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Some Modifications of Sarrus's Rule Method via Permutation for Finding Determinant of 4 by 4 Square Matrix

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Abstract. Sarrus rule is well known method for finding determinant of square matrix. This method is also known as a cross multiplication method. However this method is not applicable for $n > 3$. With this motivation, we attempt to extend this method by employing some modifications using permutation for the case of 4 by 4 square matrix

INTRODUCTION

The most commonly used techniques for finding determinant are cross multiplication, cofactor expansion, and Gaussian elimination. All these techniques were discussed in great details in many textbooks such as Anton [1], Anton and Busby [2], Brestscher [3], Hsiung and Mao [4], Perry [5], Wilde [6], and Sneider *et al.*, [7]. Among these three methods, cross multiplication method works only for a $n \times n$ square matrix for $n \leq 3$. Thus in this paper, we determine the determinant of 4×4 matrices using Sarrus's Rule by applying a new technique which based on our new permutation method [8].

REVIEW ON SARRUS'S RULE

Pierre Frédéric Sarrus (1853) introduced the cross multiplication method which also called the Sarrus's rule. The cross multiplication method is only applicable to the size of square matrix $n \leq 3$.

Example 1: Sarrus's rule for 2×2 matrix

By lines drawn on the two diagonals,

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

The multiplications of the elements on each line give the products. Then the subtraction of the products from the up-going lines from the products of the down-going lines gives the determinant.

For 3×3 matrix, one needs to append the first two columns to the right of the matrix. Then put the lines on all diagonals, the same hold as for 2×2 matrix. See the following example.

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Example 2: Sarrus's rule for 3×3 matrix

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{vmatrix}$$

Det(A)= $[a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}] - [a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33}]$

However this method can be used when the order of the square matrix is less than or equal to three ($n \leq 3$).
In spite of Sarrus's work, Hajrijaz [9] introduced three methods to determine the determinant of square matrix. For each method, six diagonals will be formed.

Let consider matrix A

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

In one of his method where elements a_{13} and $a_{33}$are placed before first row and third row respectively. Then, elements a_{11} and a_{31} placed after first row and third row respectively. See Example 3.

Example 3:

$$\begin{matrix} a_{13} & a_{11} & a_{12} & a_{13} & a_{11} \\ & a_{21} & a_{22} & a_{23} & \\ a_{33} & a_{31} & a_{32} & a_{33} & a_{31} \end{matrix}$$

The process of elements product and the sign for six diagonals are similar to Sarrus's rule for 3×3 matrix. The determinant of this matrix is equal to the determinant in Example 2. The other two methods in Hajrijaz [9] also generated six diagonals and also work for 3×3 matrix. There are several interesting methods has been discussed for finding determinant of matrix 3×3 in Assen and Rao [10]. As stated in Assen and Rao [10], Sarrus's rule and Hajrijaz's methods use applicable to square matrix with order at most 3×3 matrix. Now, we explore the relationship between circular permutation and Sarrus's rule for case $n = 3$.

Example 4:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{vmatrix}$$

From the column indices of elements in the main diagonal $[a_{11}, a_{22}, a_{33}]$, if we extract the column indices then we obtained 123. Next, to main diagonal $[a_{12}, a_{23}, a_{31}]$, the column indices is 231 and the main diagonal $[a_{13}, a_{21}, a_{32}]$, the column indices is 312. Then we do the same process for secondary diagonal and its parallel diagonal. We obtained the following permutations for main diagonal, secondary diagonal and their parallel diagonals column indices as shown below:

TABLE 1. Pairs of main diagonal and secondary diagonal column indices

Main diagonal and its parallel diagonal	Secondary diagonal and its parallel diagonal
123	321
231	132
312	213

From TABLE 1, we noticed that the circular permutation pattern appeared in the main diagonal and its parallel diagonal column indices and elements are cycled. The similar pattern also appeared in the secondary diagonal and its parallel diagonal column indices.

As shown in TABLE 1, the secondary diagonal column indices are the reverse of the main diagonal column indices. It is also apply to other parallel main diagonal and parallel secondary diagonal. From Sarrus's rule, the determinant is

$$[a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}] - [a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33}]. \tag{1}$$

Then we rearranged the result in equation (1), with respect to the circular permutation of element column indices in pair of diagonals as shown in Table 1, we had

$$[a_{11}a_{22}a_{33} - a_{13}a_{22}a_{31}] + [a_{12}a_{23}a_{31} - a_{11}a_{23}a_{32}] + [a_{13}a_{21}a_{32} - a_{12}a_{21}a_{33}]. \tag{2}$$

Thus, we extended this concept discussed in Example 4 for any 4×4 matrix. Consider the case $n = 4$, there are three distinct 4th order diagrams.

