View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Repositório Institucional da Universidade de Aveiro

Two-sided hypergenic functions*

S.L. Eriksson‡, H. Orelma‡ and N. Vieira†

Department of Mathematics, Tampere University of Technology
P.O.Box 553, FI-33101 Tampere, Finland

† CIDMA - Center for Research and Development in Mathematics and Applications
Department of Mathematics, University of Aveiro
Campus Universitário de Santiago, 3810-193 Aveiro, Portugal.
E-mail: nloureirovieira@gmail.com

Abstract

In this paper we present an analogous of the class of two-sided axial monogenic functions to the case of axial $\kappa-$hypermonogenic functions. In order to do that we will solve a Vekua-type system in terms of Bessel functions.

Keywords: Clifford algebras; Hypergenic functions; $\kappa-$hypergenic functions; Two-side hypergenic functions.

MSC2010: 30G20; 30G35; 33C10; 30A05; 30B30.

1 Introduction

There exist two generalizations of classical complex analysis to higher dimensions using geometric algebras. The first one is the theory of monogenic functions introduced R. Delanghe in [1] based on Euclidian metric, and further developed in [2]. In [7] the solutions of the Dirac equation were obtained by considering functions of axial type. This indeed gave rise to Vekua-type systems that was solved in terms of special functions.

The second one is the theory of hypermonogenic functions based on the hyperbolic model. The advantage of hypermonogenic functions is that the positive and negative powers of hypercomplex variables are included into the theory, which is not in the monogenic case. Hence elementary functions can be defined similarly as in the classical complex case. In [4]-[6], H. Leutwiler and S.L. Eriksson introduced hypermonogenic and $\kappa-$hypermonogenic functions, and studied some of their properties in Clifford analysis. Hypermonogenic functions are generalizations of the monogenic functions and the $\kappa-$monogenic functions are extensions of the hypermonogenic functions. When $\kappa = n-1$, a $\kappa-$hypermonogenic function is a hypermonogenic function, and when $\kappa = 0$, a $\kappa-$hypermonogenic function is a monogenic function.

The aim of this paper is to introduce a similar class of the axial two-sided monogenic functions presented in [7] for the case of $\kappa-$hypermonogenic functions, and hence to provide new characteristics of the two-sided $\kappa-$monogenic functions. In order to do that we structured the paper as follows: in the Preliminaries section we recall some basic notions about Clifford analysis and $\kappa-$hypermonogenic functions. In Section 3, we study the generalized Cauchy-Riemann -system for two-sided $\kappa-$hypermonogenic functions. In the last section we investigate axial symmetry for the solutions of the two-sided $\kappa-$hypermonogenic system in terms of Bessel functions.

*The final version is published in *Advances in Applied Clifford Algebras*, **27**-No.1 (2017), 111-123. It as available via the website: https://link.springer.com/article/10.1007/s00006-015-0605-2

2 Preliminaries

2.1 Theory of monogenic functions based in the Euclidian metric

We consider the n-dimensional vector space $\mathbb{R}^n$ endowed with an orthonormal basis $\{e_1, \cdots, e_n\}$. We define the universal real Clifford algebra $C\ell_{0,n}$ as the 2^n-dimensional associative algebra which obeys the multiplication rules $e_ie_j + e_je_i = -2\delta_{i,j}$. A vector space basis for $C\ell_{0,n}$ is generated by the elements $e_0 = 1$ and $e_A = e_{h_1} \cdots e_{h_k}$, where $A = \{h_1, \ldots, h_k\} \subset M = \{1, \ldots, n\}$, for $1 \leq h_1 < \cdots < h_k \leq n$. Each element $x \in C\ell_{0,n}$ can be represented by $x = \sum_A x_A e_A$, $x_A \in \mathbb{R}$. The main involution is defined by $1' = 1$, $e_j' = -e_j$ for all $j = 1, \ldots, n$, and we have $(ab)' = a'b'$.

A function $u : U \to C\ell_{0,n}$ has a representation $u = \sum_A u_A e_A$ with $C\ell_{0,n}$-valued components u_A. Properties such as continuity will be understood component-wisely. Consider the null solutions of the so-called generalized Cauchy-Riemann or Dirac operator $\partial_{\mathbf{x}} = \sum_{j=1}^n e_j \partial_{x_j}$ in $\mathbb{R}^n$. More precisely, a $C\ell_{0,n}-$valued function u defined on a open set $U \subseteq \mathbb{R}^n$, continuous and differentiable, is said to be left (resp. right) monogenic in U if $\partial_{\mathbf{x}} u = 0$ (resp. $u\partial_{\mathbf{x}} = 0$). Furthermore, functions which are both left and right monogenic are called two-sided monogenic, i.e., functions that satisfy $\partial_{\mathbf{x}} u = 0 = u\partial_{\mathbf{x}}$. An important application of this class of functions is in the resolution of many important systems such as the Riesz system, the Maxwell equations and the Hodge system. For more details about Clifford analysis we refer [1, 2].

In this paper we study the theory related to case $\mathbb{R}^{n-1}$. Let $\underline{x} \in \mathbb{R}^{n-1}$, $f : \Omega \to C\ell_{0,n-1}$ a differentiable function, where $\Omega \subset \mathbb{R}^{n-1}$. For the Dirac operator the so-called commutator relations are

$$\begin{aligned} \partial_{\underline{x}}(\underline{x}f) &= -(n-1)f - 2Ef - \underline{x}\partial_{\underline{x}}f, \\ (f\underline{x})\partial_{\underline{x}} &= -(n-1)f - 2Ef - (f\partial_{\underline{x}})\underline{x}, \end{aligned}$$

where $Ef = \sum_{j=1}^{n-1} x_j \partial_{x_j}$ is the Euler operator and the Dirac operator is decomposed as $\partial_{\mathbf{x}} = \partial_{\underline{x}} + e_n \partial_{x_n} = \sum_{j=1}^{n} e_j \partial_{x_j} + e_n \partial_{x_n}$.

