

Iterative approach of 3D datum transformation with a non-isotropic weight

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Abstract The analytical solution of 3D datum transformation with an isotropic weight has been elegantly presented based on Procrustes algorithm (singular value decomposition). But the existence of analytical solution of 3D datum transformation with a non-isotropic weight needs further investigation. Based on the Lagrangian extremum law, the paper derives the analytical formula for translation parameter and scale factor, but because the rotation matrix is unsolved, the analytical solution does not exist. For this reason, the paper presents two kinds of iterative approach of 3D datum transformation with a non-isotropic weight. One is the iterative approach dependent on the objective function value, which uses the Lagrangian minimum function in the variable of rotation matrix as the objective function, and the other is the iterative approach dependent on the derivative of function, which uses the 3D datum transformation model that eliminates the translation parameter. In order to improve the speed and reliability of iterative computation, the form of rotation matrix represented by Rodrigues matrix instead of rotation angles or unit quaternion is adopted for the two iterative approaches. A numerical experiment is demonstrated, and comparison analysis of the two iterative approaches is carried out. The result shows from the view of computing speed and reliability, the iterative approach based on derivatives is preferred.

Keywords 3D datum transformation · Lagrangian extremum · Non-isotropic weight · Iterative approach · Procrustes algorithm

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1 Introduction

Three-dimensional datum transformation is a central problem in geodesy, engineering surveying, photogrammetry, geographical information science (GIS), etc., e.g. Aktuğ (2012), Akyilmaz (2007), Burša (1967), Chen et al. (2004), Dermanis (1998), El-Mowafy et al. (2009), Ge et al. (2013), Kashani (2006), Neitzel (2010), Paláncz et al. (2013), Soler and Snay (2004), Soycan and Soycan (2008), Watson (2006), Zeng (2014, 2015). It aims to determine the transformation parameter of the transformation model. Usually, the transformation model adopts seven-parameter similarity transformation, including three translation parameters and three rotation angle parameters and one scale factor, see Leick (2004), Leick and van Gelder (1975). So far a lot of approaches for computing the seven parameters have been presented, which can be classified into two categories. One is analytical algorithm, and the other one is iterative algorithm. The former can give the exact solution quickly by the analytical formulae of the parameters. Its advantage over the latter is that it does not need the initial parameter values as well as iterative computation. Due to the difficulty of mathematical derivation, rare analytical algorithms have been put forward. E.g., the most famous analytical algorithm is the Procrustes algorithm presented by Grafarend and Awange (2003). It used the singular value decomposition technique to solve the unconstrained Lagrangian extremum problem, and obtained the computation of rotation matrix. It can deal with the case that the weight is isotropic, namely the weight is identical in different coordinate direction. Shen et al. (2006) presented a quaternion-based algorithm by means of eigenvalue–eigenvector decomposition. Han (2010) presented a step-wise approach to compute the transformation parameters considering the physical meaning of similarity transformation. Zeng and Yi (2010a) presented an analytical algorithm based on Rodrigues matrix. The latter needs the initial parameter values, and iterative computation. Although it has the initial values problem of parameter, i.e. in the case that the rotation angle is large, the iterative computation fails due to a bad initial value of parameter (see e.g. Zeng and Tao 2003; Zeng and Yi 2011), it plays the dominant role at present, which is proved by a large amount of literature published recently. E.g., Zeng and Tao (2003) investigated the feasibility of linearization of transformation model in different rotation angles. Zeng and Huang (2008) presented a kind of searching method based on genetic algorithm and pattern search method that is suitable for any size of rotation angle. El-Habiby, et al. (2009) compared four non-linear least squares methods for 3D coordinates transformation, namely Steepest Descent, Trust region, Gauss–Newton and Levenberg–Marquardt. Zeng and Yi (2011) presented a quaternion-based iterative solution of 3D coordinates transformation. It overcomes the initial value problem of parameter and is valid for 3D coordinate transformation of any rotation angle.

