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Farkas-type results for vector-valued functions with applications

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Abstract

The main purpose of this paper consists of providing characterizations of the inclusion of the solution set of a given conic system posed in a real locally convex topological space into a variety of subsets of the same space defined by means of vector-valued functions. These Farkas-type results are used to derive characterizations of the weak solutions of vector optimization problems (including multiobjective and scalar ones), vector variational inequalities, and vector equilibrium problems.

1 Introduction

In this paper we consider an optimization problem posed in a real locally convex Hausdorff topological vector space (lcHtvs in short) X, called *space of decisions*, with a vector-valued objective function f to be minimized on a feasible set $\emptyset \neq A \subset X$ with respect to a given *weak* partial ordering on a second lcHtvs Y, called *space of criteria*, enlarged with a smallest element $-\infty_Y$ and a greatest element $+\infty_Y$. The *weak* ordering on the extended space of criteria $Y^\bullet := Y \cup \{-\infty_Y, +\infty_Y\}$ is defined from a given pointed convex cone with nonempty interior $K \subset Y$ and the task "minimize" consists of computing the *weak infimum* of the set $f(A)$ in the sense of [25, p. 93] (see also [3, p. 366]). Particular cases of this vector optimization problem are the multiobjective problem, where $Y = \mathbb{R}^p$ and $K = \mathbb{R}^p_+$, with $p \geq 2$, and the scalar optimization problem, where $Y = \mathbb{R}$ and $K = \mathbb{R}_+$.

Different reasons for using a weak ordering in vector optimization are pointed out by many authors. From [16, p. 1421] we quote the following sentence: "The advantages and disadvantages of the different concepts [of solutions] are severely discussed among experts. Efficient solutions are usually motivated by applications and weakly or properly efficient solutions are motivated to be beneficial for the theory and sometimes easier to calculate". In particular, in multiobjective optimization they are characterized and computed by means of scalarization (assigning weights to the different objectives). Moreover, weak orders are

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essential in the construction of a complete lattice, giving rise to a conjugate duality approach for set-valued optimization problems which is considerably close to the conjugate duality for scalar optimization problems (see, for instance, [3, p. 360]). Conjugate maps and Farkas-type results are crucial in any duality theory, and this is why they constitute the main tool and the main objective, respectively, of our research. The state of the art in vector optimization is described, e.g., in [3], [16], [18], [23], and references therein.

The Farkas-type results are well-known basic theoretical tools in scalar optimization. The classical Farkas lemma [15] characterizes the containment of a polyhedral convex cone A into a given half-space whose boundary contains the origin. The non-homogeneous version of this famous result [24] characterizes the containment of a polyhedral convex set into a given half-space and was used in the mid 1900s to provide simple proofs of the duality theorem of linear programming and the KKT optimality theorem of non-linear programming. Since then, many Farkas-type results have been proposed to characterize the inclusion of a give set A, described by some kind of system, into another set B, typically the solution set of a single inequality, in order to obtain optimality and duality theorems in different frameworks (see, e.g. the survey papers [8], [19], [20] and references therein). A Farkas-type result is called *asymptotic* whenever the characterization of $A \subset B$ involves the closure of certain sets, it is called P_A / P_B whenever P_A and P_B are properties satisfied by A and B, e.g., convexity, non-convexity or being the inverse image by some function of finitely many complements of convex sets (reverse-convexity in brief). In particular, each convex / reverse-convex non-asymptotic Farkas' lemma provides a different optimality theorem of the KKT-type.

The objective of this paper is to provide Farkas-type results for vector optimization and to show that, like their scalar counterparts in scalar optimization, these results have interesting applications in vector optimization and other fields.

Section 2 contains the necessary preliminaries on epigraphical calculus with scalar functions, calculus rules for the extrema of sets in the sense of [25], and the definitions of conjugate and subdifferential of vector-valued maps. In Section 3 we characterize the inclusion of $A = \{x \in C : g(x) \in -S\}$, where $\emptyset \neq C \subset X$, S is a convex cone in Z (a third lcHtvs), and $g : X \to Z$, into the reverse-convex set $B := Y \diagdown (-\operatorname{int} K)$. Since A is generally non-convex and the characterizations of $A \subset B$ include closures, Theorems 3.1 and 3.2 are asymptotic non-convex / reverse-convex Farkas' lemmas. From these two main results we obtain asymptotic non-convex / linear and convex / reverse-convex Farkas' lemmas as well as a stable convex / reverse-convex Farkas' lemma, where the term *stability* means that the inclusion $A \subset B$ is preserved by arbitrary linear perturbations of the convex function defining the reverse-convex set B. Section 4 provides reverse and non-asymptotic Farkas-type results, stable or not, under alternative qualification conditions involving the data. Section 5 is devoted to the characterization of the weak solutions of vector optimization problems, paying attention to some particular types (scalar and multiobjective, constrained and unconstrained) of optimization problems. Finally, Section 6 provides applications to vector variational inequalities and vector equilibrium problems.

2 Preliminaries

Let X be a lcHtvs, whose origin is denoted by 0_X, and with topological dual space represented by X^*. The only topology we consider on dual spaces is the weak*-topology. For a set $U \subset X$, we denote by $\operatorname{cl} U$, $\operatorname{co} U$, and $\operatorname{cl} \operatorname{co} U$ the *closure*, the *convex hull*, and the *closed convex hull* of

U, respectively. Note that $\operatorname{cl}\operatorname{co} U = \operatorname{cl}(\operatorname{co}U)$. We assume that all the cones under consideration contain the origin of the corresponding space.

Given $f : X \to \mathbb{R} \cup \{\pm\infty\}$, the *epigraph* of f is the set

$$\operatorname{epi} f := \{(x, r) \in X \times \mathbb{R} : \; x \in \operatorname{dom} f, \; f(x) \leq r\},$$

where $\operatorname{dom} f := \{x \in X : \; f(x) \neq +\infty\}$. The function is said to be *proper* if $\operatorname{epi} f \neq \emptyset$ and $-\infty \notin f(X)$, it is *convex* if $\operatorname{epi} f$ is convex, and it is *lower semicontinuous* (*lsc*, in brief) if $\operatorname{epi} f$ is closed. We denote by $\Gamma(X)$ the class of lsc proper convex functions on X.

The *Legendre-Fenchel conjugate* of f is the weak*-lsc convex function $f^* : X^* \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ defined by

$$f^*(x^*) = \sup_{x \in X} (\langle x^*, x\rangle - f(x)), \; \forall x^* \in X^*. \tag{2.1}$$

Let $f_1, f_2 \in \Gamma(X)$ be such that $(\operatorname{dom} f_1) \cap (\operatorname{dom} f_2) \neq \emptyset$. Then

$$\operatorname{epi}(f_1 + f_2)^* = \operatorname{cl}(\operatorname{epi} f_1^* + \operatorname{epi} f_2^*), \tag{2.2}$$

and, if one of these functions is continuous at a point in the intersection of their domains, we actually have [26, (2.63)]

$$\operatorname{epi}(f_1 + f_2)^* = \operatorname{epi}\; f_1^* + \operatorname{epi}\; f_2^*. \tag{2.3}$$

Let $\{f_i, \; i \in I\} \subset \Gamma(X)$, where I is an arbitrary index set, and suppose that there exists $x_0 \in X$ such that $\sup\limits_{i \in I} f_i(x_0) < +\infty$. Then one has [22, Lemma 2.2]

$$\operatorname{epi}\left(\sup_{i \in I} f_i\right)^* = \operatorname{cl}\operatorname{co}\left(\bigcup_{i \in I} \operatorname{epi}\; f_i^*\right). \tag{2.4}$$

Now we extend the above concepts to vector-valued functions as it is done in [3] and [25]. Let Y be a second lcHtvs, with origin 0_Y and topological dual space Y^*. Let K be a nonempty, closed and pointed convex cone in Y with nonempty interior, i.e., $\operatorname{int} K \neq \emptyset$. We now define a *weak ordering* in Y, associated with $\operatorname{int} K$, in the following way:

$$y_1 <_K y_2 \text{ if and only if } y_1 - y_2 \in -\operatorname{int} K.$$

Equivalently, $y_1 \not<_K y_2$ if and only if $y_1 - y_2 \notin -\operatorname{int} K$.

We enlarge Y by attaching a *greatest element* $+\infty_Y$ and a *smallest element* $-\infty_Y$ with respect to $<_K$, which do not belong to Y, and we denote $Y^\bullet := Y \cup \{-\infty_Y, \; +\infty_Y\}$. By convention, $-\infty_Y <_K y$ and $y <_K (+\infty_Y)$ for any $y \in Y$. We also assume by convention that

$$\begin{aligned}
-(+\infty_Y) &= -\infty_Y, \; -(-\infty_Y) = +\infty_Y, \\
(+\infty_Y) + y &= y + (+\infty_Y) = +\infty_Y, \text{ for all } y \in Y \cup \{+\infty_Y\}, \\
(-\infty_Y) + y &= y + (-\infty_Y) = -\infty_Y, \text{ for all } y \in Y \cup \{-\infty_Y\}.
\end{aligned} \tag{2.5}$$

The sums $(-\infty_Y) + (+\infty_Y)$ and $(+\infty_Y) + (-\infty_Y)$ are not considered in this paper.

Given a vector-valued mapping $f : X \to Y^\bullet$, the *domain* of f is defined by

$$\operatorname{dom} f := \{x \in X : \; f(x) \neq +\infty_Y\},$$

and f is *proper* when $\operatorname{dom} f \neq \emptyset$ and $-\infty_Y \notin f(X)$. The $K-$*epigraph* of f, denoted by $\operatorname{epi}_K f$, is defined by

$$\operatorname{epi}_K f = \{(x,y) \in X \times Y : \ y \in f(x) + K\}.$$

Moreover, we say that f is $K-$*epi closed* when $\operatorname{epi}_K f$ is a closed set in the product space, and also that f is $K-$*convex*, i.e., $\operatorname{epi}_K f$ is a convex set (equivalently, if for any $x_1, x_2 \in X$ and $\mu \in [0,1]$ one has $f(\mu x_1 + (1-\mu)x_2) - \mu f(x_1) - (1-\mu) f(x_2) \in -K$).

Given $M \subset Y^\bullet$ we shall recall the following definitions (e.g. [3, Definition 7.4.1]):

Definition 2.1 *(a) An element $\bar{v} \in Y^\bullet$ is said to be a **weakly infimal element** of M if for all $v \in M$ we have $v \not<_K \bar{v}$ and if for any $\widetilde{v} \in Y^\bullet$ such that $\bar{v} <_K \widetilde{v}$, then there exists some $v \in M$ satisfying $v <_K \widetilde{v}$. The set of all weakly infimal elements of M is denoted by* $\operatorname{WInf} M$ *and it is called the **weak infimum** of M.*
*(b) An element $\bar{v} \in Y^\bullet$ is said to be a **weakly supremal element** of M if for all $v \in M$ we have $\bar{v} \not<_K v$, and if for any $\widetilde{v} \in Y^\bullet$ such that $\widetilde{v} <_K \bar{v}$, then there exists some $v \in M$ satisfying $\widetilde{v} <_K v$. The set of all weakly supremal elements of M is denoted by* $\operatorname{WSup} M$ *and it is called the **weak supremum** of M.*

Definition 2.2 *(a) The **weak minimum** of M is the set*

$$\operatorname{WMin} M = M \cap \operatorname{WInf} M,$$

*and its elements are the **weakly minimal elements** of M.*
*(b) The **weak maximum** of M is the set*

$$\operatorname{WMax} M = M \cap \operatorname{WSup} M,$$

*and its elements are the **weakly maximal elements** of M.*

Remark 2.1 *(a) For any $M \subset Y$, thanks to the conventions $-\infty_Y <_K y$ and $y <_K (+\infty_Y)$ for any $y \in Y$, it is easy to check that* $\operatorname{WInf} M = \operatorname{WInf}(M \cup \{+\infty_Y\})$, $\operatorname{WMin} M = \operatorname{WMin}(M \cup \{+\infty_Y\})$, $\operatorname{WSup} M = \operatorname{WSup}(M \cup \{-\infty_Y\})$ *and* $\operatorname{WMax} M = \operatorname{WMax}(M \cup \{-\infty_Y\})$.
(b) If $M \neq \emptyset$, since $v <_K (+\infty_Y)$ for all $v \in M$, $+\infty_Y \notin$ $\operatorname{WInf} M$ *(similarly, $-\infty_Y \notin$* $\operatorname{WSup} M$*). Otherwise, if $M = \emptyset$, and according to the definition,* $\operatorname{WInf} M = \{+\infty_Y\}$ *and* $\operatorname{WSup} M = \{-\infty_Y\}$ *[3, Remark 7.4.1].*
(c) $(Y^\bullet, <_K)$ turns out to be a complete lattice.
(d) If $M \neq \emptyset$, it is easy to see that if $\operatorname{WSup} M \neq \{+\infty_Y\}$ *then* $\bar{v} \in \operatorname{WSup} M$ *if and only if* $\bar{v} \in Y \setminus (M - \operatorname{int} K)$ *and* $\bar{v} - \operatorname{int} K \subset M - \operatorname{int} K$ *(similarly, if* $\operatorname{WInf} M \neq \{-\infty_Y\}$ *then* $\bar{v} \in \operatorname{WInf} M$ *if and only if* $\bar{v} \in Y \setminus (M + \operatorname{int} K)$ *and* $\bar{v} + \operatorname{int} K \subset M + \operatorname{int} K$).