One of standard class of monogenic functions are so called spherical monogenics. We denote by $P_{k,l}(\underline{x})$ a k-homogeneous monogenic polynomial with l-multivector values, i.e., $\partial_{\underline{x}} P_{k,l}(\underline{x}) = P_{k,l}(\underline{x})\partial_{\underline{x}} = 0$, $P_{k,l}(\underline{x})' = (-1)^l P_{k,l}(\underline{x})$ and $EP_{k,l}(\underline{x}) = kP_{k,l}(\underline{x})$. We will also consider the following formula presented in [7]

$$\partial_{\underline{x}}(P_{k,l}(\underline{x})\underline{x}) = (\underline{x}P_{k,l}(\underline{x}))\partial_{\underline{x}} = (-1)^l(2l - n + 1)P_{k,l}(\underline{x}). \tag{1}$$

Also if $r = |\underline{x}|$ and B is a real valued differentiable function, then $\partial_{\underline{x}} B(r^2, x_n) = 2\underline{x}\partial_1 B(r^2, x_n)$.

Suppose that $P_k(\underline{x})$, $k \in \mathbb{N}_0$, is a left monogenic homogeneous polynomial of degree k in $\mathbb{R}^n$ and with values in the real Clifford algebra $C\ell_{0,n}$. A remarkable class of monogenic functions are the so-called axial left monogenic functions of degree k (see [9, 10]). These are left monogenic functions of the form $\left(A(x_0, r) + \frac{\underline{x}}{r}\, B(x_0, r)\right)\, P_k(\underline{x})$, with A and B being $\mathbb{R}-$valued and continuously differentiable functions depending on the two variables (x_0, r). The functions A and B satisfy the Vekua system (see [11])

$$\left\{ \begin{array}{l} \partial_{x_0} A - \partial_r B = \frac{3k+m-1}{r}\, B \\ \partial_{x_0} B + \partial_r A = 0 \end{array} \right. .$$

We point out that every left monogenic function u defined in a open set of $\mathbb{R}^{n+1}$ invariant under $SO(n)$ may be written as $u = \sum_{k=0}^{\infty} M_k$, where M_k is an axial left monogenic functions of degree k (see [9]). In [7] the authors introduced a similar class in the space of two-sided monogenic functions, which they called axial two-sided monogenic functions.

2.2 Theory of hypermonogenic functions based on the hyperbolic metric

In the case of hypermonogenic functions we deal with the modified Dirac operator $\partial_x u + \frac{\kappa}{x_n} Qu$, functions $f : \Omega \to C\ell_{0,n}$ and $\Omega \subset \mathbb{R}^{n+1}$, where $'$ is the main involution such that $Q'f := (Qf)'$ and Qu is given by the decomposition $u = Pu + Que_n$ with $Pu, Qu \in C\ell_{0,n}$. The functions satisfying the preceding equation are called $\kappa-$hypermonogenic functions. In the case $k = n - 1$ they called hypermonogenic functions. The Dirac operator is defined by $\partial_x = \partial_{x_0} + e_1\partial_{x_1} + \cdots + e_n\partial_{x_n}$ and functions are with the paravector variable $f = f(x)$ with $x = x_0 + x_1e_1 + \cdots + x_ne_n$. In this paper we consider theory, which based vector variables and derivates. In this case we will talk hypergenic functions instead of hypermonogenic functions.

3 Generalized Cauchy-Riemann -systems

Consider a functions $f:\Omega\to C\ell_{0,n}$. We may always write these functions on the form $f=Pf+e_nQf$, where $Pf,Qf:\Omega\to C\ell_{0,n-1}$. In the hyperbolic function theory the fundamental operators are

$$M_\kappa^l f=\partial_{\mathbf{x}}f+\frac{\kappa}{x_n}Qf,\qquad M_\kappa^r f=f\partial_{\mathbf{x}}+\frac{\kappa}{x_n}Q'f.$$

These operators act on differentiable functions $f:\Omega\to C\ell_{0,n}$ and they are called modified Dirac operators. If $M_\kappa^l f=0$ (or $M_\kappa^r f=0$) a function is called left (or right) κ-hypergenic function. If $M_\kappa^l f=M_\kappa^r f=0$ is f called two-sided κ-hypergenic . Let us first prove some formulae related to P and Q operators.

Proposition 3.1 *If $\underline{x}\in\mathbb{R}^{n-1}$ and $f:\Omega\to C\ell_{0,n}$, then*

(a) $P(\underline{x}f)=\underline{x}Pf$ and $P(f\underline{x})=Pf\underline{x}$

(b) $Q(\underline{x}f)=-\underline{x}Qf$ and $Q(f\underline{x})=Qf\underline{x}$

(c) $P(e_nf)=-Qf$ and $Q(e_nf)=Pf$.

(d) $P(e_n\underline{x}f)=\underline{x}Qf$ and $Q(e_n\underline{x}f)=\underline{x}Pf$

(e) $P(e_n\underline{x}f\underline{x})=\underline{x}Qf\underline{x}$ and $Q(e_n\underline{x}f\underline{x})=\underline{x}Pf\underline{x}$.

Proof: All the properties follow from straightforward calculations. For example, for the second property we have

$$\underline{x}f=\underline{x}Pf+\underline{x}e_nQf=\underline{x}Pf-e_n\underline{x}Qf,\qquad f\underline{x}=Pf\underline{x}+e_nQf\underline{x}.$$

From the previous equalities follows the second property. In a similar way we proceed for the other cases.

■

Theorem 3.2 (Generalized Cauchy-Riemann -system for two-sided κ-hypermonogenic functions) *If $f=Pf+e_nQf$ is differentiable function then $M_\kappa^l f=M_\kappa^r f=0$ if and only if*

$$\begin{aligned}
&\partial_{\underline{x}}Pf-\partial_{x_n}Qf+\frac{\kappa}{x_n}Qf=0,\\
&\partial_{x_n}Pf-\partial_{\underline{x}}Qf=0,\\
&(Pf)\partial_{\underline{x}}-\partial_{x_n}Q'f+\frac{\kappa}{x_n}Q'f=0,\\
&\partial_{x_n}P'f+(Qf)\partial_{\underline{x}}=0.
\end{aligned}$$

Proof: We compute

$$M_\kappa^l f\ =(\partial_{\mathbf{x}}+\frac{\kappa}{x_n}Q)(Pf+e_nQf)\ =\partial_{\mathbf{x}}Pf-e_n\partial_{\underline{x}}Qf-\partial_{x_n}Qf+\frac{\kappa}{x_n}Qf.$$

Since $e_nA=A'e_n$, if $A\in\mathcal{C}\ell_{n-1}$, we have

$$M_\kappa^r f\ =M_\kappa^r(Pf+e_nQf)\ =(Pf)\partial_{\mathbf{x}}+e_n(Qf)\partial_{\underline{x}}-\partial_{x_n}Q'f+\frac{\kappa}{x_n}Q'f.$$

Hence, we obtain

$$\begin{aligned}
P(M_\kappa^l f)&=\partial_{\underline{x}}Pf-\partial_{x_n}Qf+\frac{\kappa}{x_n}Qf,\\
Q(M_\kappa^l f)&=\partial_{x_n}Pf-\partial_{\underline{x}}Qf,\\
P(M_\kappa^r f)&=(Pf)\partial_{\underline{x}}-\partial_{x_n}Q'f+\frac{\kappa}{x_n}Q'f,\\
Q(M_\kappa^r f)&=\partial_{x_n}P'f+(Qf)\partial_{\underline{x}}.
\end{aligned}$$

■

Let us also derive some formulae for modified Dirac operators.

