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Abstract. This paper will discuss a new method of Ball surface generation from prescribed boundaries based on the partial differential operator. In particular, we focus on the construction of a bicubic Said-Ball surface using biharmonic partial differentiation equation. The main result is that the use of biharmonic Said-Ball surface would enable the overall surface to be generated and controlled based on the boundary curves rather than a set of control points. We illustrate the new method by using several graphical examples.

Keywords: Said-Ball, Biharmonic, Partial differentiation equation
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INTRODUCTION

The PDE based on the Laplacian operator and fourth order equation are widely used for surface modeling in CAGD. For example, [1] and [2] constructed the harmonic and biharmonic Bézier surface bounded by the given boundary curves. [3] presents the construction of biharmonic and tetra harmonic Bézier surface with respect to boundary curves and the tangent conditions along them. [4] discussed the methods for designing triangular Bézier PDE surfaces for given different sets of prescribed control points and including the special cases of harmonic and biharmonic surfaces. Currently, [5] presents a new approach to construct surfaces bounded by line of curvature or geodesics based on fourth order PDE. Generally, PDE-based surface modeling has the advantages over the free-form surface modeling based on finding control points because the entire surfaces can be constructed subject to a given set of boundary curves only.

In this paper we will discuss biharmonic surface construction based on Ball surface representation instead of using Bézier representation as stated in the previous researches. We will develop a boundary based intuitive surface design technique for Said-Ball surfaces ([6], [7]) which are one of the basic types of surfaces that are used in Computer Aided Geometric Design (CAGD). The main idea is to find Said-Ball solutions to some natural PDEs which can only be controlled through the boundary control points and those adjacent to them.

The construction of a lower dimension of biharmonic Said-Ball surface of degree 3 by 3 which consists of 16 control points where twelve of them are boundary control points will be discussed. Using the biharmonic PDE, we shall derive a relationship between inner control points with the given boundary control points.

SAID-BALL SURFACE

Degree m by n Said-Ball patch S is defined as

$$S(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} b_{ij} S_i^3(u) S_j^3(v), \quad 0 \leq u,v \leq 1 \tag{1}$$

with b_{ij} are the control points and

$$
S_i^n(t) = \begin{cases}
\dbinom{\lfloor n/2 \rfloor + i}{i} t^i (1-t)^{\lfloor n/2 \rfloor + 1}, & 0 \le i \le \lceil n/2 \rceil - 1, \\[2ex]
\dbinom{n}{n/2} t^{n/2}(1-t)^{n/2}, & i = n/2, \\[2ex]
S_{n-i}^n(1-t), & \lfloor n/2 \rfloor + 1 \le i \le n.
\end{cases}
\tag{2}
$$

where $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ denote the greatest integer less than or equal to $n/2$ and the least integer greater than or equal to $n/2$, respectively.

BIHARMONIC EQUATION

Biharmonic equation for parametric surface $X(u,v)$ is defined as the differential equation obtained by applying the biharmonic operator also known as the bilaplacian, that is the differential operator defined by $\Delta^4 = (\Delta^2)^2$ where

$$
\Delta^2 = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}
$$

is the Laplacian and setting to zeros. In general, for rectangular system of coordinates can be written as,

$$
\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right)^2 X(u,v) = 0
\tag{3}
$$

or

$$
\frac{\partial^4 X}{\partial u^4} + 2\frac{\partial^4 X}{\partial u^2 \partial v^2} + \frac{\partial^4 X}{\partial u^4} = 0.
\tag{4}
$$

CONSTRUCTION OF BIHARMONIC SAID-BALL SURFACE DEGREE 3 BY 3

Let X: [0,1] x [0, 1] to be a bicubic Said-Ball surface ($m = n = 3$) as given in (1) represented in matrix form as

$$
X(u,v) = UMV
\tag{5}
$$

where

$$
U = [(1-u)^2 \quad 2u(1-u)^2 \quad 2u^2(1-u) \quad u^2], V = [(1-v)^2 \quad 2v(1-v)^2 \quad 2v^2(1-v) \quad v^2]^T
$$

and

$$
M = \begin{bmatrix}
b_{00} & b_{01} & b_{02} & b_{03} \\
b_{10} & b_{11} & b_{12} & b_{13} \\
b_{20} & b_{21} & b_{22} & b_{23} \\
b_{30} & b_{31} & b_{32} & b_{33}
\end{bmatrix}
$$

is the control points matrix.

