Universidad de Navarra

Doctoral Program in

Complex Systems

Permutability of Fuzzy Consequence and Interior Operators and their Connection with Fuzzy Relations

Neus Carmona Cervelló

Universidad de Navarra

Doctoral Program in

Complex Systems

Permutability of Fuzzy Consequence and Interior Operators and their Connection with Fuzzy Relations

Report submitted by **Neus Carmona Cervelló** in partial fulfillment for the requirements of the **Research Project** unit of the Doctoral Program

This work has been done under my supervision, and has obtained the following grade:

....

Signed on July 1, 2014

Dr. Jorge Elorza Barbajero

Declaración:
Por la presente yo, **Dª. Neus Carmona Cervelló**, declaro que esta memoria es fruto de mi propio trabajo y que en mi conocimiento, no contiene ni material previamente publicado o escrito por otra persona, ni material que sustancialmente haya formado parte de los requerimientos para obtener cualquier otro título en cualquier centro de educación superior, excepto en los lugares del texto en los que se ha hecho referencia explícita a la fuente de la información.

(I hereby declare that this submission is my own work and that, to the best of my knowledge and belief, it contains no material previously published or written by another person nor material which to a substantial extent has been accepted for the award of any other degree of the university or other institute of higher learning, except where due acknowledgment has been made in the text.)

De igual manera, autorizo al Departamento de Física y Matemática Aplicada de la Universidad de Navarra, la distribución de esta memoria y, si procede, de la "fe de erratas" correspondiente por cualquier medio, sin perjuicio de los derechos de propiedad intelectual que me corresponden.

Signed on July 1, 2014

Neus Carmona Cervelló

Acknowledgements

En primer lugar, quisiera agradecer a mis directores Jorge Elorza y Jordi Recasens la ayuda, el apoyo, las ideas y la confianza que me han brindado durante todo este tiempo. También agradecer a la Universidad de Navarra y a la Asociación de Amigos de la Universidad la posibilidad que me ofrecieron al recibirme y concederme una beca; asimismo agradecer el apoyo económico del proyecto FIS2011-28820-C02-02 del Gobierno de España.

Quisiera dar las gracias a todos los miembros del Departamento de Física y Matemática Aplicada por haberme hecho sentir tan a gusto en el departamento: Silvia Olza, LuisFer Urrea, Javier Burguete, Wenceslao González-Viñas, Angel Garcimartín, Sergio Ardanza-Trevijano, Diego Maza, Iker Zuriguel, Hćtor Mancini, Antonio Peláez, María Jesús Chasco, Carmen Palacios, Jean Bragard y Raúl Cruz. También a Martín Pastor y Andrea.

Me gustaría agradecer especialmente a los doctorandos (y a los que ya dejaron de serlo) los buenos momentos que hemos compartido: Moorthi Pichumani, José Guadarrama, Manuel Acevedo, Miguel López, Ana Simic, Celia Lozano, Raheema M. Aslam, Hugo J. Bello, Karol V. Asencio, Sara M. Rubio e Iván Cortés; Sin olvidarme de los doctorandos visitantes, Gerardo y Cristian, y Víctor, mi compañero de piso. Sin todos ellos, ni Zizur ni Pamplona hubieran sido lo mismo.

Gracias a Puri, Felipe, Sor Cristina y todos los residentes, trabajadores y voluntarios de la clínica Padre Menni, por hacer de los lunes el mejor día de la semana. También a Sofía y Tantaka, por habérmelos presentado.

Finalment, un agraïment molt especial mereixen la meva parella, Quim, per la seva inesgotable paciència i suport, i la meva familia, que sempre ha estat al meu costat: els meus pares, Teresa i Joan Josep, el meus germans Jordi i Meritxell, la meva tieta Xon i la meva àvia Paula. Sense oblidar-me dels ànims que m'han donat els meus amics de Sant Cugat i voltants, especialment l'Emma, el Manel, el Dani i el Txetxu.

Contents

Preface

Since L. Zadeh introduced fuzzy logic in 1965 as a model to the uncertainty and vagueness inherent in the human reasoning, it has been largely developed. Fuzzy sets appeared as imprecise statements or propositions, fuzzy operators were developped in order to transform these statements into others and fuzzy relations were defined as vague connections between the concepts behind the statements. From then, fuzzy logic has been useful in a variety of fields such as approximate reasoning, fuzzy mathematical morphology, image processing, fuzzy control or decision making.

Fuzzy consequence operators and fuzzy interior operators are an essential tool in most of the different frameworks were fuzzy logic appears. As prominent examples, we find approximate reasoning and fuzzy mathematical morphology. In approximate reasoning, fuzzy consequence operators are used to obtain conclusions from certain fuzzy premises and relations and fuzzy interior operators appear as their dual notion. In fuzzy mathematical morphology, fuzzy consequence and interior operators are called fuzzy closings and openings respectively and they act as morphological filters used for image processing. There is a widely studied relationship between the concepts of fuzzy operator and fuzzy relation. One can obtain fuzzy operators from fuzzy relations and viceversa through different means. As we shall see, it is specially interesting the case of fuzzy operators induced by fuzzy preorders or indistinguishabilities using Zadeh's compositional rule and the $\inf - \rightarrow$ composition.

In all these contexts there is a need of concatenating two or more operators and it is important to know when this composition preserves their properties. In this work, we have studied when the order of composition of either two fuzzy consequence operators or two interior operators does not change the result. As it will be shown, it turns out that permutability is completely connected with the preservation of the properties. That is, whether the composition of two fuzzy consequence (interior) operators is such an operator. We have analyzed the particular case of fuzzy operators induced by fuzzy relations through Zadeh's compositional rule and the $\inf - \rightarrow$ composition. We have characterized when permutability is transfered from the inducing relations to the induced operators. A deeper study has been been done for the case of preorders and indistinguishabilities.

This work is structured as follows. In Chapter 1 we briefly introduce some background and we recall the main definitions and results that will be needed. In Chapter 2 we collect several results that show connections between fuzzy relations and fuzzy operators. We also extend some of the known definitions to a more general ones. Chapter 3 is devoted to the analysis of permutability. We start analyzing certain cases of fuzzy relations and fuzzy operators. In particular, fuzzy preorders and indistinguishabilities and fuzzy consequence and interior operators. Then, we relate permutability of fuzzy relations with permutability of the operators that they induce using the different processes defined in Chapter 2. We focus in the cases of fuzzy operators induced by fuzzy preorders and similarities.

Finally, we present the conclusions and the possible future work.

Chapter 1

Introduction

1.1. Background

Let X be a non-empty classical set and let $(L, \vee, \wedge)$ be a complete lattice. A fuzzy set μ defined on X is a map $\mu : X \longrightarrow L$ which assigns to every element $x \in X$ its degree of truth (or degree of belonging to the set μ). In general, L is called *the structure of truth values*.

The most common structure of truth values is the $[0, 1]$ interval. In this structure, $\mu(x) = 0$ represents *being completely false* and $\mu(x) = 1$ *completely true*. The inclusion of fuzzy sets is defined by the pointwise order, i.e. $\mu \subseteq \nu$ if and only if $\mu(x) \leq \nu(x)$ for all $x \in X$. Intersection of fuzzy sets (conjunction) in this structure is usually defined by the use of a t-norm.

Definition 1 *A t-norm $*$ is a map $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ satisfying for all $a, b, c \in [0, 1]$:*

1. *Commutativity $a * b = b * a$*
2. *Monotonicity: $a * b \leq a * c$ for all $b \leq c$.*
3. *Associativity: $a * (b * c) = (a * b) * c$.*
4. *Identity element: $a * 1 = a$.*

There are generalizations of t-norms to more general lattices. Different properties can be required to the t-norms. We are interested in left-continuity, since it allows to define a residuum.

Definition 2 *A t-norm is called* left-continuous *when it preserves the supremum. That is, when*

$$(\sup a_i) * b = \sup(a_i * b)$$

for any collection of subindices $i \in I$.

We will use the notation sup or $\vee$ for the supremum and inf or $\wedge$ for the infimum, indistinctly.

Let $[0, 1]^X$ denote the set of all fuzzy subsets of X with truth values in $[0, 1]$ endowed with the structure of complete commutative residuated lattice (in the sense of Bělohlávek [1]). That is, $\langle [0, 1], \wedge, \vee, *, \rightarrow, 0, 1\rangle$ where:

1. $\wedge$ and $\vee$ are the usual infimum and supremum

2. $*$ is a left-continuous t-norm

3. $\rightarrow$ is the residuum of $*$ defined for $\forall a, b \in X$ as

$$a \rightarrow b = \sup\{\gamma \in [0,1] \mid \ a * \gamma \leq b\}$$

Recall that $*$ and $\rightarrow$ satisfy the adjoiness property

$$x * y \leq z \quad \Leftrightarrow \quad y \leq x \rightarrow z$$

and that $*$ is monotone in both arguments while $\rightarrow$ is antitone in the first argument and monotone in the second one.

Let us recall some properties of $\langle [0,1], \wedge, \vee, *, \rightarrow, 0, 1\rangle$. Detailed proofs can be found in [1].

Proposition 1 *The residuated lattice $\langle [0,1], \wedge, \vee, *, \rightarrow, 0, 1\rangle$ satisfies the following conditions for each index set I and for all $x, x_i, y, y_i, z \in [0,1]$ with $i \in I$:*

1. $1 \rightarrow x = x$
2. $x \leq y \Leftrightarrow x \rightarrow y = 1$
3. $x * 0 = 0$
4. $x * (x \rightarrow y) \leq y$
5. $(x * y) \rightarrow z = x \rightarrow (y \rightarrow z)$
6. $(x \rightarrow y) * (y \rightarrow z) \leq (x \rightarrow z)$
7. $x * \bigvee_{i \in I} y_i = \bigvee_{i \in I}(x * y_i)$
8. $x \rightarrow \bigwedge_{i \in I} y_i \ = \bigwedge_{i \in I}(x \rightarrow y_i)$
9. $\bigvee_{i \in I} x_i \rightarrow y = \bigwedge_{i \in I}(x_i \rightarrow y)$
10. $x * \bigwedge_{i \in I} y_i \leq \bigwedge_{i \in I}(x * y_i)$

Recall that every partially ordered set P, and therefore every lattice, gives rise to a dual (or opposite) partially ordered set which usually denoted P^δ. P^δ is defined to be the set P with the inverse order, i.e. $x \leq y$ holds in P^δ if and only if $y \leq x$ holds in P. It is easy to see that this construction allows us to translate every statement from P to a statement P^δ by replacing each occurrence of $\leq$ by $\geq$. Notice that if P is a lattice, every occurrence of $\vee$ gets replaced by $\wedge$ and viceversa [2].