Proposition 3.3 *Let $f:\Omega\to C\ell_{0,n}$ is a differentiable function. Then*

(a) $M_\kappa^l(\underline{x}f) = -(n-1)f - 2Ef - \underline{x}M_\kappa^l f$,

(b) $M_\kappa^r(f\underline{x}) = -(n-1)f - 2Ef - (M_\kappa^r f)\underline{x}$,

(c) $M_\kappa^l(e_n f) + e_n M_\kappa^l f = -2\partial_{x_n} f + \frac{\kappa}{x_n} f$,

(d) $M_\kappa^r(e_n f) + e_n M_\kappa^r f = 2e_n f\partial_{\mathbf{x}} + \frac{\kappa}{x_n} f'$.

Proof: We have

$$\begin{aligned} M_\kappa^l(\underline{x}f) &= \partial_{\mathbf{x}}(\underline{x}f) + \frac{\kappa}{x_n}Q(\underline{x}f) \\ &= \partial_{\underline{x}}(\underline{x}f) + e_n\partial_{x_n}(\underline{x}f) + \frac{\kappa}{x_n}Q(\underline{x}f) \\ &= -(n-1)f - 2Ef - \underline{x}\partial_{\underline{x}}f + e_n\partial_{x_n}(\underline{x}f) - \frac{\kappa}{x_n}\underline{x}Qf \\ &= -(n-1)f - 2Ef - \underline{x}M_\kappa^l f. \end{aligned}$$

In a very similar way

$$M_\kappa^r(f\underline{x}) = -(n-1)f - 2Ef - (M_\kappa^r f)\underline{x}.$$

We have also that

$$\begin{aligned} M_\kappa^l(e_n f) &= M_\kappa^l(e_n Pf - Qf) = \partial_{\mathbf{x}}(e_n f) + \frac{\kappa}{x_n}Pf \\ &= \partial_{\underline{x}}(e_n f) + e_n\partial_{x_n}(e_n f) + \frac{\kappa}{x_n}Pf \\ &= -e_n\partial_{\underline{x}}f - \partial_{x_n}f + \frac{\kappa}{x_n}Pf, \end{aligned}$$

and

$$e_n M_\kappa^l f = e_n\partial_{\mathbf{x}}f + \frac{\kappa}{x_n}e_n Qf = e_n\partial_{\underline{x}}f - \partial_{x_n}f + \frac{\kappa}{x_n}e_n Qf.$$

Combining the previous equalities we obtain

$$M_\kappa^l(e_n f) + e_n M_\kappa^l f = -2\partial_{x_n}f + \frac{\kappa}{x_n}f.$$

For the terms in the right-hand side of the last equality we have

$$M_\kappa^r(e_n f) = M_\kappa^r(e_n Pf - Qf) = e_n f\partial_{\mathbf{x}} + \frac{\kappa}{x_n}P'f$$

and

$$e_n M_\kappa^r f = e_n f\partial_{\mathbf{x}} + \frac{\kappa}{x_n}e_n Q'f,$$

which implies

$$M_\kappa^r(e_n f) + e_n M_\kappa^r f = e_n f\partial_{\mathbf{x}} + \frac{\kappa}{x_n}P'f + e_n f\partial_{\mathbf{x}} + \frac{\kappa}{x_n}e_n Q'f = 2e_n f\partial_{\mathbf{x}} + \frac{\kappa}{x_n}f'.$$

■

We look for solutions which are invariant around x_n-axis, i.e., depend only coordinates (r, x_n), where $r = |\underline{x}|$. Therefore we consider only the functions of the form

$$G = PG + e_n QG, \tag{2}$$

where

$$\begin{aligned} PG &= A_2(r^2, x_n)\underline{x}P_{k,l}(\underline{x}) + A_3(r^2, x_n)P_{k,l}(\underline{x})\underline{x}, \\ QG &= A_1(r^2, x_n)P_{k,l}(\underline{x}) + A_4(r^2, x_n)\underline{x}P_{k,l}(\underline{x})\underline{x}, \end{aligned}$$

and $P_{k,l}$ is k-homogeneous l-multivector valued spherical monogenic. We want to find the functions A_i such that G is a two-sided κ-hypergenic. In order to proceed we need to prove the following technical lemmata.

Lemma 3.4 *If $B = B(r^2, x_n)$ is an arbitrary differentiable function, then*

$$\begin{aligned}\partial_{\underline{x}}(B\underline{x}P_{k,l}(\underline{x})) &= (BP_{k,l}(\underline{x})\underline{x})\partial_{\underline{x}}\\ &= \big(-2\partial_1 B(r^2,x_n)r^2 - (n+2k-1)B(r^2,x_n)\big)P_{k,l}(\underline{x}),\\ \partial_{\underline{x}}(BP_{k,l}(\underline{x})\underline{x}) &= (B\underline{x}P_{k,l}(\underline{x}))\partial_{\underline{x}}\\ &= 2\partial_1 B(r^2,x_n)\underline{x}P_{k,l}(\underline{x})\underline{x} + (-1)^l(2l-n+1)B(r^2,x_n)P_{k,l}(\underline{x}).\end{aligned}$$

Proof: The commutator relations gives

$$\partial_{\underline{x}}(B\underline{x}P_{k,l}(\underline{x})) = -(n+2k-1)B(r^2,x_n)P_{k,l}(\underline{x}) - 2\partial_1 B(r^2,x_n)r^2P_{k,l}(\underline{x}).$$