Recall (e.g., from [1, Lemma 5.3]) that, given two nonempty sets $N, V \subset Y$ such that V is open, one has

$$\operatorname{cl} N + V = N + V. \tag{2.6}$$

If K is a nonempty convex closed cone in Y with nonempty interior, i.e., $\operatorname{int} K \neq \emptyset$, taking in (2.6) $N = V = \operatorname{int} K$, we get

$$K + \operatorname{int} K = \operatorname{int} K, \tag{2.7}$$

and consequently,

$$\left.\begin{array}{c} y \in K \\ y + y' \notin \operatorname{int} K \end{array}\right\} \Longrightarrow y' \notin \operatorname{int} K. \tag{2.8}$$

Definition 2.3 *Given $\emptyset \neq M \subset Y^\bullet$, we define the set $\mathcal{A}(M)$ of all* ***points above*** *M, and the set $\mathcal{B}(M)$ of all* ***points below*** *M by*

$$\mathcal{A}(M) = \{\bar{v} \in Y^\bullet : v <_K \bar{v} \text{ for some } v \in M\}$$

and

$$\mathcal{B}(M) = \{\bar{v} \in Y^\bullet : \bar{v} <_K v \text{ for some } v \in M\}\,.$$

Remark 2.2 *(a) One has*

$$\mathcal{A}(M) = \begin{cases} Y \cup \{+\infty_Y\}\,, & \text{if} -\infty_Y \in M, \\ \emptyset, & \text{if } M = \{+\infty_Y\}\,, \\ \{+\infty_Y\} \cup (M + \operatorname{int} K)\,, & \text{otherwise,} \end{cases} \tag{2.9}$$

and

$$\mathcal{B}(M) = \begin{cases} Y \cup \{-\infty_Y\}\,, & \text{if} +\infty_Y \in M, \\ \emptyset, & \text{if } M = \{-\infty_Y\}\,, \\ \{-\infty_Y\} \cup (M - \operatorname{int} K)\,, & \text{otherwise.} \end{cases} \tag{2.10}$$

(b) In particular,

$$-\infty_Y \notin M \neq \{+\infty_Y\} \implies \mathcal{A}(M) = \{+\infty_Y\} \cup (M + \operatorname{int} K), \tag{2.11}$$

$$+\infty_Y \notin M \neq \{-\infty_Y\} \implies \mathcal{B}(M) = \{-\infty_Y\} \cup (M - \operatorname{int} K)\,. \tag{2.12}$$

(c) Moreover, it is easy to check that

$$\begin{aligned} \operatorname{WSup} M = \{+\infty_Y\} &\iff \mathcal{B}(M) = Y \cup \{-\infty_Y\} \\ &\iff +\infty_Y \in M \text{ or } M - \operatorname{int} K = Y \\ &\iff \big(\forall v \in Y,\ \exists \widetilde{v} \in M \text{ such that } v <_K \widetilde{v}\big). \end{aligned} \tag{2.13}$$

Analogously, we can characterize the case $\operatorname{WInf} M = \{-\infty_Y\}$. The first equivalence in (2.13) is Proposition 2.2 (ii) in [25].

Lemma 2.1 *Let $\emptyset \neq M \subset Y$ and $\bar{v} \in Y$. Then*

$$(\bar{v} - \operatorname{int} K \subset M - \operatorname{int} K) \iff \bar{v} \in \operatorname{cl}(M - \operatorname{int} K).$$

Proof. $[\Longrightarrow]$ Assume that $\bar{v} - \operatorname{int} K \subset M - \operatorname{int} K$ and let U be a barrelled neighborhood of 0_Y. We will show that $(\bar{v} + U) \cap (M - \operatorname{int} K) \neq \emptyset$. Take $k_0 \in \operatorname{int} K$ (remember that $\operatorname{int} K \neq \emptyset$). Then, there exists $\lambda > 0$ such that $-\lambda k_0 \in U$. Since $\lambda k_0 \in \operatorname{int} K$, we get $\bar{v} - \lambda k_0 \in \bar{v} - \operatorname{int} K \subset M - \operatorname{int} K$, and hence, $\bar{v} - \lambda k_0 \in (\bar{v} + U) \cap (M - \operatorname{int} K)$.

$[\Longleftarrow]$ Assume that $\bar{v} \in \operatorname{cl}(M - \operatorname{int} K)$ and let $k \in \operatorname{int} K$. We will show that $\bar{v} - k \in M - \operatorname{int} K$. Since $0_Y \in -k + \operatorname{int} K$, there is a neighborhood U of 0_Y such that $U \subset -k + \operatorname{int} K$, and hence, $\bar{v} + U \subset \bar{v} - k + \operatorname{int} K$. Now, as $\bar{v} \in \operatorname{cl}(M - \operatorname{int} K)$, $(\bar{v} - k + \operatorname{int} K) \cap (M - \operatorname{int} K) \neq \emptyset$. Therefore, $\bar{v} - k + k' \in M - \operatorname{int} K$ for some $k' \in \operatorname{int} K$, and so

$$\bar{v} - k \in M - \operatorname{int} K - k' \subset M - \operatorname{int} K - \operatorname{int} K \subset M - \operatorname{int} K$$

(as $\operatorname{int} K$ is a convex cone), and we are done. $\square$

Proposition 2.1 *Given $\emptyset \neq M \subset Y^\bullet$ such that $+\infty_Y \notin M \neq \{-\infty_Y\}$, the following statements hold:*
(i) $\emptyset \neq \operatorname{WSup} M \subset Y \cup \{+\infty_Y\}$. *If* $\operatorname{WSup} M \neq \{+\infty_Y\}$, *then*

$$\begin{aligned} \operatorname{WSup} M &= \{\overline{v} \in Y \diagdown (M - \operatorname{int} K) : \overline{v} - \operatorname{int} K \subset M - \operatorname{int} K\} \\ &= \operatorname{cl}(M - \operatorname{int} K) \diagdown (M - \operatorname{int} K), \end{aligned} \tag{2.14}$$

in other words, $\operatorname{WSup} M$ *is the boundary in Y of the set* $M - \operatorname{int} K$.
(ii) The weak maximum of M is

$$\operatorname{WMax} M = M \diagdown (M - \operatorname{int} K), \tag{2.15}$$

so that $\operatorname{WMax} M$ *is a closed (compact) set whenever M is a closed (compact, respectively) set of Y.*
(iii)

$$Y^\bullet = \{-\infty_Y\} \cup (M - \operatorname{int} K) \cup (\operatorname{WSup} M) \cup \mathcal{A}(\operatorname{WSup} M).$$

Moreover, if $\operatorname{WSup} M \neq \{+\infty_Y\}$, *then*

$$Y = (M - \operatorname{int} K) \cup (\operatorname{WSup} M) \cup (\operatorname{WSup} M + \operatorname{int} K), \tag{2.16}$$

and the three sets in the right-hand side are disjoint.
(iv) Let $M \subset Y$ be such that $\operatorname{cl} M = \operatorname{cl}(\operatorname{int} M)$ *(e.g., if M is convex and* $\operatorname{int} M \neq \emptyset$*), then*

$$\operatorname{WSup}(\operatorname{int} M) = \operatorname{WSup} M = \operatorname{WSup}(\operatorname{cl} M). \tag{2.17}$$

Proof. The assumptions on M entail $M \cap Y \neq \emptyset$.

(i) It is obvious that $\operatorname{WSup} M \neq \emptyset$ and $-\infty_Y \notin \operatorname{WSup} M$ (see Remark 2.1(*b*)). Thus, $\emptyset \neq \operatorname{WSup} M \subset Y \cup \{+\infty_Y\}$ and the first claim holds.

Let $\operatorname{WSup} M \neq \{+\infty_Y\}$ or, equivalently (2.13), $\mathcal{B}(M) \neq Y \cup \{-\infty_Y\}$. The first equality in (2.14) comes from Remark 2.1(*d*) and the assumption $\operatorname{WSup} M \neq \{+\infty_Y\}$, while the second one comes from the first one and Lemma 2.1.

(ii) It is a straightforward consequence of $\operatorname{WMax} M = M \cap \operatorname{WSup} M$.

(iii) According to Proposition 7.4.1(b)(d) in [3], and using (2.12),

$$\begin{aligned} Y^\bullet &= (\operatorname{WSup} M) \cup \mathcal{B}(\operatorname{WSup} M) \cup \mathcal{A}(\operatorname{WSup} M) \\ &= (\operatorname{WSup} M) \cup \mathcal{B}(M) \cup \mathcal{A}(\operatorname{WSup} M) \\ &= \{-\infty_Y\} \cup (M - \operatorname{int} K) \cup (\operatorname{WSup} M) \cup \mathcal{A}(\operatorname{WSup} M). \end{aligned} \tag{2.18}$$

The first assertion in *(iii)* holds.

Assume now that $\operatorname{WSup} M \neq \{+\infty_Y\}$. Applying (2.11) to the set $\operatorname{WSup} M$ (note that $-\infty_Y \notin \operatorname{WSup} M$) we get

$$\mathcal{A}(\operatorname{WSup} M) = (\operatorname{WSup} M + \operatorname{int} K) \cup \{+\infty_Y\}. \tag{2.19}$$

According to (2.18) and (2.19),

$$Y^\bullet = \{-\infty_Y\} \cup (M - \operatorname{int} K) \cup (\operatorname{WSup} M) \cup (\operatorname{WSup} M + \operatorname{int} K) \cup \{+\infty_Y\}$$

and dropping $-\infty_Y$ and $+\infty_Y$ in both sides we get (2.16), together with the conclusion that the three sets in the right-hand side are disjoint (see again Proposition 7.4.1(d) in [3]).

(*iv*) It is a consequence of (2.14) and (2.6), applying the last one to the sets $N := \operatorname{int} M$ and $V := -\operatorname{int} K$. $\square$

Observe that (2.14), (2.15) and (2.17) remain true by replacing WSup, WMax, and $-\operatorname{int} K$ with WInf, WMin, and $\operatorname{int} K$, respectively.

We denote by $\mathcal{L}(X,Y)$ the space of linear continuous mappings from X to Y, and $0_{\mathcal{L}} \in \mathcal{L}(X,Y)$ is the zero mapping defined by $0_{\mathcal{L}}(x) = 0_Y$ for all $x \in X$. Obviously, when $Y = \mathbb{R}$ then $\mathcal{L}(X,Y) = X^*$.