Example 5:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

The three 4th diagrams were constructed based on permutation column indices, (1234), (1324) and (1342) and the first 3 columns are append to the right of matrices. The permutation column indices were obtained under exchanging two element restrictions as follows:

1 2 3 4
1 3 2 4
1 3 4 2

The three 4th order diagrams and their signs are shown below.

$$\left|\begin{matrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{matrix}\right|\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{matrix}\right|$$

TABLE 2. Signs of product diagonal for first 4th order diagram

Main Diagonal	Sign	Secondary Diagonal	Sign
1234	+	**4321**	+
2341	−	**1432**	−
3412	+	**2143**	+
4123	−	**3214**	−

$$\left|\begin{matrix} a_{11} & a_{13} & a_{12} & a_{14} \\ a_{21} & a_{23} & a_{22} & a_{24} \\ a_{31} & a_{33} & a_{32} & a_{34} \\ a_{41} & a_{43} & a_{42} & a_{44} \end{matrix}\right|\begin{matrix} a_{11} & a_{13} & a_{12} \\ a_{21} & a_{23} & a_{22} \\ a_{31} & a_{33} & a_{32} \\ a_{41} & a_{43} & a_{42} \end{matrix}\right|$$

TABLE 3. Signs of product diagonal for second 4th order diagram

Main Diagonal	Sign	Secondary Diagonal	Sign
1324	$-$	**4231**	$-$
3241	$+$	**1423**	$+$
2413	$-$	**3142**	$-$
4132	$+$	**2314**	$+$

$$\begin{vmatrix} a_{11} & a_{13} & a_{14} & a_{12} & |a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} & a_{22} & |a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} & a_{32} & |a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} & a_{42} & |a_{41} & a_{43} & a_{44} \end{vmatrix}$$

TABLE 4. Signs of product diagonal for third 4th order diagram

Main Diagonal	sign	Secondary Diagonal	sign
1342	$+$	**2431**	$+$
3421	$-$	**1243**	$-$
4213	$+$	**3124**	$+$
2134	$-$	**4312**	$-$

Every 4th diagram consists 8 terms (diagonals). Then the total terms is 24. Now, we demonstrate the Sarrus's rule with permutations by giving an example of 4×4 matrices.

Example 6:

$$A = \begin{bmatrix} 1 & 4 & 7 & 8 \\ 5 & 10 & 3 & 5 \\ 2 & 11 & 90 & 53 \\ 9 & 33 & 6 & 7 \end{bmatrix}$$

First 4th order diagram:

$$|A_1| = \begin{vmatrix} 1 & 4 & 7 & 8 & |1 & 4 & 7 \\ 5 & 10 & 3 & 5 & |5 & 10 & 3 \\ 2 & 11 & 90 & 53 & |2 & 11 & 90 \\ 9 & 33 & 6 & 7 & |9 & 33 & 6 \end{vmatrix}$$

$$\begin{aligned} |A_1| = \quad & [(1 \times 10 \times 90 \times 7) - (4 \times 3 \times 53 \times 9) + (7 \times 5 \times 2 \times 33) - (8 \times 5 \times 11 \times 6)] \\ & +[(8 \times 3 \times 11 \times 9) - (1 \times 5 \times 90 \times 33) + (4 \times 5 \times 53 \times 6) - (7 \times 10 \times 2 \times 7)] \\ = -6848 & \end{aligned}$$

Second 4th order diagram:

$$|A_2| = \begin{vmatrix} 1 & 7 & 4 & 8 & |1 & 7 & 4 \\ 5 & 3 & 10 & 5 & |5 & 3 & 10 \\ 2 & 90 & 11 & 53 & |2 & 90 & 11 \\ 9 & 6 & 33 & 7 & |9 & 6 & 33 \end{vmatrix}$$

$$\begin{aligned} |A_2| = \quad & [-(1\times 3\times 11\times 7)+(7\times 10\times 53\times 9)-(4\times 5\times 2\times 6)+(8\times 5\times 90\times 33)] \\ & +[-(8\times 10\times 90\times 9)+(1\times 5\times 11\times 6)-(7\times 5\times 53\times 33)+(4\times 3\times 2\times 7)] \\ = 26202 & \end{aligned}$$

Third 4th order diagram:

$$|A_3| = \begin{vmatrix} 1 & 7 & 8 & 4 & |1 & 7 & 8 \\ 5 & 3 & 5 & 10 & |5 & 3 & 5 \\ 2 & 90 & 53 & 11 & |2 & 90 & 53 \\ 9 & 6 & 7 & 33 & |9 & 6 & 7 \end{vmatrix}$$

$$\begin{aligned} |A_3| = \quad & [(1\times 3\times 53\times 33)-(7\times 5\times 11\times 9)+(8\times 10\times 2\times 6)-(4\times 5\times 90\times 7)] \\ & +[(4\times 5\times 90\times 9)-(6\times 53\times 10\times 1)+(7\times 11\times 5\times 7)-(33\times 2\times 3\times 8)] \\ = 4273 & \end{aligned}$$

Thus $|A| = |A_1| + |A_2| + |A_3| = 23627$

CONCLUSION

The result showed that the Sarrus's rule can be extended for finding the determinant of 4×4 square matrices by employing some modifications in term of applying permutation and their signs. Hopefully this work will be added as another alternative method for finding determinant of 4×4 square matrices.

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