Using formula (1) we obtain

$$\partial_{\underline{x}}(BP_{k,l}(\underline{x})\underline{x}) = 2\partial_1 B(r^2,x_n)\underline{x}P_{k,l}(\underline{x})\underline{x} + (-1)^l(2l-n+1)B(r^2,x_n)P_{k,l}(\underline{x}).$$

■

Lemma 3.5 *If $B = B(r^2, x_n)$ is an arbitrary differentiable function, then*

$$\begin{aligned}\partial_{\underline{x}}(B(r^2,x_n)\underline{x}P_{k,l}(\underline{x})\underline{x}) &= \big(-2\partial_1 B(r^2,x_n)r^2 - (n+2k+1)B(r^2,x_n)\big)P_{k,l}(\underline{x})\underline{x}\\ &\quad + (-1)^{l+1}(2l-n+1)B(r^2,x_n)\underline{x}P_{k,l}(\underline{x}),\end{aligned}$$

$$\begin{aligned}(B(r^2,x_n)\underline{x}P_{k,l}(\underline{x})\underline{x})\partial_{\underline{x}} &= \big(-2\partial_1 B(r^2,x_n)r^2 - (n+2k+1)B(r^2,x_n)\big)\underline{x}P_{k,l}(\underline{x})\\ &\quad + (-1)^{l+1}(2l-n+1)B(r^2,x_n)P_{k,l}(\underline{x})\underline{x}.\end{aligned}$$

Proof: Using the commutator relation we have

$$\begin{aligned}\partial_{\underline{x}}(\underline{x}P_{k,l}(\underline{x})\underline{x}) &= -(n-1)P_{k,l}(\underline{x})\underline{x} - 2E(P_{k,l}(\underline{x})\underline{x}) - \underline{x}\partial_{\underline{x}}(P_{k,l}(\underline{x})\underline{x})\\ &= -(n+2k+1)P_{k,l}(\underline{x})\underline{x} + (-1)^{l+1}(2l-n+1)\underline{x}P_{k,l}(\underline{x}),\end{aligned}$$

and similarly

$$(\underline{x}P_{k,l}(\underline{x})\underline{x})\partial_{\underline{x}} = -(n+2k+1)\underline{x}P_{k,l}(\underline{x}) + (-1)^{l+1}(2l-n+1)P_{k,l}(\underline{x})\underline{x}.$$

Then we have

$$\begin{aligned}\partial_{\underline{x}}(B(r^2,x_n)\underline{x}P_{k,l}(\underline{x})\underline{x}) &= \partial_{\underline{x}}B(r^2,x_n)\underline{x}P_{k,l}(\underline{x})\underline{x} + B(r^2,x_n)\partial_{\underline{x}}(\underline{x}P_{k,l}(\underline{x})\underline{x})\\ &= \big(-2\partial_1 B(r^2,x_n)r^2 - (n+2k+1)B(r^2,x_n)\big)P_{k,l}(\underline{x})\underline{x}\\ &\quad + (-1)^{l+1}(2l-n+1)B(r^2,x_n)\underline{x}P_{k,l}(\underline{x}).\end{aligned}$$

Similarly

$$\begin{aligned}(B(r^2,x_n)\underline{x}P_{k,l}(\underline{x})\underline{x})\partial_{\underline{x}} &= \underline{x}P_{k,l}(\underline{x})\underline{x}\partial_{\underline{x}}B(r^2,x_n) + B(r^2,x_n)(\underline{x}P_{k,l}(\underline{x})\underline{x})\partial_{\underline{x}}\\ &= \big(-2\partial_1 B(r^2,x_n)r^2 - (n+2k+1)B(r^2,x_n)\big)\underline{x}P_{k,l}(\underline{x})\\ &\quad + (-1)^{l+1}(2l-n+1)B(r^2,x_n)P_{k,l}(\underline{x})\underline{x}.\end{aligned}$$

■

Using these we may compute the following technical lemmata.

Lemma 3.6 *If PG is as in above then*

$$\begin{aligned}\partial_{\underline{x}}PG = (PG)\partial_{\underline{x}} &= \big(-2\partial_1 A_2(r^2,x_n)r^2 - (n+2k-1)A_2(r^2,x_n)\\ &\quad + (-1)^l(2l-n+1)A_3(r^2,x_n)\big)P_{k,l}(\underline{x}) + 2\partial_1 A_3(r^2,x_n)\underline{x}P_{k,l}(\underline{x})\underline{x}\end{aligned}$$

Proof: We compute

$$\begin{aligned}\partial_{\underline{x}}PG = (PG)\partial_{\underline{x}} &= \partial_{\underline{x}}(A_2(r^2,x_n)\underline{x}P_{k,l}(\underline{x})) + \partial_{\underline{x}}(A_3(r^2,x_n)P_{k,l}(\underline{x})\underline{x})\\ &= \big(-2\partial_1 A_2(r^2,x_n)r^2 - (n+2k-1)A_2(r^2,x_n)\big)P_{k,l}(\underline{x})\\ &\quad + 2\partial_1 A_3(r^2,x_n)\underline{x}P_{k,l}(\underline{x})\underline{x} + (-1)^l(2l-n+1)A_3(r^2,x_n)P_{k,l}(\underline{x})\\ &= \big(-2\partial_1 A_2(r^2,x_n)r^2 - (n+2k-1)A_2(r^2,x_n)\\ &\quad + (-1)^l(2l-n+1)A_3(r^2,x_n)\big)P_{k,l}(\underline{x}) + 2\partial_1 A_3(r^2,x_n)\underline{x}P_{k,l}(\underline{x})\underline{x}.\end{aligned}$$

■

We may also deduce that

$$P'G = A_2(r^2, x_n)\underline{x}' P_{k,l}(\underline{x})' + A_3(r^2, x_n) P_{k,l}(\underline{x})' \underline{x}' = (-1)^{l+1} QG,$$

and

$$Q'G = A_1(r^2, x_n) P_{k,l}(\underline{x})' + A_4(r^2, x_n)\underline{x}' P_{k,l}(\underline{x})' \underline{x}' = (-1)^{l} QG.$$

In this case the generalized CR-system takes the form

$$\partial_{\underline{x}} PG - \partial_{x_n} QG + \frac{\kappa}{x_n} QG = 0, \tag{3}$$

$$\partial_{x_n} PG - \partial_{\underline{x}} QG = 0, \tag{4}$$

$$\partial_{\underline{x}} PG - (-1)^{l+1} \partial_{x_n} QG + (-1)^{l+1} \frac{\kappa}{x_n} QG = 0, \tag{5}$$

$$(-1)^{l+1} \partial_{x_n} QG + (QG)\partial_{\underline{x}} = 0. \tag{6}$$

For Q-part we obtain the following symmetric formulae.