Definition 2.4 *Given $f : X \to Y^\bullet$, the set-valued map $f^* : \mathcal{L}(X,Y) \rightrightarrows Y^\bullet$ defined by*

$$f^*(L) := \operatorname{WSup}\{L(x) - f(x) :\ x \in X\} \equiv \operatorname{WSup}\{(L-f)(X)\},$$

*is called the **conjugate map** of f. The **domain** of f^* is*

$$\operatorname{dom} f^* = \{L \in \mathcal{L}(X,Y) :\ f^*(L) \neq \{+\infty_Y\}\}$$

*and the $K-$**epigraph** of f^* is*

$$\operatorname{epi}_K f^* = \{(L,y) \in \mathcal{L}(X,Y) \times Y :\ y \in f^*(L) + K\}.$$

Remark 2.3 (*a*) *In [25, Definition 3.1] and in [3, Definition 7.4.2] the conjugate map is defined for a set-valued map $F : X \rightrightarrows Y^\bullet$.*
(*b*) *In the scalar case, when $Y = \mathbb{R}$ and $K = \mathbb{R}_+$, the notion of conjugate map introduced in Definition 2.4 collapses to (2.1) just identifying $y \in \overline{\mathbb{R}}$ with $\{y\} \in 2^{\overline{\mathbb{R}}}$.*
(*c*) *According to Remark 2.1(a), $f^*(L) = \operatorname{WSup}\{(L-f)(\operatorname{dom} f)\}$. Moreover, by Proposition 2.1(i), $f^*(L)$ is the boundary of $\{(L-f)(\operatorname{dom} f)\} - \operatorname{int} K$ if f is a proper function and $\operatorname{WSup}\{(L-f)(\operatorname{dom} f)\} \neq \{+\infty_Y\}$. The necessity of the latter assumption can be shown by considering the finite-valued function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = -|x|$. In fact, given $L = x^* \in \mathbb{R}$, one has*

$$(L-f)(\operatorname{dom} f) = \{x^*x + |x| :\ x \in \mathbb{R}\} = \begin{cases} \mathbb{R}_+, & \text{if } x^* \in [-1,1], \\ \mathbb{R}, & \text{if } x^* \notin [-1,1], \end{cases}$$

so that

$$\{(L-f)(\operatorname{dom} f)\} - \operatorname{int} K = \mathbb{R},$$

with

$$\operatorname{bd}\{\{(L-f)(\operatorname{dom} f)\} - \operatorname{int} K\} = \emptyset \neq \{+\infty\} = f^*(L).$$

Proposition 2.2 *Let $h : X \to Y^\bullet$ be proper and $(L,y) \in \mathcal{L}(X,Y) \times Y$. The following implication holds*

$$y + h(x) \nless_K L(x)\ \forall x \in X \implies (L,y) \in \operatorname{epi}_K h^*,$$

or equivalently

$$y + h(x) - L(x) \notin -\operatorname{int} K\ \forall x \in X \implies y \in h^*(L) + K.$$

Proof. Let $h : X \to Y^\bullet$ be proper and $(L, y) \in \mathcal{L}(X, Y) \times Y$ be such that

$$y + h(x) - L(x) \notin -\operatorname{int} K, \quad \text{for all} \quad x \in X. \tag{2.20}$$

Observe that (2.20) is equivalent to

$$y \notin (L - h)(\operatorname{dom} h) - \operatorname{int} K, \tag{2.21}$$

and then $\operatorname{WSup}\{(L - h)(\operatorname{dom} h)\} \neq \{+\infty_Y\}$. Indeed, assume the contrary, i.e., that

$$\operatorname{WSup}\{(L - h)(\operatorname{dom} h) = \{+\infty_Y\}.$$

Then as $y <_K +\infty_Y$, by the definition of the weak supremum there exists $\bar{x} \in \operatorname{dom} h$ such that

$$y <_K L(\bar{x}) - h(\bar{x}),$$

or equivalently, there is $\bar{x} \in X$ satisfying

$$y + h(\bar{x}) - L(\bar{x}) \in -\operatorname{int} K,$$

which contradicts (2.20).

Now, since $\emptyset \neq (L - h)(\operatorname{dom} h) \subset Y$ and $\operatorname{WSup}\{(L - h)(\operatorname{dom} h)\} \neq \{+\infty_Y\}$, we get from Proposition 2.1(*iii*) the following partition of Y :

$$\begin{aligned} Y \;\; = \;\; & \{(L - h)(\operatorname{dom} h) - \operatorname{int} K\} \cup \operatorname{WSup}\{(L - h)(\operatorname{dom} h))\} \\ & \cup \{\operatorname{WSup}\{(L - h)(\operatorname{dom} h)\} + \operatorname{int} K\}. \end{aligned}$$

Then (2.21) yields

$$\begin{aligned} y \;\; & \in \operatorname{WSup}\{(L - h)(\operatorname{dom} h)\} \cup \{\operatorname{WSup}\{(L - h)(\operatorname{dom} h)\} + \operatorname{int} K\} \\ & \subset \operatorname{WSup}\{(L - h)(\operatorname{dom} h)\} + K \\ & = h^*(L) + K, \end{aligned}$$

and we are done. □

The following notion of subdifferential of a vector-valued function particularizes the corresponding one for set-valued maps given in [3, Definition 7.4.2(c)] and in [25, Definition 4.1].

Definition 2.5 *Given $f : X \to Y^\bullet$ and $\bar{x} \in \operatorname{dom} f$, we say that $L \in \mathcal{L}(X, Y)$ is a **subgradient** of f at $\bar{x}$ if*

$$L(\bar{x}) - f(\bar{x}) \in \operatorname{WSup}\{(L - f)(X)\}.$$

*The set of all subgradients of f at $\bar{x}$ is called **subdifferential** of f at $\bar{x}$, and it is denoted by $\partial f(\bar{x})$.*

When $Y = \mathbb{R}$ and $K = \mathbb{R}_+$, the above definition of subdifferential of f at $\bar{x}$ is nothing else but the classical subdifferential of f at $\bar{x}$, i.e., $x^* \in \partial f(\bar{x})$ if only if $f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle$ for all $x \in X$.

Proposition 2.3 *Given $L \in \mathcal{L}(X, Y)$ and $\bar{x} \in \operatorname{dom} f$, one has*

$$\begin{aligned} L \in \partial f(\bar{x}) \;\; & \Longleftrightarrow L(\bar{x}) - f(\bar{x}) \in \operatorname{WMax}\{(L - f)(X)\} \\ & \Longleftrightarrow L(\bar{x}) - f(\bar{x}) \in f^*(L) \\ & \Longleftrightarrow (L, L(\bar{x}) - f(\bar{x})) \in \operatorname{epi}_K f^*. \end{aligned}$$

Proof. From Definitions 2.2, 2.4 and 2.5,

$$\begin{aligned} L \in \partial f(\bar{x}) \quad & \Longleftrightarrow L(\bar{x}) - f(\bar{x}) \in \mathrm{WMax}\{L(x) - f(x) : \ x \in X\} \\ & \Longleftrightarrow L(\bar{x}) - f(\bar{x}) \in f^*(L) \\ & \Longrightarrow (L, L(\bar{x}) - f(\bar{x})) \in \mathrm{epi}_K \ f^*. \end{aligned}$$

Now we assume that $(L, L(\bar{x}) - f(\bar{x})) \in \mathrm{epi}_K \ f^*$. Then $L(\bar{x}) - f(\bar{x}) \in f^*(L) + K$, and there exists $k \in K$ such that

$$L(\bar{x}) - f(\bar{x}) - k \in f^*(L) = \mathrm{WSup}\{L(x) - f(x) : \ x \in X\}.$$

From the definition of WSup

$$L(\bar{x}) - L(x) + f(x) - f(\bar{x}) - k \notin -\mathrm{int}\, K, \ \forall x \in X,$$

and it follows from (2.8) that

$$L(\bar{x}) - L(x) + f(x) - f(\bar{x}) \notin -\mathrm{int}\, K, \ \forall x \in X.$$

Thus,

$$L(\bar{x}) - f(\bar{x}) \in \mathrm{WSup}\{L(x) - f(x) : \ x \in X\},$$

i.e. $L \in \partial f(\bar{x})$. The proof is complete. □

Strong versions of the above notions of conjugate and subdifferential of a vector-valued function can be found in the recent book [20].

3 Reverse and asymptotic Farkas-type results

Let X, Y and Z be lcHtvs, 0_Z be the zero in Z, S be a nonempty convex cone in Z, and K be a nonempty closed and pointed convex cone in Y with $\mathrm{int}\, K \neq \emptyset$. Let $\leqq_S$ be the ordering on Z induced by the cone S, i.e.,

$$z_1 \leqq_S z_2 \text{ if and only if } z_1 - z_2 \in -S.$$

We also enlarge Z by attaching a greatest element $+\infty_Z$ and a smallest element $-\infty_Z$ with respect to $\leqq_S$, which do not belong to Z, and define $Z^\bullet := Z \cup \{-\infty_Z, \ +\infty_Z\}$. In $Z^\bullet$ we adopt the same sign conventions as in (2.5).

Let $f : X \to Y \cup \{+\infty_Y\}$, $g : X \to Z \cup \{+\infty_Z\}$, and consider a nonempty set $C \subset X$.

In this paper we associate with the data triple (f, g, C) the *constraint system*

$$\{x \in C, \ g(x) \in -S\} \equiv \{x \in C, \ g(x) \leqq_S 0_Z\},$$

with associated *feasible set*

$$A := C \cap g^{-1}(-S),$$

and the *vector optimization problem*

$$\text{(VOP)} \qquad \mathrm{WMin}\left\{f(x) : \ x \in C, \ g(x) \in -S\right\}, \tag{3.1}$$

where WMin concerns the weak ordering on $Y^\bullet$ associated with K. A feasible solution $\bar{x} \in A$ is said to be a *weak solution* to (VOP) if

$$f(\bar{x}) \in \mathrm{WMin}\, f(A).$$

We assume from now on that $A \cap \mathrm{dom}\, f \neq \emptyset$, in other words, (VOP) is *feasible* and *non-trivial.*

When $Y = \mathbb{R}$ and $K = \mathbb{R}_+$, we say that the data triple (f, g, C) is *scalar.* In that case (VOP) collapses to the scalar optimization problem

$$\text{(SOP)} \qquad \mathrm{Min}\left\{ f(x) : x \in C,\ g(x) \in -S \right\}, \tag{3.2}$$

where Min stands for the task consisting of identifying standard optimal solutions to (SOP). Here $Y^\bullet$ is nothing else than the extended real line $\overline{\mathbb{R}}$ ordered by $<_{\mathbb{R}_+}$while $L \in \mathcal{L}(X, \mathbb{R})$ is usually written as $L = x^* \in X^*$.

When $Y = \mathbb{R}^p$ and $K = \mathbb{R}^p_+$, $p \geq 2$, we say that the data triple (f, g, C) is *componentwise.* Then (VOP) becomes the *multiobjective optimization* problem

$$\text{(MOP)} \qquad \text{“}\mathrm{Min}\text{''} \left\{ f(x) : x \in C,\ g(x) \in -S \right\}, \tag{3.3}$$

where “Min” stands for the task consisting of computing $\bar{x} \in A$ such that $\left(f\left(\bar{x}\right) - \mathbb{R}^p_{++}\right) \cap f\left(A\right) = \emptyset$, i.e., weakly efficient solutions to (MOP), which coincide with the weak solutions to (VOP). Here, given $y = (y_1, ..., y_p)$ and $y' = \left(y'_1, ..., y'_p\right) \in \mathbb{R}^p$,

$$y <_{\mathbb{R}^p_+} y' \Longleftrightarrow y_j < y'_j,\ \forall j \in \{1, ..., p\},$$

and, consequently,

$$y \not<_{\mathbb{R}^p_+} y' \Longleftrightarrow \exists j_0 \in \{1, ..., p\} \text{ such that } \ y_{j_0} \geq y'_{j_0}.$$

In this paper we establish Farkas-type results from which we deduce necessary and sufficient conditions for the existence of weak solutions to problem (VOP) and some particular instances as (MOP) and (SOP). With this purpose, we provide next some fundamental results which will be used in the following sections.

For $T \in \mathcal{L}(Z, Y)$, we define the composite function $T \circ g : X \to Y^\bullet$ as follows:

$$(T \circ g)(x) = \begin{cases} T(g(x)), & \text{if } g(x) \in Z, \\ +\infty_Y, & \text{if } g(x) = +\infty_Z. \end{cases}$$

The indicator map $i_D : X \to Y^\bullet$ of a set $D \subset X$ is defined by

$$i_D(x) = \begin{cases} 0_Y, & \text{if } x \in D, \\ +\infty_Y, & \text{otherwise.} \end{cases}$$

In the case $Y = \mathbb{R}$, i_D is the usual indicator function.

Let us consider

$$\mathcal{L}_+(S, K) := \{T \in \mathcal{L}(Z, Y) :\ T(S) \subset K\}.$$

If $Y = \mathbb{R}$ and $K = \mathbb{R}_+$ then $\mathcal{L}_+(S, K) = S^+$, where S^+ is the (positive) *dual cone* of S in the sense of convex analysis, i.e.,

$$S^+ = \{z^* \in Z^* :\ \langle z^*, s \rangle \geq 0 \text{ for all } s \in S\}.$$

The sets of the form $(f + i_C + T \circ g)^*(L)$, with $T \in \mathcal{L}_+(S, K)$ and $L \in \mathcal{L}(X, Y)$, play an important role in this paper. The next example, to be used later, illustrates the way to calculate analytically such sets when the three involved lctHtvs, X, Y, and Z, are finite-dimensional.

