Quantum Programs as Kleisli Maps

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Furber and Jacobs have shown in their study of quantum computation that the category of commutative C^*-algebras and *PU-maps* (positive linear maps which preserve the unit) is isomorphic to the Kleisli category of a comonad on the category of commutative C^*-algebras with *MIU-maps* (linear maps which preserve multiplication, involution and unit). [3]

In this paper, we prove a non-commutative variant of this result: the category of C^*-algebras and PU-maps is isomorphic to the Kleisli category of a comonad on the subcategory of MIU-maps.

A variation on this result has been used to construct a model of Selinger and Valiron's quantum lambda calculus using von Neumann algebras. [1]

The semantics of a non-deterministic program that takes two bits and returns three bits can be described as a multimap (= binary relation) from $\{0,1\}^2$ to $\{0,1\}^3$. Similarly, a program that takes two qubits and returns three qubits can be modelled as a positive linear unit-preserving map from $\mathrm{M}_2 \otimes \mathrm{M}_2 \otimes \mathrm{M}_2$ to $\mathrm{M}_2 \otimes \mathrm{M}_2$, where M_2 is the C^*-algebra of 2×2-matrices over $\mathbb{C}$.

More generally, the category $\mathbf{Set}_{\mathrm{multi}}$ of multimaps between sets models non-deterministic programs (running on an ordinary computer), while the opposite of the category $\mathbf{C}^*_{\mathrm{PU}}$ of *PU-maps* (positive linear unit-preserving maps) between C^*-algebras models programs running on a quantum computer. (When we write "C^*-algebra" we always mean "C^*-algebra with unit".)

A multimap from $\{0,1\}^2$ to $\{0,1\}^3$ is simply a map from $\{0,1\}^2$ to $\mathscr{P}(\{0,1\}^3)$. In the same line $\mathbf{Set}_{\mathrm{multi}}$ is (isomorphic to) the Kleisli category of the powerset monad $\mathscr{P}$ on $\mathbf{Set}$. What about $\mathbf{C}^*_{\mathrm{PU}}$?

We will show that there is a monad Ω on $(\mathbf{C}^*_{\mathrm{MIU}})^{\mathrm{op}}$, the opposite of the category $\mathbf{C}^*_{\mathrm{MIU}}$ of C^*-algebras and *MIU-maps* (linear maps that preserve the multiplication, involution and unit), such that $(\mathbf{C}^*_{\mathrm{PU}})^{\mathrm{op}}$ is isomorphic to the Kleisli category of Ω. We say that $(\mathbf{C}^*_{\mathrm{PU}})^{\mathrm{op}}$ is *Kleislian* over $(\mathbf{C}^*_{\mathrm{MIU}})^{\mathrm{op}}$. So in the same way we add non-determinism to $\mathbf{Set}$ by the powerset monad $\mathscr{P}$ yielding $\mathbf{Set}_{\mathrm{multi}}$, we can obtain $(\mathbf{C}^*_{\mathrm{PU}})^{\mathrm{op}}$ from $(\mathbf{C}^*_{\mathrm{MIU}})^{\mathrm{op}}$ by a monad Ω.

Let us spend some words on how we obtain this monad Ω. Note that since every positive element of a C^*-algebra $\mathscr{A}$ is of the form a^*a for some $a \in \mathscr{A}$ any MIU-map will be positive. Thus $\mathbf{C}^*_{\mathrm{MIU}}$ is a subcategory of $\mathbf{C}^*_{\mathrm{PU}}$. Let $U\colon\ \mathbf{C}^*_{\mathrm{MIU}} \longrightarrow \mathbf{C}^*_{\mathrm{PU}}$ be the embedding.

In Section 1 we will prove that U has a left adjoint $F\colon\ \mathbf{C}^*_{\mathrm{PU}} \longrightarrow \mathbf{C}^*_{\mathrm{MIU}}$, see Theorem 5. This adjunction gives us a comonad $\Omega := FU$ on $\mathbf{C}^*_{\mathrm{MIU}}$ (which is a monad on $(\mathbf{C}^*_{\mathrm{MIU}})^{\mathrm{op}}$) with the same counit as the adjunction. The comultiplication δ is given by $\delta_{\mathscr{A}} = F\eta_{U\mathscr{A}}$ for every object $\mathscr{A}$ from $\mathbf{C}^*_{\mathrm{MIU}}$ where η is the unit of the adjunction between F and U.

In Section 2 we will prove that $(\mathbf{C}^*_{\mathrm{PU}})^{\mathrm{op}}$ is isomorphic to $\mathscr{K}\ell(FU)$ if FU is considered a monad on $(\mathbf{C}^*_{\mathrm{MIU}})^{\mathrm{op}}$. In fact, we will prove that the *comparison functor* $L\colon\ \mathscr{K}\ell(FU) \longrightarrow (\mathbf{C}^*_{\mathrm{PU}})^{\mathrm{op}}$ (which sends a MIU-map $f\colon\ FU\mathscr{A} \longrightarrow \mathscr{B}$ to $Uf \circ \eta_{U\mathscr{A}}\colon\ U\mathscr{A} \longrightarrow U\mathscr{B}$) is an isomorphism, see Corollary 10.

The method used to show that $(\mathbf{C}^*_{\mathrm{PU}})^{\mathrm{op}}$ is Kleislian over $(\mathbf{C}^*_{\mathrm{MIU}})^{\mathrm{op}}$ is quite general and it will be obvious that many variations on $(\mathbf{C}^*_{\mathrm{PU}})^{\mathrm{op}}$ will be Kleislian over $(\mathbf{C}^*_{\mathrm{MIU}})^{\mathrm{op}}$ as well, such as the opposite of the category of subunital completely positive linear maps between C^*-algebras. The flip-side of this generality is that we discover preciously little about the monad Ω which leaves room for future inquiry (see Section 3).

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We will also see that the opposite $(\mathbf{W}^*_{\mathrm{NCPsU}})^{\mathrm{op}}$ of the category of normal completely positive subunital maps between von Neumann algebras is Kleislian over the subcategory $(\mathbf{W}^*_{\mathrm{NMIU}})^{\mathrm{op}}$ of normal unital $*$-homomorphisms. This fact is used in [1] to construct an adequate model of Selinger and Valiron's quantum lambda calculus using von Neumann algebras.

1 The Left Adjoint

In Theorem 5 we will show that U has a left adjoint, $F\colon \mathbf{C}^*_{\mathrm{MIU}} \to \mathbf{C}^*_{\mathrm{PU}}$, using a quite general method. As a result we do not get any "concrete" information about F in the sense that while we will learn that for every C^*-algebra $\mathscr{A}$ there exists an arrow $\rho\colon \mathscr{A} \to UF\mathscr{A}$ which is initial from $\mathscr{A}$ to U we will learn nothing more about ρ than this. Nevertheless, for some (very) basic C^*-algebras $\mathscr{A}$ we can describe $F\mathscr{A}$ directly, as is shown below in Example 1–3.

Example 1. Let us start easy: $\mathbb{C}$ will be mapped to itself by F, that is:
the identity $\rho\colon \mathbb{C} \longrightarrow U\mathbb{C}$ is an initial arrow from $\mathbb{C}$ to $U(-)$.
Indeed, let $\mathscr{A}$ be a C^*-algebra and let $\sigma\colon \mathbb{C} \to U\mathscr{A}$ be a PU-map. Then σ must be given by $\sigma(\lambda) = \lambda \cdot 1$ for $\lambda \in \mathbb{C}$, where 1 is the identity of $\mathscr{A}$. Thus σ is a MIU-map as well. Hence there is a unique MIU-map $\hat{\sigma}\colon \mathbb{C} \to \mathscr{A}$ (namely $\hat{\sigma} = \sigma$) such that $\hat{\sigma} \circ \rho = \sigma$. ($\mathbb{C}$ is initial in both $\mathbf{C}^*_{\mathrm{MIU}}$ and $\mathbf{C}^*_{\mathrm{PU}}$.)

Example 2. The image of $\mathbb{C}^2$ under F will be the C^*-algebra $C[0,1]$ of continuous functions from $[0,1]$ to $\mathbb{C}$. As will become clear below, this is very much related to the familiar functional calculus for C^*-algebras: given an element a of a C^*-algebra $\mathscr{A}$ with $0 \leq a \leq 1$ and $f \in C[0,1]$ we can make sense of "$f(a)$", as an element of $\mathscr{A}$.
The map $\rho\colon \mathbb{C}^2 \longrightarrow UC[0,1]$ given by, for $\lambda, \mu \in \mathbb{C}$, $x \in [0,1]$,

$$\rho(\lambda,\mu)(x) \;=\; \lambda x + \mu(1-x)$$

is an initial arrow from $\mathbb{C}^2$ to U.
Let $\sigma\colon \mathbb{C}^2 \to U\mathscr{A}$ be a PU-map. We must show that there is a unique MIU-map $\overline{\sigma}\colon C[0,1] \to \mathscr{A}$ such that $\sigma = \overline{\sigma} \circ \rho$.