Lemma 3.7 *If QG is as in above then*

$$\begin{aligned} \partial_{\underline{x}} QG = {} & 2\partial_1 A_1(r^2, x_n)\underline{x} P_{k,l}(\underline{x}) \\ & + \big(-2\partial_1 A_4(r^2, x_n) r^2 - (n+2k+1) A_4(r^2, x_n)\big) P_{k,l}(\underline{x})\underline{x} \\ & + (-1)^{l+1}(2l - n + 1) A_4(r^2, x_n)\underline{x} P_{k,l}(\underline{x}) \end{aligned}$$

and

$$\begin{aligned} (QG)\partial_{\underline{x}} = {} & 2\partial_1 A_1(r^2, x_n) P_{k,l}(\underline{x})\underline{x} \\ & + \big(-2\partial_1 A_4(r^2, x_n) r^2 - (n+2k+1) A_4(r^2, x_n)\big)\underline{x} P_{k,l}(\underline{x}) \\ & + (-1)^{l+1}(2l - n + 1) A_4(r^2, x_n) P_{k,l}(\underline{x})\underline{x}. \end{aligned}$$

4 Two-sided κ-monogenicity

Now we write the preceding Cauchy-Riemann -system in component form using the formulas in above. After a straightforward computation, equations (3) and (4) gives us the system

$$\left\{ \begin{array}{l} \partial_{x_n} A_1(r^2, x_n) - \frac{\kappa}{x_n} A_1(r^2, x_n) + 2\partial_1 A_2(r^2, x_n) r^2, \\ \quad +(n + 2k - 1) A_2(r^2, x_n) - (-1)^l (2l - n + 1) A_3(r^2, x_n) = 0, \\ 2\partial_1 A_1(r^2, x_n) - \partial_{x_n} A_2(r^2, x_n) - (-1)^l (2l - n + 1) A_4(r^2, x_n) = 0, \\ 2\partial_1 A_4(r^2, x_n) r^2 + \partial_{x_n} A_3(r^2, x_n) + (n + 2k + 1) A_4(r^2, x_n) = 0, \\ \partial_{x_n} A_4(r^2, x_n) - \frac{\kappa}{x_n} A_4(r^2, x_n) - 2\partial_1 A_3(r^2, x_n) = 0, \end{array} \right. \tag{7}$$

and equations (5) and (6) gives us the system

$$\left\{ \begin{array}{l} \partial_{x_n} A_1(r^2, x_n) - \frac{\kappa}{x_n} A_1(r^2, x_n) - (2l - n + 1) A_2(r^2, x_n), \\ \quad +(-1)^l 2\partial_1 A_3(r^2, x_n) r^2 + (-1)^l (n + 2k - 1) A_3(r^2, x_n) = 0, \\ 2\partial_1 A_1(r^2, x_n) - (-1)^l \partial_{x_n} A_3(r^2, x_n) - (-1)^l (2l - n + 1) A_4(r^2, x_n) = 0, \\ 2\partial_1 A_4(r^2, x_n) r^2 + (-1)^l \partial_{x_n} A_2(r^2, x_n) + (n + 2k + 1) A_4(r^2, x_n) = 0, \\ \partial_{x_n} A_4(r^2, x_n) - \frac{\kappa}{x_n} A_4(r^2, x_n) + (-1)^{l+1} 2\partial_1 A_2(r^2, x_n) + = 0. \end{array} \right. \tag{8}$$

Hence, in order to obtain the conditions over A_1, A_2, A_3, A_4, such that G is two-sided κ-hypergenic, we need to combine the systems (7) and (8). To do that we will need to split our study in two cases: when l is even and when l is odd.

4.1 The Case $l = 2j$

Supposing that l is even, systems (7) and (8) become, respectively, equal to

$$\begin{cases} \partial_{x_n} A_1(r^2, x_n) - \dfrac{\kappa}{x_n} A_1(r^2, x_n) + 2\partial_1 A_2(r^2, x_n) r^2, \\ \qquad +(n + 2k - 1) A_2(r^2, x_n) - (4j - n + 1) A_3(r^2, x_n) = 0, \\ 2\partial_1 A_1(r^2, x_n) - \partial_{x_n} A_2(r^2, x_n) - (4j - n + 1) A_4(r^2, x_n) = 0, \\ 2\partial_1 A_4(r^2, x_n) r^2 + \partial_{x_n} A_3(r^2, x_n) + (n + 2k + 1) A_4(r^2, x_n) = 0, \\ \partial_{x_n} A_4(r^2, x_n) - \dfrac{\kappa}{x_n} A_4(r^2, x_n) - 2\partial_1 A_3(r^2, x_n) = 0, \end{cases} \tag{9}$$

and

$$\begin{cases} \partial_{x_n} A_1(r^2, x_n) - \dfrac{\kappa}{x_n} A_1(r^2, x_n) - (4j - n + 1) A_2(r^2, x_n), \\ \qquad +2\partial_1 A_3(r^2, x_n) r^2 + (n + 2k - 1) A_3(r^2, x_n) = 0, \\ 2\partial_1 A_1(r^2, x_n) - \partial_{x_n} A_3(r^2, x_n) - (4j - n + 1) A_4(r^2, x_n) = 0, \\ 2\partial_1 A_4(r^2, x_n) r^2 + \partial_{x_n} A_2(r^2, x_n) + (n + 2k + 1) A_4(r^2, x_n) = 0, \\ \partial_{x_n} A_4(r^2, x_n) - \dfrac{\kappa}{x_n} A_4(r^2, x_n) - 2\partial_1 A_2(r^2, x_n) = 0. \end{cases} \tag{10}$$