This paper is organized as follows. In Sect. 2, the analytical solution of 3D datum transformation with a non-isotropic weight is investigated in detail based on Lagrangian extremum law. The analytical formulae of translation parameter and scale factor are obtained, but the rotation matrix is unsolved, i.e. the analytical solution does not exist. For this reason, two kinds of iterative approaches are presented. One is the iterative approach dependent on the objective function value introduced in Sect. 3, which uses the Lagrangian minimum function as the objective function, and the other is the iterative approach dependent on the derivative of function introduced in Sect. 4, which uses the 3D datum transformation model that eliminates the translation parameter. In Sect. 5, a numerical experiment is given to demonstrate the presented two approaches, and comparisons of the two approaches are carried out. Lastly conclusions are made in the last section, i.e. Sect. 6.

2 Basic mathematical model of 3D datum transformation with a non-isotropic weight

The seven-parameter similarity transformation model can be expressed as

$$A_i = \lambda RB_i + t, \quad (1)$$

subject to

$$R^T R = I, \det(R) = +1, \quad (2)$$

where $A_i = [X_i \ Y_i \ Z_i]^T$ and $B_i = [x_i \ y_i \ z_i]^T$ ($i = 1, 2, \dots, n$) are the 3D coordinates of a control point in the target and source coordinate systems of transformation, labelled as system A and system B respectively. Superscript T represents transpose, I denotes an identity matrix with the dimension of three, \det is the determinant computation of matrix. λ denotes the scale factor, $t = [\Delta X \ \Delta Y \ \Delta Z]^T$ denotes the three translation parameters, and R denotes the rotation matrix.

The computation problem of transformation parameters is sought in the principle of least squares, namely

$$L(t, \lambda, R) = \sum_{i=1}^n (A_i - \lambda RB_i - t)^T P_i (A_i - \lambda RB_i - t) = \min, \quad (3)$$

where P_i is the weight matrix corresponding to the control point i , which is non-isotropic and even correlated in different coordinate direction.

Equation (3) is essentially an optimization problem, and can usually be solved by the means of Lagrangian extremum law. If and only if the following conditions are satisfied, the Lagrangian extremum exists.

$$\frac{\partial L}{\partial t} = 0, \quad (4)$$

$$\frac{\partial L}{\partial \lambda} = 0, \quad (5)$$

$$\frac{\partial L}{\partial R} = 0. \quad (6)$$

By Eqs. (3) and (4), one gets

$$\frac{\partial L}{\partial t} = \sum_{i=1}^n -2(A_i - \lambda RB_i - t)^T P_i = 0, \quad (7)$$

further making the transpose of Eq. (7), one gets

$$\sum_{i=1}^n P_i^T (A_i - \lambda RB_i - t) = 0, \quad (8)$$

and then

$$t = \left(\sum_{i=1}^n P_i^T \right)^{-1} \sum_{i=1}^n P_i^T (A_i - \lambda RB_i). \quad (9)$$

Let

$$p_i = \left(\sum_{i=1}^n P_i^T \right)^{-1} P_i^T, \tag{10}$$

thus

$$t = \sum_{i=1}^n p_i(A_i - \lambda RB_i). \tag{11}$$

Obviously, t is the function form of λ and R .

Substituting Eq. (11) into Eq. (3), one gets

$$\begin{aligned} L(\lambda, R) &= \sum_{i=1}^n \left(A_i - \lambda RB_i - \sum_{j=1}^n p_j(A_j - \lambda RB_j) \right)^T P_i \left(A_i - \lambda RB_i - \sum_{j=1}^n p_j(A_j - \lambda RB_j) \right) \\ &= \sum_{i=1}^n \left(A_i - \sum_{j=1}^n p_j A_j - \lambda \left(RB_i - \sum_{j=1}^n p_j RB_j \right) \right)^T \\ &\quad \times P_i \left(A_i - \sum_{j=1}^n p_j A_j - \lambda \left(RB_i - \sum_{j=1}^n p_j RB_j \right) \right). \end{aligned} \tag{12}$$