Example 3.1 *Let $X=\mathbb{R}$, $Y=\mathbb{R}^2$, $Z=\mathbb{R}$, $K=\mathbb{R}^2_+$, $S=\mathbb{R}_+$, and $C=]-1,+\infty[$. Let $f:\mathbb{R}\to\mathbb{R}^2\cup\{+\infty_{\mathbb{R}^2}\}$ and $g:\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ be such that*

$$f(x)=\begin{cases}\left(0,\frac{1}{x}\right), & \text{if } x\neq 0,\\ +\infty_{\mathbb{R}^2}, & \text{otherwise,}\end{cases} \qquad \text{and} \qquad g(x)=\begin{cases}-\frac{|x+1|}{x}, & \text{if } x\neq 0,\\ +\infty, & \text{otherwise.}\end{cases}$$

The linear mappings $T\in\mathcal{L}_+(\mathbb{R}_+,\mathbb{R}^2_+)$ and $L\in\mathcal{L}(\mathbb{R},\mathbb{R}^2)$ can be represented as $T(z)=(az,bz)$ for all $z\in\mathbb{R}$, with $a,b\in\mathbb{R}_+$, and $L(x)=(cx,dx)$ for all $x\in\mathbb{R}$, with $c,d\in\mathbb{R}$. We now calculate $(f+i_C+T\circ g)^*(L)$ for one typical case where $a>0$, $0<b<1$, $c=0$, $d<0$, and $d<b-1$. One has

$$\begin{aligned}
&(f+i_C+T\circ g)^*(L)=\\
&=\operatorname{WSup}\{L(x)-f(x)-(T\circ g)(x):\ x\in C\} \qquad (3.4)\\
&=\operatorname{WSup}\left\{(0,dx)-\left(0,\frac{1}{x}\right)+\left(\frac{a|x+1|}{x},\frac{b|x+1|}{x}\right):x\in C\diagdown\{0\}\right\}\\
&=\operatorname{WSup}\left\{\left(a\left(\frac{x+1}{x}\right),dx-\frac{1}{x}+b\left(\frac{x+1}{x}\right)\right):x\in]-1,+\infty[\diagdown\{0\}\right\}\\
&=\operatorname{WSup}\left\{y\in\mathbb{R}^2:y_2=\left(\frac{b-1}{a}\right)y_1+1+\frac{da}{y_1-a},\ y_1<0\ \text{ or }\ y_1>a\right\}\\
&=\left\{y\in\mathbb{R}^2:y_2=\frac{(b-1)\,y_1}{a}+1+\frac{da}{y_1-a},\ y_1<a+\frac{ad}{1-b}\ \text{or } y_1>a+a\frac{\sqrt{-d}}{\sqrt{1-b}}\right\}\\
&\quad\bigcup\left(\left[a+\frac{ad}{1-b},0\right]\times\{1-d\}\right)\cup\left(\{0\}\times\left[b-2\sqrt{d(b-1)},1-d\right]\right)\\
&\quad\bigcup\left(\left[0,a+a\frac{\sqrt{-d}}{\sqrt{1-b}}\right]\times\left\{b-2\sqrt{d(b-1)}\right\}\right).
\end{aligned}$$

The set $(f+i_C+T\circ g)^*(L)$ in (3.4), with $a=1$, $b=\frac{1}{2}$, $c=0$, and $d=-1$ is represented in Figure 1.

We are now in the position to prove the main results of this section: two versions of reverse Farkas-lemma for vector-valued functions. Remember that we are assuming all the time that the triple (f,g,C) satisfies $A\cap\operatorname{dom}f\neq\emptyset$ with $A=C\cap g^{-1}(-S)$.

Theorem 3.1 (Reverse Farkas lemma I) *Let $(L,y)\in\mathcal{L}(X,Y)\times Y$. Then, the following statements are equivalent:*
$(\mathbf{a_1})$ $\quad g(x)\in -S,\ x\in C \implies f(x)-L(x)+y\notin -\operatorname{int}K,$
$(\mathbf{b_1})$ $\quad (L,y)\in\operatorname{epi}_K(f+i_A)^*.$

Proof. Taking $h:=f+i_A$ one has $\operatorname{dom}h=A\cap\operatorname{dom}f\neq\emptyset$. Then, by Proposition 2.2, the following implication holds:

$$y+(f+i_A)(x)-L(x)\notin -\operatorname{int}K\ \forall x\in X \implies y\in(f+i_A)^*(L)+K. \qquad (3.5)$$

$[(\mathbf{a_1})\implies(\mathbf{b_1})]$ Assume $(\mathbf{a_1})$ holds, i.e.,

$$g(x)\in -S, x\in C\implies f(x)-L(x)+y\notin -\operatorname{int}K,$$

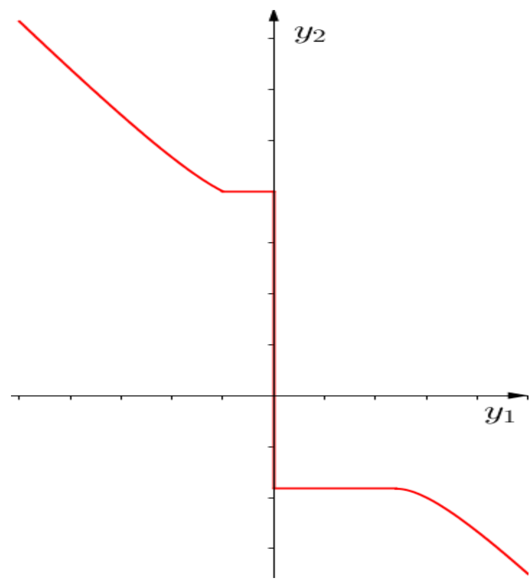

Figure 1: The set $(f + i_C + \left(1, \frac{1}{2}\right) \circ g)^*(0, -1)$

or, equivalently,

$$y + (f + i_A)(x) - L(x) \notin -\text{int}\, K, \forall x \in X.$$

It then follows from (3.5) that $(L, y) \in \text{epi}_K (f + i_A)^*$.

[$(\mathbf{b_1}) \Longrightarrow (\mathbf{a_1})$] Assume $(\mathbf{b_1})$ holds, i.e.,

$$y \in (f + i_A)^*(L) + K.$$

This accounts for the existence of $k \in K$ such that

$$y - k \in (f + i_A)^*(L) = \text{WSup}\{L(x) - (f + i_A)(x) : \ x \in X\}.$$

By the definition of WSup one has

$$y - k - (L(x) - (f + i_A)(x)) \notin -\text{int}\, K, \ \forall x \in X.$$

It follows from this and (2.8) that

$$y + (f + i_A)(x) - L(x) \notin -\text{int}\, K, \ \forall x \in X,$$

which is equivalent to

$$g(x) \in -S, \ x \in C \Longrightarrow y + f(x) - L(x) \notin -\text{int}\, K.$$

The proof is complete. □

Theorem 3.2 (Reverse Farkas lemma II) *Let $\bar{x} \in A \cap \text{dom}\, f$. The following statements are equivalent:*

$(\mathbf{c_1})$ $\quad g(x) \in -S, \ x \in C \quad \Longrightarrow \quad f(x) - f(\bar{x}) \notin -\text{int}\, K,$

$(\mathbf{d_1})$ $\quad 0_{\mathcal{L}} \in \partial(f + i_A)(\bar{x}),$

$(\mathbf{e_1})$ $\quad (0_{\mathcal{L}}, -f(\bar{x})) \in \text{epi}_K\, (f + i_A)^*.$

Proof. [($\mathbf{c_1}$) $\iff$ ($\mathbf{e_1}$)] follows from Theorem 3.1 with $L = 0_{\mathcal{L}}$ and $\overline{y} = -f(\bar{x})$.

[($\mathbf{d_1}$) $\iff$ ($\mathbf{e_1}$)] follows from Proposition 2.3 applied to $L = 0_{\mathcal{L}}$ and the function $f + i_A$. $\square$

In the absence of the vector-valued function g (or, equivalently, when $g(x) = 0_Z$ for all $x \in X$), Theorem 3.2 yields the following immediate corollary:

Corollary 3.1 *Let $\bar{x} \in C \cap \operatorname{dom} f$. Then the following statements are equivalent:*
($\mathbf{f_1}$) $f(x) - f(\bar{x}) \notin -\operatorname{int} K \ \forall x \in C$,
($\mathbf{g_1}$) $0_{\mathcal{L}} \in \partial(f + i_C)(\bar{x})$,
($\mathbf{h_1}$) $(0_{\mathcal{L}}, -f(\bar{x})) \in \operatorname{epi}_K(f + i_C)^*$.

When we apply Theorem 3.2 to the scalar optimization problem (SOP), we obtain the characterization of optimality given in the following corollary, which does not require the classical closedness and convexity assumptions on C, $f \in \Gamma(X)$, and $S-$epi closedness and $S-$convexity of g. It is worth noting that the first statement (i) in the next corollary means that $\overline{x}$ is an optimal solution of (SOP).

Corollary 3.2 *Let (f, g, C) be a given scalar triple and $\bar{x} \in A \cap \operatorname{dom} f$. Then the following statements are equivalent:*
($\mathbf{i_1}$) $g(x) \in -S,\ x \in C \implies f(x) - f(\bar{x}) \geq 0$,
($\mathbf{j_1}$) $0_{X^*} \in \partial(f + i_A)(\bar{x})$,
($\mathbf{k_1}$) $(0_{X^*}, -f(\bar{x})) \in \operatorname{epi}(f + i_A)^*$.

In the rest of this section we consider some special cases where the results above collapse to several well-known *asymptotic Farkas-type* results in the literature ([6], [9]). These results have been used to get optimality conditions, duality theorems, and set containment characterizations for (SOP). In particular, Corollary 3.2 leads us back to the following asymptotic Farkas lemma in [5] (see also [8]).

Corollary 3.3 (Asymptotic linear Farkas lemma) *Let $g \in \mathcal{L}(X, Z)$, with adjoint operator denoted by $g^\sharp$, and assume that the cone S is closed. Given $x^* \in X^*$, the following statements are equivalent:*
($\mathbf{l_1}$) $g(x) \in -S \implies \langle x^*, x\rangle \geq 0$,
($\mathbf{m_1}$) $-x^* \in \operatorname{cl}(g^\sharp(S^+))$.

Proof. The conclusion follows from Corollary 3.2. Indeed, let us take $f(\cdot) := \langle x^*, \cdot\rangle$, and $\bar{x} = 0_X \in g^{-1}(-S)$. Then $f(\bar{x}) = 0$ and it follows from Corollary 3.2 that

$$(\mathbf{l_1}) \iff (0_{X^*}, 0) \in \operatorname{epi}(f + i_A)^*, \tag{3.6}$$

where $A = g^{-1}(-S)$. It is a standard fact that $i_A = \sup_{z^* \in S^+}(z^* \circ g)$, and hence, by (2.4), one obtains

$$\operatorname{epi} i_A^* = \operatorname{cl}\operatorname{co}\left(\bigcup_{z^* \in S^+} \operatorname{epi}(z^* \circ g)^*\right).$$

We now have, by the last equality and the convexity of S^+:

$$\begin{aligned}
\operatorname{epi}(f+i_A)^* &= \operatorname{epi} f^* + \operatorname{epi} i_A^* \qquad \text{(by (2.3))}\\
&= \{x^*\}\times\mathbb{R}_+ + \operatorname{cl}\operatorname{co}\left(\bigcup_{z^*\in S^+}\operatorname{epi}(z^*\circ g)^*\right)\\
&= \{x^*\}\times\mathbb{R}_+ + \operatorname{cl}\operatorname{co}\left(\bigcup_{z^*\in S^+}\{z^*\circ g\}\times\mathbb{R}_+\right)\\
&= \{x^*\}\times\mathbb{R}_+ + \operatorname{cl}\left(\bigcup_{z^*\in S^+}\{z^*\circ g\}\times\mathbb{R}_+\right)\\
&= \{x^*\}\times\mathbb{R}_+ + \operatorname{cl}(g^\sharp(S^+))\times\mathbb{R}_+.
\end{aligned}$$

Therefore,

$$(0_{X^*},0)\in\operatorname{epi}(f+i_A)^* \Longleftrightarrow -x^*\in\operatorname{cl}(g^\sharp(S^+)).$$

The conclusion follows from the last equivalence and (3.6). □

Next we are approaching scalar asymptotic Farkas-type results for convex systems. Now, $Y=\mathbb{R}$, $K=\mathbb{R}_+$, $f\in\Gamma(X)$, and C is a nonempty closed convex set in X. Additionally, we assume that g is $S-$epi closed and $S-$convex. Note that, under these assumptions, $g^{-1}(-S)$ is a closed convex set and $i_C\in\Gamma(X)$.

As a consequence of Theorem 3.1 we now can provide an asymptotic Farkas lemma for convex systems with linear perturbations which extends some results in the literature ([7], [9]).