The function G is two-sided κ-hypergenic if the correspondent equations of (9) and (10) can be solved at the same time., i.e., if

$$\begin{cases} 2\partial_1 A_2(r^2, x_n) r^2 + (n + 2k - 1) A_2(r^2, x_n) - (4j - n + 1) A_3(r^2, x_n), \\ = -(4j - n + 1) A_2(r^2, x_n) + 2\partial_1 A_3(r^2, x_n) r^2 + (n + 2k - 1) A_3(r^2, x_n), \\ \partial_{x_n} A_2(r^2, x_n) = \partial_{x_n} A_3(r^2, x_n), \\ \partial_{x_n} A_3(r^2, x_n) = \partial_{x_n} A_2(r^2, x_n), \\ \partial_1 A_3(r^2, x_n) = \partial_1 A_2(r^2, x_n). \end{cases} \tag{11}$$

From the third and the fourth equation of (11) we get

$$\partial_{x_n}(A_3(r^2, x_n) - A_2(r^2, x_n)) = \partial_1(A_3(r^2, x_n) - A_2(r^2, x_n)) = 0,$$

and consequently $A_2(r^2, x_n) - A_3(r^2, x_n)$ is constant. The first equation gives

$$\begin{aligned} 2\partial_1(A_2(r^2, x_n) - A_3(r^2, x_n)) r^2 + (n + 2k - 1)(A_2(r^2, x_n) - A_3(r^2, x_n)) \\ + (4j - n + 1)(A_2(r^2, x_n) - A_3(r^2, x_n)) = 0, \end{aligned}$$

i.e.,

$$A_3(r^2, x_n) = A_2(r^2, x_n). \tag{12}$$

Combining (12) with system (9), we finally obtain that G is two-sided κ-hypergenic if and only if coefficients satisfies the Vekua-type system

$$\begin{cases} 2\partial_1 A_2(r^2, x_n) r^2 + \partial_{x_n} A_1(r^2, x_n) - \dfrac{\kappa}{x_n} A_1(r^2, x_n) \\ \qquad +2(n + k - 2j - 1) A_2(r^2, x_n) = 0, \\ 2\partial_1 A_1(r^2, x_n) - \partial_{x_n} A_2(r^2, x_n) - (4j - n + 1) A_4(r^2, x_n) = 0, \\ 2\partial_1 A_4(r^2, x_n) r^2 + \partial_{x_n} A_2(r^2, x_n) + (n + 2k + 1) A_4(r^2, x_n) = 0, \\ 2\partial_1 A_2(r^2, x_n) - \partial_{x_n} A_4(r^2, x_n) + \dfrac{\kappa}{x_n} A_4(r^2, x_n) = 0. \end{cases} \tag{13}$$

We look for solutions of the form $A_q(r^2, x_n) = \sum_{i=0}^{\infty} r^{2i} A_{q,i}(x_n)$, where

$$\partial_1 A_q(r^2, x_n) = \sum_{i=0}^{\infty} 2i r^{2i-1} A_{q,i}(x_n), \qquad \partial_{x_n} A_q(x_n) = \sum_{i=0}^{\infty} r^{2i} A'_{q,i}(x_n). \tag{14}$$

Substituting (14) in system (13) and making some calculations, we get

$$\begin{cases} \sum_{i=0}^{\infty} r^{2i}\left[A'_{1,i}(x_n) - \frac{\kappa}{x_n}A_{1,i}(x_n) + 2(2i+n+k-2j-1)A_{2,i}(x_n)\right] = 0, \\ \sum_{i=0}^{\infty} r^{2i}\left[\frac{4i}{r}A_{1,i}(x_n) - A'_{2,i}(x_n) - (4j-n+1)A_{4,i}(x_n)\right] = 0, \\ \sum_{i=0}^{\infty} r^{2i}\left[A'_{2,i}(x_n) + (4i+n+2k+1)A_{4,i}(x_n)\right] = 0, \\ \sum_{i=0}^{\infty} r^{2i}\left[\frac{4i}{r}A_{2,i}(x_n) + \frac{\kappa}{x_n}A_{4,i}(x_n) - A'_{4,i}(x_n)\right] = 0. \end{cases} \tag{15}$$

Since all the coefficients should be equal to zero, we get the first order system

$$\begin{cases} A'_{1,i}(x_n) - \frac{\kappa}{x_n}A_{1,i}(x_n) + 2(2i+n+k-2j-1)A_{2,i}(x_n) = 0, \\ \frac{4i}{r}A_{1,i}(x_n) - A'_{2,i}(x_n) - (4j-n+1)A_{4,i}(x_n) = 0, \\ A'_{2,i}(x_n) + (4i+n+2k+1)A_{4,i}(x_n) = 0, \\ \frac{4i}{r}A_{2,i}(x_n) + \frac{\kappa}{x_n}A_{4,i}(x_n) - A'_{4,i}(x_n) = 0. \end{cases} \tag{16}$$

Using the techniques of ordinary differential equations [8] and properties of special functions [3], we obtain our solutions of system (16) in terms of Bessel functions, namely

$$\begin{aligned} A_1(x_n) &= \frac{r(2i(2+8j-2n) - Ar)\ x_n^{\frac{k+1}{2}}}{8i^2}\left[B\ J_{\frac{k-1}{2}}(\sqrt{A}x_n) + C\ Y_{\frac{k-1}{2}}(\sqrt{A}x_n)\right], \\ A_2(x_n) &= -\frac{\sqrt{A}\ r\ x_n^{\frac{k+1}{2}}}{4i}\left[B\ J_{\frac{k+1}{2}}(\sqrt{A}x_n) + C\ Y_{\frac{k+1}{2}}(\sqrt{A}x_n)\right], \\ A_4(x_n) &= x_n^{\frac{k+1}{2}}\left[B\ J_{\frac{k-1}{2}}(\sqrt{A}x_n) + C\ Y_{\frac{k-1}{2}}(\sqrt{A}x_n)\right], \end{aligned}$$

where B, C are arbitrary real constants and $A = \frac{4i(n+2k+1+4i)}{r}$. Hence, our functions A_q take the form

$$\begin{aligned} A_1(r^2, x_n) &= \sum_{i=0}^{\infty} \frac{(2i(2+8j-2n) - Ar)\ r^{2i+1}\ x_n^{\frac{k+1}{2}}}{4i^2}\left[B\ J_{\frac{k-1}{2}}(\sqrt{A}x_n) + C\ Y_{\frac{k-1}{2}}(\sqrt{A}x_n)\right], \\ A_3(r^2, x_n) = A_2(r^2, x_n) &= -\sum_{i=0}^{\infty} \frac{\sqrt{A}\ r^{2i+1}\ x_n^{\frac{k+1}{2}}}{4i}\left[B\ J_{\frac{k+1}{2}}(\sqrt{A}x_n) + C\ Y_{\frac{k+1}{2}}(\sqrt{A}x_n)\right], \\ A_4(r^2, x_n) &= \sum_{i=0}^{\infty} r^{2i}\ x_n^{\frac{k+1}{2}}\left[B\ J_{\frac{k-1}{2}}(\sqrt{A}x_n) + C\ Y_{\frac{k-1}{2}}(\sqrt{A}x_n)\right]. \end{aligned}$$

Remark 4.1 *In an analogous way, this case can be studied for solutions of the type* $A_q(r^2, x_n) = \sum_{i=0}^{\infty} x_n A_{q,i}(r^2)$.

