_{12} = Z_1^{0,+} \sqcup Z_1^{0,-}$$

The restrictions of \tilde{f}_1 and \tilde{f}_2 to $Z_1^{0,+}$ and $Z_1^{0,-}$ are homeomorphisms, and the compositions

$$Z_1^0 \xrightarrow{\tilde{f}_2^{-1}} Z_1^{0,+} \xrightarrow{\tilde{f}_1} Z_2^0 \quad \text{and} \quad Z_1^0 \xrightarrow{\tilde{f}_2^{-1}} Z_1^{0,-} \xrightarrow{\tilde{f}_1} Z_2^0$$

differ by postcomposition with the deck transformation of Z_2^0 (equivalently, by precomposition with the deck transformation of Z_1^0). □

5.2 Reducible couplings

Definition 5.2. A coupling of spherical four-bar linkages on Figure 14 is called *reducible*, if there exist non-trivial irreducible components Z_1^0 and Z_2^0 of Z_1 and Z_2 , respectively, such that the fiber product $Z_1^0 \times_{\mathbb{CP}^1} Z_2^0$ is reducible. \square

By Lemma 5.1, a coupling is reducible if and only if the maps $f_1|_{Z_1^0}$ and $f_2|_{Z_2^0}$ are two-fold branched covers and their sets of branch points coincide: $A_1 = A_2$. Each of the components Z_i^0 can have one of the following forms, see Section 4:

- Equation (21) has no solutions; then $Z_i^0 = Z_i$ is an elliptic curve with the parametrization

$$Z_i = \{(z, w_i) \mid z = p_i F_i(t), w_i = q_i F_i(t + t_i)\}, \quad (35)$$

where $F_i(t) = \text{sn}(t, k_i)$ or $F_i(t) = \text{cn}(t, k_i)$.

- Equation (21) has exactly one solution; then $Z_i^0 = Z_i$ is a conic with the parametrization

$$Z_i = \{(z, w_i) \mid z^{m_i} = p_i \sin t, w_i^{n_i} = q_i \sin(t + t_i)\}, \quad (36)$$

where $m_i, n_i = \pm 1$ depending on the form of the solution of (21), as described in Theorem 2.6.

- The quadrilateral Q_i is either deltoid with $\alpha_i = \delta_i, \beta_i = \gamma_i$ or antideltoid with $\alpha_i + \delta_i = \pi = \beta_i + \gamma_i$, and

$$Z_i^0 = \{(z, w_i) \mid z_i^{m_i} = p_i \sin t, w_i = \varepsilon_i \sqrt{-\mu_i} e^{it}, \}, \quad (37)$$

where $m_i = 1$ if Q_i is a deltoid, and $m_i = -1$ if Q_i is antideltoid, and with ε_i and μ_i as in Theorem 2.5.

We have $|A_i| = 4$ when Q_i is elliptic, and $|A_i| = 2$ when Q_i is a conic quadrilateral or an (anti)deltoid. Therefore a reducible coupling is only possible between two elliptics, or between two conics, or between (anti)deltoid and conic, or between two (anti)deltoids.

5.2.1 Reducible couplings of elliptic quadrilaterals

Lemma 5.3. A coupling of two elliptic quadrilaterals Q_1 and Q_2 is reducible if and only if the parametrizations (35) of their configuration spaces satisfy one of the following sets of conditions:

- (1) F_1 and F_2 is the same elliptic function (sn or cn), and

$$k_1 = k_2, \quad p_1 = p_2 =: p$$

In terms of the side lengths of Q_1 and Q_2 this is equivalent to

$$\frac{\sin \alpha_1 \sin \delta_1}{\sin \bar{\alpha}_1 \sin \bar{\delta}_1} = \frac{\sin \alpha_2 \sin \delta_2}{\sin \bar{\alpha}_2 \sin \bar{\delta}_2}, \quad \frac{\sin \beta_1 \sin \gamma_1}{\sin \bar{\beta}_1 \sin \bar{\gamma}_1} = \frac{\sin \beta_2 \sin \gamma_2}{\sin \bar{\beta}_2 \sin \bar{\gamma}_2}, \quad (38)$$

The components of the configuration space Z_{12} of the coupling have the following parametrizations:

$$\begin{aligned} & \{w_1 = q_1 F(t - t_1), z = pF(t), w_2 = q_2 F(t + t_2)\} \\ \cup & \{w_1 = q_1 F(t - t_1), z = pF(t), w_2 = q_2 F(t - t_2)\} \end{aligned} \quad (39)$$

- (2) $F_1 = \text{cn}(\cdot, k_1)$ and $F_2 = \text{cn}(\cdot, k_2)$ and

$$k_1^2 + k_2^2 = 1, \quad \frac{p_1}{p_2} = \pm i \frac{k_1}{k_2}$$

The components of the configuration space of the coupling have the following parametrizations:

$$\begin{aligned} & \left\{ w_1 = q_1 \text{cn}(t - t_1, k_1), z = p_1 \text{cn}(t, k_1), w_2 = \frac{p_1 q_2}{p_2} \text{cn}(t + it_2, k_1) \right\} \\ \cup & \left\{ w_1 = q_1 \text{cn}(t - t_1, k_1), z = p_1 \text{cn}(t, k_1), w_2 = \frac{p_1 q_2}{p_2} \text{cn}(t - it_2, k_1) \right\} \quad \square \end{aligned} \quad (40)$$

Proof. Branch points are given by part (4) of Lemma 4.10. In the sn case either all four points are real or all purely imaginary, while in the cn case there are two real and two imaginary points. Thus we have two possibilities:

$$\begin{aligned} \left\{ \pm p_1, \pm \frac{p_1}{k_1} \right\} &= \left\{ \pm p_2, \pm \frac{p_2}{k_2} \right\} && \text{if } F_i = \text{sn}(\cdot, k_i) \\ \left\{ \pm p_1, \pm i p_1 \frac{k'_1}{k_1} \right\} &= \left\{ \pm p_2, \pm i p_2 \frac{k'_2}{k_2} \right\} && \text{if } F_i = \text{cn}(\cdot, k_i) \end{aligned}$$

If $p_1 = p_2$, then in both cases we have $k_1 = k_2$, which results in the case (1) of the Lemma. The parametrization (39) follows from parametrizations (35) and from the action of the involutions i_1 and i_2 on Z_1 and Z_2 .

Recall that $p_i \in \mathbb{R}_+ \cup i\mathbb{R}_+$, so that $p_1 = -p_2$ is not possible. It remains to check if the elements of A_1 can be equal to those of A_2 "crosswise."

In the sn-case because of $p_i \in \mathbb{R}_+ \cup i\mathbb{R}_+$ we have only the possibility

$$p_1 = \frac{p_2}{k_2}, \quad p_2 = \frac{p_1}{k_1}$$

which cannot occur because of $0 < k_i < 1$. In the cn-case we have

$$p_1 = \pm i p_2 \frac{k'_2}{k_2} \text{ and } p_2 = \pm i p_1 \frac{k'_1}{k_1} \Leftrightarrow k'_1 = k_2 \text{ and } \frac{p_1}{p_2} = \pm i \frac{k_1}{k_2},$$

which leads to the Case (2) of the Lemma.

To obtain a parametrization of Z_{12} in the second case, use Jacobi's imaginary transformation:

$$\text{cn}(t, k') = \frac{1}{\text{cn}(it, k)} = i \frac{k}{k'} \text{cn}(it + K + iK', k)$$

It leads to the following parametrization of Z_2 :

$$Z_2 = \left\{ (z, w_2) \mid z = ip_2 \frac{k_1}{k'_1} \text{cn } t, w_2 = iq_2 \frac{k_1}{k'_1} \text{cn}(t + it_2) \right\}$$

By making, if needed, the parameter change $t \rightarrow t + 2K$, this can be rewritten as $z = p_1 \text{cn } t, w_2 = \frac{p_1 q_2}{p_2} \text{cn}(t + it_2)$, and we obtain (40). ■

5.2.2 Reducible couplings of conic quadrilaterals

Lemma 5.4. A coupling of two conic quadrilaterals Q_1 and Q_2 is reducible if and only if the parametrizations (36) of their configuration spaces satisfy one of the following sets of conditions:

- (1) $m_1 = m_2 =: m$ and $p_1 = p_2 =: p$. In terms of the side lengths, this is equivalent to

$$\frac{\sin \alpha_1 \sin \delta_1}{\sin \beta_1 \sin \gamma_1} = \frac{\sin \alpha_2 \sin \delta_2}{\sin \beta_2 \sin \gamma_2} \tag{41}$$

with an additional condition that each of $(\alpha_i, \beta_i, \gamma_i, \delta_i)$ satisfies one of the equations

$$\alpha_i + \gamma_i = \beta_i + \delta_i, \quad \alpha_i + \beta_i = \gamma_i + \delta_i \quad (\Leftrightarrow m_i = 1)$$

or each satisfies one of the equations

$$\alpha_i + \delta_i = \beta_i + \gamma_i, \quad \alpha_i + \beta_i + \gamma_i + \delta_i = 2\pi \quad (\Leftrightarrow m_i = -1)$$

The components of the configuration space of the coupling have the following parametrizations:

$$\begin{aligned} & \{w_1^{n_1} = q_1 \sin(t - t_1), z^m = p \sin t, w_2^{n_2} = q_2 \sin(t + t_2)\} \\ & \cup \{w_1^{n_1} = q_1 \sin(t - t_1), z^m = p \sin t, w_2^{n_2} = q_2 \sin(t - t_2)\} \end{aligned} \quad (42)$$

- (2) $m_1 = -m_2$ and $p_1 = \pm \frac{1}{p_2}$. The components of the configuration space of the coupling have the following parametrizations:

$$\begin{aligned} & \left\{ w_1^{n_1} = q_1 \sin(t - t_1), z^{m_1} = p_1 \sin t, w_2^{n_2} = q_2 \frac{\cos t_2 + i \sin t_2 \cos t}{\sin t} \right\} \\ & \cup \left\{ w_1^{n_1} = q_1 \sin(t - t_1), z^{m_1} = p_1 \sin t, w_2^{n_2} = q_2 \frac{\cos t_2 - i \sin t_2 \cos t}{\sin t} \right\} \end{aligned} \quad (43) \quad \square$$

Proof. Branch points are given by part (3) of Lemma 4.10. Thus $A_1 = A_2$ is equivalent to

$$\begin{aligned} \{\pm p_1\} &= \{\pm p_2\} && \text{if } m_1 = m_2 \\ \{\pm p_1\} &= \left\{ \pm \frac{1}{p_2} \right\} && \text{if } m_1 = -m_2 \end{aligned}$$

The parametrization of Z_{12} in the first case is obvious. In the second case we must set

$$p_1 \sin t = z^{m_1} = \frac{1}{p_2 \sin t'} \Leftrightarrow \sin t = \frac{1}{\sin t'}$$

where t and t' are parameters on Z_1 and Z_2 , respectively. (If we have $p_1 = -\frac{1}{p_2}$, which happens when the amplitudes are imaginary, then make the parameter change $t \mapsto t + \pi$.) This determines two different automorphisms of $\mathbb{C}P^1 = (\mathbb{C}/2\pi\mathbb{Z}) \cup \{i\infty\}$. Since $\sin t' = \frac{1}{\sin t}$ implies $\cos t' = \pm i \cot t$, we have

$$\sin(t' \pm t_2) = \frac{\cos t_2 \pm i \sin t_2 \cos t}{\sin t},$$

which leads to the parametrization in the Lemma. ■

5.2.3 Reducible couplings involving deltoids

An (anti)deltoid Q_1 is said to be *frontally* coupled with Q_2 if the common vertex of Q_1 and Q_2 is an apex of Q_1 . Otherwise, Q_1 is said to be *laterally* coupled with Q_2 . The map $f_1|_{Z_1^0}$ in the diagram (34) is two-fold if and only if Q_1 is coupled frontally.

Lemma 5.5. A coupling (Q_1, Q_2) with Q_1 a frontally coupled (anti)deltoid is reducible if and only if one of the following sets of conditions is satisfied.

- (1) The quadrilateral Q_2 is conic, $m_1 = m_2 =: m$ and $p_1 = p_2 =: p$. In terms of the side lengths, this is equivalent to

$$\frac{\sin^2 \alpha_1}{\sin^2 \beta_1} = \frac{\sin \alpha_2 \sin \delta_2}{\sin \beta_2 \sin \gamma_2}$$

with the additional condition:

$$Q_1 \text{ deltoid} \Rightarrow \alpha_2 + \gamma_2 = \beta_2 + \delta_2 \text{ or } \alpha_2 + \beta_2 = \gamma_2 + \delta_2$$

$$Q_1 \text{ antideltoid} \Rightarrow \alpha_2 + \delta_2 = \beta_2 + \gamma_2 \text{ or } \alpha_2 + \beta_2 + \gamma_2 + \delta_2 = 2\pi$$

The components of Z_{12} can in this case be parametrized as

$$\begin{aligned} & \{w_1 = \varepsilon_1 \sqrt{-\mu_1} e^{it}, z^m = p \sin t, w_2^{n_2} = q_2 \sin(t + t_2)\} \\ & \cup \{w_1 = \varepsilon_1 \sqrt{-\mu_1} e^{it}, z^m = p \sin t, w_2^{n_2} = q_2 \sin(t - t_2)\} \end{aligned} \quad (44)$$

- (2) The quadrilateral Q_2 is conic, $m_1 = -m_2$, and $p_1 = \pm \frac{1}{p_2}$.

The components of the configuration space of the coupling have the following parametrizations:

$$\begin{aligned} & \left\{ w_1 = \varepsilon_1 \sqrt{-\mu_1} e^{it}, z^{m_1} = p_1 \sin t, w_2^{n_2} = q_2 \frac{\cos t_2 + i \sin t_2 \cos t}{\sin t} \right\} \\ & \cup \left\{ w_1 = \varepsilon_1 \sqrt{-\mu_1} e^{it}, z^m = p_1 \sin t, w_2^{n_2} = q_2 \frac{\cos t_2 - i \sin t_2 \cos t}{\sin t} \right\} \end{aligned} \quad (45)$$

- (3) Both Q_1 and Q_2 are frontally coupled deltoids or frontally coupled antideltoids (so that $m_1 = m_2 =: m$) such that $p_1 = p_2 =: p$ in (37). In terms of side lengths:

$$\frac{\sin \alpha_1}{\sin \beta_1} = \frac{\sin \alpha_2}{\sin \beta_2} \quad \text{and} \quad \begin{aligned} & \text{either } \alpha_i = \delta_i, \beta_i = \gamma_i \\ & \text{or } \alpha_i + \delta_i = \pi = \beta_i + \gamma_i \end{aligned} \quad \text{for } i = 1, 2$$

The components of the configuration space of the coupling have the following parametrizations.

$$\begin{aligned} & \{w_1 = \varepsilon_1 \sqrt{-\mu_1} e^{it}, z^m = p \sin t, w_2 = \varepsilon_2 \sqrt{-\mu_2} e^{it}\} \\ & \cup \{w_1 = \varepsilon_1 \sqrt{-\mu_1} e^{it}, z^m = p \sin t, w_2 = -\varepsilon_2 \sqrt{-\mu_2} e^{-it}\} \end{aligned}$$

- (4) Q_1 is a deltoid, Q_2 is an antideltoid such that $p_1 p_2 = 1$.