Writing $a := \sigma(1,0)$, we have $\sigma(\lambda,\mu) = \lambda a + \mu(1-a)$ for all $\lambda, \mu \in \mathbb{C}$. Note that $(0,0) \leq (1,0) \leq (1,1)$ and thus $0 \leq a \leq 1$. Let $C^*(a)$ be the C^*-subalgebra of $\mathscr{A}$ generated by a. Then $C^*(a)$ is commutative since a is positive (and thus normal). Given a MIU-map $\omega\colon C^*(a) \to \mathbb{C}$ we have $\omega(a) \in [0,1]$ since $0 \leq a \leq 1$. Thus $\omega \mapsto \omega(a)$ gives a map $j\colon \Sigma C^*(a) \to [0,1]$, where $\Sigma C^*(a)$ is the spectrum of $C^*(a)$, that is, $\Sigma C^*(a)$ is the set of MIU-maps from $C^*(a)$ to $\mathbb{C}$ with the topology of pointwise convergence. (By the way, the image of j is the spectrum of the *element* a.) The map j is continuous since the topology on $\Sigma C^*(a)$ is induced by the product topology on $\mathbb{C}^{C^*(a)}$. Thus the assignment $h \mapsto h \circ j$ gives a MIU-map $Cj\colon C[0,1] \to C\Sigma C^*(a)$. By Gelfand's representation theorem there is a MIU-isomorphism

$$\gamma\colon C^*(a) \longrightarrow C\Sigma C^*(a)$$

given by $\gamma(b)(\omega) = \omega(b)$ for all $b \in C^*(a)$ and $\omega \in \Sigma C^*(a)$. Now, define

$$\overline{\sigma} := \gamma^{-1} \circ Cj\colon\; C[0,1] \longrightarrow \mathbb{C}^*(a) \hookrightarrow \mathscr{A}.$$

(In the language of the functional calculus, $\overline{\sigma}$ maps f to $f(a)$.) We claim that $\overline{\sigma} \circ \rho = \sigma$. It suffices to show that $Cj \circ \rho \equiv \gamma \circ \overline{\sigma} \circ \rho = \gamma \circ \sigma$. Let $\lambda, \mu \in \mathbb{C}$ and $\omega \in \Sigma C^*(a)$ be given. We have

$$\begin{aligned}
(Cj \circ \rho)(\lambda,\mu)(\omega) &= (Cj)(\rho(\lambda,\mu))(\omega) \\
&= \rho(\lambda,\mu)(\,j(\omega)\,) && \text{by def. of } Cj \\
&= \lambda\, j(\omega) + \mu(1-j(\omega)) && \text{by def. of } \rho \\
&= \lambda\,\omega(a) + \mu(1-\omega(a)) && \text{by def. of } j \\
&= \omega(\,\lambda a + \mu(1-a)\,) && \text{as } \omega \text{ is a MIU-map} \\
&= \omega(\,\sigma(\lambda,\mu)\,) && \text{by choice of } a \\
&= \gamma(\,\sigma(\lambda,\mu)\,)(\omega). && \text{by def. of } \gamma \\
&= (\gamma \circ \sigma)(\lambda,\mu)(\omega).
\end{aligned}$$

It remains to be shown that $\overline{\sigma}$ is the only MIU-map $\tau\colon C[0,1] \to \mathscr{A}$ such that $U\tau \circ \rho = \sigma$. Let τ be such a map; we prove that $\tau = \overline{\sigma}$. By assumption τ and $\overline{\sigma}$ agree on the elements $f \in C[0,1]$ of the form

$$f(x) \;=\; \lambda x + \mu(1-x).$$

In particular, $\overline{\sigma}$ and τ agree on the map $h\colon [0,1] \to \mathbb{C}$ given by $h(x) = x$.

Now, since $\overline{\sigma}$ and τ are MIU-maps and h generates the C^*-algebra $C[0,1]$ (this is Weierstrass's theorem), it follows that $\overline{\sigma} = \tau$.

Example 3. The image of $\mathbb{C}^3$ under F will not be commutative, or more formally:
If $\rho : \mathbb{C}^3 \longrightarrow U\mathscr{B}$ is an initial map from $\mathbb{C}^3$ to U, then $\mathscr{B}$ is not commutative.
Suppose that $\mathscr{B}$ is commutative towards contradiction. Let $\mathscr{A}$ be a C^*-algebra in which there are positive a_1, a_2, a_3 such that $a_1 a_2 \neq a_2 a_1$ and $a_1 + a_2 + a_3 = 1$.

(For example, we can take $\mathscr{A}$ to be the set of linear operators on $\mathbb{C}^2$ and let

$$a_1 \;:=\; 1/2 P_1 \qquad\qquad a_2 \;:=\; 1/2 P_+ \qquad\qquad a_3 \;:=\; I - 1/2 P_1 - 1/2 P_+$$

where P_1 denotes the orthogonal projection onto $\{\,(0,x)\colon x \in \mathbb{C}\,\}$ and P_+ is the orthogonal projection onto $\{\,(x,x)\colon x \in \mathbb{C}\,\}$.)

Define $f\colon \mathbb{C}^3 \to \mathscr{A}$ by, for all $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$,

$$f(\lambda_1,\lambda_2,\lambda_3) \;=\; \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3.$$

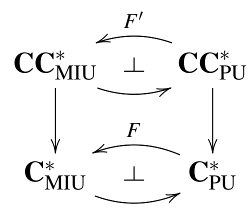
Then it is not hard to see that f a PU-map. So as $\mathscr{B}$ is the initial arrow from $\mathbb{C}^3$ to U there is a (unique) MIU-map $\overline{f}\colon \mathscr{B} \to \mathscr{A}$ such that $\overline{f} \circ \rho = f$. We have

$$\begin{aligned}
a_1 \cdot a_2 &= f(1,0,0) \cdot f(0,1,0) \\
&= \overline{f}(\rho(1,0,0)) \cdot \overline{f}(\rho(0,1,0)) \\
&= \overline{f}(\,\rho(1,0,0) \cdot \rho(0,1,0)\,) \\
&= \overline{f}(\,\rho(0,1,0) \cdot \rho(1,0,0)\,) && \text{because } \mathscr{B} \text{ is commutative} \\
&= \overline{f}(\rho(0,1,0)) \cdot \overline{f}(\rho(1,0,0)) \\
&= a_2 \cdot a_1.
\end{aligned}$$

This contradicts $a_1 \cdot a_2 \neq a_2 \cdot a_1$. Hence $\mathscr{B}$ is not commutative.

Remark 4. Before we prove that the embedding $\mathbf{C}^*_{\text{MIU}} \to \mathbf{C}^*_{\text{PU}}$ has a left adjoint F (see Theorem 5) let us compare what we already know about F with the commutative case. Let $\mathbf{CC}^*_{\text{MIU}}$ denote the category of MIU-maps between commutative C^*-algebras and let $\mathbf{CC}^*_{\text{PU}}$ denote the category of PU-maps between commutative C^*-algebras. From the work in [3] it follows that the embedding $\mathbf{CC}^*_{\text{MIU}} \longrightarrow \mathbf{CC}^*_{\text{PU}}$ has a left adjoint F' and moreover that $F'\mathscr{A} = C\text{Stat}\mathscr{A}$, where $\text{Stat}\mathscr{A}$ is the topological space of PU-maps from $\mathscr{A}$ to $\mathbb{C}$ with pointwise convergence and $C\text{Stat}\mathscr{A}$ is the C^*-algebra of continuous functions from $\text{Stat}\mathscr{A}$ to $\mathbb{C}$.

Let $x \in [0,1]$. Then the assignment $(\lambda,\mu) \mapsto x\lambda + (1-x)\mu$ gives a PU-map $\bar{x}\colon \mathbb{C}^2 \to \mathbb{C}$. It is not hard to see that $x \mapsto \bar{x}$ gives an isomorphism from $[0,1]$ to $\text{Stat}\mathbb{C}^2$. Thus $F'\mathbb{C}^2 \cong C[0,1]$. Hence on $\mathbb{C}^2$ the functor F and its commutative variant F' agree (see Example 2). However, on $\mathbb{C}^3$ the functors F and F' differ. Indeed, $F'\mathbb{C}^3$ is commutative while $F\mathbb{C}^3$ is not (see Example 3).



Roughly summarised: while in the diagram above the right adjoints commute with the vertical embeddings, the left adjoints do not.

Theorem 5. *The embedding* $U\colon \mathbf{C}^*_{\text{MIU}} \longrightarrow \mathbf{C}^*_{\text{PU}}$ *has a left adjoint.*

Proof. By Freyd's Adjoint Functor Theorem (see Theorem V.6.1 of [6]) and the fact that all limits can be formed using only products and equalisers (see Theorem V.2.1 and Exercise V.4.2 of [6]) it suffices to prove the following.

(i) The category $\mathbf{C}^*_{\text{MIU}}$ has all small products and equalisers.