4.2 The Case $l = 2j + 1$

Supposing that l is odd, systems (7) and (8) become, respectively, equal to

$$\begin{cases} \partial_{x_n}A_1(r^2,x_n) - \frac{\kappa}{x_n}A_1(r^2,x_n) + 2\partial_1 A_2(r^2,x_n)r^2 \\ \qquad +(n+2k-1)A_2(r^2,x_n) + (4j+2-n+1)A_3(r^2,x_n) = 0, \\ 2\partial_1 A_1(r^2,x_n) - \partial_{x_n}A_2(r^2,x_n) + (4j+2-n+1)A_4(r^2,x_n) = 0, \\ 2\partial_1 A_4(r^2,x_n)r^2 + \partial_{x_n}A_3(r^2,x_n) + (n+2k+1)A_4(r^2,x_n) = 0, \\ \partial_{x_n}A_4(r^2,x_n) - \frac{\kappa}{x_n}A_4(r^2,x_n) - 2\partial_1 A_3(r^2,x_n) = 0, \end{cases} \tag{17}$$

and

$$\begin{cases} \partial_{x_n}A_1(r^2,x_n) - \frac{\kappa}{x_n}A_1(r^2,x_n) - (4j+2+n+1)A_2(r^2,x_n) \\ \qquad -2\partial_1 A_3(r^2,x_n)r^2 - (n+2k-1)A_3(r^2,x_n) = 0, \\ 2\partial_1 A_1(r^2,x_n) + \partial_{x_n}A_3(r^2,x_n) + (4j+2-n+1)A_4(r^2,x_n) = 0, \\ 2\partial_1 A_4(r^2,x_n)r^2 - \partial_{x_n}A_2(r^2,x_n) + (n+2k+1)A_4(r^2,x_n) = 0, \\ \partial_{x_n}A_4(r^2,x_n) - \frac{\kappa}{x_n}A_4(r^2,x_n) - 2\partial_1 A_2(r^2,x_n) = 0. \end{cases} \tag{18}$$

The function G is two-sided κ-hypergenic if the correspondent equations of (17) and (18) can be solved at the same time., i.e., if

$$\begin{cases} 2\partial_1 A_2(r^2,x_n)r^2 + (n+2k-1)A_2(r^2,x_n) + (4j+2-n+1)A_3(r^2,x_n) \\ = -(4j+2-n+1)A_2(r^2,x_n) - 2\partial_1 A_3(r^2,x_n)r^2 - (n+2k-1)A_3(r^2,x_n), \\ -\partial_{x_n} A_2(r^2,x_n) = \partial_{x_n} A_3(r^2,x_n), \\ \partial_{x_n} A_3(r^2,x_n) = -\partial_{x_n} A_2(r^2,x_n), \\ -\partial_1 A_3(r^2,x_n) = \partial_1 A_2(r^2,x_n), \end{cases} \tag{19}$$

From the third and the fourth equation of (19) we get

$$\partial_{x_n}(A_3(r^2,x_n) + A_2(r^2,x_n)) = \partial_1(A_3(r^2,x_n) + A_2(r^2,x_n)) = 0,$$

and consequently $A_2(r^2,x_n) + A_3(r^2,x_n)$ is constant. The first equation gives

$$\begin{aligned} 2\partial_1(A_2(r^2,x_n) + A_3(r^2,x_n))r^2 + (n+2k-1)(A_2(r^2,x_n) + A_3(r^2,x_n)) \\ + (4j+2-n+1)(A_2(r^2,x_n) + A_3(r^2,x_n)) = 0, \end{aligned}$$

i.e.,

$$A_3(r^2,x_n) = -A_2(r^2,x_n). \tag{20}$$

Combining (20) with system (17), we finally obtain that G is two-sided κ-hypergenic if and only if coefficients satisfies the Vekua-type system

$$\begin{cases} 2\partial_1 A_2(r^2,x_n)r^2 + \partial_{x_n} A_1(r^2,x_n) - \frac{\kappa}{x_n} A_1(r^2,x_n) \\ \quad +2(n+k-2j-2)A_2(r^2,x_n) = 0, \\ 2\partial_1 A_1(r^2,x_n) - \partial_{x_n} A_2(r^2,x_n) + (4j-n+3)A_4(r^2,x_n) = 0, \\ 2\partial_1 A_4(r^2,x_n)r^2 - \partial_{x_n} A_2(r^2,x_n) + (n+2k+1)A_4(r^2,x_n) = 0, \\ 2\partial_1 A_2(r^2,x_n) + \partial_{x_n} A_4(r^2,x_n) - \frac{\kappa}{x_n} A_4(r^2,x_n) = 0. \end{cases} \tag{21}$$

As we had done for the even case, we look for solutions of the type $A_q(r^2,x_n) = \sum_{i=0}^{\infty} r^{2i} A_{q,i}(x_n)$, where

$$\partial_1 A_q(r^2,x_n) = \sum_{i=0}^{\infty} 2iu^{2i-1} A_{q,i}(x_n), \qquad \partial_{x_n} A_q(r^2,x_n) = \sum_{i=0}^{\infty} r^{2i} A'_{q,i}(x_n). \tag{22}$$