The components of the configuration space of the coupling have the following parametrizations.

$$\begin{aligned} & \left\{ w_1 = \varepsilon_1 \sqrt{-\mu_1} e^{it}, z^{m_1} = p_1 \sin t, w_2 = \varepsilon_2 \sqrt{-\mu_2} \frac{e^{it} - 1}{e^{it} + 1} \right\} \\ \cup & \left\{ w_1 = \varepsilon_1 \sqrt{-\mu_1} e^{it}, z^{m_1} = p_1 \sin t, w_2 = -\varepsilon_2 \sqrt{-\mu_2} \frac{e^{it} + 1}{e^{it} - 1} \right\} \end{aligned} \quad \square$$

Proof. Cases (1) and (2) are treated similar to Lemma 5.4.

In the Case (3) we have $A_1 = A_2$ if and only if $p_1 = p_2$, see part (2) of Lemma 4.10. Parametrizations of the two components of the fiber product is straightforward from (37) by taking into account the involution action $t \mapsto \pi - t$ (or, alternatively, its description in Lemma 4.16).

In the Case (4) we must adjoin to the parametrization (37) the equation $w_2 = \varepsilon_2 \sqrt{-\mu_2} e^{it'}$, where $\sin t' \sin t = 1$. The latter equation is equivalent to

$$(e^{i(t+t')} - e^{it} + e^{it'} + 1)(e^{i(t+t')} + e^{it} - e^{it'} + 1) = 0, \tag{46}$$

which implies the formula in the lemma. ■

5.3 Fiber product and resultant

The fiber product $Z_{12} = Z_1^0 \times_{\mathbb{CP}^1} Z_2^0$ lies in the space $(\mathbb{CP}^1)^3$ with coordinates (w_1, z, w_2) , and its projections to the (z, w_1) and to the (z, w_2) planes are the spaces Z_1^0 and Z_2^0 , respectively. Consider now the projection to the (w_1, w_2) -plane and denote by W its image:

$$\begin{aligned} \pi : Z_{12} & \rightarrow W, \\ (w_1, z, w_2) & \mapsto (w_1, w_2), \end{aligned} \tag{47}$$

which we call the *partial configuration space* of the coupling. Then we have

$$W := \{(w_1, w_2) \mid \exists z \in \mathbb{CP}^1 \text{ such that } (z, w_1) \in Z_1^0, (z, w_2) \in Z_2^0\}$$

If P_1^0 and P_2^0 are polynomials defining the components Z_1^0 and Z_2^0 , respectively, then the set W (at least its affine part) is the zero set of the resultant of the polynomials P_1^0 and P_2^0 viewed as polynomials in z .

If the coupling is reducible, then the resultant is a reducible polynomial, so that W consists of several irreducible components. In this case it is convenient to use the

parametrizations of the irreducible components of Z_{12} that we obtained in Section 5.2 in order to obtain the descriptions of the components of W .

Lemma 5.6. The following are the equations of the irreducible components of the space W for some of the reducible couplings.

- (1) If (Q_1, Q_2) is a reducible non-involutive coupling of elliptic quadrilaterals, then

$$a_{22}w_1^2w_2^2 + a_{20}w_1^2 + a_{02}w_2^2 + 2a_{11}w_1w_2 + a_{00} = 0$$

- (2) If (Q_1, Q_2) is a reducible coupling of conic quadrilaterals such that $m_1 = m_2$ and $p_1 = p_2$ (Case 1 of Lemma 5.4), then

$$a_{20}w_1^{2n_1} + a_{02}w_2^{2n_2} + 2a_{11}w_1^{n_1}w_2^{n_2} + a_{00} = 0$$

for some $a_{ij} \in \mathbb{R}$, except for the components corresponding to $t_1 - t_2 \equiv 0 \pmod{\pi}$.

- (3) If (Q_1, Q_2) is a reducible coupling of conic quadrilaterals such that $m_1 = -m_2$ and $p_1 = \pm \frac{1}{p_2}$ (Case 2 of Lemma 5.4), then

$$w_1^{2n_1}w_2^{2n_2} + a_{20}w_1^{2n_1} + a_{02}w_2^{2n_2} + 2a_{11}w_1^{n_1}w_2^{n_2} + a_{10}w_1^{n_1} + a_{01}w_2^{n_2} + a_{00} = 0 \quad (48)$$

for some $a_{ij} \in \mathbb{C}$ with $a_{20}, a_{02}, a_{11}, a_{10}, a_{01} \neq 0$. The polynomial is irreducible.

- (4) If Q_1 is an (anti)deltoid, and Q_2 is a conic quadrilateral such that $m_1 = m_2$ and $p_1 = p_2$ (Case 1 of Lemma 5.5), then

$$w_1 + \mu_1 e^{\pm 2it_2} w_1^{-1} = \varepsilon_1 \frac{2i\sqrt{-\mu_1} e^{\pm it_2}}{q_2} w_2^{n_2}$$

- (5) If Q_1 is an (anti)deltoid, and Q_2 is a conic quadrilateral such that $m_1 = \pm m_2$ and $p_1 = \pm \frac{1}{p_2}$ (Case 2 of Lemma 5.5), then

$$\frac{w_1^2 + 2aw_1 + b}{w_1^2 - b} = cw_2^{n_2}, \quad (49)$$

where the fraction on the left-hand side is irreducible.

- (6) If Q_1 is a deltoid, and Q_2 is an antideltoid such that $p_1 p_2 = 1$ (Case (4) of Lemma 5.5), then

$$w_1 w_2 \mp \varepsilon_2 \sqrt{-\mu_2} w_1 \pm \varepsilon_1 \sqrt{-\mu_1} w_2 + \varepsilon_1 \varepsilon_2 \sqrt{-\mu_1} \sqrt{-\mu_2} = 0 \quad \square$$

Proof. In the Case (1) each of the components of W has the parametrization of the form

$$w_1 = q_1 F(t), \quad w_2 = q_2' F(t + t_{12}),$$

where t_{12} is either $t_1 \pm t_2$ or $t_1 \pm it_2$. For an irreducible component, t_{12} is not a half-period of the elliptic function F , and the component is described by a biquadratic equation of the above form.

Similarly, in the case (2), the components of W are parametrized by

$$w_1 = q_1 \sin t, \quad w_2 = q_2 \sin(t + (t_1 \pm t_2))$$

If $t_1 \pm t_2 \notin \{0, \pi\}$, then the corresponding component is described by a quadratic equation without a linear part.

In the case (3) by Lemma 5.4, part (2) we have a parametrization of the form

$$w_1^{n_1} = a \sin t + b \cos t, \quad w_2^{n_2} = \frac{c + d \cos t}{\sin t}$$

with $a, b, c, d \neq 0$. It suffices to consider the case $n_1 = n_2 = 1$. One easily computes

$$\sin t = \frac{dw_1 + bc}{bw_2 + ad}, \quad \cos t = \frac{w_1 w_2 - ac}{bw_2 + ad}$$

By substituting this into $\sin^2 t + \cos^2 t = 1$ we obtain the equation

$$w_1^2 w_2^2 + d^2 w_1^2 - b^2 w_2^2 - 2acw_1 w_2 + 2bcdw_1 - 2abdw_2 + (a^2 c^2 + b^2 c^2 - a^2 d^2) = 0$$

The polynomial at the left-hand side is irreducible, since it describes a 2-2 correspondence, and W_1^0 is also a double cover over both Z_1 and Z_2 .

In the cases (4) and (5) the equations are found by substituting $w_1 = c_1 e^{it}$ into $\sin(t \pm t_2) = \frac{e^{i(t \pm t_2)} - e^{-i(t \pm t_2)}}{2i}$ and $\cos(t \pm t_2) = \frac{e^{i(t \pm t_2)} + e^{-i(t \pm t_2)}}{2}$ in the formulas of Lemma 5.5. The irreducibility of the rational function in the case (5) is equivalent to the non-divisibility of the numerator by $e^{it} \pm 1$. This, in turn, is equivalent to $\tan t_2 \neq i$ that was observed in Section 4.3 after the proof of Theorem 2.6.

Finally, equations in the case (6) follow from equation (46). ■

5.4 Involutive couplings

Consider the projection (47), where Z_{12} is the fiber product of non-trivial components Z_1^0 and Z_2^0 of the configuration spaces Z_1 and Z_2 . Let Z_{12}^0 be some irreducible component of

Z_{12} . Then the map π restricted to Z_{12}^0 is either an isomorphism or a two-fold (possibly branched) cover.

Definition 5.7. A coupling (Q_1, Q_2) is called *involutive*, if there exists an irreducible component Z_{12}^0 of $Z_1^0 \times_{\mathbb{C}P^1} Z_2^0$ such that the restriction of the projection (47) to Z_{12}^0 is two-fold. \square

Lemma 5.8. The coupling (Q_1, Q_2) is involutive if and only if the maps g_1 and g_2 on the diagram (34) (with Z_i^0 in place of Z_i) are two-fold, and the corresponding deck transformations $j_i: Z_i^0 \rightarrow Z_i^0$ lift to a common involution $j_{12}: Z_{12}^0 \rightarrow Z_{12}^0$. \square

Proof. If (Q_1, Q_2) is involutive, then the deck transformation of the two-fold cover $Z_{12}^0 \rightarrow W^0$ has the form

$$(w_1, z, w_2) \mapsto (w_1, z', w_2) \quad (50)$$

It follows that j_1 and j_2 are two-fold and that (50) is a common lift of their deck transformations.

In the opposite direction, if j_1 and j_2 lift to an involution j_{12} , then we have $\pi \circ j_{12} = \pi$, and since $j_{12} \neq \text{id}$, this means that π is two-fold. \blacksquare

5.4.1 Classification of involutive couplings

Lemma 5.9. Any involutive coupling (Q_1, Q_2) has one of the following forms.

- (1) The quadrilaterals Q_1 and Q_2 are orthodiagonal and have equal involution factors at their common vertex: $\lambda_1 = \lambda_2$, see Definition 2.8. (In particular this means that λ_i are defined, that is, if Q_i is an (anti)deltoid, then it is coupled laterally.)
- (2) The quadrilaterals Q_1 and Q_2 are elliptic and form a reducible elliptic coupling of the first type from Lemma 5.3. Besides, $t_1 \pm t_2$ is a half-period of the corresponding elliptic function.

In terms of side lengths this means that, in addition to equations (38), we have either

$$\frac{\sin \alpha_1 \sin \gamma_1}{\sin \bar{\alpha}_1 \sin \bar{\gamma}_1} = \frac{\sin \alpha_2 \sin \gamma_2}{\sin \bar{\alpha}_2 \sin \bar{\gamma}_2},$$

which is equivalent to $t_1 - t_2 \in \{0, 2K\}$, or

$$\frac{\sin \alpha_1 \sin \gamma_1}{\sin \bar{\alpha}_1 \sin \bar{\gamma}_1} = \frac{\sin \alpha_2 \sin \gamma_2}{\sin \bar{\beta}_2 \sin \bar{\delta}_2},$$

which is equivalent to $t_1 + t_2$ being a half-period with $\text{Im}(t_1 + t_2) = K'$.

- (3) The quadrilaterals Q_1 and Q_2 are conic and form a reducible conic coupling of the first type from Lemma 5.4. Besides, $t_1 - t_2 \equiv 0 \pmod{\pi}$.

In terms of the side lengths this is equivalent to

$$\frac{\sin \alpha_1}{\sin \beta_1} = \frac{\sin \alpha_2}{\sin \beta_2}, \quad \frac{\sin \gamma_1}{\sin \delta_1} = \frac{\sin \gamma_2}{\sin \delta_2}$$

with an additional condition that each of $(\alpha_1, \beta_1, \gamma_1, \delta_1)$ and $(\alpha_2, \beta_2, \gamma_2, \delta_2)$ satisfies one of the equations

$$\alpha + \gamma = \beta + \delta, \quad \alpha + \beta = \gamma + \delta \quad (\Leftrightarrow m = 1)$$

or each of them satisfies one of the equations

$$\alpha + \delta = \beta + \gamma, \quad \alpha + \beta + \gamma + \delta = 2\pi \quad (\Leftrightarrow m = -1) \quad \square$$

Proof. By assumption, both covers g_1 and g_2 are two-fold. Assume that the map f_1 is an isomorphism. Then, according to the classification of configuration spaces (in particular, Lemma 4.10), the quadrilateral Q_1 is an (anti)deltoid coupled to Q_2 frontally. Then, by Lemma 4.16, the deck transformation j_1 acts by

$$z \mapsto \lambda_1 z^{-1}$$

If j_1 and j_2 have a common lift (50), then j_2 must descend to an involution on $\mathbb{C}P^1$ given by the same formula. By Lemma 4.16, this happens if and only if the quadrilateral Q_2 is orthodiagonal and has the same involution factor at the $\delta\alpha$ -vertex as Q_1 . Thus we arrive at a special case of the situation described in part (1) of the Lemma, with a laterally coupled (anti)deltoid as one of the orthodiagonal quadrilaterals.

From now on assume that both f_1 and g_1 are two-fold, that is, each of $Z_1^0 = Z_1$ and $Z_2^0 = Z_2$ is either a conic or an elliptic curve. We will distinguish two cases: when the coupling (Q_1, Q_2) is reducible and when not.

If (Q_1, Q_2) is reducible, then consider case-by-case possible parametrizations of the components of Z_{12} given in Lemmas 5.3 and 5.4.

- $w_1 = q_1 F(t - t_1), \quad z = pF(t), \quad w_2 = q_2 F(t \pm t_2),$

where $F = \text{sn}$ or cn , and the choice of $+t_2$ or $-t_2$ shift yields two different components of Z_{12} . The involution j_1 acts by

$$j_1(t) = \begin{cases} 2K + 2t_1 - t, & \text{if } F = \text{sn} \\ 2t_1 - t, & \text{if } F = \text{cn} \end{cases}$$

In order for j_1 to preserve the value of w_2 , we must have

$$\text{sn}(t \pm t_2) = \text{sn}(2K + 2t_1 - t \pm t_2) \quad \text{or} \quad \text{cn}(t \pm t_2) = \text{cn}(2t_1 - t \pm t_2) \quad \forall t,$$

respectively. This is equivalent to $t_1 \pm t_2$ being a half-period of sn , respectively, cn . Hence we have the situation described in part (2) of the Lemma. Because of $\text{Im } t_1, \text{Im } t_2 \in (0, K')$ the possible half-periods are

$$t_1 - t_2 \in \{0, 2K\} \quad \text{or} \quad t_1 + t_2 \in \begin{cases} \{iK', 2K + iK'\}, & \text{in the sn case} \\ \{K + iK', 3K + iK'\}, & \text{in the cn case} \end{cases} \quad (51)$$

The corresponding conditions on the side lengths follow from the formulas for $\text{dn } t_0$ in Theorem 2.7 and from the identities $\text{dn}(iK' - t_0) = i \frac{\text{cn } t_0}{\text{sn } t_0}$ and $\text{dn}(K + iK' - t_0) = -ik' \frac{\text{sn } t_0}{\text{cn } t_0}$.