Let

$$\bar{A}_i = A_i - \sum_{j=1}^n p_j A_j, \tag{13}$$

thus Eq. (12) is written as

$$\begin{aligned} L(\lambda, R) &= \sum_{i=1}^n \left(\bar{A}_i - \lambda \left(RB_i - \sum_{j=1}^n p_j RB_j \right) \right)^T P_i \left(\bar{A}_i - \lambda \left(RB_i - \sum_{j=1}^n p_j RB_j \right) \right) \\ &= \sum_{i=1}^n \left(\begin{aligned} &\bar{A}_i^T P_i \bar{A}_i - \lambda \bar{A}_i^T P_i \left(RB_i - \sum_{j=1}^n p_j RB_j \right) - \lambda \left(RB_i - \sum_{j=1}^n p_j RB_j \right)^T P_i \bar{A}_i \\ &+ \lambda^2 \left(RB_i - \sum_{j=1}^n p_j RB_j \right)^T P_i \left(RB_i - \sum_{j=1}^n p_j RB_j \right) \end{aligned} \right) \\ &= \sum_{i=1}^n \left(\begin{aligned} &\bar{A}_i^T P_i \bar{A}_i - \lambda \bar{A}_i^T P_i \left(RB_i - \sum_{j=1}^n p_j RB_j \right) - \lambda \bar{A}_i^T P_i^T \left(RB_i - \sum_{j=1}^n p_j RB_j \right) \\ &+ \lambda^2 \left(RB_i - \sum_{j=1}^n p_j RB_j \right)^T P_i \left(RB_i - \sum_{j=1}^n p_j RB_j \right) \end{aligned} \right) \\ &= \sum_{i=1}^n \left(\begin{aligned} &\bar{A}_i^T P_i \bar{A}_i - \lambda \bar{A}_i^T (P_i + P_i^T) \left(RB_i - \sum_{j=1}^n p_j RB_j \right) \\ &+ \lambda^2 \left(RB_i - \sum_{j=1}^n p_j RB_j \right)^T P_i \left(RB_i - \sum_{j=1}^n p_j RB_j \right) \end{aligned} \right). \end{aligned} \tag{14}$$

Substituting Eq. (14) into Eq. (5), one gets

$$\begin{aligned} \frac{\partial L}{\partial \lambda} = & - \sum_{i=1}^n \left(\bar{A}_i^T (P_i + P_i^T) \left(RB_i - \sum_{j=1}^n p_j RB_j \right) \right) \\ & + 2\lambda \sum_{i=1}^n \left(\left(RB_i - \sum_{j=1}^n p_j RB_j \right)^T P_i \left(RB_i - \sum_{j=1}^n p_j RB_j \right) \right) = 0, \end{aligned} \tag{15}$$

thus

$$\lambda = \frac{\sum_{i=1}^n \left(\bar{A}_i^T (P_i + P_i^T) \left(RB_i - \sum_{j=1}^n p_j RB_j \right) \right)}{2 \sum_{i=1}^n \left(\left(RB_i - \sum_{j=1}^n p_j RB_j \right)^T P_i \left(RB_i - \sum_{j=1}^n p_j RB_j \right) \right)} \tag{16}$$

Obviously, λ is the function form of R .

Substituting Eq. (16) into Eq. (14), one gets

$$\begin{aligned} L(R) = & \sum_{i=1}^n (\bar{A}_i^T P_i \bar{A}_i) - \frac{\sum_{i=1}^n \left(\bar{A}_i^T (P_i + P_i^T) \left(RB_i - \sum_{j=1}^n p_j RB_j \right) \right)}{2 \sum_{i=1}^n \left(\left(RB_i - \sum_{j=1}^n p_j RB_j \right)^T P_i \left(RB_i - \sum_{j=1}^n p_j RB_j \right) \right)} \\ & \times \sum_{i=1}^n \left(\bar{A}_i^T (P_i + P_i^T) \left(RB_i - \sum_{j=1}^n p_j RB_j \right) \right) \\ & + \left(\frac{\sum_{i=1}^n \left(\bar{A}_i^T (P_i + P_i^T) \left(RB_i - \sum_{j=1}^n p_j RB_j \right) \right)}{2 \sum_{i=1}^n \left(\left(RB_i - \sum_{j=1}^n p_j RB_j \right)^T P_i \left(RB_i - \sum_{j=1}^n p_j RB_j \right) \right)} \right)^2 \\ & \times \sum_{i=1}^n \left(\left(RB_i - \sum_{j=1}^n p_j RB_j \right)^T P_i \left(RB_i - \sum_{j=1}^n p_j RB_j \right) \right), \end{aligned} \tag{17}$$