(ii) The functor $U\colon \mathbf{C}^*_{\text{MIU}} \longrightarrow \mathbf{C}^*_{\text{PU}}$ preserves small products and equalisers.

(iii) *Solution Set Condition.* For every C^*-algebra $\mathscr{A}$ there is a set I and for each $i \in I$ a PU-map $f_i\colon \mathscr{A} \to \mathscr{A}_i$ such that for any PU-map $f\colon \mathscr{A} \to \mathscr{B}$ there is an $i \in I$ and a MIU-map $h\colon \mathscr{A}_i \to \mathscr{B}$ such that $h \circ f_i = f$.

Conditions (i) and (ii) can be verified with routine so we will spend only a few words on them (and leave the details to the reader). To see that Condition (iii) holds requires a little more ingenuity and so we will give the proof in detail.

(Conditions (i) and (ii)) Let us first think about small products in $\mathbf{C}^*_{\text{MIU}}$ and $\mathbf{C}^*_{\text{PU}}$.

Let I be a set, and for each $i \in I$ let $\mathscr{A}_i$ be a C^*-algebra.

It is not hard to see that cartesian product $\prod_{i\in I}\mathscr{A}_i$ is a $*$-algebra when endowed with coordinate-wise operations (and it is in fact the product of the $\mathscr{A}_i$ in the category of $*$-algebras with MIU-maps, and with PU-maps).

However, $\prod_{i\in I}\mathscr{A}_i$ cannot be the product of the $\mathscr{A}_i$ as C^*-algebras: there is not even a C^*-norm on $\prod_{i\in I}\mathscr{A}_i$ unless $\mathscr{A}_i$ is trivial for all but finitely many $i \in I$. Indeed, if $\|-\|$ were a C^*-norm on $\prod_{i\in I}\mathscr{A}_i$, then we must have $\|\sigma(i)\| \leq \|\sigma\|$ for all $\sigma \in \prod_{i\in I}\mathscr{A}_i$ and $i \in I$, and so for any sequence $i_0, i_1, \ldots$ of distinct elements of I for which $\mathscr{A}_{i_0}, \mathscr{A}_{i_1}, \ldots$ are non-trivial, and for every $\sigma \in \prod_{i\in I}\mathscr{A}_i$ with $\sigma(i_n) = n\cdot 1$ for all n, we have $n = \|\sigma(i_n)\| \leq \|\sigma\|$ for all n, so $\|\sigma\| = \infty$, which is not allowed.

Nevertheless, the $*$-subalgebra of $\prod_{i\in I}\mathscr{A}_i$ given by

$$\textstyle\bigoplus_{i\in I}\mathscr{A}_i \;:=\; \{\, \sigma \in \prod_{i\in I}\mathscr{A}_i \colon\;\; \sup_{i\in I}\|\sigma(i)\| < +\infty \,\}$$

is a C^*-algebra with norm given by, for $\sigma \in \bigoplus_{i\in I} \mathscr{A}_i$,

$$\|\sigma\| \;=\; \sup_{i\in I} \|\sigma(i)\|.$$

We claim that $\bigoplus_{i\in I} \mathscr{A}_i$ is the product of the $\mathscr{A}_i$ in $\mathbf{C}^*_{\mathrm{PU}}$ (and in $\mathbf{C}^*_{\mathrm{MIU}}$).

Let $\mathscr{C}$ be a C^*-algebra, and for each $i \in I$, let $f_i \colon \mathscr{C} \to \mathscr{A}_i$ be a PU-map. We must show that there is a unique PU-map $f \colon \mathscr{C} \to \bigoplus_{i\in I} \mathscr{A}_i$ such that $\pi_i \circ f = f_i$ for all $i \in I$ where $\pi_i \colon \bigoplus_{j\in I} \mathscr{A}_j \to \mathscr{A}_i$ is the i-th projection. It is clear that there is at most one such f, and it would satisfy for all $i \in I$, and $c \in \mathscr{C}$, $f(c)(i) \;=\; f_i(c)$.

To see that such map f exists is easy if we are able to prove that, for all $c \in \mathscr{C}$,

$$\sup_{i\in I} \|f_i(c)\| \;<\; +\infty. \tag{1}$$

Let $i \in I$ be given. We claim that that $\|f_i(c)\| \leq \|c\|$ for any *positive* $c \in \mathscr{C}$. Indeed, we have $c \leq \|c\| \cdot 1$, and thus $f_i(c) \leq \|c\| \cdot f(1) = \|c\| \cdot 1$, and so $\|f_i(c)\| \leq \|c\|$. It follows that $\|f_i(c)\| \leq 4 \cdot \|c\|$ for any $c \in \mathscr{A}$ by writing $c = c_1 - c_2 + ic_3 - ic_4$ where $c_1, c_2, c_3, c_4 \in \mathscr{C}$ are all positive. (We even have $\|f(c)\| \leq \|c\|$ for all $c \in \mathscr{C}$, but this requires a bit more effort[1]) Thus, we have $\sup_{i\in I} \|f_i(c)\| \leq 4\|c\| < +\infty$. Hence Statement (1) holds.

Thus $\bigoplus_{i\in I} \mathscr{A}_i$ is the product of the $\mathscr{A}_i$ in $\mathbf{C}^*_{\mathrm{PU}}$. It is easy to see that $\bigoplus_{i\in I} \mathscr{A}_i$ is the product of the $\mathscr{A}_i$ in $\mathbf{C}^*_{\mathrm{MIU}}$ as well. Hence $\mathbf{C}^*_{\mathrm{MIU}}$ has all small products (as does $\mathbf{C}^*_{\mathrm{PU}}$) and $U \colon \mathbf{C}^*_{\mathrm{MIU}} \longrightarrow \mathbf{C}^*_{\mathrm{PU}}$ preserves small products.

Let us think about equalisers in $\mathbf{C}^*_{\mathrm{MIU}}$ and $\mathbf{C}^*_{\mathrm{PU}}$. Let $\mathscr{A}$ and $\mathscr{B}$ be C^*-algebras and let $f, g \colon \mathscr{A} \to \mathscr{B}$ be MIU-maps. We must prove that f and g have an equaliser $e \colon \mathscr{E} \to \mathscr{A}$ in $\mathbf{C}^*_{\mathrm{MIU}}$, and that e is the equaliser of f and g in $\mathbf{C}^*_{\mathrm{PU}}$ as well.

Since f and g are MIU-maps (and hence continuous), it is not hard to see that

$$\mathscr{E} \;:=\; \{\, a \in \mathscr{A} \colon\; f(a) = g(a) \,\}$$

is a C^*-subalgebra of $\mathscr{A}$. We claim that the inclusion $e \colon \mathscr{E} \to \mathscr{A}$ is the equaliser of f, g in $\mathbf{C}^*_{\mathrm{PU}}$. Let $\mathscr{D}$ be a C^*-algebra and let $d \colon \mathscr{D} \to \mathscr{A}$ be a PU-map such that $f \circ d = g \circ d$. We must show that there is a unique PU-map $h \colon \mathscr{D} \to \mathscr{E}$ such that $d = e \circ h$. Note that d maps $\mathscr{A}$ into $\mathscr{E}$. The map $h \colon \mathscr{D} \to \mathscr{E}$ is simply the restriction of $d \colon \mathscr{D} \to \mathscr{A}$ in the codomain. Hence e is the equaliser of f, g in $\mathbf{C}^*_{\mathrm{PU}}$.

Note that in the argument above h is a PU-map since d is a PU-map. If d were a MIU-map, then h would be a MIU-map too. Hence e is the equaliser of f, g in the category $\mathbf{C}^*_{\mathrm{MIU}}$ as well.

Hence $\mathbf{C}^*_{\mathrm{MIU}}$ has all equalisers and $U \colon \mathbf{C}^*_{\mathrm{MIU}} \longrightarrow \mathbf{C}^*_{\mathrm{PU}}$ preserves equalisers. Hence $\mathbf{C}^*_{\mathrm{MIU}}$ has all small limits and $U \colon \mathbf{C}^*_{\mathrm{MIU}} \longrightarrow \mathbf{C}^*_{\mathrm{PU}}$ preserves all small limits.

(Note that while we have seen that $\mathbf{C}^*_{\mathrm{PU}}$ has all small products, and it was easy to see that $\mathbf{C}^*_{\mathrm{MIU}}$ has all equalisers, it is not clear whether $\mathbf{C}^*_{\mathrm{PU}}$ has all equalisers. Indeed, if $f, g \colon \mathscr{A} \to \mathscr{B}$ are PU-maps, then the set $\{a \in \mathscr{A} \colon f(a) = g(a)\}$ need not be a C^*-subalgebra of $\mathscr{A}$.)

(Condition (iii)). Let $\mathscr{A}$ be a C^*-algebra. We must find a set I and for each $i \in I$ a PU-map $f_i \colon \mathscr{A} \to \mathscr{A}_i$ such that for every PU-map $f \colon \mathscr{A} \to \mathscr{B}$ there is a (not necessarily unique) $i \in I$ and $h \colon \mathscr{A}_i \to \mathscr{B}$ such that $f = h \circ f_i$.