Corollary 3.4 (Asymptotic convex Farkas lemma for linear perturbations) *Let (f,g,C) be a scalar triple such that $f\in\Gamma(X)$, C is a closed convex set, and g is S-convex and S-epi closed. Then, for any pair $x^*\in X^*$ and $\alpha\in\mathbb{R}$ the following statements are equivalent:*

($\mathbf{n_1}$) $g(x)\in -S,\ x\in C \implies f(x)-\langle x^*,x\rangle+\alpha\geq 0,$

($\mathbf{o_1}$) $(x^*,\alpha)\in\operatorname{cl}\left(\bigcup_{z^*\in S^+}\operatorname{epi}(f+i_C+z^*\circ g)^*\right),$

($\mathbf{p_1}$) *there exists a net $(z_i^*)_{i\in I}\subset S^+$ such that*

$$f(x)+\liminf_i(z_i^*\circ g)(x)-\langle x^*,x\rangle+\alpha\geq 0,\ \forall x\in C.$$

Proof. We apply Theorem 3.1 with $Y=\mathbb{R}$, $K=\mathbb{R}_+$, $L=x^*$ and $y=\alpha$. Then, ($\mathbf{n_1}$) is equivalent to $(x^*,\alpha)\in\operatorname{epi}(f+i_A)^*$. The equivalence of ($\mathbf{n_1}$) and ($\mathbf{o_1}$) follows from the following formula (3.7) in [2, Theorem 8.2]:

$$\operatorname{epi}(f+i_A)^*=\operatorname{cl}\left(\bigcup_{z^*\in S^+}\operatorname{epi}(f+i_C+z^*\circ g)^*\right). \tag{3.7}$$

[($\mathbf{o_1}$) $\implies$ ($\mathbf{p_1}$)] Assume that ($\mathbf{o_1}$) holds. Then, there exist nets $(z_i^*)_{i\in I}\subset S^+$, $(x_i^*,r_i)_{i\in I}\subset X^*\times\mathbb{R}$ such that $x_i^*\to x^*$ and $r_i\to\alpha$ and that

$$(f+i_C+z_i^*\circ g)^*(x_i^*)\leq r_i,\ \forall i\in I,$$

which leads to

$$f(x) + (z_i^* \circ g)(x) - \langle x_i^*, x\rangle + r_i \geq 0, \forall x \in C \text{ and } \forall i \in I.$$

Since $x_i^* \to x^*$ and $r_i \to \alpha$, $(\mathbf{p_1})$ follows from the last inequality.

$[(\mathbf{p_1}) \Longrightarrow (\mathbf{n_1})]$ For any $x \in C$ such that $g(x) \in -S$ one has $(z^* \circ g)(x) \leq 0$ for all $z^* \in S^+$. Hence, if $(\mathbf{p_1})$ holds, one has for such x, $f(x) - \langle x^*, x\rangle + \alpha \geq 0$ which means that $(\mathbf{n_1})$ holds. The proof is complete. □

Remark 3.1 *Since we also have [4, p. 328]*

$$\operatorname{epi}(f + i_A)^* = \operatorname{cl}\left(\operatorname{epi} f^* + \operatorname{epi} i_C^* + \bigcup_{z^* \in S^+} \operatorname{epi}(z^* \circ g)^*\right), \tag{3.8}$$

it follows that $(\mathbf{n_1})$ *is also equivalent to*

$$(x^*, \alpha) \in \operatorname{cl}\left(\operatorname{epi} f^* + \operatorname{epi} i_C^* + \bigcup_{z^* \in S^+} \operatorname{epi}(z^* \circ g)^*\right).$$

The Farkas lemma for linearly perturbed convex systems in Corollary 3.4 extends the sequential Farkas lemma for convex systems given in [7, Proposition 4] and in [9, Theorem 2.1], where $(x^*, \alpha) = (0_{X^*}, 0)$ (in [9], also $C = X$). When the set in the right hand side of $(\mathbf{o_1})$ is closed, Corollary 3.4 leads to the stable Farkas lemma for convex systems ([6, Theorem 3.1], [13, Corollary 4]). Extensions of this result to nonconvex systems will be established in the next section. It is worth observing that conditions (3.7) and (3.8) have been used in the framework of duality theory (see, e.g., [2] and [4]) while some of their generalizations have been used for extensions of Farkas-type results (see, [10], [13]). Moreover, when taking $x^* = 0$ in Corollary 3.4, the result collapses to an asymptotic Farkas lemma in the next corollary that extends the sequential Farkas lemma established in [9, Theorem 2.1], where $C = X$ and the map g was assumed to be continuous (assumption which is much stronger than the S-epi closedness required below).

Corollary 3.5 (Asymptotic convex Farkas lemma) *Let (f, g, C) be a scalar triple and $\alpha \in \mathbb{R}$. Assume that $f \in \Gamma(X)$, the convex set C is closed and g is S-convex and S-epi closed. Then the following statements are equivalent:*

$(\mathbf{q_1})$ $\quad g(x) \in -S,\ x \in C \quad \Longrightarrow \quad f(x) + \alpha \geq 0,$

$(\mathbf{r_1})$ $\quad (0_{X^*}, \alpha) \in \operatorname{cl}\left(\bigcup_{z^* \in S^+} \operatorname{epi}(f + i_C + z^* \circ g)^*\right),$

$(\mathbf{s_1})$ *there exists a net* $(z_i^*)_{i \in I} \subset S^+$ *such that*

$$f(x) + \liminf_i\ (z_i^* \circ g)(x) + \alpha \geq 0,\ \forall x \in C.$$

The next stable Farkas lemma for convex systems under linear perturbations [6] is a direct consequence of the previous results.

Corollary 3.6 (Stable convex Farkas lemma) [6] *Let (f,g,C) be a scalar triple such that $f \in \Gamma(X)$, the convex cone S is closed, and g is S-convex and S-epi closed. Then, the following statements are equivalent:*

($\mathbf{t_1}$) *The set* $\bigcup\limits_{z^*\in S^+} \operatorname{epi}(f+i_C+z^*\circ g)^*$ *is weak*-closed,*

($\mathbf{v_1}$) *For any pair* $(x^*,\alpha)\in X^*\times\mathbb{R}$, *it holds*

$$\begin{gathered}\{g(x)\in -S,\ x\in C \Longrightarrow f(x)-\langle x^*,x\rangle+\alpha\geq 0\}\\ \Updownarrow\\ \{\exists z^*\in S^+, \forall x\in C:\ f(x)+(z^*\circ g)(x)-\langle x^*,x\rangle+\alpha\geq 0\}\,.\end{gathered}$$

Proof. The result is a direct consequence of the equivalences in Corollary 3.4. □

4 Farkas-type results for vector-valued functions

In this section we consider the triple (f,g,C) corresponding to problem (VOP) in (3.1), with $A = C\cap g^{-1}(-S)$ such that $A\cap \operatorname{dom} f\neq\emptyset$, and we establish a version of Farkas lemma for vector-valued functions corresponding to the mentioned problem (VOP). We firstly give some preliminary lemmas.

Lemma 4.1 *It holds*

$$\bigcup_{T\in\mathcal{L}_+(S,K)} \operatorname{epi}_K(f+i_C+T\circ g)^* \subset \operatorname{epi}_K(f+i_A)^*. \tag{4.1}$$

Proof. Take arbitrarily $(L,y)\in \bigcup\limits_{T\in\mathcal{L}_+(S,K)} \operatorname{epi}_K(f+i_C+T\circ g)^*$. Then there exists $T_0\in\mathcal{L}_+(S,K)$ such that $y\in (f+i_C+T_0\circ g)^*(L)+K$. Hence, there is $k_0\in K$ such that

$$y-k_0\in (f+i_C+T_0\circ g)^*(L) = \operatorname{WSup}\{L(x)-f(x)-(T_0\circ g)(x) : x\in C\}\,.$$

By the definition of WSup one has

$$L(x)-f(x)-(T_0\circ g)(x)-y+k_0\notin \operatorname{int}K,\ \forall x\in C. \tag{4.2}$$

Observe that if $x\in A$, then $-(T_0\circ g)(x)\in K$ (as $T_0\in\mathcal{L}_+(S,K)$). From this, (4.2) and (2.8) we get

$$L(x)-f(x)-y\notin \operatorname{int}K,\ \forall x\in A,$$

or equivalently,

$$y+(f+i_A)(x)-L(x)\notin -\operatorname{int}K,\ \forall x\in X.$$

According to Proposition 2.2, we conclude

$$(L,y)\in \operatorname{epi}_K(f+i_A)^*,$$

and so the inclusion (4.1) has been proved. □

The next example shows that the inclusion (4.1) can be strict.

Example 4.1 *Let X, Y, Z, K, S, C, f, and g, be as in Example 3.1. Now we shall prove that*

$$\left(\overline{L}, (-1,-2)\right) \in \operatorname{epi}_K (f + i_A)^* \diagdown \bigcup_{T \in \mathcal{L}_+(S,K)} \operatorname{epi}_K (f + i_C + T \circ g)^*,$$

for $\overline{L} = (0,-1)$, by showing that

$$(-1,-2) \in \left[(f+i_A)^*(\overline{L}) + \mathbb{R}^2_+\right] \diagdown \bigcup_{T \in \mathcal{L}_+(S,K)} \left[(f + i_C + T \circ g)^* \left(\overline{L}\right) + \mathbb{R}^2_+\right].$$

On the one hand, since $A = C \cap g^{-1}(-S) =]0,+\infty[$, we have

$$\begin{aligned} (f+i_A)^*(\overline{L}) &= \operatorname{WSup}\left\{\overline{L}(x) - f(x) : \ x \in A\right\} = \operatorname{WSup}\left\{(0,-x) - \left(0, \frac{1}{x}\right) : x > 0\right\} \\ &= \operatorname{WSup}\left\{\left(0, -x - \frac{1}{x}\right) : x > 0\right\} = (\mathbb{R}_- \times \{-2\}) \cup (\{0\} \times]-\infty, -2]), \end{aligned}$$

so that

$$(-1,-2) \in (f+i_A)^*(\overline{L}) + \mathbb{R}^2_+ = (\mathbb{R}_- \times [-2,+\infty[) \cup (\mathbb{R}_+ \times \mathbb{R}).$$

On the other hand, recalling that $(T \circ g)(0) = T(+\infty) = +\infty_{\mathbb{R}^2}$, we can write, for any $T = (a,b) \in \mathbb{R}^2_+$,

$$\begin{aligned} (f + i_C + T \circ g)^*(\overline{L}) &= \operatorname{WSup}\left\{\overline{L}(x) - f(x) - (T \circ g)(x) : \ x \in C\right\} \\ &= \operatorname{WSup}\left\{(0,-x) - \left(0, \frac{1}{x}\right) + \left(\frac{a|x+1|}{x}, \frac{b|x+1|}{x}\right) : x \in C \diagdown \{0\}\right\} \\ &= \operatorname{WSup}\left\{\left(a\left(\frac{x+1}{x}\right), -x - \frac{1}{x} + b\left(\frac{x+1}{x}\right)\right) : x \in]-1,+\infty[\diagdown \{0\}\right\}. \end{aligned}$$

Table 1 describes $\bigcup_{T \in \mathcal{L}_+(S,K)} \left[(f + i_C + T \circ g)^* \left(\overline{L}\right) + \mathbb{R}^2_+\right]$ as a union of sets of the form $(f + i_C + (a,b) \circ g)^* \left(\overline{L}\right) + \mathbb{R}^2_+$ for $(a,b) \in \mathcal{L}_i$, where $\{\mathcal{L}_1, ..., \mathcal{L}_8\}$ is the partition of $\mathbb{R}^2_+$ in the second column of Table 1. Observe that (3.4) allows to express $(f + i_C + (a,b) \circ g)^* \left(\overline{L}\right) + \mathbb{R}^2_+$ as it appears in row 6, column 3 of Table 1, corresponding to the harder case that $(a,b) \in \mathcal{L}_6$. Similar calculations provide $(f + i_C + (a,b) \circ g)^* \left(\overline{L}\right) + \mathbb{R}^2_+$ for $i = 1, ..., 8$, $i \neq 6$.

i	$\mathcal{L}_i$	$(f + i_C + (a,b) \circ g)^* \left(\overline{L}\right) + \mathbb{R}^2_+$
1	$\{(0,0)\}$	$\mathbb{R}_+ \times \mathbb{R}$
2	$\{0\} \times]0,1[$	$\mathbb{R}_+ \times \mathbb{R}$
3	$\{(0,1)\}$	$(\mathbb{R}_- \times [2,+\infty[) \cup (\mathbb{R}_+ \times \mathbb{R})$
4	$\{0\} \times]1,+\infty[$	$\mathbb{R}_+ \times \mathbb{R}$
5	$]0,+\infty[\times \{0\}$	$\left\{y \in \mathbb{R}^2 : y_2 \geq -\frac{a}{y_1 - a} - \frac{y_1 - a}{a}, y_1 \notin]0,2a[\right\} \cup ([0,2a] \times [-2,+\infty[)$
6	$]0,+\infty[\times]0,1[$	$\left\{y \in \mathbb{R}^2 : y_2 \geq \left(\frac{b-1}{a}\right) y_1 + 1 - \frac{a}{y_1 - a},\ y_1 \notin \left]a - \frac{a}{1-b}, a + \frac{a}{\sqrt{1-b}}\right[\right\}$ $\cup \left(\left[a - \frac{a}{1-b}, 0\right] \times [2,+\infty[\right) \cup \left(\left[0, a + \frac{a}{\sqrt{1-b}}\right] \times \left[b - 2\sqrt{1-b}, +\infty\right[\right)$
7	$]0,+\infty[\times \{1\}$	$(\mathbb{R}_- \times [2,+\infty[) \cup (\mathbb{R}_+ \times [1,+\infty[)$
8	$]0,+\infty[\times]1,+\infty[$	$\{+\infty_{\mathbb{R}^2}\}$

Table 1

The conclusion follows from the fact that no set in column 3 of Table 1 contains $(-1,-2)$.