1.2. Fuzzy operators

A fuzzy operator is a map $C : [0,1]^X \longrightarrow [0,1]^X$. We denote Ω' the set of all fuzzy operators on the referential set X. Recall that Ω' is a lattice with order given by $C \leq C'$ if and only if $C(\mu) \subseteq C'(\mu)$ for all $\mu \in [0,1]^X$. All the operations are pointwise inherited from the structure given to $[0,1]$.

Definition 3 *A fuzzy operator $C \in \Omega'$ is called a fuzzy consequence operator or fuzzy closure operator (FCO for short) when it satisfies for all $\mu, \nu \in [0,1]^X$:*

(C1) Inclusion $\mu \subseteq C(\mu)$

(C2) Monotonicity $\mu \subseteq \nu \Rightarrow C(\mu) \subseteq C(\nu)$

(C3) Idempotence $C(C(\mu)) = C(\mu)$

Ω *will denote the set of all fuzzy consequence operators of* $[0,1]^X$*.*

Definition 4 *A fuzzy operator* $C \in \Omega'$ *is called a fuzzy interior operator (FIO for short) when it satisfies for all* $\mu, \nu \in [0,1]^X$*:*

(I1) Anti-inclusion $C(\mu) \subseteq \mu$

(I2) Monotonicity $\mu \subseteq \nu \Rightarrow C(\mu) \subseteq C(\nu)$

(I3) Idempotence $C(C(\mu)) = C(\mu)$

Λ *will denote the set of all fuzzy interior operators of* $[0,1]^X$*.*

Fuzzy consequence operators were introduced by Pavelka in 1979 as an extension of Tarski's consequence operators to fuzzy sets [3]. In approximate reasoning, they perform the role of deriving consequences from certain premises and relations [3–6].

From an algebraic point of view, fuzzy consequence operators are the closure operators the lattice $[0,1]^X$ [7]. Fuzzy interior operators appear as a dual notion of fuzzy closure operators [8]. They can be seen as fuzzy consequence operators in the dual lattice Ω'^{δ}. One can prove that Ω' and Ω'^{δ} are isomorphic through the function $\varphi : \Omega' \longrightarrow \Omega'^{\delta}$ defined as $\varphi(C) = 1 - C$ where $1 - C$ is the fuzzy operator defined as $(1-C)(\mu)(x) = 1 - C(\mu)(x)$ for every $\mu \in [0,1]^X$ and $x \in X$. Notice that C is a fuzzy consequence operator in Ω' if and only if $\varphi(C)$ is a fuzzy consequence operator in Ω'^{δ}. The same is true for fuzzy interior operators, C is a fuzzy interior operator in Ω' if and only if $\varphi(C)$ is an fuzzy interior operator in Ω'^{δ}. Therefore, every result stated for fuzzy consequence operators in Ω' have its dual statement, true for fuzzy interior operators in Ω'^{δ} which becomes also true for fuzzy interior operators in Ω' via φ^{-1}.

In fuzzy mathematical morphology, both kinds of operators act as morphological filters for image processing [9, 10]. They have been extensively studied in several contexts [11–13] and they have been used to transfer results from the field of approximate reasoning to the field of fuzzy mathematical morphology [14].

Let us recall the definition of the fuzzy closure of a fuzzy operator. This notion was first defined for general lattices [7] and later translated to the fuzzy context by Pavelka [3]. It can be thought as the best upper approximation by a fuzzy consequence operator to a given operator.

Definition 5 *Let* $C : [0,1]^X \longrightarrow [0,1]^X$ *be a fuzzy operator. We define the fuzzy closure* $\overline{C}$ *of* C *as the fuzzy operator given by*

$$\overline{C} = \inf_{\substack{\phi \in \Omega \\ C \leq \phi}} \{\phi\} \ . \tag{1.1}$$

The fuzzy closure is a fuzzy consequence operator and it is uniquely determined since the infimum of fuzzy consequence operators so is. Dually, one can consider the greatest fuzzy interior operator which is smaller than or equal to a given operator; that is the best lower approximation of a fuzzy operator C by a fuzzy interior operator.

Definition 6 *Let* $C : [0,1]^X \longrightarrow [0,1]^X$ *be a fuzzy operator. We define the fuzzy interior* $\underline{C}$ *of* C *as the fuzzy operator given by*

$$\underline{C} = \sup_{\substack{\phi \in \Lambda \\ C \geq \phi}} \{\phi\} \ . \tag{1.2}$$

1.3. Fuzzy relations

Fuzzy (binary) relations on X are fuzzy subsets of the cartesian product $X \times X$. For every pair $(x, y) \in X \times X$, $R(x, y)$ represents the degree in which x is related to y. We denote Γ' the set of fuzzy binary relations defined on X.

Definition 7 *A fuzzy relation $R : X \times X \longrightarrow [0, 1]$ is called a fuzzy $*$-preorder if it satisfies:*

1. *Reflexivity: $R(x, x) = 1 \quad \forall x \in X$*
2. *$*$-Transitivity: $R(x, y) * R(y, z) \leq R(x, z) \quad \forall x, y, z \in X$*

A fuzzy preorder is called a fuzzy $*$-indistinguishability relation or fuzzy $*$-similarity if it also satisfies

3. *Symmetry: $R(x, y) = R(y, x) \quad \forall x, y \in X$*

Recall that for R and $S \in \Gamma'$, we say that $R \leq S$ if $R(x, y) \leq S(x, y)$ for all $x, y \in X$.

Fuzzy preorders and indistinguishabilities are interesting since they are the fuzzy generalization of preorder and equivalence relations respectively. $*$-Indistinguishability relations model how similar the objects in the universe are. $*$- preorders establish a kind of *previouness* between the elements. As we shall see in the next section, there is a strong connection between this *previouness* that preorders model and the *consequences* that FCOs describe.

Observe that both definitions depend on the definition that we choose for the conjunction (the t-norm) since we need it to define transitivity. If certain element a is somehow previous to a certain other element b and this b is previous to another element c, we need to *add up* both statements in order to find how previous is a to c. This *adding up* is what clearly depends on the definition of the t-norm.

The importance of transitivity gives rise to the definition of the transitve closure of a fuzzy relation. The transitive closure of a fuzzy relation R is the smallest upper approximation of R which is $*$-transitive [15]. More precisely,

Definition 8 *Let R be a fuzzy relation. We define the transitive closure $\overline{R}$ of R as the fuzzy relation given by*

$$\overline{R} = \inf_{\substack{S \in \widehat{\Gamma} \\ R \leq S}} \{S\} \tag{1.3}$$

where $\widehat{\Gamma}$ denotes the set of all $*$-transitive fuzzy relations on X.

The explicit formula for the transitive closure is given by $\overline{R} = \sup_{n \in \mathbb{N}} R^n$ where the power of R is defined using the sup-$*$ composition [15]. It is the smallest transitive relation greater than or equal to R. The $*$-transitive closure preserves reflexivity and symmetry. Hence, the transitive closure of a reflexive fuzzy relation is fuzzy preorder and the transitive closure of a reflexive and symmetric relation is an indistinguishability relation.

We can define certain kind of compatibility between fuzzy relations and fuzzy sets. This allows to select which fuzzy sets *respect* the relation.

Definition 9 *Let $R \in \Gamma'$ be fuzzy relation and $\mu \in [0,1]^X$ a fuzzy subset of X. Then, μ is called $*$-compatible with R if*

$$\mu(x) * R(x,y) \leq \mu(y)$$

for all $x, y \in X$.

From its logical implications, these sets are also called true-sets or closed under modus ponens. This notion gets special interest when R is a preorder [16]. When R is not only a preorder but also an indistinguishability relation, these sets are called *extensional sets* and they have been largely studied since they represent the fuzzy equivalence classes under the chosen indistinguishability [17].

We will consider an operation in Γ'. The sup-$*$ composition between fuzzy relations was introduced by L. Zadeh [18].

Definition 10 *Let $R, S \in \Gamma'$ be fuzzy relations on a set X and $*$ a t-norm. The sup-$*$ composition of R and S is the fuzzy relation defined for all $x, y \in X$ by*

$$R \circ S(x,y) = \sup_{w \in X} \{R(x,w) * S(w,y)\} \tag{1.4}$$

Chapter 2

Connections between fuzzy relations and fuzzy operators

Concepts of fuzzy relations and fuzzy operators are closely related. There are several ways to obtain fuzzy operators from fuzzy relations. We shall focus on the fuzzy operators C_R^* and $C_R^{\rightarrow}$ induced by a fuzzy relation R.

2.1. The operator C_R^*

Every fuzzy relation induces a fuzzy operator through the well-known Zadeh's rule of inference [19].

Definition 11 *Let $R \in \Gamma'$ be a fuzzy relation on X. The fuzzy operator induced by R through Zadeh's compositional rule is defined by*

$$C_R^*(\mu)(x) = \sup_{w \in X} \{\mu(w) * R(w, x)\} \tag{2.1}$$

Notice that from a logical point of view, C_R^* can be understood as the operator that sends every fuzzy set μ to the fuzzy set containing all the elements which are related to some element w in μ by means of the relation R. This operator is widely used as the IF-THEN rule in fuzzy control.