thus

$$L(R) = \sum_{i=1}^n (\bar{A}_i^T P_i \bar{A}_i) - \frac{\left(\sum_{i=1}^n \left(\bar{A}_i^T (P_i + P_i^T) \left(RB_i - \sum_{j=1}^n p_j RB_j \right) \right) \right)^2}{4 \sum_{i=1}^n \left(\left(RB_i - \sum_{j=1}^n p_j RB_j \right)^T P_i \left(RB_i - \sum_{j=1}^n p_j RB_j \right) \right)} \tag{18}$$

Substituting Eq. (18) into Eq. (6), one gets a very complex equation from which the analytical formula of R is impossible. In other words, the analytical solution of R does not exist when the rotation is non-isotropic and even correlated in different coordinate direction. Therefore, to overcome the problem, two kinds of iterative approaches are

presented in this paper. One is dependent on the objective function value, while the other is dependent on the derivative of function.

3 Iterative approach dependent on function value

Iterative approach dependent on function value is a family of iterative approach, also called direct search method that does not depend on the derivative information of function. It is very useful in the cases that the deduce process of derivative of function is very difficult and even derivative of function is not existing. The direct search concept is firstly presented by Hooke and Jeeves (1961). Pattern search method is a kind of very popular direct search method, and for its more detail, the readers are referred to e.g. Torczon (1997), Lewis, et al. (2000), Dolan, et al. (2003), Al-Sumaita, et al. (2007), Zeng and Yi (2010b). In this paper pattern search method is supposed to solve the Lagrangian minimum problem of Eq. (18).

The variable in Eq. (18) namely R is a matrix with 3 rows and 3 columns, usually represented by rotation angles (see e.g. El-Habiby et al. 2009; Zeng and Yi, 2011), unit quaternion (see e.g. Shen et al. 2006; Zeng and Yi 2011), and Rodrigues matrix (see e.g. Zeng and Yi 2010a). If the form of rotation matrix represented by rotation angles is used in searching the solution of the minimum problem, there are a large number of computations of trigonometric function, which slows down the speed of computation and even leads to a failure of iterative computation. If the form of rotation matrix represented by unit quaternion is used, there are four variables and a constraint condition that the sum of squares of variables equals to 1. Searching the solution of the minimum problem in this case is more difficult than that in the case which has three variables space and is unconstrained. So the form of rotation matrix represented by Rodrigues matrix is adopted for its simple algebraic computation.

Rodrigues matrix can be written as

$$R = (I + r)(I - r)^{-1}, \quad (19)$$

where r is an anti-symmetric matrix formed with a Gibbs vector $v = (v_x \ v_y \ v_z)$ as

$$r = \begin{bmatrix} 0 & -v_z & -v_y \\ v_z & 0 & -v_x \\ v_y & v_x & 0 \end{bmatrix}. \quad (20)$$

4 Iterative approach dependent on derivatives

Substituting Eq. (11) into Eq. (1) and utilizing Eq. (13), one gets the 3D datum transformation model that eliminates the translation parameters. The form of rotation matrix represented by Rodrigues matrix is adopted with the same consideration as Sect. 3.