Substituting (22) in system (21) and making some calculations, we get

$$\begin{cases} \sum_{i=0}^{\infty} r^{2i} \left[A'_{1,i}(x_n) - \frac{\kappa}{x_n} A_{1,i}(x_n) + 2(2i+n+k-2j-2)A_{2,i}(x_n) \right] = 0, \\ \sum_{i=0}^{\infty} r^{2i} \left[\frac{4i}{r} A_{1,i}(x_n) - A'_{2,i}(x_n) + (4j-n+3)A_{4,i}(x_n) \right] = 0, \\ \sum_{i=0}^{\infty} r^{2i} \left[-A'_{2,i}(x_n) + (4i+n+2k+1)A_{4,i}(x_n) \right] = 0, \\ \sum_{i=0}^{\infty} r^{2i} \left[\frac{4i}{r} A_{2,i}(x_n) - \frac{\kappa}{x_n} A_{4,i}(x_n) + A'_{4,i}(x_n) \right] = 0. \end{cases} \tag{23}$$

Since all the coefficients should be equal to zero, we get the first order system

$$\begin{cases} A'_{1,i}(x_n) - \frac{\kappa}{x_n} A_{1,i}(x_n) + 2(2i+n+k-2j-2)A_{2,i}(x_n) = 0, \\ \frac{4i}{r} A_{1,i}(x_n) - A'_{2,i}(x_n) + (4j-n+3)A_{4,i}(x_n) = 0, \\ -A'_{2,i}(x_n) + (4i+n+2k+1)A_{4,i}(x_n) = 0, \\ \frac{4i}{r} A_{2,i}(x_n) - \frac{\kappa}{x_n} A_{4,i}(x_n) + A'_{4,i}(x_n) = 0. \end{cases} . \tag{24}$$

Using the techniques of ordinary differential equations [8] and properties of special functions [3], we obtain our solutions of system (24) in terms of Bessel functions, namely

$$\begin{aligned} A_1(x_n) &= \frac{r(4i(n-3-4j) + Ar)\ x_n^{\frac{k+1}{2}}}{8i^2} \left[B\ J_{\frac{k-1}{2}}(\sqrt{A}x_n) + C\ Y_{\frac{k-1}{2}}(\sqrt{A}x_n) \right], \\ A_2(x_n) &= \frac{\sqrt{A}\ r\ x_n^{\frac{k+1}{2}}}{4i} \left[B\ J_{\frac{k+1}{2}}(\sqrt{A}x_n) + C\ Y_{\frac{k+1}{2}}(\sqrt{A}x_n) \right], \\ A_4(x_n) &= x_n^{\frac{k+1}{2}} \left[B\ J_{\frac{k-1}{2}}(\sqrt{A}x_n) + C\ Y_{\frac{k-1}{2}}(\sqrt{A}x_n) \right], \end{aligned}$$

where B, C are arbitrary real constants and $A = \frac{4i(n+2k+1+4i)}{r}$. Hence, our functions A_q take the form

$$A_1(r^2, x_n) = \sum_{i=0}^{\infty} \frac{(4i(n-3-4j) + Ar)\ r^{2i+1}\ x_n^{\frac{k+1}{2}}}{8i^2} \left[B\ J_{\frac{k-1}{2}}(\sqrt{A}x_n) + C\ Y_{\frac{k-1}{2}}(\sqrt{A}x_n)\right],$$

$$A_2(r^2, x_n) = \sum_{i=0}^{\infty} \frac{\sqrt{A}\ r^{2i+1}\ x_n^{\frac{k+1}{2}}}{4i} \left[B\ J_{\frac{k+1}{2}}(\sqrt{A}x_n) + C\ Y_{\frac{k+1}{2}}(\sqrt{A}x_n)\right],$$

$$A_3(r^2, x_n) = -\sum_{i=0}^{\infty} \frac{\sqrt{A}\ r^{2i+1}\ x_n^{\frac{k+1}{2}}}{4i} \left[B\ J_{\frac{k+1}{2}}(\sqrt{A}x_n) + C\ Y_{\frac{k+1}{2}}(\sqrt{A}x_n)\right],$$

$$A_4(r^2, x_n) = \sum_{i=0}^{\infty} r^{2i}\ x_n^{\frac{k+1}{2}} \left[B\ J_{\frac{k-1}{2}}(\sqrt{A}x_n) + C\ Y_{\frac{k-1}{2}}(\sqrt{A}x_n)\right].$$

Remark 4.2 *In an analogous way, this case can be studied for solutions of the type* $A_q(r^2, x_n) = \sum_{i=0}^{\infty} x_n A_{q,i}(r^2)$.

Acknowledgement: N. Vieira was supported by Portuguese funds through the CIDMA - Center for Research and Development in Mathematics and Applications, and the Portuguese Foundation for Science and Technology ("FCT–Fundação para a Ciência e a Tecnologia"), within project PEst-OE/MAT/UI4106/2014.

References

[1] F. Brackx, R. Delanghe and F. Sommen, *Clifford Analysis*, Pitman, Boston-London-Melbourne, 1982.

[2] R. Delanghe, F. Sommen and V. Souček, *Clifford algebras and spinor-valued functions*, Kluwer Academic Publishers, 1992.

[3] A. Erdelyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions-Vol.I,II* , Bateman Manuscript Project, McGraw-Hill, New York, 1953.

[4] S.L. Eriksson, *Hyperbolic extensions of integral formulas*, Adv. Appl. Clifford Algebra, **20**, (2010), 575-586.

[5] S.L. Eriksson, *Integral formulas for hypermonogenic functions*, Bull. Belg. Math. Soc., **11**, (2004), 705-718.

[6] S.L. Eriksson and H. Leutwiler, *Hypermonogenic functions in Clifford analysis and their applications in mathematical physics*, Clifford Anal., **2**, (2000), 287-302.

[7] D. Peña Peña and F. Sommen, *Vekua-type systems related to two-sided monogenic functions*, Complex. Anal. Oper. Theory, **6**-No.2, (2012), 397-405.

[8] A.D. Polyanin and V.F. Zaitsev, *Handbook of exact solutions for ordinary differential equations. 2nd ed.*, CRC Press, Boca Raton, FL, 2003.

[9] F. Sommen, *Plane elliptic systems and monogenic functions in symetric domains*, Red. Circ. Mat. Palermo, **62**-No.2, (1984), 259-269.

[10] F. Sommen, *Special functions in Clifford analysis and axial symetry*, J. Math. Anal. Appl., **130**-No.1, (1988), 110-133.

[11] I.N. Vekua, *Generalized analytic functions*, Pergamon Press, London, 1962.