We shall need the following technical lemmas:

Lemma 4.2 *Let* $(L, y) \in \mathcal{L}(X, Y) \times Y$ *and* $T \in \mathcal{L}(Z, Y)$. *The following implication holds:*

$$y + (f + i_C + T \circ g)(x) \nless_K L(x) \ \forall x \in X \quad \Longrightarrow \quad (L, y) \in \mathrm{epi}_K(f + i_C + T \circ g)^*,$$

or equivalently,

$$y + (f + i_C + T \circ g)(x) - L(x) \notin -\mathrm{int}\, K \quad \forall x \in X \quad \Longrightarrow \quad y \in (f + i_C + T \circ g)^*(L) + K,$$

Proof. It comes from Proposition 2.2 by taking $h := f + i_C + T \circ g$ and observing that $\mathrm{dom}(T \circ g) = g^{-1}(Z)$, so that

$$\mathrm{dom}(f + i_C + T \circ g) = (\mathrm{dom}\, f) \cap C \cap g^{-1}(Z) \supset (\mathrm{dom}\, f) \cap A \neq \emptyset.$$

□

Lemma 4.3 *Let* $(L, y) \in \mathcal{L}(X, Y) \times Y$*, and consider the following statements:*
$(\mathbf{a_1})$ $g(x) \in -S, x \in C \Longrightarrow f(x) - L(x) + y \notin -\mathrm{int}\, K$,
$(\mathbf{a_2})$ $\exists\, T \in \mathcal{L}_+(S, K)$ *such that*

$$(L, y) \in \mathrm{epi}_K(f + i_C + T \circ g)^*.$$

$(\mathbf{b_2})$ $\exists\, T \in \mathcal{L}_+(S, K)$ *such that*

$$f(x) + (T \circ g)(x) - L(x) + y \notin -\mathrm{int}\, K, \ \forall x \in C.$$

We have the following relationships among them:

$$(\mathbf{a_1}) \ \Longleftarrow \ (\mathbf{a_2}) \ \Longleftrightarrow \ (\mathbf{b_2}).$$

Proof. $[(\mathbf{a_1}) \Longleftarrow (\mathbf{a_2})]$ It follows from Lemma 4.1 and Theorem 3.1

$$\begin{aligned} (\mathbf{a_2}) \quad &\Longleftrightarrow \quad (L, y) \in \bigcup\nolimits_{T \in \mathcal{L}_+(S,K)} \mathrm{epi}_K(f + i_C + T \circ g)^* \\ &\Longrightarrow \quad (L, y) \in \mathrm{epi}_K(f + i_A)^* \ \Longleftrightarrow \ (\mathbf{a_1}). \end{aligned} \tag{4.3}$$

$[(\mathbf{a_2}) \implies (\mathbf{b_2})]$ Assume that $(\mathbf{a_2})$ holds; in other words, there exist $T \in \mathcal{L}_+(S, K)$ and $k \in K$ such that

$$y - k \in (f + i_C + T \circ g)^*(L).$$

Therefore

$$y - k - L(x) + f(x) + i_C(x) + (T \circ g)(x) \ \notin -\mathrm{int}\, K, \ \forall x \in X.$$

Now, again by (2.8), we get

$$y - L(x) + f(x) + i_C(x) + (T \circ g)(x) \ \notin -\mathrm{int}\, K, \ \forall x \in X, \tag{4.4}$$

which is nothing else but $(\mathbf{b_2})$.

$[(\mathbf{b_2}) \Longrightarrow (\mathbf{a_2})]$ This implication follows from Lemma 4.2. □

Next we present the main result in this section.

Theorem 4.1 (Stable Reverse Farkas lemma)

$$\left\{ \mathrm{epi}_K(f+i_A)^* = \bigcup_{T\in\mathcal{L}_+(S,K)} \mathrm{epi}_K(f+i_C+T\circ g)^* \right\}$$
$$\Updownarrow$$
$$\{\forall (L,y)\in\mathcal{L}(X,Y)\times Y, \quad (\mathbf{a_1}) \Longleftrightarrow (\mathbf{a_2}) \Longleftrightarrow (\mathbf{b_2})\}.$$

Proof. ($\Downarrow$) Now the implication in (4.3) is an equivalence.

($\Uparrow$) The implication ($\mathbf{a_1}$)$\Longrightarrow$($\mathbf{a_2}$) yields

$$\mathrm{epi}_K(f+i_A)^* \subset \bigcup_{T\in\mathcal{L}_+(S,K)} \mathrm{epi}_K(f+i_C+T\circ g)^*,$$

and the proof finishes by applying (4.1). $\square$

Remark 4.1 *The equivalence* ($\mathbf{a_1}$) $\Longleftrightarrow$ ($\mathbf{a_2}$) *in Theorem 4.1 is called* ***stable*** *as it holds for all* $(L,y)\in\mathcal{L}(X,Y)\times Y$.

Remark 4.2 *When we are confined to the convex (SOP) (i.e.* $f\in\Gamma(X)$, C *is a closed convex set, and* g *is* S*-convex and* S*-epi closed), the equality*

$$\mathrm{epi}_K(f+i_A)^* = \bigcup_{T\in\mathcal{L}_+(S,K)} \mathrm{epi}_K(f+i_C+T\circ g)^* \tag{4.5}$$

is equivalent to the weak-closedness of* $\bigcup_{z^*\in S^+} \mathrm{epi}(f+i_C+z^*\circ g)^*$. *This condition is necessary and sufficient for the stable Farkas lemma and stable Lagrange duality for (SOP) in [6] (see also [13]). The following example illustrates the fulfilment of (4.5).*

Example 4.2 *Let* $X=\mathbb{R}$, $Y=\mathbb{R}^2$, $Z=\mathbb{R}$, $K=\mathbb{R}^2_+$, $S=\mathbb{R}_+$, $C=]0,1[$, $f(x)=(x,x)$, *and* $g(x)=-x$. *We add to* $Y=\mathbb{R}^2$ *a greatest and smallest elements with respect to the ordering defined by* $K=\mathbb{R}^2_+$, *denoted by* $-\infty_{\mathbb{R}^2}$ *and* $+\infty_{\mathbb{R}^2}$, *i.e.,* $Y^\bullet=\mathbb{R}^2\cup\{-\infty_{\mathbb{R}^2}\}\cup\{+\infty_{\mathbb{R}^2}\}$. *Observe first that* $A=C\cap g^{-1}(-S)=]0,1[$. *Let* $L\in\mathcal{L}(X,Y)=\mathcal{L}(\mathbb{R},\mathbb{R}^2)$ *be defined by* $L(x)=(\alpha x,\beta x)$ *for all* $x\in X=\mathbb{R}$ $(\alpha,\beta\in\mathbb{R})$. *Then one has,*

$$(f+i_A)^*(L) \quad = \mathrm{WSup}\left\{((\alpha-1)x,(\beta-1)x)\in\mathbb{R}^2: \ x\in A=]0,1[\right\}.$$

On the other hand, for any $T\in\mathcal{L}_+(S,K)=\mathcal{L}_+(\mathbb{R}_+,\mathbb{R}^2_+)$ *(it is easy to see that* $T(z)=(az,bz)$ *for all* $z\in Z=\mathbb{R}$ *with* $a\geq 0$ *and* $b\geq 0$*), one has*

$$(f+i_C+T\circ g)^*(L)=\mathrm{WSup}\left\{((\alpha+a-1)x,(\beta+b-1)x): x\in]0,1[\right\}.$$

Routine calculations show that condition (4.5) holds.

Theorem 4.2 (Partially-stable Reverse Farkas lemma) *The following statements are equivalent:*

($\mathbf{c_2}$) $\mathrm{epi}_K(f+i_A)^*\cap(\{0_{\mathcal{L}}\}\times Y)=\left(\bigcup_{T\in\mathcal{L}_+(S,K)} \mathrm{epi}_K(f+i_C+T\circ g)^*\right)\cap(\{0_{\mathcal{L}}\}\times Y)$.

($\mathbf{d_2}$) *For any* $y\in Y$,

$$\{g(x)\in -S, x\in C \implies f(x)+y\notin -\mathrm{int}\,K\}$$
$$\Updownarrow$$
$$\{\exists\ T\in\mathcal{L}_+(S,K) \text{ such that } y+f(x)+(T\circ g)(x)\notin -\mathrm{int}\,K\ \ \forall x\in C\}.$$

Proof. It is similar to the proofs of Lemma 4.3 and Theorem 4.1, but taking $L = 0_{\mathcal{L}}$. $\square$

Remark 4.3 *Again for the convex (SOP) (i.e. $f \in \Gamma(X)$, C is a closed convex set, and g is S-convex and S-epi closed), condition* $(\mathbf{c_2})$ *accounts for the closedness of* $\bigcup_{z^* \in S^+} \operatorname{epi}(f + i_C + z^* \circ g)^*$ *regarding the set $\{0_{X^*}\} \times \mathbb{R}$ (recall that a set A is said to be **closed regarding to** the set B if $B \cap \operatorname{cl} A = B \cap A$, see e.g. [2, p. 56]), and this condition is sufficient for generalized Farkas lemma for systems involving extended real-valued functions (see, e.g., [6], [13], and [10]).*

The following example illustrates the fulfilment of $(\mathbf{c_2})$.

Example 4.3 *Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $Z = \mathbb{R}$, $K = \mathbb{R}^2_+$, $S = \mathbb{R}_+$, $C =]-1, 1[$, $f(x) = (x, x^2)$, and $g(x) = -x$. We add to $Y = \mathbb{R}^2$ a greatest and smallest elements with respect to the ordering defined by $K = \mathbb{R}^2_+$, denoted by $-\infty_{\mathbb{R}^2}$ and $+\infty_{\mathbb{R}^2}$, i.e., $Y^\bullet = \mathbb{R}^2 \cup \{-\infty_{\mathbb{R}^2}\} \cup \{+\infty_{\mathbb{R}^2}\}$. Observe firstly that $A = C \cap g^{-1}(-S) = [0, 1[$ and*

$$\begin{aligned}(f + i_A)^*(0_{\mathcal{L}}) &= \operatorname{WSup}\left\{(-x, -x^2) : \ x \in A = [0, 1[\right\} \\ &= (]-\infty, 0] \times \{0\}) \cup (\{0\} \times]-\infty, 0]) \,.\end{aligned}$$

Therefore,

$$(f + i_A)^*(0_{\mathcal{L}}) + \mathbb{R}^2_+ = \mathbb{R}^2 \diagdown (-\mathbb{R}^2_{++}).$$

So,

$$\operatorname{epi}_K (f + i_A)^* \cap (\{0_{\mathcal{L}}\} \times Y) = \{0_{\mathcal{L}}\} \times \left(\mathbb{R}^2 \diagdown (-\mathbb{R}^2_{++})\right). \tag{4.6}$$

On the other hand, given $T = (a, b) \in \mathcal{L}_+(\mathbb{R}_+, \mathbb{R}^2_+) = \mathbb{R}^2_+$, one has

$$(f + i_C + T \circ g)^*(0_{\mathcal{L}}) = \operatorname{WSup}\left\{\left((a-1)x, bx - x^2\right) : x \in]-1, 1[\right\}.$$

New routine calculations, together with (4.6), show that $(\mathbf{c_2})$ *holds.*

For problem (SOP), Theorems 4.1 and 4.2 yield respectively the following versions of well-known Farkas-type results where we succeeded to eliminate superfluous convexity and lower semicontinuity assumptions. In particular, in [14] the authors require convexity of the involved sets and functions but, in page 1313, they claim that "most results remain valid even if one drops the convexity assumptions". Of course, for problem (SOP) in (3.2), we also assume that $A \cap \operatorname{dom} f \neq \emptyset$.

Corollary 4.1 *[14, Theorem 6.7] For problem (SOP) in (3.2) the following statements are equivalent:*

$(\mathbf{e_2})$ $\operatorname{epi}(f + i_A)^* = \bigcup_{z^* \in S^+} \operatorname{epi}(f + i_C + z^* \circ g)^*$.

$(\mathbf{f_2})$ *For any $x^* \in X^*$ and any $\alpha \in \mathbb{R}$,*

$$\begin{gathered}\{g(x) \in -S, x \in C \implies f(x) - \langle x^*, x\rangle + \alpha \geq 0\} \\ \Updownarrow \\ \{\exists\, z^* \in S^+ \text{ such that } f(x) + (z^* \circ g)(x) - \langle x^*, x\rangle + \alpha \geq 0, \ \forall x \in C\}\,.\end{gathered}$$

Corollary 4.2 *[14, Theorem 6.6] For problem (SOP) in (3.2) the following statements are equivalent:*

($\mathbf{g_2}$) $\operatorname{epi}(f+i_A)^* \cap (\{0_{X^*}\}\times\mathbb{R}) = \left(\bigcup_{z^*\in S^+} \operatorname{epi}(f+i_C+z^*\circ g)^*\right) \cap (\{0_{X^*}\}\times\mathbb{R})$.