Assume that, boundary control points b_{i0}, b_{0j}, b_{i3}, b_{3j} $i, j = 0,...,3$ are given. We need to determine the function X which satisfy the biharmonic condition as in (4). In order to fullfil these conditions, we will derive the inner points, b_{11}, b_{12}, b_{21}, b_{22} of bicubic Said-ball surface in term of boundary control points. For the bicubic Said-Ball surface, the first and third terms in (4) will be eliminated, leaving only the second term to be considered. Thus, from (4) and (5), we obtain,

$$
RMS = 0
\tag{6}
$$

where

$$
R = [2 \quad 12u - 8 \quad 4 - 12u \quad 2], S = [2 \quad 12v - 8 \quad 4 - 12v \quad 2]^T.
$$

By expanding (6), we obtain the system of four linear equations in four unknowns and is given as follows:

$$AY=B \tag{7}$$

where

$$A = \begin{bmatrix} 64 & -32 & -32 & 16 \\ -96 & 48 & 96 & -48 \\ -96 & 96 & 48 & -48 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

is a full rank coefficient matrix, $Y = [b_{11} \ b_{12} \ b_{21} \ b_{22}]^T$ and B is a constant column vector given by $B = [b_1 \ b_2 \ b_3 \ 0]^T$ with

$$b_1 = -(4b_{00} - 16b_{01} + 8b_{02} + 4b_{03} - 16b_{10} - 16b_{13} + 8b_{20} + 8b_{23} + 4b_{30} - 16b_{31} + 8b_{32} + 4b_{33})$$

$$b_2 = -(24b_{10} + 24b_{13} - 24b_{20} - 24b_{23})$$

and

$$b_3 = -(24b_{01} - 24b_{02} + 24b_{31} - 24b_{32}).$$

Thus, the solution of (7) is

$$b_{11} = (-b_{00} + 2b_{01} - b_{03} + 2b_{10} + 2b_{13} - b_{30} + 2b_{31} - b_{33})/4$$
$$b_{12} = (-b_{00} + 2b_{02} - b_{03} + 2b_{10} + 2b_{13} - b_{30} + 2b_{32} - b_{33})/4$$
$$b_{21} = (-b_{00} + 2b_{01} - b_{03} + 2b_{20} + 2b_{23} - b_{30} + 2b_{31} - b_{33})/4$$
$$b_{22} = (-b_{00} + 2b_{02} - b_{03} + 2b_{20} + 2b_{23} - b_{30} + 2b_{32} - b_{33})/4.$$

The above result shows that the boundaries of a biharmonic Said-Ball surface enables us to fully construct the entire surface.

GRAPHICAL EXAMPLES

To generate the biharmonic Said-Ball surface, we have chosen three sets of boundary conditions. First set of control points is taken from [3],

$b_{00} = (0,0,0)$, $b_{10} = (3/2,0,-3)$, $b_{20} = (3/2,0,3)$, $b_{30} = (3,0,0)$, $b_{01} = (0,3/2,-3)$, $b_{31} = (3,1,1)$
$b_{02} = (0,3/2,3)$, $b_{32} = (3,2,1)$, $b_{03} = (0,3,0)$, $b_{13} = (1,3,-1)$, $b_{23} = (2,3,-1)$, $b_{33} = (3,3,0)$.

Using the above boundary points, the inner control points of biharmonic Said-Ball surface are,
$b_{11} = (5/4, 5/4, -3)$, $b_{12} = (5/4, 7/4, 0)$, $b_{21} = (7/4, 5/4, 0)$ and $b_{22} = (7/4, 7/4, 3)$.

Second example, two boundary curves are defined approximately as semi-circular arcs with radius 1 and 0.5 respectively, while two more boundary curves define on the curves connecting both arcs as given by the following control points.

$b_{00} = (0.5,0,0)$, $b_{10} = (0.2577,0.4,0)$, $b_{20} = (-0.5,0.36,0)$, $b_{30} = (-0.5,0,0)$, $b_{01} = (0.425,0,0.45)$,
$b_{31} = (-0.425, 0, 0.45)$, $b_{02} = (0.575,0,0.75)$, $b_{32} = (-0.575,0,0.75)$, $b_{03} = (1,0,1)$,
$b_{13} = (0.5153,1,1)$, $b_{23} = (-1,0.72,1)$, $b_{33} = (-1,0,1)$.

The inner control points are calculated as, b_{11} = (0.3865, 0.7, 0.45), b_{12} = (0.3865, 0.7, 0.75), b_{21}= (-0.75, 0.54, 0.45) and b_{22} = (-0.75, 0.54, 0.75).

For the last example, we use the following control points,

$b_{00} = (0 - 0.9271, 2.853, 0), b_{10} = (-3, 0, 0), b_{20} = (-2.427, -1.763, 0), b_{30} = (0.9271, -2.853, 0),$
$b_{01} = (-0.4364, 1.343, 0.35), b_{31} = (0.5225, -1.608, 0.3), b_{02} = (-0.368, 1.133, 0.6),$
$b_{32} = (0.3241, 0.9976, 0.55), b_{03} = (-0.9271, 2.853, 1), b_{13} = (-2.853, -0.9271, 1),$
$b_{23} = (-0.9271, -2.853, 1), b_{33} = (0.9271, -2.853, 1).$

and the inner control points are obtained as,

b_{11} = (-2.8834, -0.59605, 0.325), b_{12} = (-2.9485, -0.39585, 0.575),

b_{21}= (-1.634, -2.4405, 0.325) and b_{22} = (-1.699, -2.2403, 0.575).

Figures 1(a)-3(a) show the control polygons together with its common boundary cuves for Said-Ball of degree 3, while Figures 1(b)- 3(b) show the bicubic Said-Ball surfaces resulting from the biharmonic solution along with the boundary control points. Clearly seen that, the shape of constructing surfaces reproduce the shapes of the boundary curves.