Note that if $f \colon \mathscr{A} \to \mathscr{B}$ is a PU-map, then the range of the PU-map f need not be a C^*-subalgebra of $\mathscr{B}$. (If the range of PU-maps would have been C^*-algebras, then we could have taken I to be the set of all ideals of $\mathscr{A}$, and $f_J \colon \mathscr{A} \to \mathscr{A}/J$ to be the quotient map for any ideal J of $\mathscr{A}$.)

[1] See Corollary 1 of [7].

Nevertheless, given a PU-map $f\colon \mathscr{A} \to \mathscr{B}$ there is a smallest C^*-subalgebra, say $\mathscr{B}'$, of $\mathscr{B}$ that contains the range of f. **We claim that** $\#\mathscr{B}' \leq \#(\mathscr{A}^{\mathbb{N}})$ where $\#\mathscr{B}'$ is the cardinality of $\mathscr{B}'$ and $\#(\mathscr{A}^{\mathbb{N}})$ is the cardinality of $\mathscr{A}^{\mathbb{N}}$.[2]

If we can find proof for our claim, the rest is easy. Indeed, to begin note that the collection of all C^*-algebras is not a small set. However, given a set U, the collection of all C^*-algebras $\mathscr{C}$ whose elements come from U (so $\mathscr{C} \subseteq U$) is a small set. Now, let $\kappa := \#(\mathscr{A}^{\mathbb{N}})$ be the cardinality of $\mathscr{A}^{\mathbb{N}}$ (so κ is itself a set) and take

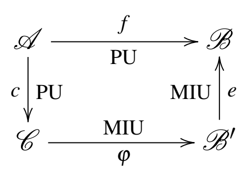
$$I \;:=\; \{\, (\mathscr{C},c)\colon\; \mathscr{C} \text{ is a } C^*\text{-algebra on a subset of } \kappa \text{ and } c\colon \mathscr{A} \to \mathscr{C} \text{ is a PU-map} \,\}.$$

Since the collection of C^*-algebras $\mathscr{C}$ with $\mathscr{C} \subseteq \kappa$ is small, and since the collection of PU-maps from $\mathscr{A}$ to $\mathscr{C}$ is small for any C^*-algebra $\mathscr{C}$, it follows that I is small.

For each $i \in I$ with $i \equiv (\mathscr{C},c)$ define $\mathscr{A}_i := \mathscr{C}$ and $f_i := c$.

Let $f\colon \mathscr{A} \to \mathscr{B}$ be a PU-map. We must find $i \in I$ and a MIU-map $h\colon \mathscr{A}_i \to \mathscr{B}$ such that $h \circ f_i = f$. Let $\mathscr{B}'$ be the smallest C^*-subalgebra that contains the range of f. By our claim we have $\#\mathscr{B}' \leq \#(\mathscr{A}^{\mathbb{N}}) \equiv \kappa$. By renaming the elements of $\mathscr{B}'$ we can find a C^*-algebra $\mathscr{C}$ isomorphic to $\mathscr{B}'$ whose elements come from κ. Let $\varphi\colon \mathscr{C} \to \mathscr{B}'$ be the isomorphism.

Note that $c := \varphi^{-1} \circ f\colon \mathscr{A} \to \mathscr{C}$ is a PU-map. So we have $i := (\mathscr{C},c) \in I$. Further, the inclusion $e\colon \mathscr{B}' \to \mathscr{B}$ is a MIU-map, as is φ. So we have:



Now, $h := e \circ \varphi\colon \mathscr{C} \to \mathscr{B}$ is a MIU-map with $f = h \circ f_i$. Hence Cond. (iii) holds.

Let us proof our claim. Let $\mathscr{A}$ and $\mathscr{B}$ be C^*-algebras and let $f\colon \mathscr{A} \to \mathscr{B}$ be a PU-map. Let $\mathscr{B}'$ be the smallest C^*-subalgebra that contains the range of f.
We must show that $\#\mathscr{B}' \leq \#(\mathscr{A}^{\mathbb{N}})$.

Let us first take care of pathological case. Note that if $\mathscr{A}$ is trivial, i.e. $\mathscr{A} = \{0\}$, then $\mathscr{B}' = \{0\}$, so $\#(\mathscr{A}^{\mathbb{N}}) = 1 = \#\mathscr{B}'$. Now, let us assume that $\mathscr{A}$ is not trivial. Then we have an injection $\mathbb{C} \to \mathscr{A}$ given by $\lambda \mapsto \lambda \cdot 1$, and thus $\#\mathbb{C} \leq \#\mathscr{A}$.

The trick to prove $\#\mathscr{B}' \leq \#(\mathscr{A}^{\mathbb{N}})$ is to find a more explicit description of $\mathscr{B}'$. Let T be the set of terms formed using a unary operation $(-)^*$ (involution) and two binary operations, $\cdot$ (multiplication) and $+$ (addition), starting from the elements of $\mathscr{A}$. Let $f_T\colon T \longrightarrow \mathscr{B}'$ be the map (recursively) given by, for $a \in \mathscr{A}$, and $s,t \in T$,

$$\begin{aligned} f_T(a) &= f(a) \\ f_T(s^*) &= (f_T(s))^* \\ f_T(s \cdot t) &= f_T(s) \cdot f_T(t) \\ f_T(s+t) &= f_T(s) + f_T(t). \end{aligned}$$

[2] Although it has no bearing on the validity of the proof one might wonder if the simpler statement $\#\mathscr{B}' \leq \#\mathscr{A}$ holds as well. Indeed, if $\#\mathscr{A} = \#\mathbb{C}$ or $\#\mathscr{A} = \#(2^X)$ for some infinite set X, then we have $\#\mathscr{A} = \#(\mathscr{A}^{\mathbb{N}})$, and so $\#\mathscr{B}' \leq \#\mathscr{A}$. However, not every uncountable set is of the form 2^X for some infinite set X, and in fact, if $\#\mathscr{A} = \aleph_\omega$, then $\#(\mathscr{A}^{\mathbb{N}}) > \#\mathscr{A}$ by Corollary 3.9.6 of [2]

Note that the range of f_B, let us call it $\mathrm{Ran} f_B$, is a $*$-subalgebra of $\mathscr{B}'$. We will prove that $\#\mathrm{Ran} f_B \leq \#\mathscr{A}$. Since f_B is a surjection of T onto $\mathrm{Ran} f_B$ it suffices to prove that $\#T \leq \#\mathscr{A}$. In fact, we will show that $\#T = \#\mathscr{A}$.

First note that $\mathscr{A}$ is infinite, and $\mathscr{A} \subseteq T$, so T is infinite as well. To prove that $\#T = \#\mathscr{A}$ we write the elements of T as words (with the use of brackets). Indeed, with $Q := \mathscr{A} \cup \{``\cdot", ``+", ``*", ``)", ``("\}$ there is an obvious injection from T into the set Q^* of words over Q. Since $\mathscr{A}$ is infinite, and $Q \backslash \mathscr{A}$ is finite we have $\#Q = \#\mathscr{A}$ by Hilbert's hotel. Recall that $Q^* = \bigcup_{n=0}^{\infty} Q^n$. Since Q is infinite, we also have $\#(\mathbb{N} \times Q) = \#Q$ and even $\#(Q \times Q) = \#Q$ (see Theorem 3.7.7 of [2]), so $\#Q = \#(Q^n)$ for all $n > 0$. It follows that

$$\begin{aligned} \#(Q^*) &= \#(\textstyle\bigcup_{n=0}^{\infty} Q^n) \\ &= \#(1 + \textstyle\bigcup_{n=1}^{\infty} Q) \\ &= \#(1 + \mathbb{N} \times Q) \\ &= \#Q. \end{aligned}$$

Since there is an injection from T to Q^* we have $\#\mathscr{A} \leq \#T \leq \#(Q^*) = \#Q = \#\mathscr{A}$ and so $\#T = \#\mathscr{A}$. Hence $\#\mathrm{Ran} f_B \leq \#\mathscr{A}$.

Since $\mathrm{Ran} f_B$ is a $*$-algebra that contains $\mathrm{Ran} f$, the closure $\overline{\mathrm{Ran} f_B}$ of $\mathrm{Ran} f_B$ with respect to the norm on $\mathscr{B}'$ is a C^*-algebra that contains $\mathrm{Ran} f$. As $\mathscr{B}'$ is the smallest C^*-subalgebra that contains $\mathrm{Ran} f$, we see that $\mathscr{B}' = \overline{\mathrm{Ran} f_B}$.

Let S be the set of all Cauchy sequences in $\mathrm{Ran} f_B$. As every point in $\mathscr{B}'$ is the limit of a Cauchy sequence in $\mathrm{Ran} f_B$, we get $\#\mathscr{B}' \leq \#S$. Thus:

$$\begin{aligned} \#\mathscr{B}' &\leq \#S \\ &\leq \#(\mathrm{Ran} f_B)^{\mathbb{N}} && \text{as } S \subseteq (\mathrm{Ran} f_B)^{\mathbb{N}} \\ &\leq \#(\mathscr{A}^{\mathbb{N}}) && \text{as } \#\mathrm{Ran} f_B \leq \#\mathscr{A}. \end{aligned}$$

Thus we have proven our claim.