- $w_1 = q_1 \text{cn}(t - t_1), \quad z = p_1 \text{cn } t, \quad w_2 = \frac{p_1 q_2}{p_2} \text{cn}(t \pm it_2)$

Similar to the previous case, $t_1 \pm it_2$ must be a half-period of cn , that is belong to $2K\mathbb{Z} + (K + iK')\mathbb{Z}$. This is impossible, since $\text{Im } t_1 \in (0, K')$ and $\text{Im}(it_2)$ is a multiple of K' . Thus this coupling cannot be involutive.

- $w_1^{n_1} = q_1 \sin(t - t_1), \quad z^m = p \sin t, \quad w_2^{n_2} = q_2 \sin(t \pm t_2)$

Arguing as before, we see that $t_1 \pm t_2$ must be a multiple of π . Since t_1 and t_2 have positive imaginary parts, we have $t_1 - t_2 = n\pi$. By the formula for $\tan t_i$ from 2.6 this is equivalent to

$$\frac{\sin \beta_1 \sin \delta_1}{\sin \alpha_1 \sin \gamma_1} = \frac{\sin \beta_2 \sin \delta_2}{\sin \alpha_2 \sin \gamma_2}$$

Together with (41), this is equivalent to the condition on the side lengths in the Lemma.

- $w_1^{n_1} = q_1 \sin(t - t_1), \quad z^{m_1} = p_1 \sin t, \quad w_2^{n_2} = q_2 \frac{\cos t_2 + i \sin t_2 \cos t}{\sin t}$

The lift on involution j_1 acts by $t \mapsto \pi - 2t_1 - t$ which does not preserve the value of $w_2(t)$, so this coupling cannot be involutive.

If the coupling (Q_1, Q_2) is irreducible, then we have

$$Z_{12} = \{(t, t') \in Z_1 \times Z_2 \mid f_1(t) = f_2(t')\},$$

where we identified Z_1 and Z_2 with their parameter domains. The common lift j_{12} of the involutions j_1 and j_2 is the restriction to Z_{12} of the map

$$(j_1, j_2): Z_1 \times Z_2 \rightarrow Z_1 \times Z_2$$

On the other hand, we have another map of Z_{12} to itself:

$$(i_1, \text{id}): Z_{12} \rightarrow Z_{12},$$

which changes w_1 while preserving z and w_2 . We have

$$(i_1, \text{id}) \circ (j_1, j_2)(t, t') = (t + 2t_1, j_2(t')),$$

hence

$$((i_1, \text{id}) \circ (j_2, j_2))^2(t, t') = (t + 4t_1, t')$$

Since this maps Z_{12} to itself, we have $f_1(t + 4t_1) = f_1(t)$ for all t . That is, the phase shift t_1 is a quarter-period of the function parametrizing Z_1 . By an argument from Lemma 4.12, this implies that Q_1 is orthodiagonal. Similarly, by using (id, i_2) in place of (i_1, id) , we show that Q_2 is orthodiagonal. But then, by Lemma 4.16, the involutions j_1 and j_2 descend to \mathbb{CP}^1 . They descend to the same involution $z \mapsto z'$ if and only if $\lambda_1 = \lambda_2$. Thus we are in the situation of the part (1) of the Lemma. ■

5.4.2 The partial configuration space of an involutive coupling

Lemma 5.10. Let (Q_1, Q_2) be an involutive coupling of orthodiagonal quadrilaterals. Then the quotient space $W := Z_{12}/j_{12}$ is the solution set of the following equation.

- (1) If neither Q_1 nor Q_2 is a deltoid, then

$$w_1 + \mu_1 w_1^{-1} = \frac{\nu_1}{\nu_2} (w_2 + \mu_2 w_2^{-1}), \tag{52}$$

- (2) If Q_2 is an (anti)deltoid laterally coupled to Q_1 that is not an (anti)deltoid, then

$$w_1 + \mu_1 w_1^{-1} = \frac{\nu_1}{\xi_2} w_2^{-n_2} \quad (53)$$

- (3) If Q_1 and Q_2 are laterally coupled (anti)deltoids, then

$$w_1^{n_1} = \frac{\xi_2}{\xi_1} w_2^{n_2}$$

Here μ_i are the involution factors from Definition 2.8, and ν_i, ξ_i, n_i are as in Corollary 4.15. □

Proof. Follows directly from the equations of the configuration spaces given in Corollary 4.15. ■

Lemma 5.11. Let (Q_1, Q_2) be a reducible involutive coupling as described in parts (2) and (3) of Lemma 5.9, and let Z_{12}^0 be the component of its configuration space carrying the involution j_{12} . Then the quotient space $W^0 := Z_{12}^0/j_{12}$ has the following form.

- (1) If (Q_1, Q_2) is elliptic, and $t_1 - t_2$ is a half-period, then

$$w_1 = \begin{cases} cw_2, & \text{if } \sin \sigma_1 \sin \sigma_2 > 0 \\ -cw_2, & \text{if } \sin \sigma_1 \sin \sigma_2 < 0 \end{cases}, \quad \text{where } c = \sqrt{\frac{\frac{\sin \gamma_1 \sin \delta_1}{\sin \bar{\gamma}_1 \sin \bar{\delta}_1} - 1}{\frac{\sin \gamma_2 \sin \delta_2}{\sin \bar{\gamma}_2 \sin \bar{\delta}_2} - 1}}$$

- (2) If (Q_1, Q_2) is conic such that

$$\alpha_i + \gamma_i = \beta_i + \delta_i, \quad i = 1, 2,$$

and $t_1 - t_2 \in \{0, \pi\}$, then

$$w_1 = \begin{cases} cw_2, & \text{if } \sin \sigma_1 \sin \sigma_2 > 0 \\ -cw_2, & \text{if } \sin \sigma_1 \sin \sigma_2 < 0 \end{cases}, \quad \text{where } c = \sqrt{\frac{\frac{\sin \gamma_1 \sin \delta_1}{\sin \alpha_1 \sin \beta_1} - 1}{\frac{\sin \gamma_2 \sin \delta_2}{\sin \alpha_2 \sin \beta_2} - 1}}$$

- (3) If (Q_1, Q_2) is conic such that

$$\alpha_i + \beta_i + \gamma_i + \delta_i = 2\pi, \quad i = 1, 2,$$

and $t_1 - t_2 \in \{0, \pi\}$, then

$$w_1 = \begin{cases} c^{-1}w_2, & \text{if } \sin \sigma_1 \sin \sigma_2 > 0 \\ -c^{-1}w_2, & \text{if } \sin \sigma_1 \sin \sigma_2 < 0 \end{cases},$$

where c is given by the same formula as in the previous case. \square

In a reducible equimodular coupling of conic quadrilaterals, the side lengths can satisfy conditions other than those in parts (2) and (3) (see Lemma 5.4), but we will not need the equation of W in these cases.

Proof. Let $t_1 - t_2 \in \{0, 2K\}$ be a real half-period of an elliptic function F . Then Z_{12}^0 is the second component in (39), and its quotient (obtained by forgetting the coordinate w_2) is described by the equation

$$w_1 = \begin{cases} \frac{q_1}{q_2}w_2, & \text{if } t_1 - t_2 = 0 \\ -\frac{q_1}{q_2}w_2, & \text{if } t_1 - t_2 = 2K \end{cases}$$

If the coupling is geometrically realizable (i.e., $W \cap (\mathbb{R}P^1)^2$ is a curve), then q_1 and q_2 are either both real or both imaginary. Since we also have $p_1 = p_2$, formulas in Theorem 2.7 determining $\text{Re } t_0$ imply that $\text{Re } t_1 = \text{Re } t_2$ if and only if either $\sigma_1, \sigma_2 < \pi$ or $\sigma_1, \sigma_2 > \pi$. By observing finally that $\frac{\sqrt{x}}{\sqrt{y}} = \sqrt{\frac{x}{y}}$ if x and y have the same sign, we arrive at the formula in the Lemma.

The argument in the conic case is the same. \blacksquare

5.5 Lateral coupling of deltoids

Lemma 5.12. Let O_1 and O_2 be laterally coupled deltoids:

$$\alpha_i = \beta_i, \gamma_i = \delta_i, i = 1, 2$$

If their coupling is non-involutive, then the configuration space Z_{12} is described by the equation

$$aw_1^2 + 2bw_1w_2 + cw_2^2 + d = 0, \tag{54}$$

where

$$a = \frac{1 - d_2^2}{\cos^2 \delta_1}, \quad b = -\frac{1 - d_1d_2}{\cos \delta_1 \cos \delta_2}, \quad c = \frac{1 - d_1^2}{\cos^2 \delta_2}, \quad d = (d_1 - d_2)^2$$

and $d_1 = \frac{\tan \delta_1}{\tan \alpha_1}, d_2 = \frac{\tan \delta_2}{\tan \alpha_2}$. \square

Proof. The spaces Z_1 and Z_2 have equations of the form $w_i = a_i z + b_i z^{-1}$, $i = 1, 2$, see Corollary 4.15. The coupling is non-involutive if and only if

$$\det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \neq 0$$

If this is the case, then z and z^{-1} can be expressed as linear functions of w_1 and w_2 , whose substitution in $zz^{-1} = 1$ yields equation (54). ■

6 Proof of the Classification Theorem

For every Kokotsakis polyhedron, the dihedral angles at the edges of its central face determine its shape uniquely. These angles were denoted by ψ_1 , φ , ψ_2 , and θ , and the tangents of their halves by w_1 , z , w_2 , and u , respectively. See Section 2.1 and the beginning of Section 2.2. Therefore a continuous isometric deformation is represented by a non-constant map

$$\begin{aligned} I &\rightarrow (\mathbb{S}^1)^4 \\ t &\mapsto (\psi_1(t), \varphi(t), \psi_2(t), \theta(t)), \end{aligned} \tag{55}$$

where $I \subset \mathbb{R}$ is a segment. Recall that the dihedral angles at each pair of adjacent edges are related through an equation that is polynomial in the corresponding tangents of half-angles, see (4).

6.1 Trivially flexible polyhedra

A deformation will be called trivial, if one of the functions $\psi_1(t)$, $\varphi(t)$, $\psi_2(t)$, or $\theta(t)$ is constant. Let us classify trivial deformations.

In every trivial deformation there is a pair of adjacent dihedral angles one of which remains constant while the other one varies. Let $\varphi(t) = \text{const}$ and $\psi_1(t)$ be changing. From the classification of the configuration spaces of spherical quadrilaterals in Section 4 it follows that either $\varphi(t) = 0$ or $\varphi(t) = \pi$; in the former case, Q_1 is a deltoid with $\alpha_1 = \delta_1$ and $\beta_1 = \gamma_1$, in the latter case Q_1 is an antideltoid with $\alpha_1 + \delta_1 = \pi = \beta_1 + \gamma_1$. A case-by-case analysis of the behavior of the dihedral angles $\psi_2(t)$ and $\theta(t)$ leaves us with the following possibilities.

- Both $\theta(t)$ and $\psi_2(t)$ are changing;
- $\theta(t)$ is changing, $\psi_2(t) = 0$ or π ;

- $\theta(t) = 0$ or π , while $\psi_2(t)$ is changing;
- $\theta(t) = 0$ or π , while $\psi_2(t) = \text{const} \notin \{0, \pi\}$.

The corresponding flexible polyhedra are described in Section 3.8.

6.2 A diagram of branched covers

Assume that the deformation (55) is non-trivial, that is neither of the angles remains constant during the deformation. Make the substitution (3) and consider the irreducible components of the configuration spaces Z_i containing the deformation (55):

$$\begin{aligned} (w_1(t), z(t)) \in Z_1^0, \quad (z(t), w_2(t)) \in Z_2^0, \\ (w_2(t), u(t)) \in Z_3^0, \quad (u(t), w_1(t)) \in Z_4^0 \quad \forall t \in I \end{aligned}$$

The components Z_i^0 are well defined since by choosing, if needed, a subinterval of I we may assume that the path avoids singular points of all Z_i .

Denote further by Z_{12}^0 the irreducible component of the fiber product (see Section 5.1)

$$Z_1^0 \times_{\mathbb{C}P^1} Z_2^0 = \{(w_1, z, w_2) \mid P_1(w_1, z) = 0, P_2(z, w_2) = 0\}$$

that contains the path $(w_1(t), z(t), w_2(t))$ (again we might need to choose a subinterval for Z_{12}^0 to be well defined). Define similarly Z_{23}^0, Z_{34}^0 , and Z_{41}^0 . Then define Z_{123}^0 as the irreducible component of

$$Z_{12}^0 \times_{Z_2^0} Z_{23}^0 = \{(w_1, z, w_2, u) \mid P_1(w_1, z) = 0, P_2(z, w_2) = 0, P_3(w_2, u) = 0\},$$

and similarly Z_{234}^0, Z_{341}^0 , and Z_{412}^0 . All these algebraic sets have dimension 1, because they are branched covers of Z_{ij}^0 , which are branched covers of Z_i^0 . On the other hand, the intersection of any two of them contains a non-constant path. It follows that

$$Z_{123}^0 = Z_{234}^0 = Z_{341}^0 = Z_{412}^0 =: Z_{\text{all}}^0$$

The set Z_{all}^0 is thus a one-dimensional irreducible component of the solution set of (4). We obtain the diagram of branched covers on Figure 15.

6.3 Involutive polyhedra and involutive couplings

Each of the maps on Figure 15 is either an isomorphism or a two-fold branched cover. Let us concentrate on the multiplicities of the maps h_{ij} in the center of the diagram.