CONCLUSION

In this work, we have constructed a biharmonic Said-Ball surface of degree 3, where the inner control points depend on boundary control points fullfill the harmonicity conditions of PDE. We show that for a given cubic Said-Ball boundary curve, there exists a unique solution of the biharmonic equation with that boundary. The use of biharmonic Said-Ball surface would enable the overall surface to be generated and controlled based on the boundary curves rather than a set of control points.

ACKNOWLEDGMENTS

REFERENCES

1. J. Monterde and H. Ugail, *International Journal of Computers and Applications* **31 (2)**, 202-2314 (2009).
2. J. Monterde and H. Ugail, *Computer Aided Geometric Design* **21**, 697-715 (2004).
3. P. Centella, J. Monterde, F. Moreno and R. Oset, *Computer Aided Geometric Design* **26(2)**, 152-173 (2009).
4. A. Arnal, A. Lluch and J. Monterde, *Journal of Computational and Applied Mathematics* **235(5)**, 1098-1113 (2011).
5. W. X. Huang, H.J.L. Wu and G. J. Wang, *Computers and Mathematics with Applications* **65**, 673–681 (2013).
6. H.B. Said, *ACM Transaction on Graphics* **8(4),** 360-371 (1989).
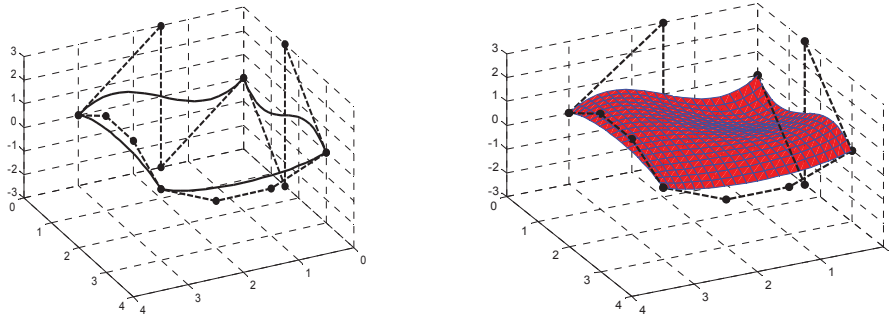7. S.M. Hu, G.Z. Wang, and T.G. Jin (1996), *Computer Aided Design* **28(2)**, 125–133.

FIGURE 1: a) Control polygon together with its boundary curves (Control points set 1) b) A biharmonic Said-Ball surface of degree 3 for the associated boundary curves in (a)
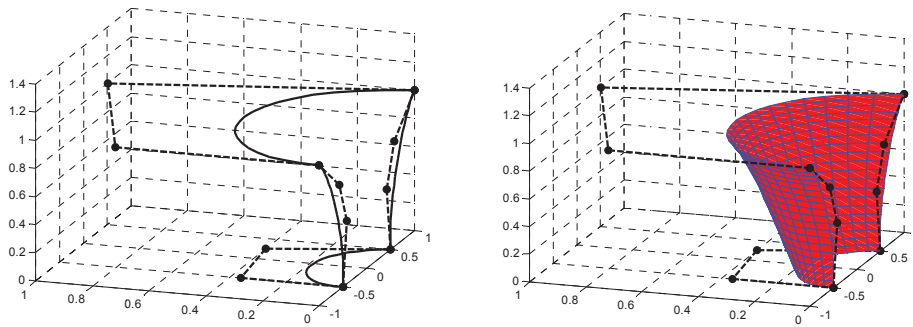

FIGURE 2: a) Control polygon together with its boundary curves (Control points set 2) b) A biharmonic Said-Ball surface of degree 3 for the associated boundary curves in (a)
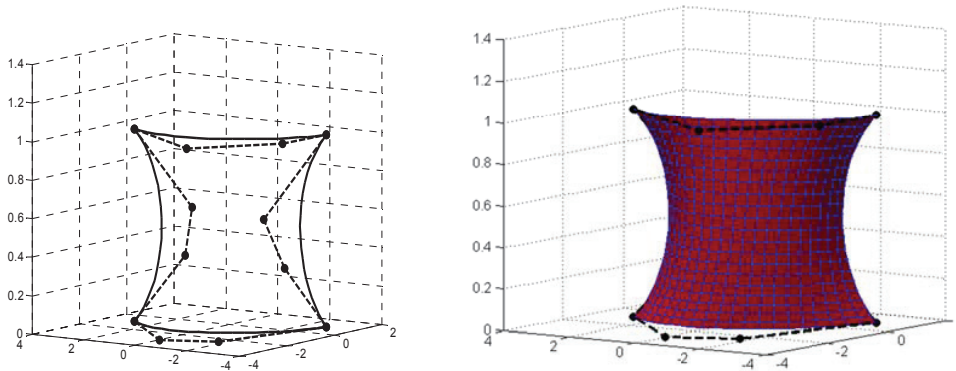

FIGURE 3: a) Control polygon together with its boundary curves (Control points set 3) b) A biharmonic Said-Ball surface of degree 3 for the associated boundary curves in (a)