Hence Conditions (i)–(iii) hold and $U\colon \mathbf{C}^*_{\mathrm{MIU}} \longrightarrow \mathbf{C}^*_{\mathrm{PU}}$ has a left adjoint. □

We have seen that $U\colon \mathbf{C}^*_{\mathrm{MIU}} \longrightarrow \mathbf{C}^*_{\mathrm{PU}}$ has a left adjoint $F\colon \mathbf{C}^*_{\mathrm{PU}} \longrightarrow \mathbf{C}^*_{\mathrm{MIU}}$. This adjunction gives a comonad FU on $\mathbf{C}^*_{\mathrm{MIU}}$, which in turns gives us two categories: the Eilenberg–Moore category $\mathscr{E}\mathscr{M}(FU)$ of FU-coalgebras and the Kleisli category $\mathscr{K}\ell(FU)$. We claim that $\mathbf{C}^*_{\mathrm{PU}}$ is isomorphic to $\mathscr{K}\ell(FU)$ since $\mathbf{C}^*_{\mathrm{MIU}}$ is a subcategory of $\mathbf{C}^*_{\mathrm{PU}}$ with the same objects.

This is a special case of a more general phenomenon which we discuss in the next section (in terms of monads instead of comonads), see Theorem 9.

2 Kleislian Adjunctions

Beck's Theorem (see [6], VI.7) gives a criterion for when an adjunction $F \dashv U$ "is" an adjunction between $\mathbf{C}$ and $\mathscr{E}\mathscr{M}(UF)$. We give a similar (but easier) criterion for when an adjunction "is" an adjunction between $\mathbf{C}$ and $\mathscr{K}\ell(UF)$. The criterion is not new; e.g., it is mentioned in [5] (paragraph 8.6) without proof or reference, and it can be seen as a consequence of Exercise VI.5.2 of [6] (if one realises that an equivalence which is bijective on objects is an isomorphism). Proofs can be found in the appendix.

Notation 6. *Let $F\colon \mathbf{C} \longrightarrow \mathbf{D}$ be a functor with right adjoint U. Denote the unit of the adjunction by $\eta\colon \mathrm{id}_{\mathbf{D}} \to UF$, and the counit by $\varepsilon\colon FU \to \mathrm{id}_{\mathbf{C}}$.*

Recall that UF is a monad with unit η and as multiplication, for C from $\mathbf{C}$,

$$\mu_C \;:=\; U\varepsilon_{FC}\colon\; UFUFC \longrightarrow UFC.$$

Let $\mathscr{K}\ell(UF)$ be the Kleisli category of the monad UF. So $\mathscr{K}\ell(UF)$ has the same objects as $\mathbf{C}$, and the morphisms in $\mathscr{K}\ell(UF)$ from C_1 to C_2 are the morphism in $\mathbf{C}$ from C_1 to UFC_2. Given C from $\mathbf{C}$ the identity in $\mathscr{K}\ell(UF)$ on C is η_C. If C_1, C_2, C_3, $f\colon C_1 \to C_2$, $g\colon C_2 \to C_3$ from $\mathbf{C}$ are given, g after f in $\mathscr{K}\ell(UF)$ is

$$g \odot f \;:=\; \mu_{C_3} \circ UFg \circ f.$$

Let $V\colon \mathbf{C} \longrightarrow \mathscr{K}\ell(UF)$ be given by, for $f\colon C_1 \longrightarrow C_2$ from $\mathbf{C}$,

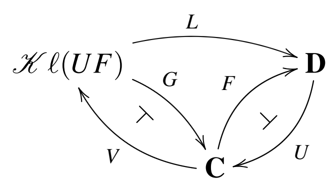
$$Vf \;:=\; \eta_{C_2} \circ f\colon \quad C_1 \longrightarrow UFC_2.$$

Let $G\colon \mathscr{K}\ell(UF) \longrightarrow \mathbf{C}$ be given by, for $f\colon C_1 \longrightarrow UFC_2$ from $\mathbf{C}$,

$$Gf \;:=\; \mu_{C_2} \circ UFf\colon \quad UFC_1 \longrightarrow UFC_2.$$

The following is Exercise VI.5.1 of [6].

Lemma 7. *Let $F\colon \mathbf{C} \longrightarrow \mathbf{D}$ be a functor with a right adjoint U. Then there is a unique functor $L\colon \mathscr{K}\ell(UF) \longrightarrow \mathbf{D}$ (called the* comparison functor*) such that $U \circ L = G$ and $L \circ V = F$ (see Notation 6).*



Definition 8. Let $\mathbf{C}$ and $\mathbf{D}$ be categories.

(i) A functor $F\colon \mathbf{C} \longrightarrow \mathbf{D}$ is called *Kleislian* when it has a right adjoint $U\colon \mathbf{D} \to \mathbf{C}$, and the functor $L\colon \mathscr{K}\ell(UF) \longrightarrow \mathbf{D}$ from Lemma 7 is an isomorphism.

(ii) We say that $\mathbf{D}$ *is Kleislian over* $\mathbf{C}$ when there is a Kleislian functor $F\colon \mathbf{C} \longrightarrow \mathbf{D}$.

Theorem 9. *Let $F\colon \mathbf{C} \longrightarrow \mathbf{D}$ be a functor with a right adjoint U. The following are equivalent.*

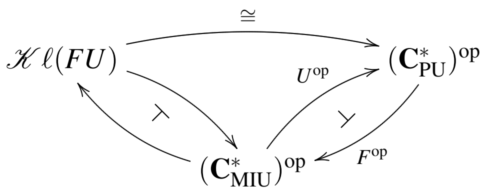
(i) F is Kleislian (see Definition 8).

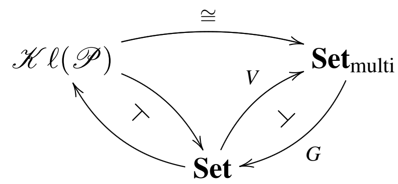
(ii) F is bijective on objects (i.e. for every object D from $\mathbf{D}$ there is a unique object C from $\mathbf{C}$ such that $FC = D$).

Corollary 10. *The embedding $U^{\mathrm{op}}\colon (\mathbf{C}^*_{\mathrm{MIU}})^{\mathrm{op}} \longrightarrow (\mathbf{C}^*_{\mathrm{PU}})^{\mathrm{op}}$ is Kleislian (see Def. 8).*

Proof. By Theorem 9 we must show that U^{op} has a left adjoint and is bijective on objects. Since the embedding $U\colon \mathbf{C}^*_{\mathrm{MIU}} \to \mathbf{C}^*_{\mathrm{PU}}$ has a *left* adjoint $F\colon \mathbf{C}^*_{\mathrm{PU}} \to \mathbf{C}^*_{\mathrm{MIU}}$ it follows that $F^{\mathrm{op}}\colon (\mathbf{C}^*_{\mathrm{PU}})^{\mathrm{op}} \to (\mathbf{C}^*_{\mathrm{MIU}})^{\mathrm{op}}$ is the *right* adjoint of U^{op}. Thus U^{op} has a left adjoint. Further, as $\mathbf{C}^*_{\mathrm{MIU}}$ and $\mathbf{C}^*_{\mathrm{PU}}$ have the same objects, U is bijective on objects, and so is U^{op}. Hence U^{op} is Kleislian. □

In summary, the embedding $U\colon \mathbf{C}^*_{\mathrm{MIU}} \longrightarrow \mathbf{C}^*_{\mathrm{PU}}$ has a left adjoint F (and so $F^{\mathrm{op}}\colon (\mathbf{C}^*_{\mathrm{MIU}})^{\mathrm{op}} \to (\mathbf{C}^*_{\mathrm{PU}})^{\mathrm{op}}$ is *right* adjoint to U^{op}), and the unique functor from the Kleisli category $\mathscr{K}\ell(FU)$ of the monad FU on $(\mathbf{C}^*_{\mathrm{MIU}})^{\mathrm{op}}$ to $(\mathbf{C}^*_{\mathrm{PU}})^{\mathrm{op}}$ that makes the two triangles in the diagram below on the left commute is an isomorphism.





For the category $\mathbf{Set}_{\mathrm{multi}}$ of multimaps between sets used in the introduction to describe the semantics of non-deterministic programs the situation is the same, see the diagram above to the right.

(The functor V is the obvious embedding. The right adjoint G of V sends a multimap f from X to Y to the function $Gf\colon \mathscr{P}(X) \to \mathscr{P}(Y)$ that assigns to a subset $A \in \mathscr{P}(X)$ the image of A under f. Note that $GV = \mathscr{P}$.)