Definition 6.1. A deformation of a Kokotsakis polyhedron is called *involutive*, if at least one of the maps h_{ij} is two-fold. Otherwise, the deformation is called *non-involutive*. \square

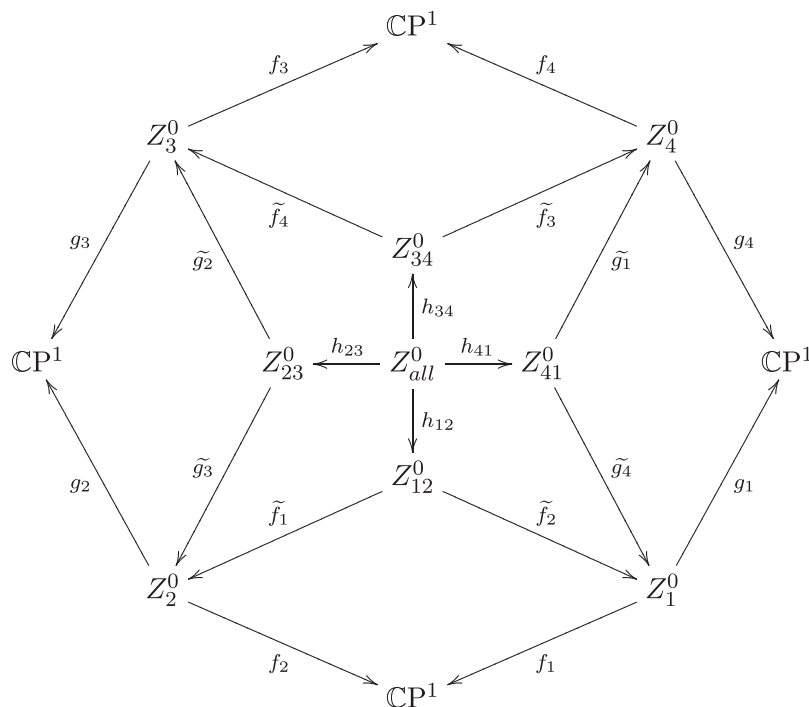


Fig. 15. A diagram of branched covers associated with a (non-trivially) flexible Kokotsakis polyhedron.

(By abuse of terminology, we will sometimes say that the polyhedron is involutive or non-involutive, although *a priori* it is possible that the same polyhedron has an involutive and a non-involutive deformation.)

If the deformation is non-involutive, then Z_{all}^0 can be identified with each of Z_{ij}^0 according to h_{ij} , and the big diagram collapses to a smaller one on Figure 31 in Section 6.6, where the non-involutive case is dealt with.

Lemma 6.2. An involutive polyhedron contains an involutive coupling. □

Proof. Let h_{34} be two-fold. Consider the commutative square

$$\begin{array}{ccc}
 & Z_{all}^0 & \\
 h_{12} \swarrow & & \searrow h_{34} \\
 Z_{12}^0 & & Z_{34}^0 \\
 \pi_{12} \searrow & & \swarrow \pi_{34} \\
 & W^0 &
 \end{array}
 \tag{56}$$

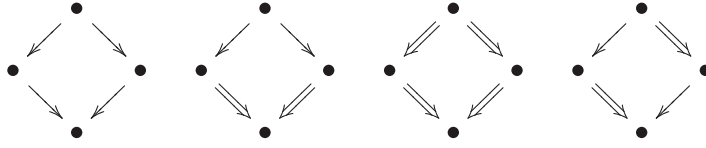


Fig. 16. Possible multiplicities of maps in a fiber product.

where π_{ij} is as in (47) and W^0 is an irreducible component of the partial configuration space

$$W = \{(w_1, w_2) \in (\mathbb{CP}^1)^2 \mid \exists z, u \in \mathbb{CP}^1 \text{ such that } (w_1, z, w_2, u) \in Z_{\text{all}}^0\}$$

This is the diagram of a fiber product, and therefore if h_{34} is two-fold, so is π_{12} . By Definition 5.7 this means that the coupling (Q_1, Q_2) is involutive. ■

The multiplicities of the maps in a fiber product diagram can be only as shown on Figure 16. Here a double arrow represents a two-fold branched cover, and a simple arrow represents an isomorphism.

The case when both (Q_1, Q_2) and (Q_3, Q_4) are involutive is studied in Section 6.4. In this case the commutative square (56) might have the form of the second square on Figure 16, so that *a priori* there may be non-involutive polyhedra with involutive couplings.

If (Q_1, Q_2) is involutive, and (Q_3, Q_4) is not, then h_{34} is double and h_{12} is simple. This case is dealt with in Section 6.5.

6.4 Combination of two involutive couplings

If (Q_1, Q_2) and (Q_3, Q_4) are involutive, then we have

$$Z_{12}^0/j_{12} = W^0 = Z_{34}^0/j_{34}$$

Equations describing the quotient of the configuration space of an involutive coupling are given in Lemmas 5.10 and 5.11. Consider them case by case.

6.4.1 Cubic case

The space W^0 is described by an equation of the form

$$a_1 w_1 + b_1 w_1^{-1} = a_2 w_2 + b_2 w_2^{-1}$$

This is the Case (1) of Lemma 5.10. Thus both couplings (Q_1, Q_2) and (Q_3, Q_4) consist of compatible orthodiagonal quadrilaterals neither of which is an (anti)deltoid. By comparing the coefficients at the corresponding terms, we obtain

$$\mu_1 = \mu_4, \quad \mu_2 = \mu_3, \quad \nu_1 \nu_3 = \nu_2 \nu_4$$

The first two equations say that the involution factors of Q_1 and Q_4 , respectively, of Q_2 and Q_3 , at their common vertex are equal. The third equation is equivalent to $\cos \delta_1 \cos \delta_3 = \cos \delta_2 \cos \delta_4$. Thus we get the orthodiagonal involutive type 3.1.1.

6.4.2 Rational case

The space W^0 is described by an equation of the form

$$aw_1 + bw_1^{-1} = w_2^{\pm 1}$$

This means that Q_1 and Q_2 form a compatible pair of orthodiagonal quadrilaterals, Q_2 is an (anti)deltoid while Q_1 not, and the same is true for the pair (Q_4, Q_3) . By comparing the coefficients, we obtain

$$\mu_1 = \mu_4, \quad n_2 = n_3, \quad \nu_1 \xi_3 = \nu_4 \xi_2$$

Thus Q_1 and Q_4 have equal involution factors at their common vertex, and Q_2, Q_3 are either both deltoids or both antideltoids. By Definition 4.18 this means that all pairs of adjacent quadrilaterals are compatible. The last equation implies $\cos \delta_1 \cos \delta_3 = \cos \delta_2 \cos \delta_4$ so that we get again the orthodiagonal involutive type 3.1.1.

6.4.3 Linear case

The space W^0 is described by an equation of the form

$$w_1 w_2 = \text{const or } \frac{w_1}{w_2} = \text{const}$$

By switching, if needed, the right boundary strip (i.e., replacing $\beta_1, \gamma_1, \beta_4$, and γ_4 by their complements to π) the former case can be reduced to the latter. We thus have a linear compound as defined in Section 3.5. By Lemmas 5.10 and 5.11 each of the couplings (Q_1, Q_2) and (Q_3, Q_4) has one of the following forms.

- two deltoids or two antideltoids, coupled laterally and with equal involution factors at the common vertex;

- a reducible coupling of two elliptic quadrilaterals with the shift difference $t_1 - t_2 \in \{0, 2K\}$;
- a reducible coupling of two conic quadrilaterals with the shift difference $t_1 - t_2 \in \{0, \pi\}$.

The first and the second linear couplings are listed in Sections 3.5.2 and 3.5.4. Let us show that the third one can be reduced to the cases described in Section 3.5.5. We have $m_1 = m_2$ by Lemma 5.9 and $n_1 = n_2$ because of $w_1 = cw_2$, hence the side lengths of both Q_1 and Q_2 satisfy the same of the four possible relations $\alpha_i \pm \beta_i \pm \gamma_i \pm \delta_i \equiv 0 \pmod{2\pi}$. By switching, if needed, the lower boundary strip, these four restrict to two possibilities (16a) and (16b). The same can be done with the coupling (Q_3, Q_4) .

6.5 Combination of an involutive coupling with a non-involutive

Assume that (Q_1, Q_2) is involutive, and (Q_3, Q_4) is not. The polyhedron is flexible if and only if $W_z = W_u$, where

$$W_z := \pi_{12}(Z_{12}^0) = Z_{12}^0/j_{12}, \quad W_u := \pi_{34}(Z_{34}^0)$$

We will consider all of the forms that (Q_1, Q_2) can take and study the multiplicities of the maps in the diagram on Figure 15.

6.5.1 The involutive coupling is equimodular

That is, (Q_1, Q_2) has the form described in parts (2) and (3) of Lemma 5.9. By switching, if needed, the boundary strips we can assume that (Q_1, Q_2) is one of the linear couplings described in Sections 3.5.4 and 3.5.5.

The maps between configuration spaces have multiplicities as shown on Figure 17, where dotted lines stand for the maps with unknown multiplicities. However, with the help of Figure 16 we can determine most of them, see Figure 17.

The two remaining dotted arrows are either both simple or both double. Thus (Q_3, Q_4) is either a coupling of (anti)isograms or a reducible coupling of (anti)deltoids. Theorem 2.4 and Lemma 5.5 imply that such a coupling yields a linear dependence $w_1 = cw_2$ if and only if Q_3 and Q_4 are of the same type (e.g., both are antideltooids) and allow us to compute the coefficient c . We thus have a polyhedron of linear compound type, where (Q_3, Q_4) is as in Section 3.5.1 or 3.5.3.

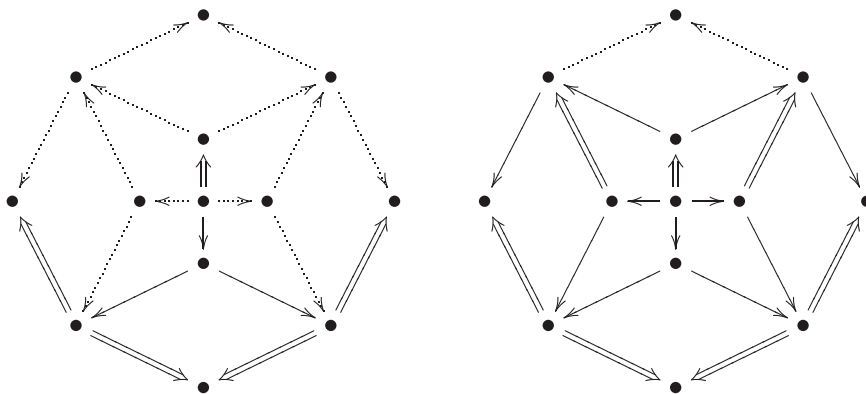


Fig. 17. Solving the diagram for Section 6.5.1.

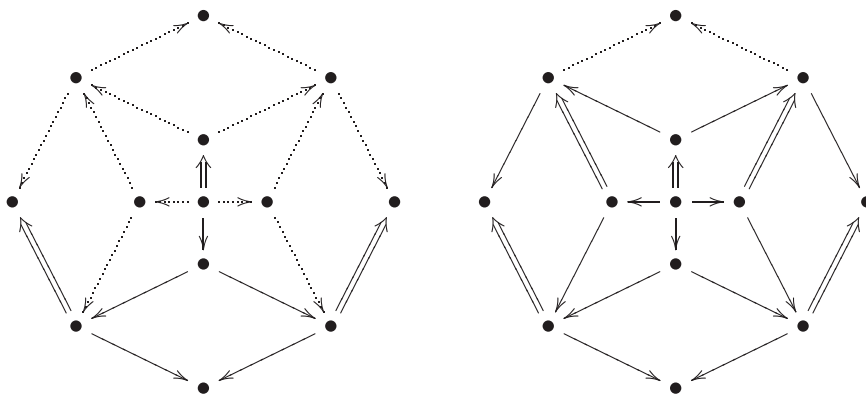


Fig. 18. Solving the diagram for Section 6.5.2.

6.5.2 The involutive coupling consists of two (anti)deltoids

Modulo switching, (Q_1, Q_2) is as described in Section 3.5.2. Again, the diagram can be solved as shown on Figure 18.

It follows that (Q_3, Q_4) has the same form as in the previous case, and we have a linear compound coupling.

6.5.3 The involutive coupling is orthodiagonal with one (anti)deltoid

Without loss of generality, let Q_2 be an (anti)deltoid, Q_1 orthodiagonal elliptic. By switching, if needed, the left boundary strip, we can transform Q_2 to antideltoid. Then, according to Lemma 5.10 we have

$$W_z = \left\{ (w_1, w_2) \mid w_1 + \mu_1 w_1^{-1} = \frac{\nu_1}{\xi_2} w_2 \right\} \tag{57}$$

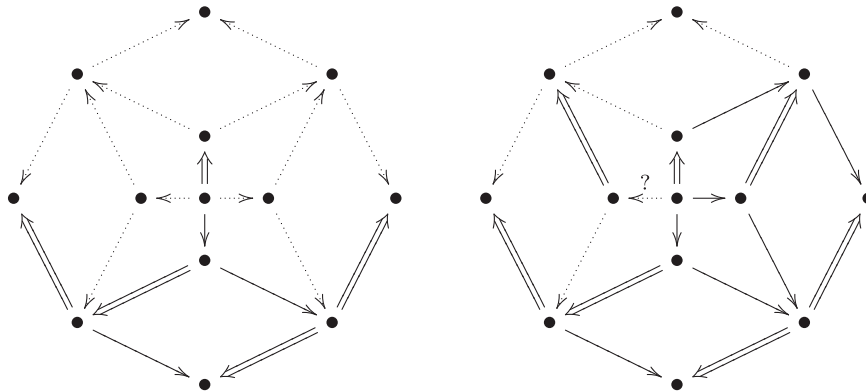


Fig. 19. Solving the diagram for Section 6.5.3.

Figure 19 shows the map multiplicities that we obtain by diagram chasing. The further case distinction depends on the multiplicity of the question mark map

Case 1. The question mark map on Figure 19 is two-fold.

This yields the multiplicities on Figure 20, left. Thus Q_4 is an (anti)deltoid compatibly coupled with an orthodiagonal non-deltoid Q_1 (since the map h_{23} is two-fold), and Q_3 is an (anti)isogram. By switching, if needed, the upper boundary strip we can transform Q_4 to an antideltoid. Let us find the equation of the space W_u .

By Theorem 2.5 and Corollary 4.15 we have

$$Z_3^0 = \{(u, w_2) \mid u^{\pm 1} = \kappa_3 w_2\}, \quad Z_4^0 = \{(u, w_1) \mid w_1 + \mu_4 w_1^{-1} = \zeta_4 u^{-1}\}$$

The equation of W_u is obtained by substituting the first equation in the second one. We have $W_z = W_u$ if and only if the resulting equation is proportional to (57). Thus we must have $u^{-1} = \kappa_3 w_2$, which means that Q_3 is an isogram, and

$$\mu_1 = \mu_4, \quad \kappa_3 \xi_2 \zeta_4 = \nu_1$$

The first equation holds automatically, since Q_1 and Q_4 are compatible. The second one is an additional restriction. Altogether we obtain a polyhedron described in Section 3.7.2.

Case 2. The question mark map on Figure 19 is an isomorphism.

This yields the multiplicities on Figure 20, right, where the dotted lines are either both simple or both double. We make a further case distinction.

Case 2a. The dotted lines on Figure 20, right, are simple.

Thus Q_3 is an (anti)deltoid, and Q_4 is an (anti)isogram. Let us find the equation of the space W_u .