$$\bar{A}_i = \lambda RB_i - \lambda \sum_{j=1}^n p_j RB_j. \quad (21)$$

Carrying out the Linearization of Eq. (21), one gets

$$V_i = \left[\frac{\partial \bar{A}_i}{\partial \lambda} \quad \frac{\partial \bar{A}_i}{\partial v_x} \quad \frac{\partial \bar{A}_i}{\partial v_y} \quad \frac{\partial \bar{A}_i}{\partial v_z} \right] \bigg|_{u = u^0} [\delta \lambda \quad \delta v_x \quad \delta v_y \quad \delta v_z]^T - l_i, \quad (22)$$

where

$$u = [\lambda \quad v_x \quad v_y \quad v_z]^T \quad (23)$$

is the unknown, u^0 is its approximate value and

$$\delta u = [\delta \lambda \quad \delta v_x \quad \delta v_y \quad \delta v_z]^T \quad (24)$$

is the correction of the unknown.

$$\frac{\partial \bar{A}_i}{\partial \lambda} = RB_i - \sum_{j=1}^n p_j RB_j, \quad (25)$$

$$\frac{\partial \bar{A}_i}{\partial v_x} = \lambda \frac{\partial R}{\partial v_x} B_i - \lambda \sum_{j=1}^n p_j \frac{\partial R}{\partial v_x} B_j, \quad (26)$$

$$\frac{\partial \bar{A}_i}{\partial v_y} = \lambda \frac{\partial R}{\partial v_y} B_i - \lambda \sum_{j=1}^n p_j \frac{\partial R}{\partial v_y} B_j, \quad (27)$$

$$\frac{\partial \bar{A}_i}{\partial v_z} = \lambda \frac{\partial R}{\partial v_z} B_i - \lambda \sum_{j=1}^n p_j \frac{\partial R}{\partial v_z} B_j, \quad (28)$$

$$l_i = \bar{A}_i - \lambda RB_i + \lambda \sum_{j=1}^n p_j RB_j, \quad (29)$$

and

$$\begin{aligned} \frac{\partial R}{\partial v_x} &= \frac{\partial r}{\partial v_x} (I - r)^{-1} + (I + r)(I - r)^{-1} \frac{\partial r}{\partial v_x} (I - r)^{-1} \\ &= \frac{\partial r}{\partial v_x} (I - r)^{-1} + R \frac{\partial r}{\partial v_x} (I - r)^{-1} \\ &= (I + R) \frac{\partial r}{\partial v_x} (I - r)^{-1}, \end{aligned} \quad (30)$$

$$\frac{\partial r}{\partial v_x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}. \quad (31)$$

Similarly

$$\frac{\partial R}{\partial v_y} = (I + R) \frac{\partial r}{\partial v_y} (I - r)^{-1}, \quad (32)$$

$$\frac{\partial r}{\partial v_y} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad (33)$$

$$\frac{\partial R}{\partial v_z} = (I + R) \frac{\partial r}{\partial v_z} (I - r)^{-1}, \quad (34)$$

$$\frac{\partial r}{\partial v_z} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (35)$$

Let

$$M_i = \left[\frac{\partial \bar{A}_i}{\partial \lambda} \quad \frac{\partial \bar{A}_i}{\partial v_x} \quad \frac{\partial \bar{A}_i}{\partial v_y} \quad \frac{\partial \bar{A}_i}{\partial v_z} \right] \Big|_{u = u^0}, \quad (36)$$

and then Eq. (22) can be written as

$$V_i = M_i \delta u - l_i. \quad (37)$$

Further let

$$V = [V_1 \quad V_2 \quad \cdots \quad V_n]^T, \quad (38)$$

$$M = [M_1^T \quad M_2^T \quad \cdots \quad M_n^T]^T, \quad (39)$$

$$l = [l_1 \quad l_2 \quad \cdots \quad l_n]^T, \quad (40)$$

thus

$$V = M \delta u - l. \quad (41)$$

According to the least squares, the solution is

$$\delta u = (M^T P M)^{-1} M^T P l, \quad (42)$$

where

$$P = \begin{bmatrix} P_1 & & & \\ & P_2 & & \\ & & \ddots & \\ & & & P_n \end{bmatrix}. \quad (43)$$

Because the approximation of the unknown is usually tough, iterative computations are needed, i.e., one firstly gives the approximation u^0 , then solve the correction δu by the above approach, next give the approximation of next iteration as $u^0 + \delta u$, and then repeat the above operation until every element of δu is less than a given tolerance.