($\mathbf{h_2}$) *For any* $\alpha\in\mathbb{R}$,

$$\{g(x)\in -S, x\in C \implies f(x)+\alpha\geq 0\}$$
$$\Updownarrow$$
$$\{\exists\ z^*\in S^+ \text{ such that } f(x)+(z^*\circ g)(x)+\alpha\geq 0,\ \forall x\in C\}.$$

Condition ($\mathbf{e_2}$) is called in [14] *weak conical epigraph hull property relative to* f, whereas ($\mathbf{f_2}$) is called *stable Farkas rule with respect to* f. The mentioned paper does not assume the lower semicontinuity of the involved functions. In [14] the following condition, similar to ($\mathbf{e_2}$), and called *conical epigraph hull property relative to* f, is also exploited:

($\mathbf{e_2'}$) $\qquad \operatorname{epi}(f+i_A)^* = \operatorname{epi} f^* + \operatorname{epi} i_C^* + \bigcup_{z^*\in S^+} \operatorname{epi}(z^*\circ g)^*.$

The conditions in Corollaries 4.1 and 4.2 are the weakest ones (necessary and sufficient conditions) for such Farkas-type results. They are conditions ($\mathbf{e_2}$) and ($\mathbf{g_2}$), which correspond to the scalar versions of (4.5) and ($\mathbf{c_2}$), respectively, but without convexity (see Remark 4.2).

5 Applications to vector optimization

This section focuses on the vector optimization problem (VOP) in (3.1):

$$\text{(VOP)} \qquad \operatorname{WMin}\{f(x):\ x\in C,\ g(x)\in -S\},$$

assuming once again $A\cap\operatorname{dom} f\neq\emptyset$, where $A=C\cap g^{-1}(-S)$ is the feasible set. Recall that an element $\bar{x}\in A$ is said to be a *weak solution* to (VOP) if

$$f(\bar{x})\in \operatorname{WMin} f(A).$$

By Proposition 2.1(*ii*),

$$\begin{aligned} \bar{x} \text{ is a weak solution of (VOP)} &\iff f(x)-f(\bar{x})\notin -\operatorname{int} K,\ \forall x\in A \\ &\iff \{g(x)\in -S,\ x\in C \Longrightarrow f(x)-f(\bar{x})\notin -\operatorname{int} K\} \end{aligned} \tag{5.1}$$

The next result is a straightforward consequence of Theorem 3.2.

Proposition 5.1 *Let* $\bar{x}\in A\cap\operatorname{dom} f$. *The following statements are equivalent:*
($\mathbf{a_3}$) $\bar{x}$ *is a weak solution to (VOP),*
($\mathbf{b}_3$) $0_{\mathcal{L}}\in\partial(f+i_A)(\bar{x})$,
($\mathbf{c_3}$) $(0_{\mathcal{L}}, -f(\bar{x}))\in \operatorname{epi}_K(f+i_A)^*$.

Example 5.1 *([21, Example 8.6]) Consider the multiobjective optimization problem* (MOP) *in (3.3), with* $C=X=Y=\mathbb{R}^2$, $Z=\mathbb{R}$, $K=\mathbb{R}^2_+$, $S=\mathbb{R}_+$, $f(x_1,x_2)=(x_1,x_2)$, *and* $g(x_1,x_2)=\max\{-x_1,0\}-x_2$. *We add to* $Y=\mathbb{R}^2$ *a greatest and smallest elements with respect to the ordering defined by* $K=\mathbb{R}^2_+$, *denoted by* $-\infty_{\mathbb{R}^2}$ *and* $+\infty_{\mathbb{R}^2}$, *i.e.,* $Y^\bullet=\mathbb{R}^2\cup$

$\{-\infty_{\mathbb{R}^2}\}\cup\{+\infty_{\mathbb{R}^2}\}$. *Obviously, the elements of* $\mathcal{L}(X,Y)$ *can be identified with* 2×2 *matrices and* $0_{\mathcal{L}}$ *with the null matrix. It is clear that* $A=\{x\in\mathbb{R}^2 : x_2\geq 0, x_1+x_2\geq 0\}$, *and hence*

$$\begin{aligned}(f+i_A)^*(0_{\mathcal{L}}) &= \operatorname{WSup}\left\{-(f+i_A)(x) : x\in\mathbb{R}^2\right\} = \operatorname{WSup}\{-A\}\\ &= \big(]-\infty,0]\times\{0\}\big)\cup\{x\in\mathbb{R}^2 : x_1\geq 0, x_1+x_2=0\}.\end{aligned}$$

So, given $\bar{x}\in A$, $(0_{\mathcal{L}},-f(\bar{x}))\in \operatorname{epi}_K(f+i_A)^*$ *if and only if*

$$-f(\bar{x})=-\bar{x}\in(f+i_A)^*(0_{\mathcal{L}})+\mathbb{R}^2_+=\{x\in\mathbb{R}^2 : x_2\geq 0 \ \text{or}\ x_1+x_2\geq 0\}$$

if and only if $\bar{x}$ *is a boundary point of* A. *Thus, by Proposition 5.1, we see in this example that the set of weak solutions to (MOP) is nothing else than the boundary of* A. *Observe that these boundary points satisfy*

$$0_{\mathcal{L}}(\bar{x})-f(\bar{x})=-\bar{x}\in \operatorname{WSup}\left\{0_{\mathcal{L}}(x)-(f+i_A)(x) : x\in\mathbb{R}^2\right\},$$

or equivalently,

$$0_{\mathcal{L}}\in\partial(f+i_A)(\bar{x}).$$

Next we establish our main result for (VOP).

Theorem 5.1 *Consider the problem* (VOP) *in (3.1), and let* $\bar{x}\in A\cap\operatorname{dom}f$. *Then the following statements are equivalent:*

$(\mathbf{d_3})$ $\operatorname{epi}_K(f+i_A)^*\cap\{(0_{\mathcal{L}},-f(\bar{x}))\}=\left(\bigcup_{T\in\mathcal{L}_+(S,K)}\operatorname{epi}_K(f+i_C+T\circ g)^*\right)\cap\{(0_{\mathcal{L}},-f(\bar{x}))\}.$

$(\mathbf{e_3})$ $\bar{x}$ *is a weak solution of (VOP) if and only if there exists* $T\in\mathcal{L}_+(S,K)$ *such that*

$$-f(\bar{x})\in(f+i_C+T\circ g)^*(0_{\mathcal{L}})+K.$$

$(\mathbf{f_3})$ $\bar{x}$ *is a weak solution of (VOP) if and only if there exists* $T\in\mathcal{L}_+(S,K)$ *such that*

$$f(x)+(T\circ g)(x)-f(\bar{x})\notin-\operatorname{int}K,\ \forall x\in C.$$

Moreover, if one of the three statements holds then the linear operator $T\in\mathcal{L}_+(S,K)$ *whose existence is stated in* $(\mathbf{e_3})$ *and* $(\mathbf{f_3})$ *can be chosen such that* $-(T\circ g)(\bar{x})\in K\setminus\operatorname{int}K$.

Proof. $[(\mathbf{d_3})\iff(\mathbf{e_3})]$ It follows from Proposition 5.1 that

$$\bar{x}\text{ is a weak solution of (VOP)}\iff(0_{\mathcal{L}},-f(\bar{x}))\in\operatorname{epi}_K(f+i_A)^*. \tag{5.2}$$

On the other hand, it is clear that

$$\begin{array}{c}(0_{\mathcal{L}},-f(\bar{x}))\in\bigcup_{T\in\mathcal{L}_+(S,K)}\operatorname{epi}_K(f+i_C+T\circ g)^*\\ \Updownarrow\\ \exists\, T\in\mathcal{L}_+(S,K):\quad -f(\bar{x})\in(f+i_C+T\circ g)^*(0_{\mathcal{L}})+K,\end{array} \tag{5.3}$$

and hence, the equivalence of $(\mathbf{d_3})$ and $(\mathbf{e_3})$ follows from (5.2) and (5.3).

$[(\mathbf{e_3})\implies(\mathbf{f_3})]$ Assume that $(\mathbf{e_3})$ holds and $\bar{x}$ is a weak solution of (VOP). Then there exists $T\in\mathcal{L}_+(S,K)$ such that $-f(\bar{x})\in(f+i_C+T\circ g)^*(0_{\mathcal{L}})+K$. Then, there exists $k\in K$

such that $-f(\bar{x})-k\in(f+i_C+T\circ g)^*(0_{\mathcal{L}})$. By the definition of the conjugate function, one has

$$(f+i_C+T\circ g)(x)-f(\bar{x})-k\notin-\operatorname{int}K,\ \forall x\in X,$$

so that

$$(f+i_C+T\circ g)(x)-f(\bar{x})\notin-\operatorname{int}K,\ \forall x\in X,$$

or equivalently,

$$f(x)+(T\circ g)(x)-f(\bar{x})\notin-\operatorname{int}K,\ \forall x\in C. \tag{5.4}$$

Conversely, let us take $T\in\mathcal{L}_+(S,K)$ such that (5.4) holds. Now if $x\in C$ and $g(x)\in-S$ then $-(T\circ g)(x)\in K$ (as $T\in\mathcal{L}_+(S,K)$) and it follows from (2.8) and (5.4) that $f(x)-f(\bar{x})\notin-\operatorname{int}K$, which shows that $\bar{x}$ is a weak solution of (VOP).

[($\mathbf{f_3}$) $\implies$ ($\mathbf{e_3}$)] It follows from Lemma 4.2 with $L=0_{\mathcal{L}}$ and $y=-f(\bar{x})$.

Lastly, by substituting $x=\bar{x}$ into (5.4) we get $-(T\circ g)(\bar{x})\notin\operatorname{int}K$. On the other hand, $g(\bar{x})\in-S$, $T\in\mathcal{L}_+(S,K)$ yields $-(T\circ g)(\bar{x})\in K$, and so, $-(T\circ g)(\bar{x})\in K\setminus\operatorname{int}K$. The proof is complete. □

Remark 5.1 *If we consider the (SOP) problem in (3.2) with the assumptions that $f\in\Gamma(X)$, the convex set C is closed, g is S-convex and S-epi closed, Proposition 5.1 offers an asymptotic optimality condition for (SOP): $\bar{x}$ is an optimal solution of (SOP) if and only if*

$$(0_{X^*},-f(\bar{x}))\in\operatorname{epi}(f+i_A)^*=\operatorname{cl}\left(\bigcup_{z^*\in S^+}\operatorname{epi}(f+i_C+z^*\circ g)^*\right)$$

(the last equality follows from [2, Theorem 8.2], see also (3.7)). So in this case, the weak-closedness of the set $\bigcup_{z^*\in S^+}\operatorname{epi}(f+i_C+z^*\circ g)^*$ implies that ($\mathbf{d_3}$) holds at $\bar{x}$. Moreover, in this specific case, one get a non-asymptotic optimality condition for (SOP): $\bar{x}$ is an optimal solution of (SOP) if and only if there is $z^*\in S^+$ such that $(0_{X^*},-f(\bar{x}))\in\operatorname{epi}(f+i_C+z^*\circ g)^*$. This simple example illustrates the use and significance of condition ($\mathbf{d_3}$).*

In the case $g\equiv 0_Z$ and $C=X$, the problem (VOP) becomes the unconstrained vector optimization problem

$$\text{(UVOP)}\qquad \operatorname{WMin}\big\{f(x):x\in X\big\}. \tag{5.5}$$

Corollary 5.1 *Let $\bar{x}\in\operatorname{dom}f$. Then $\bar{x}$ is a weak solution of the problem* (UVOP)*, if and only if $0_{\mathcal{L}}\in\partial f(\bar{x})$.*

Proof. The conclusion follows from Proposition 5.1 with $g\equiv 0_Z$ and $C=X$. □

If we take $Y=\mathbb{R}$ and $K=\mathbb{R}_+$, then the problem (UVOP) collapses to the unconstrained scalar optimization problem

$$\text{(USOP)}\qquad \operatorname{Min}\{f(x):x\in X\}.$$

Then, according to Corollary 5.1, $\bar{x}\in X$ is an optimal solution to (USOP) if and only if $0_{X^*}\in\partial f(\bar{x})$.

We now turn back to the (SOP) problem in (3.2). The optimality conditions above lead us to the corresponding ones for (SOP), which are new and interesting in the sense that they are obtained in absence of assumptions on convexity, lower semicontinuity of functions/mappings and closedness of the constraint set.