Proposition 2 *[4] Let $\sigma : \Gamma' \longrightarrow \Omega'$ be the function that sends every fuzzy relation R to the operator C_R^* induced by means of equation (2.1). Then, σ is injective.*

In other words, injectivity of σ states that for any two fuzzy relations R and S, we have $C_R^* = C_S^*$ if and only if $R = S$. The relationship between fuzzy preorders and fuzzy consequence operators was well established [6] [16].

Proposition 3 *Let R be a fuzzy relation. Then C_R^* is a fuzzy consequence operator if and only if R is a fuzzy $*$-preorder.*

It is worth recalling that not all FCO can be obtained from fuzzy preorders by means of Zadeh's compositional rule. When the starting relation is a fuzzy indistinguishability relation, the induced operator is not only a FCO but satisfies the following properties [17].

Proposition 4 *Let E be a fuzzy $*$-indistinguishability relation and let C_E^* be the fuzzy operator induced through Zadeh's compositional rule. Then,*

1. *C_E^* is a fuzzy consequence operator.*
2. *$C_E^*(\bigvee_{i\in I} \mu_i) = \bigvee_{i\in I} C_E^*(\mu_i)$ for any index set I and all $\mu_i \in [0,1]^X$.*
3. *$C_E^*(\{x\})(y) = C_E^*(\{y\})(x)$ for all $x, y \in X$ where $\{x\}$ denotes the singleton of x.*
4. *$C_E^*(\alpha * \mu) = \alpha * C_E^*(\mu)$ for any constant $\alpha \in [0,1]$ and $\mu \in [0,1]^X$.*

Moreover, every operator satisfying the previous properties can be obtained from a fuzzy indistinguishability relation by means of Zadeh's compositional rule.

Proposition 5 *There is a bijection between the set of $*$-indistinguishability relations and the set of fuzzy operators satisfying the conditions of Proposition 4.*

2.2. The operator $C_R^{\rightarrow}$

Instead of using the supremum and the t-norm, one can induce a fuzzy operator from a fuzzy relation using the infimum and the adjoined implication (residuum).

Definition 12 *Let $R \in \Gamma'$ be a fuzzy relation on X. We define the fuzzy operator induced by R through the* inf- $\rightarrow$ *composition as*

$$C_R^{\rightarrow}(\mu)(x) = \inf_{w\in X}\{R(x,w) \rightarrow \mu(w)\} \tag{2.2}$$

Given a fuzzy set μ, $C_R^{\rightarrow}(\mu)$ is the fuzzy subset containing the elements x such that whenever x is in relation through R with an element w, then w belongs to μ [20].

Proposition 6 *The mapping $\theta : \Gamma' \longrightarrow \Omega'$ that sends every fuzzy relation R to the operator $C_R^{\rightarrow}$ induced by means of equation (2.2) is decreasing. That is, if $R \leq S$ then $C_R^{\rightarrow} \geq C_S^{\rightarrow}$.*

Proof It follows from the fact that $\rightarrow$ is antitone in the first argument. □

The map that sends every fuzzy relation to its induced operator is an injective map also in this case.

Proposition 7 *The function $\theta : \Gamma' \longrightarrow \Omega'$ that sends every fuzzy relation R to the operator $C_R^{\rightarrow}$ induced by means of equation (2.2) is injective. That is, if $C_R^{\rightarrow} = C_S^{\rightarrow}$ then $R = S$.*

Proof We shall prove the contra-positive form that is, if $R \neq S$ necessarily $C_R^{\rightarrow} \neq C_S^{\rightarrow}$. Assume $R \neq S$. Then, there exists $x, y \in X$ such that $R(x,y) \neq S(x,y)$. We can suppose without loss of generality that $R(x,y) > S(x,y)$. Let us define the fuzzy set μ_x as $\mu_x(w) = S(x,w)$. Then,

$$C_S^{\rightarrow}(\mu_x)(x) = \inf_{w\in X}\{S(x,w) \rightarrow \mu_x(w)\} = \inf_{w\in X}\{S(x,w) \rightarrow S(x,w)\} = 1$$

but

$$\begin{aligned} C_R^{\rightarrow}(\mu_x)(x) &= \inf_{w\in X}\{R(x,w) \rightarrow \mu_x(w)\} \\ &= \inf_{w\in X}\{R(x,w) \rightarrow S(x,w)\} \leq R(x,y) \rightarrow S(x,y) < 1 \end{aligned}$$

by property 2 from Proposition 1. □

Again, fuzzy operators induced by fuzzy preorders or fuzzy indistinguishabilities satisfy certain special properties. It is known that the operator $C_R^{\rightarrow}$ is a FIO whenever R is a $*$-indistinguishability relation. The following result shows that it is enough that R is a fuzzy preorder.

Proposition 8 *Let R be a preorder, then $C_R^{\rightarrow}$ defined as in (2.2) is a fuzzy interior operator.*

Proof Let us first proof anti-inclusion and monotonicity.

$$C_R^{\rightarrow}(\mu)(x) = \inf_{w\in X}\{R(x,w) \rightarrow \mu(w)\} \le R(x,x) \rightarrow \mu(x) \le 1 \rightarrow \mu(x) = \mu(x)$$

Let $\mu, \nu \in [0,1]^X$ and assume $\mu \le \nu$. Since $\rightarrow$ is monotone in the second argument, we have

$$R(x,w) \rightarrow \mu(w) \le R(x,w) \rightarrow \nu(w) \quad \forall w \in X.$$

Therefore

$$C_R^{\rightarrow}(\mu)(x) = \inf_{w\in X}\{R(x,w) \rightarrow \mu(w)\} \le \inf_{w\in X}\{R(x,w) \rightarrow \nu(w)\} = C_R^{\rightarrow}(\nu)(x)$$

To prove idempotence notice that

$$\begin{aligned}
C_R^{\rightarrow}(C_R^{\rightarrow}(\mu))(x) &= \inf_{w\in X}\{\ R(x,w) \rightarrow C_R^{\rightarrow}(\mu)(w)\ \} \\
&= \inf_{w\in X}\{\ R(x,w) \rightarrow (\inf_{y\in X}\{R(w,y) \rightarrow \mu(y)\})\ \} \\
&= \inf_{w\in X}\inf_{y\in X}\{\ R(x,w) \rightarrow (R(w,y) \rightarrow \mu(y))\ \} \\
&= \inf_{w\in X}\inf_{y\in X}\{\ (R(x,w) * R(w,y)) \rightarrow \mu(y)\ \} \\
&\ge \inf_{w\in X}\inf_{y\in X}\{R(x,y) \rightarrow \mu(y)\} \\
&= \inf_{y\in X}\{R(x,y) \rightarrow \mu(y)\} = C_R^{\rightarrow}(\mu)(x).
\end{aligned}$$

The other inclusion follows from the anti-inclusion property. □

Again, operators induced by fuzzy $*$-indistinguishability relations behave specially well.

Proposition 9 *[17] Let $E \in \Gamma'$ be a fuzzy $*$-indistinguishability relation and let $C_E^{\rightarrow}$ be the fuzzy operator induced by means of equation 2.2. Then $C_E^{\rightarrow}(\mu)(x)$ satisfies the following properties:*

1. *$C_E^{\rightarrow}$ is a fuzzy interior operator.*
2. *$C_E^{\rightarrow}(\bigwedge_{i\in I}\mu_i) = \bigwedge_{i\in I} C_E^{\rightarrow}(\mu_i)$ for any index set I and all $\mu_i \in [0,1]^X$.*
3. *$C_E^{\rightarrow}(\{x\} \rightarrow \alpha)(y) = C_E^{\rightarrow}(\{y\} \rightarrow \alpha)(x)$ for all $x, y \in X$ and any constant $\alpha \in [0,1]$ where $\{x\}$ denotes the singleton of x.*
4. *$C_E^{\rightarrow}(\alpha \rightarrow \mu) = \alpha \rightarrow C_E^{\rightarrow}(\mu)$ for any constant $\alpha \in [0,1]$ and $\mu \in [0,1]^X$.*

The converse of Proposition 9 also holds.

Proposition 10 *[17] There exists a bijection between the set of fuzzy $*$-indistinguishability relations and the set of fuzzy operators satisfying all the properties from proposition 9. That is, if $C \in \Omega'$ is a fuzzy operator satisfying all the properties from proposition 9, then there exists a fuzzy $*$-indistinguishability relation E such that $C = C_E^{\rightarrow}$.*

2.3. Generalization of C_R^* and $C_R^{\rightarrow}$

We generalize the operator induced by a fuzzy relation through Zadeh's compositional rule to a fuzzy operator induced by a fuzzy relation and another fuzzy operator.

Definition 13 *Let $g \in \Omega'$ be a fuzzy operator and let $R \in \Gamma'$ be a fuzzy relation on X. We define the operator C_R^g induced by g and R as*

$$C_R^g(\mu)(x) = \sup_{w \in X}\{g(\mu)(w) * R(w,x)\} \tag{2.3}$$

R and g are called the generators of C_R^g.

The operator g used as generator performs a selection in order to apply Zadeh's usual operator only to the fuzzy subsets of its image. Notice that taking $g = id$, where id denotes the identity operator on $[0,1]^X$, we obtain $C_R^{id} = C_R^*$.

Proposition 11 *For every $g \in \Omega'$, the mapping $\sigma_g : \Gamma' \longrightarrow \Omega'$ that sends every fuzzy relation R to the operator C_R^g induced by R and g by means of equation (2.4) is increasing. That is, if $R \leq S$ then $C_R^g \leq C_S^g$.*

Corollary 1 *The mapping $\sigma : \Gamma' \longrightarrow \Omega'$ that sends every fuzzy relation R to the operator C_R^* induced by Zadeh's compositional rule (equation (2.1)) is increasing.*

Our interest lies in the obtention of fuzzy consequence operators. For this, we need certain individual properties of the generators and also some conditions involving both generators, operators and relations. More precisely, let us define the concordance between a fuzzy operator and a fuzzy relation.

Definition 14 *Let g be a fuzzy operator and R a fuzzy relation. We will say that g is* $*$-concordant with *R if all the subsets from the image of g are $*$-compatible with R. That is,*

$$g(\mu)(x) * R(x,y) \leq g(\mu)(y)$$

for all $x, y \in X$ and all $\mu \in [0,1]^X$.