3 Discussion

3.1 Variations

Example 11 (Subunital maps)**.** Let $\mathbf{C}^*_{\mathrm{PsU}}$ be the category of C^*-algebras and the positive linear maps f between them that are *subunitial*, i.e. $f(1) \leq 1$. The morphisms of $\mathbf{C}^*_{\mathrm{PsU}}$ are called *PsU-maps*.

It is not hard to see that the products in $\mathbf{C}^*_{\mathrm{PsU}}$ are the same as in $\mathbf{C}^*_{\mathrm{MIU}}$, and that the equaliser in $\mathbf{C}^*_{\mathrm{MIU}}$ of a pair f, g of MIU-maps is the equaliser of f, g in $\mathbf{C}^*_{\mathrm{PsU}}$ as well. Thus the embedding $U\colon \mathbf{C}^*_{\mathrm{MIU}} \longrightarrow \mathbf{C}^*_{\mathrm{PsU}}$ preserves limits. Using the same argument as in Theorem 5 but with "PU-map" replaced by "PsU-map" one can show that U satisfies the Solution Set Condition. Hence U has a left adjoint by Freyd's Adjoint Function Theorem, say $F\colon \mathbf{C}^*_{\mathrm{PsU}} \longrightarrow \mathbf{C}^*_{\mathrm{MIU}}$.

Since $\mathbf{C}^*_{\mathrm{PsU}}$ has the same objects as $\mathbf{C}^*_{\mathrm{MIU}}$ (namely the C^*-algebras) the functor $U^{\mathrm{op}}\colon (\mathbf{C}^*_{\mathrm{MIU}})^{\mathrm{op}} \longrightarrow (\mathbf{C}^*_{\mathrm{PsU}})^{\mathrm{op}}$ is bijective on objects and thus Kleislian (by Th. 9).

Hence $(\mathbf{C}^*_{\mathrm{PsU}})^{\mathrm{op}}$ is Kleislian over $(\mathbf{C}^*_{\mathrm{MIU}})^{\mathrm{op}}$.

Example 12 (Bounded linear maps)**.** Let $\mathbf{C}^*_{\mathrm{P}}$ be the category of positive bounded linear maps between C^*-algebras. We will show that $(\mathbf{C}^*_{\mathrm{P}})^{\mathrm{op}}$ is *not* Kleislian over $(\mathbf{C}^*_{\mathrm{MIU}})^{\mathrm{op}}$. Indeed, if it were then $(\mathbf{C}^*_{\mathrm{P}})^{\mathrm{op}}$ would be cocomplete, but it is not: there is no ω-fold product of $\mathbb{C}$ in $\mathbf{C}^*_{\mathrm{P}}$. To see this, suppose that there is a ω-fold product $\mathscr{P}$ in $\mathbf{C}^*_{\mathrm{P}}$ with projections $\pi_i\colon \mathscr{P} \to \mathbb{C}$ for $i \in \omega$. Since π_i is a bounded linear map for $i \in \omega$, it has finite operator norm, say $\|\pi_i\|$. By symmetry, $\|\pi_i\| = \|\pi_j\|$ for all $i, j \in \omega$. Write $K := \|\pi_0\| = \|\pi_1\| = \|\pi_2\| = \cdots$. Define $f_i\colon \mathbb{C} \to \mathbb{C}$ by $f_i(z) = iz$ for all $z \in \mathbb{C}$ and $i \in \omega$. Then f_i is a positive bounded linear map for each $i \in \omega$. Since $\mathscr{P}$ is the ω-fold product of $\mathbb{C}$, there is a (unique positive) bounded linear map $f\colon \mathbb{C} \to \mathscr{P}$ such that $\pi_i \circ f = f_i$ for all $i \in \omega$. For each $N \in \omega$ we have

$$N \;=\; \|f_N(1)\| \;\leq\; \|f_N\| \;=\; \|\pi_N \circ f\| \;\leq\; \|\pi_N\|\,\|f\| = K\,\|f\|.$$

Thus $K\|f\|$ is greater than any number, which is absurd.

Example 13 (Completely positive maps)**.** For clarity's sake we recall what it means for a linear map f between C^*-algebras to be completely positive (see [8]). For this we need some notation. Given a C^*-algebra $\mathscr{A}$, and $n \in \mathbb{N}$ let $M_n(\mathscr{A})$ denote the set of $n \times n$-matrices with entries from $\mathscr{A}$. We leave it to the reader to check that $M_n(\mathscr{A})$ is a $*$-algebra with the obvious operations. In fact, it turns out that $M_n(\mathscr{A})$ is a C^*-algebra, but some care must be taken to define the norm on $M_n(\mathscr{A})$ as we will see below. Now, a linear map $f\colon \mathscr{A} \longrightarrow \mathscr{B}$ is called *completely positive* when $M_n f$ is positive for each $n \in \mathbb{N}$, where $M_n f\colon M_n(\mathscr{A}) \longrightarrow M_n(\mathscr{B})$ is the map obtained by applying f to each entry of a matrix in $M_n(\mathscr{A})$. Of course, "$M_n f$ is positive" only makes sense once we know that $M_n(\mathscr{A})$ and $M_n(\mathscr{B})$ are C^*-algebras.

Let $\mathscr{A}$ be a C^*-algebra. We will put a C^*-norm on $M_n(\mathscr{A})$. Let $\mathscr{H}$ be a Hilbert space and let $\pi\colon \mathscr{A} \longrightarrow \mathscr{B}(\mathscr{H})$, be an isometric MIU-map. We get a norm $\|-\|_\pi$ on $M_n(\mathscr{A})$ given by for $A \in M_n(\mathscr{A})$,

$$\|A\|_\pi \;=\; \|\xi(\,(M_n\pi)(A)\,)\|,$$

where $\xi(\,(M_n\pi)(A)\,)\colon \mathscr{H}^{\oplus n} \to \mathscr{H}^{\oplus n}$ is the bounded linear map represented by the matrix $(M_n\pi)(A)$, and $\|\xi(\,(M_n\pi)(A)\,)\|$ is the operator norm of $\xi(\,(M_n\pi)(A)\,)$ in $\mathscr{B}(\mathscr{H}^{\oplus n})$.

It is easy to see that $\|-\|_\pi$ satisfies the C^*-identity, $\|A^*A\|_\pi = \|A\|_\pi^2$ for all $A \in M_n(\mathscr{A})$. It is less obvious that $M_n(\mathscr{A})$ is complete with respect to $\|-\|_\pi$. To see this, first note that $\|A_{ij}\| \leq \|A\|_\pi$ for all i,j. So given a Cauchy sequence $A_1, A_2, \dots$ in $M_n(\mathscr{A})$ we can form the entrywise limit A, that is, $A_{ij} = \lim_{m\to\infty} A_{ij}$. We leave it to the reader to check that A_{ij} is the limit of $A_1, A_2, \dots$, and thus $M_n(\mathscr{A})$ is complete with respect to $\|-\|_\pi$. Hence $M_n(\mathscr{A})$ is a C^*-algebra with norm $\|-\|_\pi$.

The C^*-norm $\|-\|_\pi$ does not depend on π. Indeed, let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces and let $\pi_1\colon \mathscr{A} \longrightarrow \mathscr{B}(\mathscr{H}_1)$ and $\pi_2\colon \mathscr{A} \longrightarrow \mathscr{B}(\mathscr{H}_2)$ be isometric MIU-maps; we will show that $\|-\|_{\pi_1} = \|-\|_{\pi_2}$. Recall that the norm $\|-\|_{\pi_i}$ induces an order $\leq_{\pi_i}$ on $M_n(\mathscr{A})$ given by $0 \leq_{\pi_i} A$ iff $\|A - \|A\|_{\pi_i}\|_{\pi_i} \leq \|A\|_{\pi_i}$ where $A \in M_n(\mathscr{A})$. Since $\|A\|_{\pi_i}^2 = \inf\{\, \lambda \in [0,\infty)\colon\; A^*A \leq_{\pi_i} \lambda \,\}$ for all $A \in M_n(\mathscr{A})$, to prove $\|-\|_{\pi_1} = \|-\|_{\pi_2}$ it suffices to show that the orders $\leq_{\pi_1}$ and $\leq_{\pi_2}$ coincide. But this is easy when one recalls that $A \in M_n(\mathscr{A})$ is positive iff A is of the form B^*B for some $B \in M_n(\mathscr{A})$.

The completely positive linear maps that preserve the unit are called *CPU-maps*. Let $\mathbf{C}^*_{\mathrm{CPU}}$ be the category of CPU-maps between C^*-algebras. Since $M_n(f)$ is a MIU-map when f is a MIU-map and a MIU-map is positive, we see that any MIU-map is completely positive. Thus $\mathbf{C}^*_{\mathrm{MIU}}$ is a subcategory of $\mathbf{C}^*_{\mathrm{CPU}}$. We claim that $(\mathbf{C}^*_{\mathrm{CPU}})^{\mathrm{op}}$ is Kleislian over $(\mathbf{C}^*_{\mathrm{MIU}})^{\mathrm{op}}$.