Corollary 5.2 *Let $\bar{x} \in A \cap \operatorname{dom} f$. The following statements are equivalent:*

($\mathbf{g_3}$) $(\operatorname{epi}(f + i_A)^*) \cap \{(0_{X^*}, -f(\bar{x}))\} = \left(\bigcup_{z^* \in S^+} \operatorname{epi}(f + i_C + z^* \circ g)^* \right) \cap \{(0_{X^*}, -f(\bar{x}))\}$.

($\mathbf{h_3}$) *$\bar{x}$ is an optimal solution to (SOP) if and only if there exists $z^* \in S^+$ such that*

$$0_{X^*} \in \partial(f + i_C + z^* \circ g)(\bar{x}) \quad \textit{and} \quad (z^* \circ g)(\bar{x}) = 0.$$

($\mathbf{i_3}$) *$\bar{x}$ is an optimal solution to (SOP) if and only if there exists $z^* \in S^+$ such that*

$$f(x) + (z^* \circ g)(x) - f(\bar{x}) \geq 0, \ \forall x \in C.$$

Proof. The conclusion follows from Theorem 5.1, taking into account the equivalence between ($\mathbf{h_3}$) and ($\mathbf{e_3}$) (see, e.g. Proposition 2.4.2(iii) in [26]) since $(z^* \circ g)(\bar{x}) = 0$ as $K \setminus \operatorname{int} K = \mathbb{R}_+ \setminus (\operatorname{int} \mathbb{R}_+) = \{0\}$. □

6 Other applications

In this last section we apply the Farkas-type results for vector-valued functions established in Section 4 to vector variational inequalities and vector equilibrium problems. We are under the same assumptions of the previous sections, i.e., X, Y are lcHtvs, C is a nonempty subset in X, and K is a pointed convex cone in Y such that $\operatorname{int} K \neq \emptyset$.

6.1 Vector variational inequalities

Now we consider the so-called *extended vector variational inequality* problem

$$\text{(EVVI)} \quad \text{Find } x \in C \text{ such that } \ F(x)(z - x) \not<_K H(x) - H(z) \ \text{ for all } z \in C,$$

where $F : X \to \mathcal{L}(X, Y)$ and $H : X \to Y$.

If $H = 0$, we obtain the *vector variational inequality* problem

$$\text{(VVI)} \quad \text{Find } x \in C \text{ such that } F(x)(z - x) \not<_K 0_Y \ \text{ for all } z \in C.$$

Remark 6.1 (*a*) (EVVI) *was introduced in [3, p.356] with the efficient ordering in Y generated by K ($y_1 \leq_K y_2 \iff y_2 - y_1 \in K \setminus \{0_Y\}$) in a more general form (with H being a set-valued mapping).*
(*b*) *When $Y = \mathbb{R}$ and $K = \mathbb{R}^+$, then the problem* (EVVI) *becomes the general variational inequality problem*

$$\textit{Find } x \in C \textit{ such that } F(x)(z - x) \geq H(x) - H(z) \ \textit{ for all } z \in C.$$

Such a model covers the special case when F is a continuous linear operator from X to X^*, and H is a proper, lsc and convex function considered in [11], [17]. In this case, the problem (VVI) *collapses to the ordinary variational inequality problem*

$$\textit{Find } x \in C \textit{ such that } F(x)(z - x) \geq 0 \ \textit{ for all } z \in C.$$

For a fixed $\bar{x} \in C$, we consider the vector optimization problem

$$(\mathrm{VOP}(F, H, \bar{x})) \qquad \mathrm{WMin}\,\{F(\bar{x})(z-\bar{x}) + H(z) - H(\bar{x})\,:\,z \in C\}\,.$$

It is worth observing that $\bar{x} \in C$ is a solution of the problem (EVVI) if only if $\bar{x}$ is a weak solution to the vector optimization problem $(\mathrm{VOP}(F, H, \bar{x}))$.

Theorem 6.1 *Let $\bar{x} \in C$. Then the following statements are equivalent:*
$(\mathbf{a_4})$ *$\bar{x}$ is a solution of* (EVVI)*,*
$(\mathbf{b}_4)$ *$-F(\bar{x}) \in \partial(H + i_C)(\bar{x})$,*
$(\mathbf{c_4})$ *$-(F(\bar{x}), H(\bar{x}) + F(\bar{x})(\bar{x})) \in \mathrm{epi}_K(H + i_C)^*$.*

Proof. Let $f_{\bar{x}} : X \to Y$ be the vector-valued function defined by

$$f_{\bar{x}}(\cdot) := F(\bar{x})(\cdot - \bar{x}) + H(\cdot) - H(\bar{x}). \tag{6.1}$$

It is obvious that

$$(\mathbf{a_4}) \iff f_{\bar{x}}(\bar{x}) = 0_Y \in \mathrm{WMin}\, f_{\bar{x}}(C).$$

$[(\mathbf{a_4}) \Longleftrightarrow (\mathbf{c_4})]$ In order to apply Proposition 5.1 we need to make some previous calculations. By the definition of conjugate function, we get

$$\begin{aligned}(f_{\bar{x}} + i_C)^*(L) &= \mathrm{WSup}\,\{L(z) - f_{\bar{x}}(z) - i_C(z) : z \in X\}\\ &= \mathrm{WSup}\,\{(L - F(\bar{x}))(z) - (H + i_C)(z) : z \in X\} + H(\bar{x}) + F(\bar{x})(\bar{x})\\ &= (H + i_C)^*\,(L - F(\bar{x})) + H(\bar{x}) + F(\bar{x})(\bar{x}).\end{aligned}$$

(The second equality above holds by [25, Proposition 3.2(i)]). Hence,

$$\mathrm{epi}_K(f_{\bar{x}} + i_C)^* = \mathrm{epi}_K(H + i_C)^* + (F(\bar{x}), H(\bar{x}) + F(\bar{x})(\bar{x}))\,. \tag{6.2}$$

Proposition 5.1 yields the equivalence between $(\mathbf{a_4})$ and the condition $(0_{\mathcal{L}}, 0_Y) \in \mathrm{epi}(f_{\bar{x}}+i_C)^*$, but from (6.2)

$$(0_{\mathcal{L}}, 0_Y) \in \mathrm{epi}(f_{\bar{x}} + i_C)^* \iff -(F(\bar{x}), H(\bar{x}) + F(\bar{x})(\bar{x})) \in \mathrm{epi}_K(H + i_C)^*, \tag{6.3}$$

which is $(\mathbf{c_4})$.

$[(\mathbf{a_4}) \Longleftrightarrow (\mathbf{b_4})]$ It also follows from Proposition 5.1, which establishes

$$(\mathbf{a_4}) \iff 0_{\mathcal{L}} \in \partial(f_{\bar{x}} + i_C)(\bar{x}),$$

or, equivalently,

$$\begin{aligned}0_Y &= 0_{\mathcal{L}}(\bar{x}) - (f_{\bar{x}} + i_C)(\bar{x}) \in (f_{\bar{x}} + i_C)^*\,(0_{\mathcal{L}}) \qquad (6.4)\\ &= (H + i_C)^*\,(-F(\bar{x})) + H(\bar{x}) + F(\bar{x})(\bar{x}). \qquad (6.5)\end{aligned}$$

This is also equivalent to

$$-F(\bar{x})(\bar{x}) - H(\bar{x}) - i_C(\bar{x}) \in (H + i_C)^*\,(-F(\bar{x}))\,,$$

which accounts for (see Proposition 2.3)

$$-F(\bar{x}) \in \partial\,(H + i_C)\,(\bar{x}),$$

and the equivalence is proved. The proof is complete. □

Example 6.1 *Consider the following problem (VVI) with $X = Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$:*

$$\text{Find } x \in C \text{ such that } \quad \begin{pmatrix} x_1 & x_1 \\ x_2 & 0 \end{pmatrix} \begin{pmatrix} z_1 - x_1 \\ z_2 - x_2 \end{pmatrix} \not<_{\mathbb{R}^2_+} \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \forall z \in C,$$

where $C = \{z \in \mathbb{R}^2 : z_1 + z_2 \leq 2\}$, and again $Y^\bullet = \mathbb{R}^2 \cup \{-\infty_{\mathbb{R}^2}\} \cup \{+\infty_{\mathbb{R}^2}\}$. Now we show that $\bar{x} = (1, 0)$ is a solution of (VVI). Indeed,

$$\begin{aligned} i_C^*(-F(\bar{x})) &= \text{WSup}\left\{-F(\bar{x})(z) - i_C(z) : z \in \mathbb{R}^2\right\} \\ &= \text{WSup}\left\{-\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} : z_1 + z_2 \leq 2\right\} \\ &= \text{WSup}\left\{\begin{pmatrix} -z_1 - z_2 \\ 0 \end{pmatrix} : z_1 + z_2 \leq 2\right\} \\ &= \text{WSup}\left\{[-2, +\infty[\times\{0\}\right\} \\ &= \mathbb{R} \times \{0\}. \end{aligned}$$

It is easy to see that $-F(\bar{x})(\bar{x}) - i_C(\bar{x}) = (-1, 0) \in i_C^*(-F(\bar{x}))$, which is equivalent to $-F(\bar{x}) \in \partial i_C(\bar{x})$. By Theorem 6.1, $\bar{x} = (1, 0)$ is a solution of (VVI) (here $H = 0$).

6.2 Vector equilibrium problem

Let $F : X \times X \to Y^\bullet$ be a bifunction satisfying $F(x, x) = 0_Y$ for all $x \in C$. We consider the *vector equilibrium problem*

$$\text{(VEP)} \quad \text{Find } x \in C \text{ such that } \; F(x, z) \not<_K 0_Y \; \text{ for all } z \in C.$$

Remark 6.2 (a) (VEP) *was introduced in [3, p.380] for a set-valued map $F : X \times X \rightrightarrows Y \cup \{+\infty_Y\}$.*
(b) *When $Y = \mathbb{R}$ and $K = \mathbb{R}_+$,* (VEP) *collapses to the equilibrium problem*

$$\text{Find } x \in C \text{ such that } F(x, z) \geq 0 \; \text{ for all } z \in C.$$

For a fixed $\bar{x} \in C$, we consider the vector optimization problem associated with (VEP):

$$(\text{VOEP}(F, \bar{x})) \quad \text{WMin}\left\{F(\bar{x}, z) : z \in C\right\}.$$

Observe that $\bar{x} \in C$ is a solution of (VEP) if and only if $\bar{x}$ is a weak solution of (VOEP$(F, \bar{x})$). The following result is a straightforward consequence of Proposition 5.1.

Theorem 6.2 *Let $\bar{x} \in C$. Then the following statements are equivalent:*
($\mathbf{a_5}$) *$\bar{x}$ is a solution of* (VEP)*,*
($\mathbf{b}_5$) $0_{\mathcal{L}} \in \partial(F(\bar{x}, \cdot) + i_C)(\bar{x})$,
($\mathbf{c_5}$) $(0_{\mathcal{L}}, 0_Y) \in \text{epi}_K(F(\bar{x}, \cdot) + i_C)^*$.

Example 6.2 *Consider the following (VEP) problem, with $X = Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$:*

$$\text{(VEP1)} : \text{Find } x \in C \text{ such that } F(x, z) = \begin{pmatrix} x_1^2 - x_1 z_1 \\ x_2 - z_2 \end{pmatrix} \not<_{\mathbb{R}^2_+} \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \; \forall z \in C,$$

where $C = [-1,1] \times [-1,1]$. *Obviously,* $F(x,x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ *for all* $x \in \mathbb{R}^2$. *We shall show that* $\bar{x} = (1,0)$ *is a solution of (VEP). Indeed, observe that*

$$\begin{aligned} \left(F(\bar{x},\cdot)+i_C\right)^*(0_{\mathcal{L}}) &= \operatorname{WSup}\Big\{ -F(\bar{x},z) - i_C(z) : z \in \mathbb{R}^2 \Big\} \\ &= \operatorname{WSup}\left\{ \begin{pmatrix} z_1 - 1 \\ z_2 \end{pmatrix} : z \in [-1,1] \times [-1,1] \right\} \\ &= \operatorname{WSup}\big\{[-2,0] \times [-1,1]\big\} \\ &= \big(]-\infty,0] \times \{1\}\big) \cup \big(\{0\} \times]-\infty,1]\big). \end{aligned}$$

It is easy to see that $0_{\mathcal{L}}(\bar{x}) - \big(F(\bar{x},\bar{x}) + i_C(\bar{x})\big) = (0,0) \in \big(F(\bar{x},\cdot)+i_C\big)^*(0_{\mathcal{L}})$, *which is equivalent to* $0_{\mathcal{L}} \in \partial\big(F(\bar{x},\cdot)+i_C\big)(\bar{x})$. *It follows from Theorem 6.2 that* $\bar{x} = (1,0)$ *is a solution of the problem* (VEP1).

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