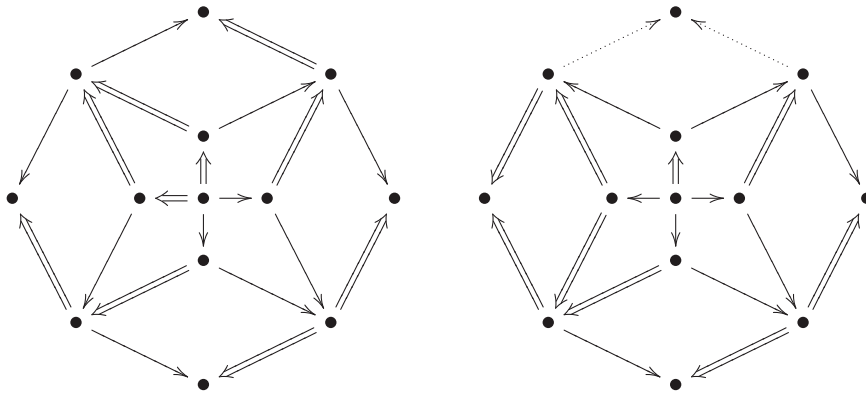


Fig. 20. Cases 1 and 2 of Section 6.5.3.

By Theorem 2.4 and Corollary 4.15 we have

$$Z_3^0 = \{(u, w_2) \mid u + \lambda_3 u^{-1} = \xi_3 w_2^{n_2}\}, \quad Z_4^0 = \{(u, w_1) \mid u^{\pm 1} = \kappa_4 w_1\}$$

By eliminating the variable u we must obtain an equation of the form (57). It follows that $n_2 = 1$, so that Q_3 is a deltoid. By switching, if needed, the upper boundary strip, we can transform Q_4 to an antiisogram, so that $u = \kappa_4 w_1$. By making this substitution and comparing the coefficients with those in (57), we obtain the necessary and sufficient conditions for flexibility:

$$\frac{\lambda_3}{\kappa_4^2} = \mu_1, \quad \frac{\xi_3}{\kappa_4} = \frac{\nu_1}{\xi_2}$$

As a result, we obtain a polyhedron described in Section 3.7.3.

Case 2b. The dotted lines on Figure 20, right, are double.

Thus Q_4 is an (anti)deltoid forming a reducible coupling with Q_3 . It follows that Q_3 is conic. The two situation when (Q_3, Q_4) is reducible are described in Lemma 5.5, and the equations for the corresponding space W are given in Lemma 5.6.

If $m_3 = m_4, p_3 = p_4$, then the equation of W_u is

$$w_1 + \mu_4 e^{\pm 2it_3} w_1^{-1} = \varepsilon_4 \frac{2i\sqrt{-\mu_4} e^{\pm it_3}}{q_3} w_2^{n_3}$$

which has the same form as (57) if and only if $n_3 = 1$. By switching, if needed, the upper boundary strip, we can achieve $m_3 = m_4 = 1$. Then the conic quadrilateral Q_3 is circumscribed, and Q_4 is a deltoid. By equating the coefficients at w_1^{-1} and w_2 , we obtain necessary and sufficient conditions of flexibility, as described in Section 3.7.10.

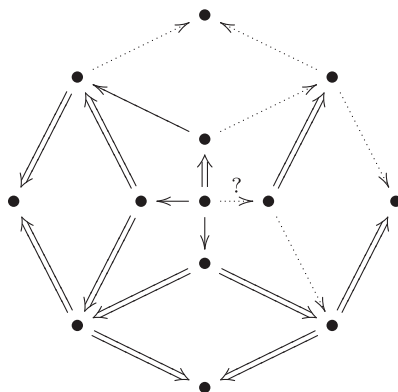


Fig. 21. The diagram for Section 6.5.4.

If $m_3 = -m_4$, $p_3 = \pm \frac{1}{p_4}$, then the set W_u is described by equation of the form (49), thus $W_u = W_z$ cannot take place, and there is no flexible polyhedron in this case.

6.5.4 The only involutive coupling is orthodiagonal without (anti)deltoids

By Lemma 5.10 we have

$$W_z = \{(w_1, w_2) \mid w_1 + \mu_1 w_1^{-1} = \frac{v_1}{v_2} (w_2 + \mu_2 w_2^{-1})\} \tag{58}$$

If both h_{23} and h_{41} are two-fold, then the couplings (Q_2, Q_3) and (Q_4, Q_1) are involutive. Up to a rotation of the diagram, this situation was considered in Section 6.4. Thus assume without loss of generality that h_{23} is an isomorphism. Then we obtain the map multiplicities as on Figure 21.

Case 1. The question mark map on Figure 21 is two-fold.

This yields the multiplicities on Figure 22, left. The quadrilateral Q_3 is either conic or elliptic, thus its configuration space $Z_3^0 = Z_3$ is an irreducible curve described by an equation of the form (19). Since Q_4 is an isogram, the equation of W_u is obtained from that of Z_3^0 by a linear substitution $u = cw_2$ or $u = cw_2^{-1}$. Therefore it cannot have the form (58). Thus there are no flexible Kokotsakis polyhedra with this diagram.

Case 2. The question mark map on Figure 21 is an isomorphism.

This yields the multiplicities on Figure 22, right, and makes a further case distinction necessary.

Case 2a. The dotted lines on Figure 22, right, are simple.

Then (Q_3, Q_4) is a lateral coupling of (anti)deltoids. By switching, if needed, the left and/or the right boundary strips we can transform them to deltoids. Then W_u has

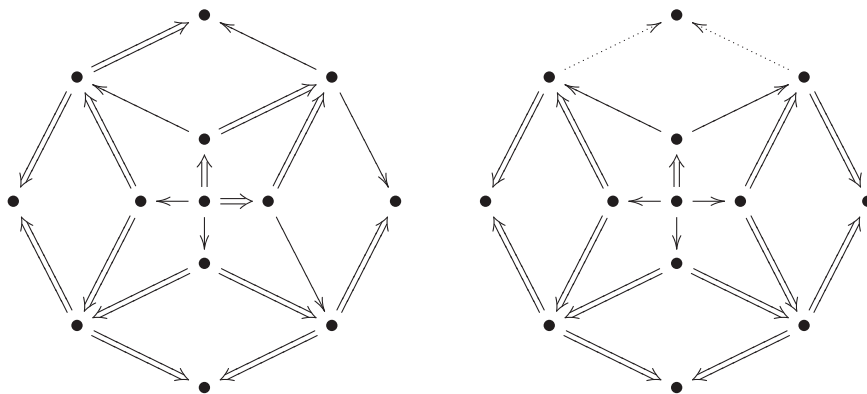


Fig. 22. Cases 1 and 2 of Section 6.5.4.

equation (54) which contradicts (58) (the former polynomial cannot be a factor of the latter). Thus there are no flexible Kokotsakis polyhedra with this diagram.

Case 2b. The dotted lines on Figure 22, right, are double.

Then (Q_3, Q_4) is a reducible non-involutive coupling of conic or elliptic quadrilaterals. Equations of the reducible components of the space W_u are described in Lemma 5.6 and are all different from (58). Hence this diagram also does not produce any flexible Kokotsakis polyhedra.

6.6 Flexible polyhedra without involutive couplings

By the argument in Section 6.3, if a deformation contains no involutive couplings, then the diagram on Figure 15 collapses to the diagram on Figure 31. The further classification is based on the multiplicities of the maps h_i .

If h_i is two-fold and Z_i is conic or elliptic, then both couplings (Q_{i-1}, Q_i) and (Q_i, Q_{i+1}) are irreducible. This case is analyzed in Section 6.6.1. Here we find the only flexible polyhedron whose vertices have elliptic configuration spaces coupled non-involutively and non-reducibly (which means that all resultant polynomials R_{ij} are irreducible).

The cases when h_i is two-fold, and the polyhedron Q_i is either (anti)deltoid or (anti)isogram, are dealt with in Sections 6.6.2 and 6.6.3. In Section 6.6.4 we study the case when all h_i are isomorphisms.

6.6.1 Irreducible coupling with a conic or elliptic quadrilateral

Lemma 6.3. Assume that the quadrilateral Q_1 is conic or elliptic and that \tilde{f}_2 and \tilde{g}_4 are equivalent two-fold branched covers. Then Q_1, Q_2 , and Q_4 are all elliptic.

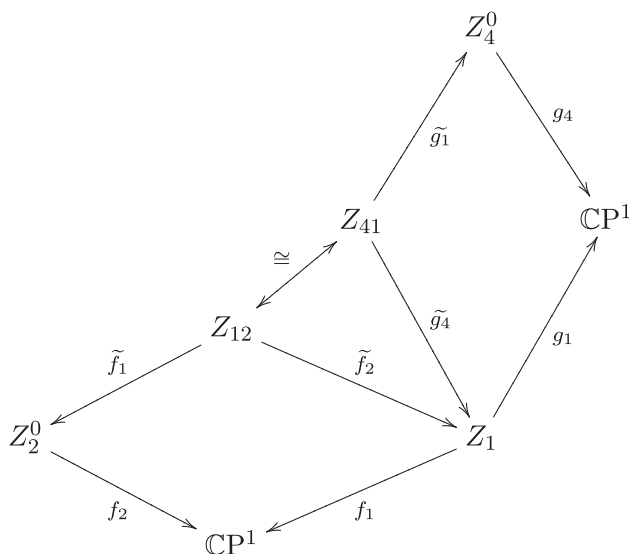


Fig. 23. A half of the diagram from Figure 15, under the assumption $\tilde{f}_2 \simeq \tilde{g}_4$.

Besides, the corresponding phase shift t_1 satisfies either $\text{Im } t_1 = \frac{iK'_1}{3}$ or $\text{Im } t_1 = \frac{iK'_1}{2}$ and the branch set C_1 of \tilde{f}_2 and \tilde{g}_4 has one of the forms depicted on Figures 24 and 25. \square

Proof. If \tilde{f}_2 is equivalent to \tilde{g}_4 , then we have the situation depicted on Figure 23. Since \tilde{f}_2 is two-fold, the map f_2 is also two-fold. The same holds for the map g_4 . The two-fold branched covers \tilde{f}_2 and \tilde{g}_4 are equivalent if and only if they have the same branch set. By Lemma 5.1, this means

$$f_1^{-1}(A_2) = g_1^{-1}(B_4) =: C_1, \tag{59}$$

where A_2 and B_4 are the branch sets of the maps f_2 and g_4 , respectively.

As a full preimage under f_1 , the set C_1 is invariant under the action of the deck transformation i_1 of the two-fold cover f_1 . Similarly, C_1 is invariant under the deck transformation j_1 of g_1 :

$$i_1(C_1) = C_1 = j_1(C_1)$$

Consider a parametrization of Z_1 obtained in Theorems 2.6 and 2.7:

$$Z_1 = \{(p_1 F(t), q_1 F(t + t_1))\},$$

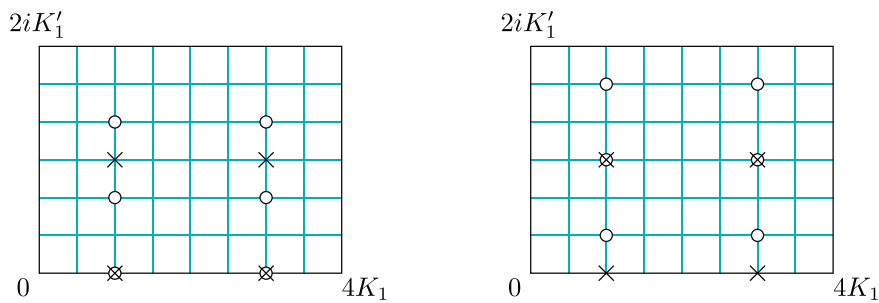


Fig. 24. The sets C_1 and $C_1 + t_1$ (marked with \circ) for $\text{Im } t_1 = \frac{iK'_1}{3}$ and $F = \text{sn}$. The sets contain two branch points of sn (marked with \times) each.

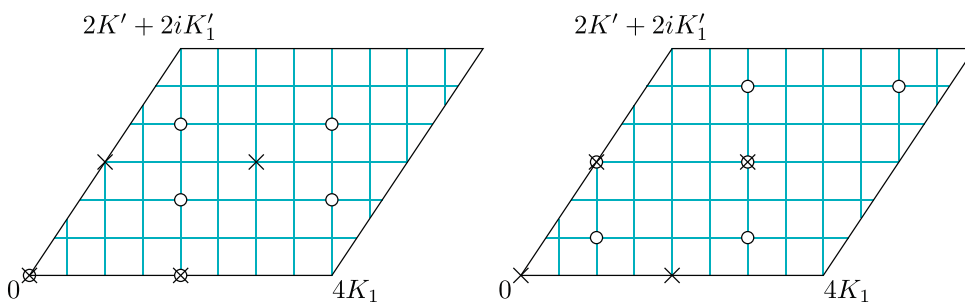


Fig. 25. The sets C_1 and $C_1 + t_1$ for $\text{Im } t_1 = \frac{iK'_1}{3}$ and $F = \text{cn}$. The sets contain two branch points of cn each.

where the function F is \sin , sn , or cn . By identifying Z_1 with its parameter domain, we have

$$f_1(t) = p_1 F(t), \quad g_1(t) = q_1 F(t + t_1)$$

Because of

$$i_1 \circ j_1(t) = t + 2t_1,$$

the set C_1 is invariant under the shift by $2t_1$. This implies that $F \neq \sin$, because the orbit of the shift by $2t_1$ is infinite in $\mathbb{C}/2\pi\mathbb{Z}$ (recall that $\text{Im } t_1 \neq 0$), while the set C_1 is finite.

It follows that Q_1 is elliptic, $F = \text{sn}(\cdot, k_1)$ or $\text{cn}(\cdot, k_1)$. Let

$$\Lambda = \begin{cases} 4K_1\mathbb{Z} + 2iK'_1\mathbb{Z}, & \text{if } F = \text{sn} \\ 4K_1\mathbb{Z} + (2K + 2iK'_1)\mathbb{Z}, & \text{if } F = \text{cn} \end{cases}$$

be the period lattice of F . We have $Z_1 = \mathbb{C}/\Lambda$. On the torus \mathbb{C}/Λ , the orbit of the shift by $2t_1$ is finite if and only if $\text{Im } t_1 = \frac{iK'_1}{n}$ for some $n \geq 2$ (recall that $\text{Im } t_1 \in (0, iK'_1)$ and $\text{Re } t_1$ is a multiple of K_1).