Let us show that U preserves limits. To show that U preserves equalisers, let $f,g\colon \mathscr{A} \longrightarrow \mathscr{B}$ be MIU-maps. Then $\mathscr{E} := \{x \in \mathscr{A}\colon\, f(x) = g(x)\}$ is a C^*-subalgebra of $\mathscr{A}$ and the embedding $e\colon \mathscr{E} \to \mathscr{A}$ is an isometric MIU-map. Then e is the equalisers of f,g in $\mathbf{C}^*_{\mathrm{MIU}}$; we will show that e is the equaliser of f,g in $\mathbf{C}^*_{\mathrm{CPU}}$. Let $\mathscr{C}$ be a C^*-algebra, and let $c\colon \mathscr{C} \to \mathscr{A}$ be a CPU-map such that $f \circ c = g \circ c$ Let $d\colon \mathscr{C} \to \mathscr{E}$ be the restriction of c. It turns out we must prove that d is completely positive. Let $n \in \mathbb{N}$ be given. We must show that $M_n d\colon M_n\mathscr{C} \to M_n\mathscr{E}$ is positive. Note that $M_n e$ is an injective MIU-map and thus an isometry. So in order to prove that $M_n d$ is positive it suffices to show that $M_n e \circ M_n d = M_n(e \circ d) = M_n c$ is positive, which it is since c is completely positive. Thus e is the equaliser of f,g in $\mathbf{C}^*_{\mathrm{CPU}}$. Hence U preservers equalisers.

To show that U preserves products, let I be a set and for each $i \in I$ let $\mathscr{A}_i$ be a C^*-algebra. We will show that $\bigoplus_{i\in I} \mathscr{A}_i$ is the product of the $\mathscr{A}_i$ in $\mathbf{C}^*_{\mathrm{CPU}}$. Let $\mathscr{C}$ be a C^*-algebra, and for each $i \in I$, let $f_i\colon \mathscr{C} \to \mathscr{A}_i$ be a CPU-map. As before, let $f\colon \mathscr{C} \to \bigoplus_{i\in I} A_i$ be the map given by $f(x)(i) = f_i(x)$ for all $i \in I$ and $x \in \mathscr{C}$. Leaving the details to the reader it turns out that it suffices to show that f is completely positive. Let $n \in \mathbb{N}$ be given. We must prove that $M_n f\colon M_n(\mathscr{C}) \longrightarrow M_n(\bigoplus_{i\in I} \mathscr{A}_i)$ is positive. Let $\varphi\colon M_n(\bigoplus_{i\in I} \mathscr{A}_i) \longrightarrow \bigoplus_{i\in I} M_n(\mathscr{A}_i)$ be the unique MIU-map such that $\pi_i \circ \varphi = M_n\pi_i$ for all $i \in I$. Then φ is a MIU-isomorphism and thus to prove that $M_n f$ is positive, it suffices to show that $\varphi \circ M_n f$ is positive. Let $i \in I$ be given. We must prove that $\pi_i \circ \varphi \circ M_n f$ is positive. But we have $\pi_i \circ \varphi \circ M_n f = M_n\pi_i \circ M_n f = M_n(\pi_i \circ f) = M_n f_i$, which is positive since f is completely positive. Thus $\bigoplus_{i\in I} \mathscr{A}_i$ is the product of the $\mathscr{A}_i$ in $\mathbf{C}^*_{\mathrm{CPU}}$ and hence U preserves limits.

With the same argument as in Theorem 9 the functor U satisfies the Solution Set Condition and thus U has a left adjoint. It follows that $U^{\mathrm{op}}\colon (\mathbf{C}^*_{\mathrm{MIU}})^{\mathrm{op}} \longrightarrow (\mathbf{C}^*_{\mathrm{CPU}})^{\mathrm{op}}$ is Kleislian.

Example 14 (W^*-algebras)**.** Let $\mathbf{W}^*_{\mathrm{NMIU}}$ be the category of von Neumann algebras (also called W^*-algebras) and the MIU-maps between them that are normal, i.e., preserve suprema of upwards directed sets of self-adjoint elements. Let $\mathbf{W}^*_{\mathrm{NPU}}$ be the category of von Neumann and normal PU-maps. Note that $\mathbf{W}^*_{\mathrm{NMIU}}$ is a subcategory of $\mathbf{W}^*_{\mathrm{NPU}}$. We will prove that $(\mathbf{W}^*_{\mathrm{NPU}})^{\mathrm{op}}$ is Kleislian over $(\mathbf{W}^*_{\mathrm{NMIU}})^{\mathrm{op}}$.

It suffices to show that U has a left adjoint. Again we follow the lines of the proof of Theorem 5. Products and equalisers in $\mathbf{W}^*_{\mathrm{NMIU}}$ are the same as in $\mathbf{C}^*_{\mathrm{MIU}}$. It is not hard to see that the embedding $U\colon \mathbf{W}^*_{\mathrm{NMIU}} \longrightarrow \mathbf{W}^*_{\mathrm{NPU}}$ preserves limits. To see that U satisfies the Solution Set Condition we use the same method as before: given a von Neumann algebra $\mathscr{A}$, find a suitable cardinal κ such that the following is a solution set.

$$I \;:=\; \{\, (\mathscr{C},c)\colon\; \mathscr{C} \text{ is a von Neumann algebra on a subset of } \kappa \\ \text{and } c\colon \mathscr{A} \longrightarrow \mathscr{C} \text{ is a normal PU-map} \,\},$$

Only this time we take $\kappa = \#(\,\wp(\wp(\mathscr{A}))\,)$ instead of $\kappa = \#(\,\mathscr{A}^{\mathbb{N}}\,)$. We leave the details to the reader, but it follows from the fact that given a subset X of a von Neumann algebra $\mathscr{B}$ the smallest von Neumann subalgebra $\mathscr{B}'$ that contains X has cardinality at most $\#(\,\wp(\wp(X))\,)$. Indeed, if $\mathscr{H}$ is a Hilbert space such that $\mathscr{B} \subseteq \mathscr{B}(\mathscr{H})$ (perhaps after renaming the elements of $\mathscr{B}$), then $\mathscr{B}'$ is the closure (in the weak operator topology on $\mathscr{B}(\mathscr{H})$) of the smallest $*$-subalgebra containing X. Thus any element of $\mathscr{B}'$ is the limit of a filter — a special type of net, see paragraph 12 of [9] — of $*$-algebra terms over X, of which there are no more than $\#(\,\wp(\wp(X))\,)$.

By a similar reasoning one sees that the opposite $(\mathbf{W}^*_{\mathrm{NCPsU}})^{\mathrm{op}}$ of the category of normal completely positive subunital linear maps between von Neumann algebras is Kleislian over $(\mathbf{W}^*_{\mathrm{NMIU}})^{\mathrm{op}}$. The existence of the adjoint to the inclusion $\mathbf{W}^*_{\mathrm{NMIU}} \to \mathbf{W}^*_{\mathrm{NCPsU}}$ is key in our construction of a model of Selinger and Valiron's quantum lambda calculus by von Neumann algebras, see [1].

3.2 Concrete description

In this note we have shown that the embedding $U\colon \mathbf{C}^*_{\mathrm{MIU}} \longrightarrow \mathbf{C}^*_{\mathrm{PU}}$ has a left adjoint F, but we miss a concrete description of $F\mathscr{A}$ for all but the simplest C^*-algebras $\mathscr{A}$. What constitutes a "concrete description" is perhaps a matter of taste or occasion, but let us pose that it should at least enable us to describe the Eilenberg–Moore category $\mathscr{E}\mathscr{M}(FU)$ of the comonad FU. More concretely, it should settle the following problem.

Problem 15. *Writing* **BOUS** *for the category of positive linear maps that preserve the unit between Banach order unit spaces, determine whether* $\mathscr{E}\mathscr{M}(FU) \cong \mathbf{BOUS}$.

(An order unit space *is an ordered vector space V over $\mathbb{R}$ with an element* 1*, the* order unit*, such that for all $v \in V$ there is $\lambda \in [0,\infty)$ such that $-\lambda \cdot 1 \leq v \leq \lambda \cdot 1$. The smallest such λ is denoted by $\|v\|$. See [4] for more details. If $v \mapsto \|v\|$ gives a complete norm, V is called a* Banach order unit space*.)*

3.3 MIU versus PU

A second "problem" is to give a physical description (if there is any) of what it means for a quantum program's semantics to be a MIU-map (and not just a PU-map). A step in this direction might be to define for a C^*-algebra $\mathscr{A}$, a PU-map $\varphi\colon \mathscr{A} \to \mathbb{C}$, and $a,b \in \mathscr{A}$ the quantity

$$\mathrm{Cov}_{\varphi}(a,b) \;:=\; \varphi(a^*b) - \varphi(a)^*\varphi(b)$$

and interpret it as the covariance between the observables a and b in state φ of the quantum system $\mathscr{A}$. Let $T\colon \mathscr{A} \longrightarrow \mathscr{B}$ be a PU-map between C^*-algebras (so perhaps T is the semantics of a quantum program). Then it is not hard to verify that T is a MIU-map if and only if T preserves covariance, that is,

$$\mathrm{Cov}_\varphi(Ta,Tb) \;=\; \mathrm{Cov}_{\varphi\circ T}(a,b) \qquad \text{for all } a,b\in\mathscr{A}.$$

4 Acknowledgements

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A Additional Proofs

Proof of Lemma 7. Define $LC := FC$ for all objects C of $\mathscr{K}\ell(UF)$ and

$$Lf \;:=\; \varepsilon_{FC_2}\circ Ff$$

for $f\colon C_1 \longrightarrow UFC_2$ from $\mathbf{C}$. We claim this gives a functor $L\colon \mathscr{K}\ell(UF)\longrightarrow \mathbf{D}$.