Theorem 1 *Let $R \in \Gamma'$ be a reflexive fuzzy relation and let $g \in \Omega'$ be a FCO. Suppose that g is $*$-concordant with R. Then, the operator C_R^g induced by g and R is also a FCO.*

Proof Let us start proving the inclusion and monotonicity properties. From the reflexivity of R, it follows that

$$C_R^g(\mu)(x) = \sup_{w \in X}\{g(\mu)(w) * R(w,x)\} \geq g(\mu)(x) * R(x,x) = g(\mu)(x).$$

Since g is a FCO and therefore inclusive, we get

$$C_R^g(\mu)(x) \geq g(\mu)(x) \geq \mu(x)$$

Let $\mu_1, \mu_2 \in [0,1]^X$ such that $\mu_1 \subseteq \mu_2$. From the monotonicity of g it follows that $g(\mu_1)(x) \leq g(\mu_2)(x)$ for all $x \in X$. Therefore,

$$\begin{aligned} C_R^g(\mu_1)(x) &= \sup_{w\in X}\{g(\mu_1)(w) * R(w,x)\} \\ &\leq \sup_{w\in X}\{g(\mu_2)(w) * R(w,x)\} = C_R^g(\mu_2)(x). \end{aligned}$$

It only remains to prove the idempotence. To prove the first inclusion notice that, since $g(\mu)$ belongs to $\mathrm{Im}(g)$, it is $*$-compatible with R. That is,

$$g(\mu)(y) * R(y,x) \leq g(\mu)(x)$$

for all $y, x \in X$. Hence,

$$\sup_{y\in X}\{g(\mu)(y) * R(y,x)\} \leq g(\mu)(x)$$

for all $x \in X$. Using this fact, the monotonicity and idempotence of g and the monotonicity of $*$ we get

$$\begin{aligned} C_R^g(C_R^g(\mu))(x) &= \sup_{w\in X}\{g(C_R^g(\mu))(w) * R(w,x)\} \\ &= \sup_{w\in X}\{g(\sup_{y\in X}\{g(\mu)(y) * R(y,w)\}) * R(w,x)\} \\ &\leq \sup_{w\in X}\{g(g(\mu)(w)) * R(w,x)\} \\ &= \sup_{w\in X}\{g(\mu)(w) * R(w,x)\} = C_R^g(\mu)(x) \end{aligned}$$

The other inclusion follows immediately from the inclusion property. □

Following the same idea of using an operator in order to perform a previous selection to the fuzzy sets to which we shall apply an operator, we generalize the operator induced by a fuzzy relation through the inf-$\longrightarrow$ product to a fuzzy operator induced by a fuzzy relation and another fuzzy operator.

Definition 15 *Let $g \in \Omega'$ be a fuzzy operator and let $R \in \Gamma'$ be a fuzzy relation on X. We define the operator $\tilde{C}_R^g$ induced by g and R as*

$$\tilde{C}_R^g(\mu)(x) = \inf_{w\in X}\{R(x,w) \rightarrow g(\mu)(w)\} \tag{2.4}$$

R and g are called the generators of $\tilde{C}_R^g$.

Again, taking $g = id$ we obtain $C_R^{id} = C_R^{\rightarrow}$.

However, we shall leave the study of the properties of this operator for future work.

Chapter 3

Permutability

In general, neither the usual composition of fuzzy operators nor the sup-$*$ composition of fuzzy relations is a commutative operation. Therefore, it is interesting to know when the order of composition does not change the result. As we shall see throughout this chapter, there are cases where permutability appears. That is, cases for which the order of composition does not change the outcome. We characterize some of them here.

3.1. Permutability of fuzzy preoders and fuzzy indistinguishability relations

Definition 16 *Let $R, S \in \Gamma'$ be fuzzy relations. We say that R and S are permutable or that R and S permute if $R \circ S = S \circ R$ where $\circ$ is the sup-$*$ composition defined as in equation (1.4).*

Permutability of preorders is closely related to the transitive closure of a fuzzy relation. It was proved in [21] that two fuzzy $*$-indistinguishability relations defined on a finite set X permute if and only if $E \circ F$ is an $*$-indistinguishability relation. In this case, $E \circ F$ coincides with the $*$-transitive closure of their maximum, that is $E \circ F = \overline{\max(E, F)}$. We extend this result to general fuzzy preorders and any set X, finite or not. We need the following lemma.

Lemma 1 *Let R and P be two fuzzy $*$-preorders on a set X. Then, $R \circ P \leq \overline{\max(R, P)}$.*

Proof $R \circ P \leq \max(R, P) \circ \max(R, P) \leq \sup_{n \in \mathbb{N}} (\max(R, P))^n = \overline{\max(R, P)}$. □

Theorem 2 *Let R and P be two fuzzy $*$-preorders on X. Then, R and P are permutable if and only if $R \circ P$ and $P \circ R$ are fuzzy $*$-preorders. Moreover, $R \circ P$ is a fuzzy $*$-preorder if and only if it coincides with the $*$-transitive closure $\overline{\max(R, P)}$ of $\max(R, P)$.*

Proof Let us first prove the second statement. That is, $R \circ P$ is a fuzzy $*$-preorder if and only if it coincides with $\overline{\max(R, P)}$. Suppose that $R \circ P$ is a fuzzy $*$-preorder. Since $R \circ P \geq R$ and $R \circ P \geq P$ we have that $R \circ P \geq \max(R, P)$. As $R \circ P$ is a fuzzy preorder, it follows that $R \circ P \geq \overline{\max(R, P)}$. From Lemma 1, we get $R \circ P = \overline{\max(R, P)}$. The other implication follows from the fact that $\max(R, P)$ is reflexive and therefore $\overline{\max(R, P)}$ is a preorder.

Now, let us prove that R and P are permutable if and only if $R \circ P$ and $P \circ R$ are fuzzy $*$-preorders. Assume that $R \circ P = P \circ R$ and let us show that they are fuzzy preorders.

- Reflexivity:
  $R \circ P(x,x) = \sup_{w \in X}\{R(x,w) * P(w,x)\} \geq R(x,x) * P(x,x) = 1$

- $*$-Transitivity: Since R is $*$-transitive, $\sup_{w \in X}\{R(x,w) * R(w,y)\} \leq R(x,y)$. The same holds for P. Thus,

$$\begin{aligned}
R \circ P(x,y) * R \circ P(y,z) = \\
&= \sup_{w \in X}\{R(x,w) * P(w,y)\} * \sup_{h \in X}\{R(y,h) * P(h,z)\} = \\
&= \sup_{w,h \in X}\{R(x,w) * P(w,y) * R(y,h) * P(h,z)\} = \\
&\leq \sup_{w,h \in X}\{R(x,w) * (P \circ R)(w,h) * P(h,z)\} = \\
&= \sup_{w,h \in X}\{R(x,w) * (R \circ P)(w,h) * P(h,z)\} = \\
&= \sup_{w,h,y \in X}\{R(x,w) * R(w,y) * P(y,h) * P(h,z)\} = \\
&= \sup_{y \in X}\{\sup_{w \in X}\{R(x,w) * R(w,y)\} * \sup_{h \in X}\{P(y,h) * P(h,z)\}\} = \\
&\leq \sup_{y \in X}\{R(x,y) * P(y,z)\} \ = \ R \circ P(x,z).
\end{aligned}$$

Hence, it follows that $R \circ P = \overline{\max(R,P)} = P \circ R$.

The other direction is straightforward. □

For fuzzy indistinguishability relations, the symmetric property facilitates the way. We need just to find that one of the compositions is an indistinguishability relation to get both of them.

Corollary 2 *Let E and F be two $*$-indistinguishability relations on X. Then, E and F are permutable if and only if $E \circ F$ is a $*$-indistinguishability relation. Moreover, this occurs if and only if $E \circ F$ coincides with the $*$-transitive closure $\overline{\max(E,F)}$ of $\max(E,F)$.*

Proof Since E and F are fuzzy preorders, Theorem 2 ensures that they permute if and only if $E \circ F = \overline{\max(E,F)} = F \circ E$. Since $\max(E,F)$ is reflexive and symmetric, $\overline{\max(E,F)}$ is an indistinguishability relation. □

Let us illustrate with some examples that there exist fuzzy indistinguishabilities and preorders that do permute and others that do not permute.

Example 1 *Let R and P be fuzzy min-preorders (but not similarities) defined as follows:*

$$R = \begin{pmatrix} 1 & 0.3 & 0.6 \\ 0.7 & 1 & 0.75 \\ 0.4 & 0.3 & 1 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 0.7 & 0.8 \\ 0.55 & 1 & 0.7 \\ 0.4 & 0.4 & 1 \end{pmatrix}$$

R and P ***permute.***

Example 2 *Let Q and S be fuzzy min-preorders (but not similarities) defined as follows.*

$$Q = \begin{pmatrix} 1 & 0.4 & 0.5 \\ 0.6 & 1 & 0.5 \\ 0.3 & 0.3 & 1 \end{pmatrix} \qquad S = \begin{pmatrix} 1 & 0.3 & 0.6 \\ 0.7 & 1 & 0.75 \\ 0.4 & 0.3 & 1 \end{pmatrix}$$

Their compositions are given by

$$Q \circ S = \begin{pmatrix} 1 & 0.4 & 0.6 \\ 0.7 & 1 & 0.75 \\ 0.4 & 0.3 & 1 \end{pmatrix}$$

$$S \circ Q = \begin{pmatrix} 1 & 0.4 & 0.6 \\ 0.7 & 1 & 0.75 \\ 0.4 & 0.4 & 1 \end{pmatrix}$$

Notice that $\overline{\max(S, Q)} = S \circ Q$ *but* Q *and* S ***do not permute***.