Equation (59) implies that $C_1 \subset \mathbb{C}/\Lambda$ is invariant also under the involution

$$t \mapsto -t, \quad \text{if } F = \text{sn}, \quad t \mapsto 2K_1 - t, \quad \text{if } F = \text{cn}$$

Indeed, the set A_2 is symmetric with respect to 0, and we have $-\text{sn } t = \text{sn}(-t)$ and $-\text{cn } t = \text{cn}(2K_1 - t)$. Combined with the invariance with respect to involutions i_1 and j_1 this implies that C_1 is invariant under the group generated by three point reflections

$$t \mapsto -t, \quad t \mapsto 2K_1 - t, \quad t \mapsto 2t_1 - t$$

Another set of generators for the same group consists of two shifts and one point reflection:

$$t \mapsto t + 2K_1, \quad t \mapsto t + 2t_1, \quad t \mapsto -t$$

Thus we have

$$C_1 = ((x + \Lambda_n) \cup (-x + \Lambda_n)) / \Lambda \tag{60}$$

for some $x \in \mathbb{C}$, where

$$\Lambda_n := 2K_1\mathbb{Z} + 2t_1\mathbb{Z} = 2K_1\mathbb{Z} + \frac{2iK'_1}{n}\mathbb{Z}$$

Since $|\Lambda_n/\Lambda| = 2n$, it follows that

$$|C_1| = \begin{cases} 4n, & \text{if } x \notin \frac{1}{2}\Lambda_n \\ 2n, & \text{if } x \in \frac{1}{2}\Lambda_n \end{cases} \tag{61}$$

From the classification of the configuration spaces we know that $|A_2| = 2$ (if Q_2 is an (anti)deltoid or conic quadrilateral) or $|A_2| = 4$ (if Q_2 is an elliptic quadrilateral). Since f_1 is a two-fold cover, equation (59) implies that $|C_1| \leq 8$. Thus, due to (61) we have $n \leq 4$.

Let us show that $n \neq 4$. If $n = 4$, then by (61) x is a half-period of Λ_n . Without loss of generality,

$$x \in \{0, K_1, -t_1, -t_1 + K_1\}$$

Consider first the case $F = \text{sn}$. If $x = 0$, then $0 \in C_1$, hence $0 = f_1(0) \in A_2$, which is impossible, see Lemma 4.10. If $x = K_1$, then we have $|A_2| > 4$ because K_1 is a branch point for the function $\text{sn}(\cdot, k_1)$ and

$$|C_1| = 2|A_2| \Leftrightarrow C_1 \text{ contains no branch points of } f_1$$

Similarly, $x = -t_1$ would imply $0 = g_1(-t_1) \in B_4$, and $x = -t_1 + K_1$ would imply that C_1 contains a branch point of g_1 . Summarizing, there is no half-period shift of Λ_4 whose image under f_1 and g_1 would consist of four points, all different from 0. In the case $F = \text{cn}$ the argument is similar. Thus $n \neq 4$.

Let us look at the case $n = 3$. Because of (61) and $|C_1| \leq 8$, the point x must be a half-period of Λ_3 :

$$x \in \left\{ 0, K_1, \frac{iK'_1}{3}, K_1 + \frac{iK'_1}{3} \right\}$$

If $F = \text{sn}$, then $x = 0$ would lead to $0 \in A_2$, and $x = \frac{iK'_1}{3}$ would lead to $\infty \in A_2$. It follows that C_1 is one of the sets depicted on Figure 24, with $C_1 + t_1$ being the other one. The images of C_1 and $C_1 + t_1$ under the map $t \mapsto p_1 \text{sn}(t, k_1)$, respectively, $t \mapsto q_1 \text{sn}(t, k_1)$, consist of four points each. Thus we have $|A_2| = |B_4| = 4$, and therefore both Q_2 and Q_4 are elliptic quadrilaterals.

For $F = \text{cn}$ the situation is similar. The sets C_1 and $C_1 + t_1$ are shown in Figure 25.

Finally, consider the case $n = 2$. We have $\text{Im } t_1 = \frac{K'_1}{2}$, therefore by Lemma 4.12 the space Z_1 is parametrized by $F_1 = \text{sn}$. If $x \in \frac{1}{2}\Lambda_2$, then we have either $\text{Im } x \equiv 0 \pmod{K'_1}$ or $\text{Im}(x + t_1) \equiv 0 \pmod{K'_1}$. Without loss of generality, assume the former to be the case. Then we have either

$$C_1 = x + \Lambda_2 = \{0, 2K_1, iK'_1, 2K_1 + iK'_1\},$$

in which case $A_2 = \{0, \infty\}$, which is a contradiction, or

$$C_1 = x + \Lambda_2 = \{K_1, 3K_1, K_1 + iK'_1, 3K_1 + iK'_1\},$$

in which case $A_2 = A_1$ and the coupling (Q_1, Q_2) is reducible, so that \tilde{f}_2 is not two-fold. This contradiction shows that $x \notin \frac{1}{2}\Lambda_2$, so that $|C_1| = 8$ and hence $|A_2| = |B_4| = 4$, that is, Q_2 and Q_4 are elliptic.

This finishes the proof of the Lemma. ■

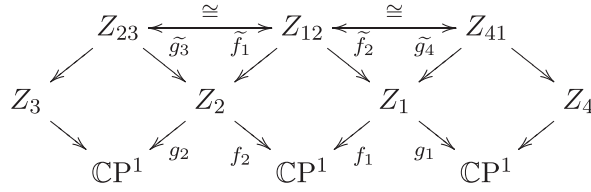


Fig. 26. The diagram for Lemma 6.4.

Lemma 6.4. Assume that the quadrilaterals Q_1 and Q_2 are elliptic and that \tilde{f}_2 and \tilde{g}_4 are equivalent two-fold branched covers over Z_1 , and \tilde{f}_1 and \tilde{g}_3 are equivalent two-fold branched covers over Z_2 , see Figure 26. Then the quadrilaterals Q_1 and Q_2 are orthodiagonal, and their involution factors at the common vertex (see Definition 2.8) are either equal or opposite: $\lambda_1 = \pm\lambda_2$. \square

Proof. By Lemma 6.3, the imaginary part of the phase shift t_1 of the configuration space Z_1 equals either $\frac{K'_1}{3}$ or $\frac{K'_1}{2}$. In the former case, equation (59) and Figures 24 and 25 provide us with the following information about the branch set A_2 .

If $\text{Im } t_1 = \frac{K'_1}{3}$ and $F_1 = \text{sn}(\cdot, k_1)$, then one of the following holds:

$$A_2 = \left\{ \pm p_1, \pm p_1 \text{sn} \left(K_1 + \frac{2iK'_1}{3}, k_1 \right) \right\} \tag{62a}$$

$$A_2 = \left\{ \pm \frac{p_1}{k_1}, \pm p_1 \text{sn} \left(K_1 + \frac{iK'_1}{3}, k_1 \right) \right\} \tag{62b}$$

If $\text{Im } t_1 = \frac{K'_1}{2}$ and $F_1 = \text{cn}(\cdot, k_1)$, then one of the following holds:

$$A_2 = \left\{ \pm p_1, \pm p_1 \text{cn} \left(\frac{2iK'_1}{3}, k_1 \right) \right\} \tag{63a}$$

$$A_2 = \left\{ \pm ip_1 \frac{k'_1}{k_1}, \pm p_1 \text{cn} \left(K_1 + \frac{iK'_1}{3}, k_1 \right) \right\} \tag{63b}$$

If $\text{Im } t_1 = \frac{K'_1}{2}$, then by Lemma 4.12 $F_1 = \text{sn}(\cdot, k_1)$. Hence, equations (59) and (60) imply

$$A_2 = \left\{ \pm p_1 \text{sn}(x, k_1), \pm \frac{p_1}{k_1 \text{sn}(x, k_1)} \right\} \tag{64}$$

On the other hand, by Lemma 4.10 the set A_2 consists of four real or four purely imaginary points if $F_2 = \text{sn}(\cdot, k_2)$, and from two real and two purely imaginary points if

$F_2 = \text{cn}(\cdot, k_2)$. Since

$$\text{sn}\left(K + \frac{2iK'}{3}\right), \text{sn}\left(K + \frac{iK'}{3}\right), \text{cn}\left(\frac{2iK'}{3}\right) \in \mathbb{R}, \quad \text{cn}\left(K + \frac{iK'}{3}\right) \in i\mathbb{R},$$

each of the quadruples in (62) and (63) consists either of four real or four imaginary points. In (64) also, we cannot have two real and two purely imaginary points, but only four of the same kind. It follows that $F_2 = \text{sn}(\cdot, k_2)$.

By reversing the roles of Q_1 and Q_2 in the above argument we see that $F_1 = \text{sn}(\cdot, k_1)$, so that the Case (63) falls out of consideration. Moreover, by Lemma 4.10 we have

$$A_2 = \left\{ \pm p_2, \pm \frac{p_2}{k_2} \right\} \tag{65}$$

Claim 1. If $\text{Im } t_1 = \frac{K'}{3}$, then $k_1 < k_2$ and either $p_1 = p_2$ or $\frac{p_1}{k_1} = \frac{p_2}{k_2}$.

Indeed, since $0 < k_1 < 1$, we have $|p_2| < \left| \frac{p_2}{k_2} \right|$. On the other hand, due to the monotonicity of sn on the segment $[K_1, K_1 + iK'_1]$ we have

$$1 < \text{sn}\left(K_1 + \frac{2iK'_1}{3}, k_1\right) < \frac{1}{k_1}$$

Thus, if (62a) occurs, then the first entry in (65) equals the first entry in (62a), and the second one equals the second one:

$$p_2 = p_1, \quad \frac{1}{k_2} = \text{sn}\left(K_1 + \frac{2iK'_1}{3}, k_1\right) < \frac{1}{k_1},$$

which proves the claim in the situation of (62a). Similarly, if (62b) occurs, then we have

$$\frac{p_1}{k_1} = \frac{p_2}{k_2}, \quad p_1 \text{sn}\left(K_1 + \frac{iK'_1}{3}, k_1\right) = p_2,$$

which also implies $k_1 < k_2$.

Claim 2. If $\text{Im } t_1 = \frac{K'}{2}$, then $\frac{p_1^2}{k_1} = \pm \frac{p_2^2}{k_2}$.

This follows from taking the products of entries in (64) and (65).

By reversing the roles of Q_1 and Q_2 we obtain analogs of Claims 1 and 2 with k_1 and k_2 as well as p_1 and p_2 interchanged.

It follows that we cannot have $\text{Im } t_1 = \frac{K'}{3}$ and $\text{Im } t_2 = \frac{K'_2}{3}$, because by Claim 1 this would imply $k_1 < k_2$ and $k_2 < k_1$. We cannot have $\text{Im } t_1 = \frac{K'_1}{3}$ and $\text{Im } t_2 = \frac{K'_2}{2}$ (or vice

versa), because then $\frac{p_1^2}{k_1} = \pm \frac{p_2^2}{k_2}$ together with $p_1 = p_2$ or $\frac{p_1}{k_1} = \frac{p_2}{k_2}$ implies $k_1 = k_2$, which contradicts $k_1 < k_2$.

Hence we have $\text{Im } t_1 = \frac{K'_1}{2}$ and $\text{Im } t_2 = \frac{K'_2}{2}$. By Lemma 4.12 both quadrilaterals Q_1 and Q_2 are then orthodiagonal. Their involution factors are equal or opposite by Claim 2 and Lemma 4.17. ■

Lemma 6.5. Let (Q_1, Q_2, Q_3, Q_4) represent a flexible Kokotsakis polyhedron without involutive couplings. Assume that Q_1 is elliptic or conic and that the coupling (Q_1, Q_2) is irreducible. Then the polyhedron belongs to the orthodiagonal antiinvolutive type described in Section 3.1.2. □

Proof. If (Q_1, Q_2) is irreducible while Q_1 is conic or elliptic, then the map $\tilde{f}_2: Z_{12} \rightarrow Z_1$ is two-fold. Since there are no involutive coupling components, the coverings \tilde{f}_2 and \tilde{g}_4 are equivalent, see the beginning of Section 6.6. Thus we are in the situation of Lemma 6.3.

Lemma 6.3 implies that Q_1, Q_2 , and Q_4 are elliptic, and their configuration spaces are all doubly covered. By non-involutivity, the double covers over Z_2 and Z_4 are equivalent, which implies that Q_3 is also elliptic. It follows that we can apply Lemma 6.4 to any pair of adjacent quadrilaterals.

Thus all Q_i are elliptic orthodiagonal quadrilaterals, and the involution factors at their common vertices are equal or opposite. If two involution factors are equal, then the corresponding coupling is involutive, which contradicts our assumption. Thus we have

$$\lambda_1 = -\lambda_2, \quad \mu_2 = -\mu_3, \quad \lambda_3 = -\lambda_4, \quad \mu_4 = -\mu_1$$

By Corollary 4.15, the configuration space of the polyhedron is the solution set of the system

$$\begin{aligned} (z + \lambda_1 z^{-1})(w_1 + \mu_1 w_1^{-1}) &= v_1 \\ (z - \lambda_1 z^{-1})(w_2 - \mu_3 w_2^{-1}) &= v_2 \\ (u + \lambda_3 u^{-1})(w_2 + \mu_3 w_2^{-1}) &= v_3 \\ (u - \lambda_3 u^{-1})(w_1 - \mu_1 w_1^{-1}) &= v_4 \end{aligned}$$

By computing the resultants or by making trigonometric substitutions, we obtain the necessary and sufficient conditions for this system to have a one-parameter set of

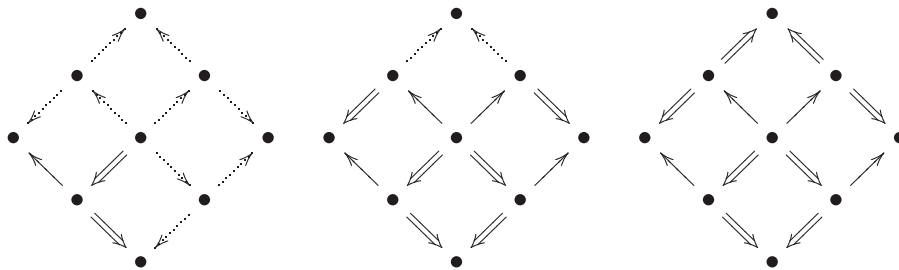


Fig. 27. Solving the diagram for Section 6.6.2.

solutions:

$$\frac{\nu_1^2}{\lambda_1\mu_1} = \frac{\nu_3^2}{\lambda_3\mu_3}, \quad \frac{\nu_2^2}{\lambda_1\mu_3} = \frac{\nu_4^2}{\lambda_3\mu_1}, \quad \frac{\nu_1^2}{\lambda_1\mu_1} + \frac{\nu_2^2}{\lambda_1\mu_3} = 1$$

Thus the polyhedron has the form described in Section 3.1.2. ■

6.6.2 A double cover over an (anti)deltoid

Without loss of generality, the map h_2 is two-fold, and the quadrilateral Q_2 is an (anti)deltoid. Besides, we may assume that Q_2 is coupled to Q_1 frontally, so that the map f_2 is two-fold and the map g_2 an isomorphism. Completing fiber product squares according to Figure 16 and using the assumption that no conic or elliptic is doubly covered, we determine multiplicities of all but two of the other maps, see Figure 27, left and middle.