(L preserves the identity) Let C be an object of $\mathscr{K}\ell(UF)$, that is, an object of $\mathbf{C}$. Then the identity on C in $\mathscr{K}\ell(UF)$ is η_C. We have $L(\eta_C) = \varepsilon_{FC}\circ F\eta_C = \mathrm{id}_{FC}$.

(L preserves composition) Let $f\colon C_1 \longrightarrow UFC_2$ and $g\colon C_2 \longrightarrow UFC_3$ from $\mathbf{C}$ be given. We must prove that $L(g \odot f) = Lg \circ Lf$. We have:

$$\begin{aligned} L(g \odot f) &= L(\mu_{C_3} \circ UFg \circ f) && \text{by def. of } g \odot f \\ &= \varepsilon_{FC_3} \circ F\mu_{C_3} \circ FUFg \circ Ff && \text{by def. of } L \\ &= \varepsilon_{FC_3} \circ FU\varepsilon_{FC_3} \circ FUFg \circ Ff && \text{by def. of } \mu_{C_3} \\ &= \varepsilon_{FC_3} \circ Fg \circ \varepsilon_{FC_2} \circ Ff && \text{by nat. of } \eta \\ &= Lg \circ Lf && \text{by def. of } L \end{aligned}$$

Hence L is a functor from $\mathscr{K}\ell(UF)$ to $\mathbf{D}$.

Let us prove that $U \circ L = G$. For $f\colon C_1 \longrightarrow UFC_2$ from $\mathbf{C}$ we have

$$\begin{aligned} ULf &= U(\varepsilon_{FC_2} \circ Ff) && \text{by def. of } L \\ &= U\varepsilon_{FC_2} \circ UFf && \\ &= \mu_{C_2} \circ UFf && \text{by def. of } \mu_{C_2} \\ &= Gf && \text{by def. of } Gf. \end{aligned}$$

Let us prove that $L \circ V = F$. For $f\colon C_1 \longrightarrow C_2$ from $\mathbf{C}$ be given, we have

$$\begin{aligned} LVf &= L(\eta_{C_2} \circ f) && \text{by def. of } V \\ &= \varepsilon_{FC_2} \circ F\eta_{C_2} \circ Ff && \text{by def. of } L \\ &= Ff && \text{by counit–unit eq.} \end{aligned}$$

We have proven that there is a functor $L\colon \mathscr{K}\ell(UF) \to \mathbf{D}$ such that $U \circ L = G$ and $L \circ V = F$. We must still prove that it is as such unique.

Let $L'\colon \mathscr{K}\ell(UF) \to \mathbf{D}$ be a functor such that $U \circ L' = G$ and $L' \circ V = F$. We must show that $L = L'$. Let us first prove that L' and L agree on objects. Let C be an object of $\mathscr{K}\ell(UF)$, i.e., C is an object of $\mathbf{C}$. Since $L' \circ V = F$ and $VC = C$ we have $L'C = L'VC = FC = LC$. Now, let $f\colon C_1 \to UFC_2$ from $\mathbf{C}$ be given (so f is a morphism in $\mathscr{K}\ell(UF)$ from C_1 to C_2). We must show that $L'f = LU \equiv \varepsilon_{FC_2} \circ Ff$. Note that since F is the left adjoint of U there is a unique morphism $\overline{f}\colon FC_1 \longrightarrow FC_2$ in $\mathbf{D}$ such that $U\overline{f} \circ \eta_{C_1} = f$. To prove that $L'f = Lf$, we show that both Lf and $L'f$ have this property. We have

$$\begin{aligned} UL'f \circ \eta_{C_1} &= Gf \circ \eta_{C_1} && \text{as } U \circ L' = G \text{ by assump.} \\ &= \mu_{C_2} \circ UFf \circ \eta_{C_1} && \text{by def. of } G \\ &= \mu_{C_2} \circ \eta_{UFC_2} \circ f && \text{by nat. of } \eta \\ &= f && \text{as } UF \text{ is a monad.} \end{aligned}$$

By a similar argument we get $ULf \circ \eta_{C_1} = f$. Hence $Lf = L'f$. □

Proof of Theorem 9. We use the symbols from Notation 6.

(i)$\Longrightarrow$ (ii) Suppose that L is an isomorphism. We must prove that F is bijective on objects. Note that $F = L \circ V$, so it suffices to show that both L and V are bijective on objects. Clearly, L is bijective on objects as L is an isomorphism, and $V\colon \mathbf{C} \longrightarrow \mathscr{K}\ell(UF)$ is bijective on objects since the objects of $\mathscr{K}\ell(UF)$ are those of $\mathbf{C}$ and $VC = C$ for all C from $\mathbf{C}$.

(ii)$\Longrightarrow$ (i) Suppose that (ii) holds. We prove that L is an isomorphism by giving its inverse. Let D be an object from $\mathbf{D}$. Note that since F is bijective on objects there is a unique object C from $\mathbf{C}$ such that $FD = C$. Define $KC := D$.

Let $g\colon D_1 \to D_2$ from $\mathbf{D}$ be given. Note that by definition of K we have:

$$KD_1 \xrightarrow{\eta_{KD_1}} UFKD_1 =\!=\!= UD_1 \xrightarrow{Ug} UD_2 =\!=\!= UFKD_2$$

Now, define $Kg\colon KD_1 \to UFKD_2$ in $\mathbf{D}$ by $Kg := Ug \circ \eta_{KD_1}$.

We claim that this gives a functor $K\colon \mathbf{D} \longrightarrow \mathscr{K}\ell(UF)$.

(K preserves the identity) For an object D of $\mathbf{D}$ we have

$$K\mathrm{id}_D = U\mathrm{id}_D \circ \eta_{KD} = \eta_{KD},$$

and η_{KD} is the identity on KD in $\mathscr{K}\ell(UF)$.

(K preserves composition) Let $f\colon D_1 \longrightarrow D_2$ and $g\colon D_2 \longrightarrow D_3$ from $\mathbf{D}$ be given. We must prove that $K(g \circ f) = K(g) \odot K(f)$. We have

$$\begin{aligned}
K(g) \odot K(f) &= \mu_{KD_3} \circ UFKg \circ Kf && \text{by def. of } \odot \\
&= \mu_{KD_3} \circ UFUg \circ UF\eta_{KD_2} \circ Uf \circ \eta_{KD_1} && \text{by def. of } K \\
&= U\varepsilon_{D_3} \circ UFUg \circ UF\eta_{KD_2} \circ Uf \circ \eta_{KD_1} && \text{by def. of } \mu \\
&= Ug \circ U\varepsilon_{D_2} \circ UF\eta_{KD_2} \circ Uf \circ \eta_{KD_1} && \text{by nat. of } \varepsilon \\
&= Ug \circ Uf \circ \eta_{KD_1} && \text{by counit–unit eq.} \\
&= K(g \circ f) && \text{by def of } K.
\end{aligned}$$

Hence K is a functor from $\mathbf{D}$ to $\mathscr{K}\ell(UF)$. We will show that K is the inverse of L. For this we must prove that $K \circ L = \mathrm{id}_{\mathbf{D}}$ and $L \circ K = \mathrm{id}_{\mathscr{K}\ell(UF)}$.

For a morphism $g\colon D_1 \longrightarrow D_2$ from $\mathbf{D}$, we have

$$\begin{aligned}
LKg &= L(Ug \circ \eta_{KD_1}) && \text{by def. of } K \\
&= \varepsilon_{FKD_2} \circ FUg \circ F\eta_{KD_1} && \text{by def. of } L \\
&= g \circ \varepsilon_{FKD_1} \circ F\eta_{KD_1} && \text{by nat. of } \varepsilon \\
&= g && \text{by counit–unit eq.}
\end{aligned}$$

For a morphism $f\colon C_1 \longrightarrow UFC_2$ in $\mathbf{C}$ we have

$$\begin{aligned}
KLf &= K(\varepsilon_{FC_2} \circ Ff) && \text{by def. of } L \\
KLfdd &= U\varepsilon_{FC_2} \circ UFf \circ \eta_{KFC_1} && \text{by def. of } K \\
&= U\varepsilon_{FC_2} \circ \eta_{UFC_2} \circ f && \text{by nat. of } \eta \\
&= f && \text{by counit–unit eq.}
\end{aligned}$$

Hence K is the inverse of L, so L is an isomorphism. □