Example 3 *Let* E *and* F *be fuzzy min-indistinguishability relations defined as follows.*

$$E = \begin{pmatrix} 1 & 0.8 & 0.7 & 0.7 \\ 0.8 & 1 & 0.7 & 0.8 \\ 0.7 & 0.7 & 1 & 0.7 \\ 0.7 & 0.8 & 0.7 & 1 \end{pmatrix}$$

$$F = \begin{pmatrix} 1 & 0.6 & 0.5 & 0.8 \\ 0.6 & 1 & 0.5 & 0.6 \\ 0.5 & 0.5 & 1 & 0.5 \\ 0.8 & 0.6 & 0.5 & 1 \end{pmatrix}$$

Notice that E *and* F ***permute***.

$$\overline{\max(E, F)} = E \circ F = F \circ E = \begin{pmatrix} 1 & 0.8 & 0.7 & 0.8 \\ 0.8 & 1 & 0.7 & 0.8 \\ 0.7 & 0.7 & 1 & 0.7 \\ 0.8 & 0.8 & 0.7 & 1 \end{pmatrix}$$

Example 4 *Let* E *and* F *be fuzzy min-indistinguishability relations defined as follows.*

$$E = \begin{pmatrix} 1 & 0.4 & 0.4 & 0.4 \\ 0.4 & 1 & 0.8 & 0.7 \\ 0.4 & 0.8 & 1 & 0.7 \\ 0.4 & 0.7 & 0.7 & 1 \end{pmatrix}$$

$$F = \begin{pmatrix} 1 & 0.5 & 0.7 & 0.8 \\ 0.5 & 1 & 0.5 & 0.5 \\ 0.7 & 0.5 & 1 & 0.7 \\ 0.8 & 0.5 & 0.7 & 1 \end{pmatrix}$$

*E and F **do not permute**.*

$$F \circ E = \begin{pmatrix} 1 & 0.7 & 0.7 & 0.8 \\ 0.5 & 1 & 0.8 & 0.7 \\ 0.7 & 0.8 & 1 & 0.7 \\ 0.8 & 0.7 & 0.7 & 1 \end{pmatrix}$$

$$E \circ F = \begin{pmatrix} 1 & 0.5 & 0.7 & 0.8 \\ 0.7 & 1 & 0.8 & 0.7 \\ 0.7 & 0.8 & 1 & 0.7 \\ 0.8 & 0.7 & 0.7 & 1 \end{pmatrix}$$

3.2. Permutability of FCO and FIO

The aim of this section is to study when two fuzzy consequence operators or two fuzzy interior operators permute. Permutability of fuzzy operators is considered with the usual composition. That is,

Definition 17 *Let C, C' be fuzzy operators. We say that C and C' are permutable or that C and C' permute if $C \circ C' = C' \circ C$ where $\circ$ denotes the usual composition.*

In order to study permutability for these two cases we need to recall the definition of the power of a fuzzy operator and several of its properties.

Definition 18 *Let $C : [0,1]^X \longrightarrow [0,1]^X$ be a fuzzy operator. We define C^k for $k \in \mathbb{N}$ as the fuzzy operator defined recursively as:*

1. *$C^1 = C$ i.e. $C^1(\mu)(x) = C(\mu)(x) \quad \forall \mu \in [0,1]^X$ and $\forall\ x \in X$.*
2. *$C^k = C(C^{k-1})$ i.e. $C^k(\mu)(x) = C(C^{k-1}(\mu))(x) \quad \forall \mu \in [0,1]^X, \forall x \in X$ and $k \geq 2$.*

That is, C^k is the usual composition of the operator C with itself k times.

The following lemma is straightforward. It allows us to define the limit operator for the sequence of powers of either an inclusive or anti-inclusive operator.

Lemma 2 *Let $C : [0,1]^X \longrightarrow [0,1]^X$ be a fuzzy operator. Then,*

1. *If C is inclusive, then C^k is inclusive for all $k \in \mathbb{N}$.*
2. *If C is anti-inclusive, then C^k is anti-inclusive for all $k \in \mathbb{N}$.*
3. *If C is monotone, then C^k is monotone for all $k \in \mathbb{N}$.*

Proposition 12 *Let $C : [0,1]^X \longrightarrow [0,1]^X$ be a fuzzy operator.*

1. *If C is inclusive, then the sequence $\left\{C^k\right\}_{k\in\mathbb{N}}$ is increasing and convergent. That is , $C^k \leq C^{k+1}$ for all $k \in \mathbb{N}$ and there exists a fuzzy operator $U \in \Omega'$ such that $U = \lim_{n\in\mathbb{N}} C^n = \sup_{n\in\mathbb{N}} C^n$.*

2. *If C is anti-inclusive, then the sequence $\left\{C^k\right\}_{k\in\mathbb{N}}$ is decreasing and convergent. That is , $C^{k+1} \leq C^k$ for all $k \in \mathbb{N}$ and there exists a fuzzy operator $L \in \Omega'$ such that $L = \lim_{n\in\mathbb{N}} C^n = \inf_{n\in\mathbb{N}} C^n$.*

Proof

1. Since 1 is an upper bound for $C^k(\mu)(x)$ for all $\mu \in [0,1]^X$, all $x \in X$ and all $k \in \mathbb{N}$, the sequences $\left\{C^k(\mu)(x)\right\}_{k\in\mathbb{N}}$ are increasing and bounded, thus they converge. Hence, the limit operator exists and it is pointwise defined by

$$U(\mu)(x) = \lim_{n\to\infty} C^n(\mu)(x) = \sup_{n\in\mathbb{N}} C^n(\mu)(x) \ . \tag{3.1}$$

2. Dual to the previous one. In this case, 0 is a lower bound for $C^k(\mu)(x)$ for all $\mu \in [0,1]^X$, all $x \in X$ and all $k \in \mathbb{N}$. Therefore, the limit operator exists and it is pointwise defined by

$$L(\mu)(x) = \lim_{n\to\infty} C^n(\mu)(x) = \inf_{n\in\mathbb{N}} C^n(\mu)(x) \ . \tag{3.2}$$

□

Permutability of fuzzy consequence operators

Now we are ready to characterize permutability for fuzzy consequence operators. We shall see that the closure of an operator plays an essential role for permutability. Similarly to the transitive closure of a fuzzy relation, the closure of certain operators can be defined from its sequence of powers.

Theorem 3 *Let $C : [0,1]^X \longrightarrow [0,1]^X$ be an inclusive and monotone fuzzy operator. Then, $\lim\limits_{n\to\infty} C^n = \overline{C}$.*

Proof First of all, let us show that $C^k \leq \overline{C}$ for all $k \in \mathbb{N}$ by induction on k.

- For $k = 1$ it is clear that $C \leq \overline{C}$.

- Assume that $C^k \leq \overline{C}$ for a certain k. Then, $C^k(\mu) \subseteq \overline{C}(\mu)$ for all $\mu \in [0,1]^X$. Since $C \leq \overline{C}$ and $\overline{C}$ is monotone and idempotent, it follows that

$$C(C^k(\mu)) \subseteq \overline{C}(C^k(\mu)) \subseteq \overline{C}(\overline{C}(\mu)) = \overline{C}(\mu) \ .$$

Since $C^n \leq \overline{C}$ for all $n \in \mathbb{N}$, it follows that $\lim_{n\in\mathbb{N}} C^n \leq \overline{C}$.

To prove that $\lim_{n\to\infty} C^n \geq \overline{C}$ let us show that $\lim_{n\to\infty} C^n$ is a closure operator. Since C is inclusive and monotone, Lemma 2 ensures the inclusion and monotonicity of $\lim_{n\to\infty} C^n$. For the idempotence, it is straightforward that

$$\lim_{n\to\infty} C^n(\lim_{n\to\infty} C^n(\mu))(x) = \lim_{n\to\infty} C^n(\mu)(x) \ .$$

Therefore, $\lim\limits_{n\to\infty} C^n = \sup\limits_{n\in\mathbb{N}} C^n = \overline{C}$. □

Lemma 3 *Let* $C, C' : [0,1]^X \longrightarrow [0,1]^X$ *be fuzzy consequence operators. Then,*

$$C \circ C' \geq \max(C, C') \ .$$

Proof It directly follows from the inclusion and monotonicity properties. Since C is inclusive $C'(\mu) \subseteq C(C'(\mu))$ for all $\mu \in [0,1]^X$ and $C \circ C' \geq C'$. Since C' is inclusive $\mu \subseteq C'(\mu)$ and adding the monotonicity of C we get that $C(\mu) \subseteq C(C'(\mu))$ for all $\mu \in [0,1]^X$ and $C \circ C' \geq C$. Therefore, $C \circ C' \geq \max(C, C')$. □

Lemma 4 *Let* $C, C' : [0,1]^X \longrightarrow [0,1]^X$ *be two fuzzy consequence operators. Then,* $\max(C, C')$ *is an inclusive and monotone fuzzy operator.*

Proof The proof is straightforward. As C and C' are inclusive, $\max(C, C')$ is also inclusive. For the monotonicity, note that $\mu_1 \subseteq \mu_2$ implies $C(\mu_1)(x) \leq C(\mu_2)(x)$ and $C'(\mu_1)(x) \leq C'(\mu_2)(x)$ for all $\mu \in [0,1]^X$ and $x \in X$. Hence, $\max(C, C')(\mu_1)(x)) \leq \max(C, C')(\mu_2)(x))$ for all $\mu \in [0,1]^X$ and $x \in X$. □

Remark 1 *Notice that the two lemmas above hold even if* C *and* C' *are not FCO, but only inclusive and monotone. We did not use idempotence at any point of the proof.*

The importance of the closure arises from the fact that preservation of the operator type through composition is the key for permuting two operators. We are ready to prove that there is only one case where the composition of two fuzzy consequence operators is again a fuzzy consequence operator.

Proposition 13 *Let* C, C' *be fuzzy consequence operators. Then,* $C \circ C'$ *is a fuzzy consequence operator if and only if* $C \circ C' = \overline{\max(C, C')}$.

Proof It is sufficient to prove that if $C \circ C'$ is a FCO then $C \circ C' = \overline{\max(C, C')}$. The other implication follows from the the fact that the closure of an operator is a FCO.