Since the maps h_1 and h_2 are two-fold, the coupling (Q_1, Q_2) is irreducible, that is the branch sets of f_1 and f_2 are different: $A_1 \neq A_2$. Then the branch set $f_2^{-1}(A_1)$ of h_2 consists of four points. Since the covering g_3 is equivalent to h_2 , it branches also over four points. It follows that the quadrilateral Q_3 is elliptic. Similarly, Q_4 is also elliptic, so that f_3 and f_4 are two-fold, see Figure 27. Note that (Q_3, Q_4) is a reducible coupling.

The spaces Z_1^0 and Z_2^0 have the equations

$$w_1 + \mu_1 w_1^{-1} = \zeta_1 z^{m_1}, \quad w_2 + \mu_2 w_2^{-1} = \zeta_2 z^{m_2}$$

If $m_1 = m_2$, then the space W_z is the solution set of the equation

$$w_1 + \mu_1 w_1^{-1} = \frac{\zeta_1}{\zeta_2} (w_2 + \mu_2 w_2^{-1}),$$

which must be irreducible, because the coupling (Q_1, Q_2) is irreducible by assumption. Thus we have $W_z \neq W_u^0$, because the latter is described by an irreducible equation from

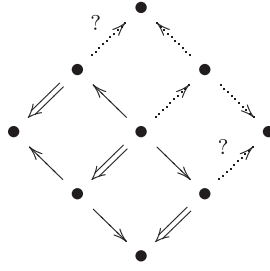


Fig. 28. The diagram in the Case 2.

part (1) of Lemma 5.6. Thus we have $m_1 = -m_2$, that is one of Q_1, Q_2 is a deltoid, and the other an antideltoid.

For $m_1 = -m_2$ the equation of W_z has the form

$$(w_1 + \mu_1 w_1^{-1})(w_2 + \mu_2 w_2^{-1}) = \zeta_1 \zeta_2 \quad (66)$$

At the same time, by Lemma 5.3 W_u^0 has the parametrization

$$w_1 = q_1 F(t), \quad w_2 = q_2' F(t + t_{12}),$$

where $t_{12} = t_1 \pm t_2$ or $t_{12} = t_1 \pm it_2$, and $q_2' = q_2$, respectively, $\frac{p_1 q_2}{p_2}$. If this curve satisfies an equation of the form (66), then the involution $(w_1, w_2) \mapsto (w_1', w_2)$ descends to \mathbb{CP}^1 , and thus t_{12} is a quarter-period of F , by an argument from Section 4.5. If $F = \text{cn}$, then μ_1 and μ_2 in (66) must be purely imaginary, which is not the case. Thus we have $F = \text{sn}$ and $t_{12} = t_1 \pm t_2$. By changing the sign of t_{12} , if needed, we can assume

$$t_{12} = \pm t_1 \pm t_2 \in K\mathbb{Z} + i\frac{K'}{2}$$

It follows that W_u^0 has equation of the form $(z + \lambda z^{-1})(w + \mu w^{-1}) = \nu$ with coefficients given by Lemma 4.17, where (q_1, q_2) must be substituted for (p, q) . By equating the coefficients to those in (66), we obtain conditions described in Section 3.7.6.

6.6.3 A double cover over an (anti)isogram

Without loss of generality, the map h_2 is two-fold, and the quadrilateral Q_2 is an (anti)isogram.

This yields the multiplicities on Figure 28, and we make a case distinction according to the multiplicities of the two question mark maps.

Case 1. Both question mark maps are isomorphisms.

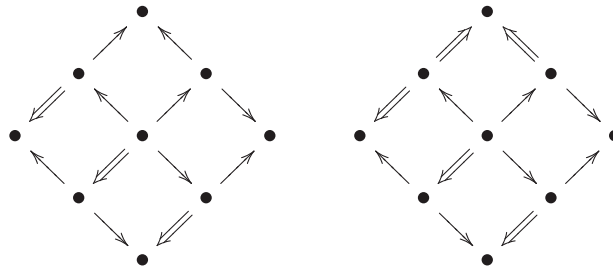


Fig. 29. Cases 1 and 2 of Section 6.6.3.

This leads to the diagram on Figure 29, left. Thus Q_1 and Q_3 are antideltoids, and Q_2 and Q_4 are (anti)isograms. By switching, if needed, the lower and/or the left boundary strips, we can transform Q_1 and Q_3 to deltoids, so that their configuration spaces have equations

$$\begin{aligned} w_1 + \mu_1 w_1^{-1} &= \zeta_1 z^{-1} \\ u + \lambda_3 u^{-1} &= \xi_3 w_2^{-1} \end{aligned} \tag{67}$$

Configuration spaces of (anti)isograms correspond to linear substitutions:

$$u^{\pm 1} = \kappa_4 w_1, \quad z^{\pm 1} = \kappa_2 w_2$$

In order for these substitutions to transform one of the equations (67) to the other, Q_2 must be an antiisogram: $z = \kappa_2 w_2$. But Q_4 may be either isogram or antiisogram (and its type can be changed by switching the upper boundary strip). This leads us to linearly conjugate antideltoids described in Section 3.6.1.

Case 2. One question mark map is two-fold, the other an isomorphism.

Without loss of generality, we obtain the diagram on Figure 29, right.

Then (Q_3, Q_4) is a reducible coupling, and Q_4 is an (anti)deltoid. It follows that Q_3 is conic.

There are two types of reducible couplings between a conic quadrilateral and an (anti)deltoid, see Lemma 5.5. For the second type, equation of an irreducible component Z_{34}^0 has the form (49), which differs from the equation of Z_{12}^0 obtained by a linear substitution in the equation of an (anti)deltoid. Thus Q_3 and Q_4 satisfy the conditions described in part (1) of Lemma 5.5: $m_3 = m_4, p_3 = p_4$.

By Lemma 5.6, the two irreducible components of Z_{34} are described by equations

$$w_1 + \mu_4 e^{\pm 2it_3} w_1^{-1} = \frac{2\varepsilon_4 i \sqrt{-\mu_4} e^{\pm it_3}}{q_2} w_2^{n_3}$$

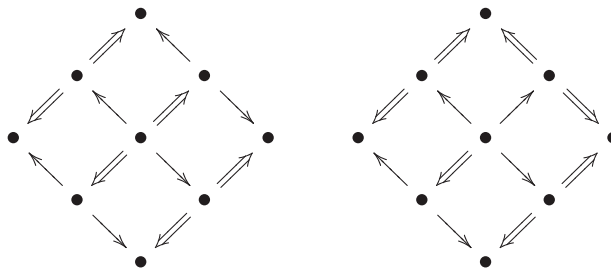


Fig. 30. Subcases of Case 3, Section 6.6.3.

At the same time, the non-trivial irreducible component of Z_{12} is given by a linear substitution $z = cw_2^{\pm 1}$ in

$$w_1 + \mu_1 w_1^{-1} = \zeta_1 z^{m_1}$$

By switching, if needed, the lower and/or the upper boundary strips, we can transform Q_1 and Q_4 to antideltooids: $m_1 = -1, m_3 = m_4 = -1$. By switching the left boundary strip, we achieve $n_3 = -1$, so that Q_3 has perimeter 2π . Then we must have $z = cw_2$, that is Q_2 is an antiisogram.

By performing the substitution, we obtain the conditions on the coefficients described in Section 3.7.7.

Case 3. Both question mark maps are two-fold.

This leaves two possibilities shown on Figure 30.

Case 3a. Figure 30, left.

Quadrilaterals Q_1 and Q_3 are conic or elliptic, Q_2 and Q_4 are (anti)isograms. That is, irreducible components of Z_2 and Z_4 correspond linear substitutions

$$z^{\pm 1} = \kappa_2 w_2, \quad u^{\pm 1} = \kappa_4 w_1 \tag{68}$$

that should transform the equation of Z_1 into the equation of Z_3 (both Z_1 and Z_3 are irreducible curves). It follows that either both Q_1 and Q_3 are conic or both are elliptic.

If Q_1 and Q_3 are conic, then by switching boundary strips we can make them both to have perimeter 2π . This means that the equations of Z_1 and Z_3 have the form (19) with $c_{00} = 0$ and all other coefficients different from 0. Substitutions (68) preserve this property if and only if $\pm 1 = 1$ in both cases, that is, if and only if Q_2 and Q_4 are antiisograms. The substitutions (68) must establish bijections between the branch sets of f_1 and g_3 , and, respectively, those of g_1 and f_3 . This implies that the following

conditions are necessary:

$$q_3 = \pm\kappa_2 p_1, \quad q_1 = \pm\kappa_4 p_3$$

It can be seen that they are also sufficient, provided that the shifts t_1 and t_3 are accordingly related, see Section 3.6.2.

If Q_1 and Q_3 are elliptic, then we use switching to transform Q_2 and Q_4 into antiisograms, so that both exponents in (68) equal 1. Then we have

$$W_z = \left\{ w_1 = q_1 F_1(t), w_2 = \frac{p_1}{\kappa_2} F_1(t + t_1) \right\}$$

$$W_u = \left\{ w_1 = \frac{p_3}{\kappa_4} F_3(t), w_2 = q_3 F_3(t + t_3) \right\}$$

It follows that F_1 and F_3 is the same elliptic function with the same modulus. If $\kappa_2, \kappa_4 > 0$, then we have $W_u = W_z$ if and only if the amplitudes and the shifts in the two parametrizations coincide. (Recall that the amplitudes belong by definition to $\mathbb{R}_+ \cup i\mathbb{R}_+$.) If $\kappa_2 < 0$, then we have

$$W_z = \{w_1 = q_1 F_1(t), w_2 = -\frac{p_1}{\kappa_2} F_1(t + t_1 + 2K)\}$$

Dealing with the other combinations of signs of κ_2 and κ_4 similarly, we obtain conditions described in Section 3.6.3.

Case 3b. Figure 30, right.

Quadrilaterals Q_1, Q_3, Q_4 are conic or elliptic, Q_2 is an (anti)isogram. Both couplings (Q_3, Q_4) and (Q_4, Q_1) are reducible. Therefore either all of these quadrilaterals are conic or all elliptic.

Let us show that in the conic case each of the reducible couplings is of the first of the two types described in Lemma 5.4. Assume the converse; then without loss of generality (Q_3, Q_4) is of the second type. Then the space W_u^0 is the solution set of an irreducible equation of the form (48). The equation of W_z has a different form, since it is obtained by a linear substitution $z^{\pm 1} = \kappa_2 w_2$ in the equation of Z_1 . Hence $W_z \neq W_u^0$. It follows that the couplings (Q_3, Q_4) and (Q_4, Q_1) satisfy $m_3 = m_4, n_4 = n_1$. By switching boundary strips, we can achieve

$$m_i = n_i = -1, \quad \text{for } i = 3, 4, 1$$

It is easy to see that Q_2 must be an antiisogram (cf. Case 1 of this Section). Thus we have

$$W_z = \{w_1^{-1} = q_1 \sin t, w_2^{-1} = \kappa_2 p_1 \sin(t + t_1)\}$$

$$W_u^0 = \{w_1^{-1} = q_4 \sin t, w_2^{-1} = q_3 \sin(t + t_3 \pm t_4)\}$$

If $\kappa_2 > 0$, then the amplitudes must be equal, and the shifts are either equal or opposite. If $\kappa_2 < 0$, then the shift t_1 must be changed by π . We obtain a polyhedron from Section 3.7.8.

If Q_3, Q_4 , and Q_1 are elliptic, then the reducibility of the couplings (Q_3, Q_4) and (Q_4, Q_1) implies that either all three quadrilaterals are of sn-type, have equal moduli and equal amplitudes at common vertices, or all three are of cn-type with equal or conjugate moduli, see Section 5.2.1. Let us show that the moduli cannot be conjugate. Assume the converse. Then, using the parametrizations from Lemma 5.3, one can show that some linear combination of t_1, t_3, t_4 with coefficients $\pm 1, \pm i$, involving at least one ± 1 and at least one $\pm i$, must be a real half-period. But we cannot have, say,

$$t_1 \pm it_3 \pm it_4 \in \{0, 2K\},$$

since $\text{Im } t_1 \in (0, K')$ and $\text{Im}(it_3), \text{Im}(it_4) \in \{0, K', 2K', 3K'\}$. In the case of one $\pm i$ coefficient, one looks at the real parts. Thus all moduli $k_i, i = 3, 4, 1$, are equal.

By switching, the quadrilateral Q_2 can be transformed to an antiisogram: $z = \kappa_2 w_2$. Then we have

$$W_z = \left\{ w_1 = q_1 F(t), w_2 = \frac{p_1}{\kappa_2} F(t + t_1) \right\}$$

$$W_u^0 = \{w_1 = q_4 F(t), w_2 = q_3 F(t + t_3 \pm t_4)\},$$

where $F = \text{sn}(\cdot, k)$ or $F = \text{cn}(\cdot, k)$. We obtain a polyhedron from Section 3.7.9.

6.6.4 All maps h_i are isomorphisms

Then for every \mathbb{CP}^1 on Figure 31 the two covers over it have the same multiplicity. Up to rotation of the diagram, there are six possibilities shown on Figures 32 and 33.

Case 1. There is no doubly covered \mathbb{CP}^1 .

Then all Q_i are (anti)isograms, and non-trivial irreducible components of their configuration spaces are described by the equations

$$z^{\pm 1} = \kappa_1 w_1, \quad z^{\pm 1} = \kappa_2 w_2, \quad u^{\pm 1} = \kappa_3 w_2, \quad u^{\pm 1} = \kappa_4 w_1$$

In order for this system to have a one-parameter set of solutions, the number of $+1$ among the exponents must be even. By switching a boundary strip, we change the exponents

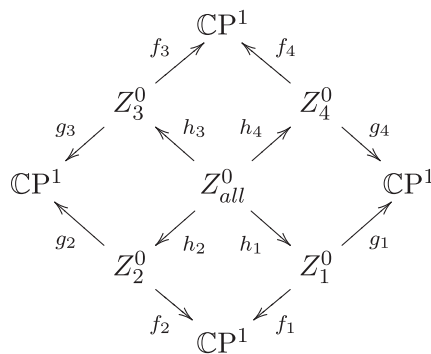


Fig. 31. A diagram of branched covers associated with a deformation without involutive couplings.

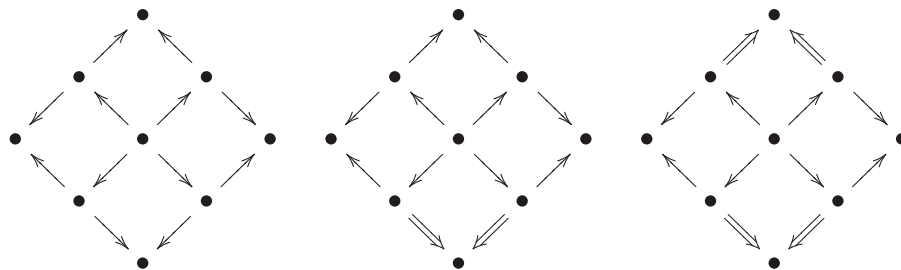


Fig. 32. Cases 1, 2, and 3 of Section 6.6.4.

in two consecutive equations. Therefore we can transform all exponents to +1 and thus all quadrilaterals to antiisograms. As a result, we obtain a polyhedron described in Section 3.2.