5 Numerical experiments and discussion

The experimental data is adopted from Grafarend and Awange (2003). The coordinates of 7 control points in the system B (local system) and A (WGS-84 system) are listed in Tables 1 and 2.

Table 1 Coordinates and variances in system B (local system)

Station name	System B (local system) (m)			Variance (m ²)		
	<i>x</i>	<i>y</i>	<i>z</i>	σ_x^2	σ_y^2	σ_z^2
Solitude	4,157,222.543	664,789.307	4,774,952.099	0.14330000	0.16126012	0.25780367
Buoch Zeil	4,149,043.336	688,836.443	4,778,632.188	0.15510000	0.19971861	0.18798665
Hohenneuffen	4,172,803.511	690,340.078	4,758,129.701	0.15030000	0.12979970	0.18605368
Kuehlenberg	4,177,148.376	642,997.635	4,760,764.800	0.14000000	0.24000544	0.12159004
Ex Mergelaec	4,137,012.190	671,808.029	4,791,128.215	0.14590000	0.14040930	0.20278356
Ex Hof Asperg	4,146,292.729	666,952.887	4,783,859.856	0.14690000	0.19497925	0.15982531
Ex Kaisersbach	4,138,759.902	702,670.738	4,785,552.196	0.12200000	0.14330598	0.17612573

Table 2 Coordinates and variances in system A (WGS-84)

Station name	System A (WGS-84) (m)			Variance (m ²)		
	<i>X</i>	<i>Y</i>	<i>Z</i>	σ_x^2	σ_y^2	σ_z^2
Solitude	4,157,870.237	664,818.678	4,775,416.524	0.01030000	0.01625193	0.02285900
Buoch Zeil	4,149,691.049	688,865.785	4,779,096.588	0.00380000	0.00395319	0.00737509
Hohenneuffen	4,173,451.354	690,369.375	4,758,594.075	0.00060000	0.00100625	0.00156868
Kuehlenberg	4,177,796.064	643,026.700	4,761,228.899	0.01140000	0.01788546	0.02858824
Ex Mergelaec	4,137,659.549	671,837.337	4,791,592.531	0.00680000	0.00506161	0.01762072
Ex Hof Asperg	4,146,940.228	666,982.151	4,784,324.099	0.00002000	0.00001245	0.00002713
Ex Kaisersbach	4,139,407.506	702,700.227	4,786,016.645	0.00410000	0.00458562	0.00975938

The numerical experiments have two parts. One is the special case that the weight matrix is identity one. The other is the common case of arbitrary form of weight matrix. It is supposed in the common case that the precisions of three coordinate directions in the local system are consistent, and then σ_x^2 adopts the same value as Grafarend and Awange (2003), and σ_y^2 , σ_z^2 are generated with ‘randn’ command in Matlab. Additionally, the precisions of three coordinate directions in the WGS-84 system are inconsistent, namely the *X* and *Y* coordinates have the same precisions, but *Z* coordinates has lower precisions than planar *X* and *Y* coordinates, and here $\sigma_z^2 = 2\sigma_x^2$ is assumed. σ_x^2 adopts the same value as Grafarend and Awange (2003), and σ_y^2 , σ_z^2 are generated with ‘randn’ command in Matlab. The correlation of each point is not taken into account. The generated variances are listed in Tables 1 and 2. The weight matrix is generated with the same method as Grafarend and Awange (2003), and the result is list in Table 3.