Assume that $C \circ C'$ is a FCO. From Lemma 3, $C \circ C' \geq \max(C, C')$. Therefore, $C \circ C' \geq \overline{\max(C, C')}$. In addition, we have

$$C \circ C' \leq \max(C, C') \circ \max(C, C') = \max^2(C, C') \leq \overline{\max(C, C')}$$

where the last inequality holds due to Theorem 3 and Lemma 4. Hence, $C \circ C' = \overline{\max(C, C')}$. □

At this point, we are ready to characterize permutability of fuzzy consequence operators.

Theorem 4 *Let* C, C' *be fuzzy consequence operators. Then,* C *and* C' *permute if and only if* $C \circ C'$ *and* $C' \circ C$ *are fuzzy consequence operators.*

Proof First, let us show that if C and C' permute, then $C \circ C'$ and $C' \circ C$ are FCO.

- Inclusion: From Lemmas 3 and 4, $C \circ C' \geq \max(C, C')$ which is inclusive.

- Monotonicity: Suppose $\mu_1 \subseteq \mu_2$. From the monotonicity of C' it follows that $C'(\mu_1) \subseteq C'(\mu_2)$ and from the monotonicity of C, $C(C'(\mu_1)) \subseteq C(C'(\mu_2))$.

- Idempotence:

$$(C \circ C')((C \circ C')(\mu))(x) = (C \circ C')((C' \circ C)(\mu))(x) = C(C'(C'(C(\mu))))(x)$$
$$= C(C'(C(\mu)))(x) = C(C(C'(\mu)))(x) = C(C'(\mu))(x) = (C \circ C')(\mu)(x) \ .$$

The same arguments hold for $C' \circ C$.

The other implication directly follows from Proposition 13. □

Let us see an example of two FCO that permute.

Example 5 *Let X be a non empty classical set. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$ and let C' and C be defined as*

$$C'(\mu)(x) = \begin{cases} 1 & if \ x = x_1 \\ \\ \mu(x) & otherwise \end{cases}$$

$$C(\mu)(x) = \begin{cases} 1 & if \ x = x_2 \\ \\ \mu(x) & otherwise \end{cases}$$

*Notice that C and C' are FCO and they **permute**, i.e.*

$$C \circ C' = C' \circ C(\mu)(x) = \begin{cases} 1 & if \quad x = x_1 \quad or \quad x = x_2 \\ \\ \mu(x) & \qquad otherwise \end{cases}$$

Therefore, $C \circ C' = C' \circ C$ is a FCO.

Remark 2 *Observe that there are cases of fuzzy consequence operators C and C' such that $C' \circ C$ is a FCO (and therefore $C' \circ C = \overline{\max(C, C')}$) but C and C' do not permute.*

Let us illustrate this fact with an example.

Example 6 *Let X be a non empty classical set and let $\alpha, \beta \in \mathbb{R}$ such that $0 < \beta < \alpha < 1$. Let C' and C be FCO defined as follows:*

$$C'(\mu)(x) = \begin{cases} 1 & if \quad \mu(x) > \beta \\ \\ \beta & if \quad \mu(x) \leq \beta \end{cases} \qquad C(\mu)(x) = \begin{cases} 1 & if \quad \mu(x) > \alpha \\ \\ \alpha & if \quad \mu(x) \leq \alpha \ . \end{cases}$$

Note that $C' \circ C = \overline{\max(C, C')} = X$ where $X(x) = 1$ for all $x \in X$, but $C' \circ C \neq C \circ C'$. Indeed, one has

$$(C \circ C')(\mu)(x) \begin{cases} 1 & if \quad \mu(x) > \beta \\ \\ \alpha & if \quad \mu(x) \leq \beta \end{cases}$$

which is not a FCO.

Permutability of fuzzy interior operators

Dual results can be obtained for fuzzy interior operators. In this case, preservation of the type of operator is related to the interior of the minimum.

Theorem 5 *Let* $C : [0,1]^X \longrightarrow [0,1]^X$ *be an anti-inclusive and monotone fuzzy operator. Then,* $\lim_{n\to\infty} C^n = \underline{C}$.

Proof The proof is dual to Theorem 3, therefore we will only give a sketch of it. By induction on k, it can be proved that $C^k \geq \underline{C}$ for all $k \in \mathbb{N}$. Thus, $\lim_{n\to\infty} C^n \geq \underline{C}$.

To prove the other inequality we need to show that $\lim_{n\to\infty} C^n$ is an interior operator. Lemmas 2 and 3 ensure the anti-inclusion and monotonicity properties. The idempotence is obtained using the definition of limit as done in Theorem 3. Hence,

$$\lim_{n\to\infty} C^n = \inf_{n\in\mathbb{N}} C^n = \underline{C}$$

. □

Lemma 5 *Let* $C, C' : [0,1]^X \longrightarrow [0,1]^X$ *be fuzzy interior operators. Then,*

$$C \circ C' \leq \min(C, C') \ .$$

Lemma 6 *Let* $C, C' : [0,1]^X \longrightarrow [0,1]^X$ *be fuzzy interior operators. Then,* $\min(C, C')$ *is an anti-inclusive and monotone fuzzy operator.*

Again, permutability is connected to the preservation of the type of operator through composition. The composition of two fuzzy interior operators is again a fuzzy interior operator in one case. This determines when permutability appears.

Proposition 14 *Let* C, C' *be fuzzy interior operators. Then,* $C \circ C'$ *is a fuzzy interior operator if and only if* $C \circ C' = \underline{\min(C, C')}$.

Proof The proof is analogous to Proposition 13. It is sufficient to prove that if $C \circ C'$ is a FIO then $C \circ C' = \underline{\min(C, C')}$. The other implication follows from the fact that the fuzzy interior of an operator is a FIO.

Suppose that $C \circ C'$ is a fuzzy interior operator. From Lemma 5, we know that $C \circ C' \leq \min(C, C')$. Therefore, $C \circ C' \leq \underline{\min(C, C')}$.
In addition, one has,

$$C \circ C' \ \geq \ \min(C, C') \circ \min(C, C') \ = \ \min^2(C, C') \ \geq \ \underline{\min(C, C')}$$

where the last inequality holds due to Theorem 5 and Lemma 6. Hence,

$$C \circ C' = \underline{\min(C, C')}$$

□

Theorem 6 *Let* C, C' *be fuzzy interior operators. Then,* C *and* C' *permute if and only if* $C \circ C'$ *and* $C' \circ C$ *are fuzzy interior operators.*

Proof The proof is analogous to the proof of Theorem 4. First of all, let us show that if C and C' permute, then $C \circ C'$ and $C' \circ C$ are fuzzy interior operators. Monotonicity and idempotence are proved exactly in the same way than in Theorem 4. Inclusion follows from Lemmas 5 and 6. Since $C \circ C' \leq \min(C, C')$ and $\min(C, C')$ is anti-inclusive, so is $C \circ C'$. The same argument holds for $C' \circ C$.

The other implication directly follows from Proposition 14. □

3.3. Permutability of fuzzy operators induced by fuzzy relations

It is natural to think that permutability of fuzzy relations is connected to the permutability of their induced operators. We shall study these connections for the fuzzy operators C_R^* and $C_R^{\rightarrow}$ introduced in Chapter 2. Let us start with the study of the operator induced through Zadeh's compositional rule.

Permutability of fuzzy operators induced by fuzzy relations through Zadeh's compositional rule

The composition of two fuzzy operators induced through Zadeh's compositional rule can be expressed in terms of the sup-$*$ composition of the inducing relations.

Proposition 15 *Let R, S be two fuzzy relations and let C_R^* and C_S^* be the corresponding fuzzy operators induced through Zadeh's compositional rule. Then,*

$$C_R^* \circ C_S^* \;=\; C_{S \circ R}^* \;=\; C_R^{C_S^*} \tag{3.3}$$

where $S \circ R$ denotes the sup-$*$ product composition of fuzzy relations.

Proof For all $\mu \in [0,1]^X$ and all $x \in X$ we have

$$C_R^* \circ C_S^*(\mu)(x) \;=\; C_R^*(C_S^*(\mu))(x) \;=\; \sup_{w \in X} \{\; C_S^*(\mu)(w) \;*\; R(w,x)\;\} = \; C_R^{C_S^*}$$

which gives us the second equality. For the first one,

$$\begin{aligned}
C_R^{C_S^*} &= \sup_{w \in X} \{\; C_S^*(\mu)(w) \;*\; R(w,x)\;\} \\
&= \sup_{w \in X} \{\; \sup_{z \in X}\{\; \mu(z) \;*\; S(z,w)\} \;*\; R(w,x)\;\} \\
&= \sup_{w,z \in X} \{\; \mu(z) \;*\; S(z,w) \;*\; R(w,x)\;\} \\
&= \sup_{z \in X} \{\; \mu(z) \;*\; \sup_{w \in X}\{\; S(z,w) \;*\; R(w,x)\;\}\;\} \\
&= \sup_{z \in X} \{\; \mu(z) \;*\; S \circ R(z,x)\;\} \;=\; C_{S \circ R}^*(\mu)(x).
\end{aligned}$$

□

The relation between permutability of fuzzy relations and permutability of their induced operators can be summarized in the following theorem.

Theorem 7 *Let R, S be two fuzzy relations and let C^*_R and C^*_S be the corresponding fuzzy operators induced through Zadeh's compositional rule. Then, C^*_R and C^*_S permute if and only if R and S permute.*

Proof It follows directly from the fact that the function that sends each fuzzy relation R to its induced operator C^*_R is injective. Hence,

$$C^*_{S\circ R} = C^*_{R\circ S} \quad \Leftrightarrow \quad S\circ R = R\circ S$$

□

As we have shown in the previous section, permutability of fuzzy consequence operators is related to the preservation of the type of operator. For fuzzy consequence operators induced by fuzzy preorders by means of equation (2.1) this occurs if and only if composition of the fuzzy preorders also preserves the type, i.e. it is again a fuzzy preorder.