Case 2. One doubly covered \mathbb{CP}^1 .

By switching, quadrilaterals Q_3 and Q_4 can be transformed to antiisograms. Then the space W_u^0 has an equation of the form $w_1 = cw_2$. The coupling (Q_1, Q_2) is a reducible coupling of (anti)deltoids. There are two classes of such couplings, see parts (3) and (4) of Lemma 5.5. It is only in the Case 3 that W_z^0 is described by a linear equation, that is, if Q_1 and Q_2 are both deltoids or both antideloids. Thus we obtain a compound of linear couplings from Sections 3.5.1 and 3.5.3.

Case 3. Two doubly covered, non-adjacent \mathbb{CP}^1 .

If Q_1 and Q_2 are both deltoids or both antideloids, then the space W_z^0 is described by a linear equation. Thus Q_3 and Q_4 must also be either both deltoids or both antideloids, and we have a compound of two linear couplings from Section 3.5.3.

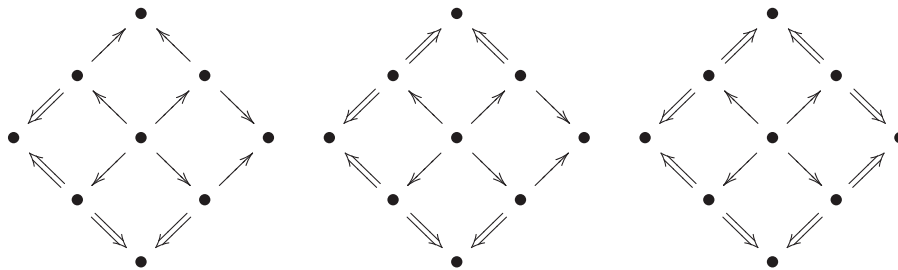


Fig. 33. Cases 4, 5, and 6 of Section 6.6.4.

If Q_1 is a deltoid, and Q_2 an antideltoid (or vice versa, which is related by switching the lower boundary strip), then by the part (6) of Lemma 5.6 the spaces W_z^0 and W_u^0 are described by equations

$$W_z^0 = \{w_1 w_2 \mp \varepsilon_2 \sqrt{-\mu_2} w_1 \pm \varepsilon_1 \sqrt{-\mu_1} w_2 + \varepsilon_1 \varepsilon_2 \sqrt{-\mu_1} \sqrt{-\mu_2} = 0\}$$

$$W_u^0 = \{w_1 w_2 \mp \varepsilon_3 \sqrt{-\mu_3} w_1 \pm \varepsilon_4 \sqrt{-\mu_4} w_2 + \varepsilon_3 \varepsilon_4 \sqrt{-\mu_3} \sqrt{-\mu_4} = 0\}$$

It follows that we must have $\mu_2 = \mu_3$ and $\mu_1 = \mu_4$, that is, the couplings (Q_2, Q_3) and (Q_4, Q_1) are involutive. This contradicts our assumption that the polyhedron contains no involutive couplings.

Case 4. Two doubly covered, adjacent \mathbb{CP}^1 .

Similar to the Case 2 of Section 6.6.3, the reducible coupling between Q_1 and Q_2 , as well as that between Q_2 and Q_3 must have the form described in part (1) of Lemma 5.5. By switching, we can achieve $m_1 = m_2 = -1$ and $n_2 = n_3 = -1$, so that Q_2 has perimeter 2π , and Q_1 and Q_3 are antideltoids.

By Lemma 5.6 the space W_z^0 is described by one of the equations

$$w_1 + \mu_1 e^{\pm 2it_2} w_1^{-1} = \varepsilon_1 \frac{2i\sqrt{-\mu_1} e^{\pm it_2}}{q_2} w_2^{-1}$$

The equation of W_u is obtained by substituting in

$$u + \lambda_3 u^{-1} = \xi_3 w_2^{-1}$$

either $u = \kappa_4 w_1$, if Q_4 is an antiisogram, or $u^{-1} = \kappa_4 w_1$, if Q_4 is an isogram. The equation of W_u has then the same form as equation of W_z^0 . The equality of the coefficients at w_2^{-1} implies the equality of the coefficients at w_1^{-1} due to $q_2 = q_3$. We obtain flexible polyhedra from Sections 3.7.4 and 3.7.5.

Case 5. Three doubly covered \mathbb{CP}^1 .

Then each of (Q_1, Q_2) and (Q_4, Q_3) is a reducible coupling of an (anti)deltoid with a conic quadrilateral. Lemma 5.6 implies that we have either $m_1 = m_2$ and $m_3 = m_4$ or $m_1 = -m_2$ and $m_3 = -m_4$, and furthermore $n_2 = n_3$ and $q_2 = q_3$. Thus by switching we can achieve that Q_2 and Q_3 have perimeter 2π ; then Q_1 and Q_4 are either both deltoids or both antideltoids.

If Q_1 and Q_4 are antideltoids, then we have

$$\begin{aligned} W_z^0 &= \{w_1 = \varepsilon_1 \sqrt{-\mu_1} e^{it}, \quad w_2^{-1} = q_2 \sin(t \pm t_2)\} \\ W_u^0 &= \{w_1 = \varepsilon_4 \sqrt{-\mu_4} e^{it'}, \quad w_2^{-1} = q_3 \sin(t' \pm t_3)\} \end{aligned}$$

The two parametrizations of w_1 differ by a shift: $t' = t + t_0$, where

$$\varepsilon_1 \sqrt{-\mu_1} = \varepsilon_4 \sqrt{-\mu_4} e^{it_0} \tag{69}$$

By substituting $t' = t + t_0$ in the second parametrization of w_2 , we obtain $t_0 = \pm t_2 \pm t_3$. This leads to a polyhedron described in Section 3.7.1.

If Q_1 and Q_4 are deltoids, then we have

$$\begin{aligned} W_z^0 &= \left\{ w_1 = \varepsilon_1 \sqrt{-\mu_1} e^{it}, \quad w_2^{-1} = q_2 \frac{\cos t_2 + i \sin t_2 \cos t}{\sin t} \right\} \\ W_u^0 &= \left\{ w_1 = \varepsilon_4 \sqrt{-\mu_4} e^{it}, \quad w_2^{-1} = q_3 \frac{\cos t_3 + i \sin t_3 \cos t}{\sin t} \right\} \end{aligned}$$

Again, by looking at w_1 we obtain $t' = t + t_0$. But comparing the zeros of w_2 in both parametrizations we arrive at $t_0 \in \{0, \pi\}$. Then (69) implies that $\mu_1 = \mu_4$, that is, the coupling (Q_1, Q_4) is involutive. This contradicts our assumption. (But we obtain a flexible polyhedron which is a compound of couplings from 3.5.2 and 3.5.5.)

Case 6. All \mathbb{CP}^1 are doubly covered.

In this case every coupling must be reducible, hence either all quadrilaterals are elliptic or all of them are conic.

Case 6a. All quadrilaterals are elliptic.

By Lemma 4.10, in the sn case the branch points are either all real or all imaginary, while in the cn case two of them are real, and two imaginary. Hence either all quadrilaterals are of sn type or all of them are of cn type.

In the sn case by Lemma 5.3 we have the following parametrizations of a component $W_z^0 = W_u^0$.

$$W_z^0 = \{w_1 = q_1 \operatorname{sn} t, \quad w_2 = q_2 \operatorname{sn}(t + t_1 \pm t_2)\}$$

$$W_u^0 = \{w_1 = q_4 \operatorname{sn} t', \quad w_2 = q_3 \operatorname{sn}(t' + t_4 \pm t_3)\}$$

The parameters t and t' are either equal or related by $t' = 2K - t$. The second substitution leads to a reparametrization of W_u^0 with $w_2 = q_3 \operatorname{sn}(t - t_4 \mp t_3)$. It follows that the spaces W_z and W_u have a common irreducible component if and only if $t_1 \pm t_2 \pm t_3 \pm t_4$ is a period of sn for some of the eight possible choices of signs. Together with the reducibility conditions $p_1 = p_2, q_2 = q_3, p_3 = p_4, q_4 = q_1$ this leads to the description of the equimodular type of flexible polyhedra given in Section 3.3.1.

Let all quadrilaterals be of cn type. A reducible coupling of elliptic quadrilaterals of cn type can have one of two forms described in Lemma 5.3. If all couplings are of the first type, then the situation is similar to that in the sn case, and the polyhedron is of equimodular type. Otherwise let $k = k_1$ be the Jacobi modulus of Q_1 . Each of the other moduli k_i equals k or $k' = \sqrt{1 - k^2}$. From the parametrization (40) the following necessary condition for flexibility can be deduced:

$$t_1 + \varepsilon_2 t_2 + \varepsilon_3 t_3 + \varepsilon_4 t_4 \in \{0, 2K_1\},$$

where $\varepsilon_i = \pm 1$ if $k_i = k$, and $\varepsilon_i = \pm i$ if $k_i = k'$. By looking at the real or imaginary part of this linear combination, we conclude that exactly two of the moduli k_i must be conjugate to k . Without loss of generality, there are two possibilities:

$$k_1 = k_3 = k, \quad k_2 = k_4 = k' \tag{70a}$$

$$k_1 = k_4 = k, \quad k_2 = k_3 = k' \tag{70b}$$

In both cases we have

$$\frac{p_1}{p_2} = \pm i \frac{k_1}{k_2}, \quad \frac{p_3}{p_4} = \pm i \frac{k_3}{k_4}$$

If (70a) takes place, then we have the parametrizations

$$W_z^0 = \left\{ w_1 = q_1 \operatorname{cn} t, \quad w_2 = \frac{p_1 q_2}{p_2} \operatorname{cn}(t + t_1 \pm it_2) \right\}$$

$$W_u^0 = \left\{ w_1 = \frac{p_3 q_4}{p_4} \operatorname{cn} t', \quad w_2 = q_3 \operatorname{cn}(t' + t_3 \pm it_4) \right\}$$

It follows that $t' = \pm t$ or $t' = \pm t + 2K$. In the former case we have $q_1 = \frac{p_3 q_4}{p_4}$, in the latter $q_1 = -\frac{p_3 q_4}{p_4}$. By comparing the two representations of $w_2(t)$, we arrive at the description given in Section 3.4.1.

If (70b) takes place, then we have $q_1 = q_4$ and $q_2 = q_3$, because the corresponding couplings are reducible and quadrilaterals have equal moduli. The parametrizations are

$$W_z^0 = \left\{ w_1 = q_1 \operatorname{cn} t, \quad w_2 = \frac{p_1 q_2}{p_2} \operatorname{cn}(t + t_1 \pm it_2) \right\}$$

$$W_u^0 = \left\{ w_1 = q_4 \operatorname{cn} t', \quad w_2 = \frac{p_4 q_3}{p_3} \operatorname{cn}(t' + t_4 \pm it_3) \right\}$$

It follows that either $t' = t$ or $t' = -t$, so that we finally arrive at

$$\frac{p_1}{p_2} \operatorname{cn}(t + t_1 \pm it_2) = \frac{p_4}{p_3} \operatorname{cn}(t \pm t_4 \pm it_3)$$

Thus either $\frac{p_1}{p_2} = \frac{p_4}{p_3}$ and $t_1 \pm it_2 \pm it_3 \pm t_4 = 0$ or $\frac{p_1}{p_2} = -\frac{p_4}{p_3}$ and $t_1 \pm it_2 \pm it_3 \pm t_4 = 2K$. This type is described in Section 3.4.2.

Case 6b. All quadrilaterals are conic.

If all couplings are of the type described in part (1) of Lemma 5.4, then we can achieve by switching that $m_i = -1$ and $n_i = -1$ for all i . We have the following parametrizations:

$$W_z^0 = \{w_1 = q_1 \sin t, \quad w_2 = q_2 \sin(t + t_1 \pm t_2)\}$$

$$W_u^0 = \{w_1 = q_4 \sin t', \quad w_2 = q_3 \sin(t' + t_4 \pm t_3)\}$$

As $q_1 = q_4$, we have either $t' = t$ or $t' = \pi - t$. By equating the two expressions for w_2 , we obtain $t_1 \pm t_2 \pm t_3 \pm t_4 = 0 \pmod{2\pi}$. Thus we have a polyhedron of conic equimodular type, see Section 3.3.2.

If the coupling (Q_1, Q_2) is of the second kind, then the space W_z^0 is described by an equation of the form given in part (3) of Lemma 5.6. By comparing this with the equation in part ((2)) of the same Lemma, we conclude that the coupling (Q_3, Q_4) must also be of the second kind. We claim that then (Q_2, Q_3) and (Q_4, Q_1) must be of the first kind. Indeed, if all couplings are of the second kind, then we can achieve by switching $n_1 = n_2 = 1$ and $n_3 = n_4 = -1$. Then the spaces W_z^0 and W_u^0 are described by equations of the form

$$a_{22} w_1^2 w_2^2 + a_{20} w_1^2 + a_{02} w_2^2 + 2a_{11} w_1 w_2 + a_{10} w_1 + a_{01} w_2 + a_{00} = 0$$

$$a'_{00} w_1^2 w_2^2 + a'_{01} w_1^2 w_2 + a'_{10} w_1 w_2^2 + \dots = 0,$$

respectively, where $a_{01} \neq 0$. This contradicts the assumption $W_z^0 = W_u^0$.

Thus the only possibility, up to a cyclic shift of indices, is that (Q_1, Q_2) and (Q_3, Q_4) are of the second kind, while (Q_2, Q_3) and (Q_4, Q_1) are of the first kind. By switching, we can achieve $n_1 = n_4 = -1$ and $n_2 = n_3 = -1$, so that we have the parametrizations

$$W_z^0 = \left\{ w_1^{-1} = q_1 \sin t, \quad w_2 = q_2 \frac{\cos t_2 \pm i \sin t_2 \cos(t + t_1)}{\sin(t + t_1)} \right\}$$

$$W_u^0 = \left\{ w_1^{-1} = q_4 \sin t', \quad w_2 = q_3 \frac{\cos t_3 \pm i \sin t_3 \cos(t' + t_4)}{\sin(t' + t_4)} \right\}$$

Since $q_1 = q_4$, we have either $t' = t$ or $t' = \pi - t$. In the first case, by looking at the zeros of w_2 , we obtain $t_1 = t_4$, so that the coupling (Q_4, Q_1) is involutive that contradicts the assumption of this section. The parameters cannot be related by $t' = \pi - t$, since this would lead to $t_1 = -t_4$, which contradicts $\text{Im } t_i > 0$.

Funding

This work was supported by the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ERC Grant agreement no. 247029-SDModels.

Acknowledgments

The author thanks Wolfgang Schief, Hellmuth Stachel, and Yuri Suris for motivating discussions and the unknown referee for very useful remarks.

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