Theorem 8 *Let R, P be fuzzy $*$-preorders and let C^*_R and C^*_P their corresponding fuzzy consequence operators induced through Zadeh's compositional rule. Then, C^*_R and C^*_P permute if and only if $R\circ P$ and $P\circ R$ are fuzzy $*$-preorders.*

Proof From Theorem 7, $C^*_R \circ C^*_P = C^*_P \circ C^*_R \Leftrightarrow R\circ P = P\circ R$ and from Theorem 2, $R\circ P = P\circ R$ if and only if both are fuzzy preorders. □

Corollary 3 *Let R, P be fuzzy $*$-preorders and let C^*_R and C^*_P their corresponding fuzzy consequence operators induced through Zadeh's compositional rule. Then, C^*_R and C^*_P permute if and only if $R\circ P = P\circ R = \overline{\max(P,R)}$.*

For permutability of fuzzy operators induced by fuzzy indistinguishability relations the following result holds.

Theorem 9 *Let E, F be fuzzy $*$-indistinguishability relations and let C^*_E and C^*_F be their corresponding fuzzy consequence operators induced through Zadeh's compositional rule. Then, C^*_E and C^*_F permute if and only if $E\circ F$ is a fuzzy $*$-indistinguishability relation.*

Proof It directly follows from Corollary 2 and Theorem 8. □

Corollary 4 *Let E, F be fuzzy $*$-indistinguishability relations and let C^*_E and C^*_F be their corresponding fuzzy consequence operators induced through Zadeh's compositional rule. Then, C^*_E and C^*_F permute if and only if $E\circ F = \overline{\max(E,F)}$.*

Corollary 5 *Let C, C' be fuzzy operators satisfying all the conditions of Proposition 4. That is,*

1. *They are fuzzy consequence operators.*
2. *They satisfy $C(\bigvee_{i\in I}\mu_i) = \bigvee_{i\in I} C(\mu_i)$ for any index set I and all $\mu_i \in [0,1]^X$.*
3. *They satisfy $C(\{x\})(y) = C(\{y\})(x)$ for all $x, y \in X$ where $\{x\}$ denotes the singleton of x.*
4. *They satisfy $C(\alpha * \mu) = \alpha * C(\mu)$ for any constant $\alpha \in [0,1]$ and $\mu \in [0,1]^X$.*

Then, C and C' permute if and only if $C\circ C'$ also satisfies all these conditions.

In Proposition 15, two different ways of writing the composition of fuzzy operators were presented. We shall see another approach to permutability that can be obtained using the second expression. This allows a sufficient condition for permutability in terms of the concordance between fuzzy relations and fuzzy operators, notion that we introduced in Definition 14.

Proposition 16 *Let R, P be fuzzy preorders and let C_R^* and C_P^* be their respective induced FCO by means of equation (2.1). If C_R^* is $*$-concordant with P and C_P^* is $*$-concordant with R, then P and R permute and therefore C_R^* and C_P^* also permute.*

Proof It directly follows from Theorems 1 and 8. □

The following theorem is adapted from [22]:

Theorem 10 *Let $\{\mu_i\}_{i\in I} \subseteq [0,1]^X$ be an arbitrary family of fuzzy subsets. Then,*

$$R(x,y) = \inf_{i\in I}\{\mu_i(x) \to \mu_i(y)\} \tag{3.4}$$

is the largest fuzzy preorder for which every fuzzy subset of the family $\{\mu_i\}_{i\in I}$ is $*$-compatible with.

Notice that $\{\mu_i\}_{i\in I}$ is also $*$-compatible with S for every fuzzy relation S smaller than or equal to (3.4). Using this result, we define the largest fuzzy preorder for which a given operator C can be $*$-concordant with.

Definition 19 *Let C be a fuzzy operator in Ω'. The fuzzy relation R_c^c induced by C is given by*

$$R_c^c(x,y) = \inf_{\mu\in[0,1]^X}\{C(\mu)(x) \to C(\mu)(y)\} \tag{3.5}$$

According to Theorem 10, the fuzzy preorder R_c^c defined above gives an upper bound which is sufficient for a relation to be $*$-concordant with the given operator C. Hence, if a fuzzy relation S is smaller than or equal to R_c^c for a certain fuzzy operator C, every fuzzy subset of the image of C will be compatible with S.

Proposition 17 *Let S be a fuzzy relation such that $S \leq R_c^c$ for a certain $C \in \Omega'$. Then, C is $*$-concordant with S.*

Proof Straightforward. □

Corollary 6 *Let R, P be fuzzy preorders and let C_R^* and C_P^* be their respective induced FCO. If*

$$R \leq R_{c_P^*}^{c_P^*} \quad \text{and} \quad P \leq R_{c_R^*}^{c_R^*},$$

then R and P permute. Therefore, so do C_R^* and C_P^*.

Proof It directly follows from Propositions 16, 17 and corollary 3. □

Permutability of fuzzy operators induced by fuzzy relations through $\inf - \rightarrow$ composition

Composition of operators induced by means of the inf-$\rightarrow$ composition as defined by (2.2) can be written in terms of the sup-$*$ composition of the inducing relations.

Proposition 18 *Let R, S be two fuzzy relations and let $C_R^{\rightarrow}$ and $C_S^{\rightarrow}$ be the corresponding fuzzy operators induced through the* $\inf - \rightarrow$ *composition. Then,*

$$C_R^{\rightarrow} \circ C_S^{\rightarrow} \;=\; C_{R\circ S}^{\rightarrow} \tag{3.6}$$

where $S \circ R$ denotes the sup-$*$ product composition of fuzzy relations.

Proof For all $\mu \in [0,1]^X$ and all $x \in X$ we have

$$\begin{aligned} C_R^{\rightarrow} \circ C_S^{\rightarrow}(\mu)(x) &= \inf_{w\in X}\{R(x,w) \rightarrow C_S^{\rightarrow}(\mu)(x)\} \\ &= \inf_{w\in X}\{\; R(x,w) \rightarrow \{\inf_{y\in X}\{S(w,y) \rightarrow \mu(y)\}\}\;\} \\ &= \inf_{w,y\in X}\{R(x,w) \rightarrow \{S(w,y) \rightarrow \mu(y)\}\} \\ &= \inf_{y\in X}\inf_{w\in X}\{\{R(x,w) * S(w,y)\} \rightarrow \mu(y)\} \\ &= \inf_{y\in X}\{\sup_{w\in X}\{R(x,w) * S(w,y)\} \rightarrow \mu(y)\} \\ &= \inf_{y\in X}\{(R \circ S)(x,y) \rightarrow \mu(y)\} \;=\; C_{R\circ S}^{\rightarrow}(\mu)(x) \end{aligned}$$

where most of the equalities follow from the properties in Proposition 1. □

As a consequence, we obtain similar results to the ones obtained for the operators induced by Zadeh's compositional rule.

Theorem 11 *Let R, S be two fuzzy relations and let $C_R^{\rightarrow}$ and $C_S^{\rightarrow}$ be the corresponding fuzzy operators induced through the* inf- $\rightarrow$ *composition. Then, $C_R^{\rightarrow}$ and $C_S^{\rightarrow}$ permute if and only if R and S permute.*

Proof One the one side, assume that $S \circ R = R \circ S$. Then, from the previous proposition it follows that

$$C_R^{\rightarrow} \circ C_S^{\rightarrow}(\mu)(x) \;=\; C_{R\circ S}^{\rightarrow}(\mu)(x) = \; C_{S\circ R}^{\rightarrow}(\mu)(x) = C_S^{\rightarrow} \circ C_R^{\rightarrow}(\mu)(x)$$

On the other side, from Proposition 7, $C_{R\circ S}^{\rightarrow} = \; C_{S\circ R}^{\rightarrow}$ implies $R \circ S = S \circ R$. □

Theorem 12 *Let R, P be fuzzy $*$-preorders and let $C_R^{\rightarrow}$ and $C_P^{\rightarrow}$ their corresponding fuzzy interior operators induced through the inf- $\rightarrow$ composition by means of (2.2). Then, $C_R^{\rightarrow}$ and $C_P^{\rightarrow}$ permute if and only if $R \circ P$ and $P \circ R$ are fuzzy $*$-preorders.*

Proof It directly follows from Theorems 2 and 11. □

Corollary 7 *Let R, P be fuzzy $*$-preorders and let $C_R^{\rightarrow}$ and $C_P^{\rightarrow}$ the corresponding fuzzy interior operators induced by means of (2.2). Then, $C_R^{\rightarrow}$ and $C_P^{\rightarrow}$ permute if and only if $R \circ P \;=\; P \circ R = \max(P, R)$.*

For permutability of fuzzy operators induced by fuzzy indistinguishability relations the following holds.

Theorem 13 *Let E, F be fuzzy $*$-indistinguishability relations and let $C_E^{\rightarrow}$ and $C_F^{\rightarrow}$ be their corresponding fuzzy interior operators induced by means of (2.2). Then, $C_E^{\rightarrow}$ and $C_F^{\rightarrow}$ permute if and only if $E \circ F$ is a fuzzy $*$-indistinguishability relation.*

Proof It directly follows from Corollary 2 and Theorem 12. □

Corollary 8 *Let E, F be fuzzy $*$-indistinguishability relations and let $C_E^{\rightarrow}$ and $C_F^{\rightarrow}$ be their corresponding fuzzy interior operators induced by means of (2.2). Then, $C_E^{\rightarrow}$ and $C_F^{\rightarrow}$ permute if and only if $E \circ F = \overline{\max(E, F)}$.*

Corollary 9 *Let C, C' be fuzzy operators satisfying all the conditions of Proposition 9. That is,*

1. *They are fuzzy interior operators.*
2. *They satisfy $C(\bigwedge_{i\in I} \mu_i) = \bigwedge_{i\in I} C(\mu_i)$ for any index set I and all $\mu_i \in [0,1]^X$.*
3. *They satisfy $C(\{x\} \to \alpha)(y) = C(\{y\} \to \alpha)(x)$ for all $x, y \in X$ and any constant $\alpha \in [0,1]$ where $\{x\}$ denotes the singleton of x.*
4. *They satisfy $C(\alpha \to \mu) = \alpha \to C(\mu)$ for any constant $\alpha \in [0,1]$ and $\mu \in [0,1]^X$.*

Then, C and C' permute if and only if $C \circ C'$ also satisfies all these conditions.

Conclusions and Outlook

Conclusions

Permutability of fuzzy consequence operators and fuzzy interior operators does not always occur. However, we have proved that there are cases for which the order of composition does not affect the result. We have shown that this fact is completely connected to the preservation of the operator type through composition. That is, when the composition of two fuzzy consequence (interior) operators is again a fuzzy consequence (interior) operator.

For the particular cases of fuzzy consequence operators induced through Zadeh's compositional rule and fuzzy interior operators induced using the inf-$\rightarrow$ composition we proved that permutability of the relations is connected to permutability of the induced operators. In fact, pemutability of the starting relations appears to be a necessary and sufficient condition in order to obtain permutability of the induced operators.

To conclude, we summarize the most important results that we have obtained. First of all, we enumerate the results about permutability of fuzzy preorders and fuzzy indistinguishability relations that have been the key to analyze permutability of their induced fuzzy operators:

1. R, P fuzzy $*$-preorders. Then, $R \circ P = P \circ R \;\Leftrightarrow\; R \circ P$ and $P \circ R$ are fuzzy $*$- preorders.

2. E, F fuzzy $*$-similarities. Then, $E \circ F = F \circ E \;\Leftrightarrow\; E \circ F$ is a fuzzy $*$-similarity.

3. $R \circ P$ preserves type (similarity or preorder) $\Leftrightarrow R \circ P = \overline{\max(R, P)}$.

Permutability of general fuzzy consequence and interior operators can be summarized in the following two results.

4. C, C' FCO. C and C' permute if and only if $C \circ C' = C' \circ C = \overline{\max(C, C')}$.

5. C, C' FIO. C and C' permute if and only if $C \circ C' = C' \circ C = \underline{\min(C, C')}$.

Several results about permutability of fuzzy operators induced by fuzzy relations have been obtained. Results about permutability of fuzzy preorders and similarities allow some of the following characterizations. The notion of concordance between fuzzy relations and fuzzy operators also plays a relevant role (see item 11).

6. $R,\ S$ fuzzy relations. $C^*_R \circ C^*_S = C^*_{S \circ R} = C_R^{C^*_S}$.

7. $R,\ S$ fuzzy relations. $C^{\rightarrow}_R \circ C^{\rightarrow}_S = C^{\rightarrow}_{R \circ S}$.

8. $R,\ S$ fuzzy relations. Then, $C^*_R \circ C^*_S = C^*_S \circ C^*_R \quad \Leftrightarrow \quad R \circ S = S \circ R$.

9. R, S fuzzy relations. Then, $C_R^{\rightarrow} \circ C_S^{\rightarrow} = C_S^{\rightarrow} \circ C_R^{\rightarrow} \quad \Leftrightarrow \quad R \circ S = S \circ R$.

10. R, S fuzzy $*$-preorders. Then, $C_R^* \circ C_S^* = C_S^* \circ C_R^* \quad \Leftrightarrow \quad R \circ S$ and $S \circ R$ are fuzzy $*$-preorders.

11. R, P fuzzy $*$-preorders. C_R^* and P $*$-concordant. C_P^* and R $*$-concordant. Then, $C_R^* \circ C_P^* = C_P^* \circ C_R^*$.

12. E, F fuzzy $*$-indistinguishabilities. Then, $C_E^* \circ C_F^* = C_F^* \circ C_E^* \quad \Leftrightarrow \quad E \circ F$ is a fuzzy $*$-indistinguishability.

13. R, S fuzzy $*$-preorders. Then, $C_R^{\rightarrow} \circ C_S^{\rightarrow} = C_S^{\rightarrow} \circ C_R^{\rightarrow} \quad \Leftrightarrow \quad R \circ S$ and $S \circ R$ are fuzzy $*$-preorders.

14. E, F fuzzy $*$-indistinguishabilities. Then, $C_E^{\rightarrow} \circ C_F^{\rightarrow} = C_F^{\rightarrow} \circ C_E^{\rightarrow} \quad \Leftrightarrow \quad E \circ F$ is a fuzzy $*$-indistinguishability.

Outlook

In this work, we have studied permutability of the composition of two operators of the same nature. For instance, two FCO or two FIO. From the theoretical point of view, it would be interesting to analyze permutability in a wider range of operators. For example, when the composition of one FCO with one FIO is permutable. Other kinds of operators could be also introduced in the study.

The analysis of how permutability is translated from the inducing relations could be extended to other operators induced by fuzzy relations using different definitions. As we already said, more properties of the operator induced by a fuzzy relation through the $\inf - \rightarrow$ composition should be studied.

From a more practical point of view, further analysis of the applicability of these results into image processing should be done. The aim would be to concatenate several morphological filters controlling how the order of application affects the outcome.

Bibliography

[1] R. Bělohlávek. *Fuzzy Relational Systems: Foundations and Principles.* IFSR international series on systems science and engineering. Kluwer Academic/Plenum Publishers (2002).

[2] B. Davey, H. Priestley. *Introduction to Lattices and Order.* Cambridge mathematical text books. Cambridge University Press (2002).

[3] J. Pavelka. *On fuzzy logic I: Many-valued rules of inference.* Mathematical Logic Quarterly **25 (3-6)**, (1979), pp. 45–52.

[4] J. Elorza, P. Burillo. *Connecting fuzzy preorders, fuzzy consequence operators and fuzzy closure and co-closure systems.* Fuzzy Sets and Systems **139 (3)**, (2003), pp. 601–613.

[5] J. Castro, E. Trillas. *Tarski's fuzzy consequences.* In *Proceedings of the International Fuzzy Engineering Symposium '91*, pp. 70–81 (1991).

[6] J. Elorza, P. Burillo. *On the relation between fuzzy preorders and fuzzy consequence operators.* International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems **7 (3)**, (1999), pp. 219–234.

[7] M. Ward. *The closure operators of a lattice.* Annals of Mathematics **43 (2)**, (1942), pp. 191–196.

[8] R. Bělohlávek, T. Funioková. *Fuzzy interior operators.* International Journal of General Systems **33 (4)**, (2004), pp. 415–430.

[9] B. De Baets, E. Kerre, M. Gupta. *The fundamentals of fuzzy mathematical morphology. part 1: Basic concepts.* International Journal of General Systems **23 (2)**, (1995), pp. 155–171.

[10] T.-Q. Deng, H. J. A. M. Heijmans. *Grey-scale morphology based on fuzzy logic.* Journal of Mathematical Imaging and Vision **16 (2)**, (2002), pp. 155–171.

[11] I. Bloch. *Lattices of fuzzy sets and bipolar fuzzy sets, and mathematical morphology.* Information Sciences **181 (10)**, (2011), pp. 2002–2015.

[12] P. Maragos. *Lattice image processing: A unification of morphological and fuzzy algebraic systems.* Journal of Mathematical Imaging and Vision **22 (2-3)**, (2005), pp. 333–353.

[13] C. Ronse, H. J. A. M. Heijmans. *The algebraic basis of mathematical morphology II. Openings and closings.* CVGIP: Image Understanding **54 (1)**, (1991), pp. 74–97.

[14] J. Elorza, R. Fuentes-González, J. Bragard, P. Burillo. *On the relation between fuzzy closing morphological operators, fuzzy consequence operators induced by fuzzy preorders and fuzzy closure and co-closure systems.* Fuzzy Sets and Systems **218**, (2013), pp. 73–89.

[15] W. Bandler, L. J. Kohout. *Special properties, closures and interiors of crisp and fuzzy relations.* Fuzzy Sets and Systems **26 (3)**, (1988), pp. 317–331.

[16] J. Castro, M. Delgado, E. Trillas. *Inducing implication relations.* International Journal of Approximate Reasoning **10 (3)**, (1994), pp. 235–250.

[17] J. Recasens. *Indistinguishability Operators: Modelling Fuzzy Equalities and Fuzzy Equivalence Relations*, vol. 260 of *Studies in Fuzziness and Soft Computing.* Springer (2010).

[18] L. Zadeh. *Fuzzy sets.* Information and Control **8 (3)**, (1965), pp. 338–353.

[19] L. A. Zadeh. *The concept of a linguistic variable and its application to approximate reasoning - III.* Information Sciences **9 (1)**, (1975), pp. 43–80.

[20] D. Dubois, H. Prade. *Upper and lower images of a fuzzy set induced by a fuzzy relation: Applications to fuzzy inference and diagnosis.* Information Sciences **64 (3)**, (1992), pp. 203–232.

[21] J. Recasens. *Permutable indistinguishability operators, perfect vague groups and fuzzy subgroups.* Information Sciences **196**, (2012), pp. 129–142.

[22] U. Bodenhofer, M. De Cock, E. E. Kerre. *Openings and closures of fuzzy preorderings: Theoretical basics and applications to fuzzy rule-based systems.* International Journal of General Systems **32 (4)**, (2003), pp. 343–360.

Summary

Fuzzy operators are an essential tool in many fields and the operation of composition is often needed. In general, composition is not a commutative operation. However, it is very useful to have operators for which the order of composition does not affect the result. In this work, we have analyzed when permutability appears. That is, when the order of application of the operators does not change the outcome. We have characterized permutability in the case of the composition of two fuzzy consequence operators and the dual case of fuzzy interior operators. We have proved that for these cases, permutability is completely connected to the preservation of the operator type through composition. That is, when the composition of two FCO is again a FCO or the composition of two FIO gives again a FIO.

We have also studied the particular case of fuzzy operators induced by fuzzy relations through Zadeh's compositional rule and the inf- $\rightarrow$ composition. For this cases, we have connected permutability of the fuzzy relations with permutability of the induced operators. We have showed that permutability of the fuzzy relations is a necessary and sufficient condition to obtain permutability of the induced operators. Special attention has been paid to the cases of operators induced by fuzzy preorders and similarities since the operators obtained are FCO and FIO.