The iterative approach dependent on function value and iterative approach dependent on derivatives are employed to recover the 7 transformation parameters. In the process of the former, the initial value of *v* is set as (0 0 0), the stopping criteria is that the mesh tolerance or function tolerance is less than 10^{-12} , and through 84 iterative times for the special case and 82 iterative times for the common case, the iterations are both convergent and the results are listed in Tables 4, 5, 6 and 7 respectively. In the process of the latter, the approximation of the unknown value *u* is set as (1 0 0 0), the stopping criteria is that every element of δu is less than 10^{-12} , and through both 2 iterative times for the special case and the common case, the iterations are both

Table 3 Weight matrix

Weight matrix	Values			Weight matrix	Values		
P1	6.51034885	0.00005093	-0.00004319	P5	6.54871861	0.00005901	-0.00004779
	0.00005093	5.63336324	0.00000938		0.00005901	6.87415224	0.00000942
	-0.00004319	0.00000938	3.56295957		-0.00004779	0.00000942	4.53707026
P2	6.29319762	0.00004981	-0.00005080	P6	6.80634928	0.00005418	-0.00005986
	0.00004981	4.90980598	-0.00000143		0.00005418	5.12836620	-0.00000546
	-0.00005080	-0.00000143	5.11865458		-0.00005986	-0.00000546	6.25569978
P3	6.62683154	0.00006515	-0.00005464	P7	7.93012846	0.00006479	-0.00005873
	0.00006515	7.64482735	0.00001110		0.00006479	6.76163613	0.00000578
	-0.00005464	0.00001110	5.32979632		-0.00005873	0.00000578	5.37961018
P4	6.60495162	0.00004390	-0.00005242				
	0.00004390	3.87756843	-0.00001480				
	-0.00005242	-0.00001480	6.65869204				

Table 4 Result of 3D datum transformation in the special case

	Iteration based on function value			Iteration based on derivative		
Iterative times	84			2		
<i>v</i>	0.0000024308	0.0000021681	-0.0000024065	0.0000024204	0.0000021664	-0.0000024073
<i>R</i>	1.0000000000	0.0000048131	-0.0000043362	1.0000000000	0.0000048146	-0.0000043328
	-0.0000048131	1.0000000000	-0.0000048615	-0.0000048146	1.0000000000	-0.0000048409
	0.0000043362	0.0000048615	1.0000000000	0.0000043327	0.0000048409	1.0000000000
ΔX (m)	641.8981			641.8804		
ΔY (m)	68.7474			68.6553		
ΔZ (m)	416.3697			416.3982		
λ	1.000005583			1.000005583		

Table 5 Transformation residuals in the special case

Station name	Iteration based on function value (m)			Iteration based on derivative (m)		
	<i>X</i>	<i>Y</i>	<i>Z</i>	<i>X</i>	<i>Y</i>	<i>Z</i>
Solitude	0.0940	0.1351	0.1404	0.0940	0.1351	0.1402
Buoch Zeil	0.0588	-0.0496	0.0134	0.0588	-0.0497	0.0137
Hohenneuffen	-0.0399	-0.0883	-0.0084	-0.0399	-0.0879	-0.0081
Kuehlenberg	0.0201	-0.0223	-0.0868	0.0202	-0.0220	-0.0874
Ex Mergelacc	-0.0918	0.0143	-0.0054	-0.0919	0.0139	-0.0055
Ex Hof Asperg	-0.0118	0.0067	-0.0544	-0.0118	0.0065	-0.0546
Ex Kaisersbach	-0.0293	0.0043	0.0012	-0.0294	0.0041	0.0017
σ_m (m)	0.0773			0.0773		

Table 6 Result of 3D datum transformation in the common case

	Iteration based on function value			Iteration based on derivative		
Iterative times	82			2		
v	0.0000025108	0.0000021756	-0.0000024401	0.0000025040	0.0000021746	-0.0000024332
R	1.0000000000	0.0000048801	-0.0000043512	1.0000000000	0.0000048664	-0.0000043491
	-0.0000048801	1.0000000000	-0.0000050217	-0.0000048664	1.0000000000	-0.0000050079
	0.0000043511	0.0000050217	1.0000000000	0.0000043491	0.0000050080	1.0000000000
ΔX (m)	641.8754			641.8770		
ΔY (m)	69.7813			69.6590		
ΔZ (m)	416.1342			416.1543		
λ	1.000005594			1.000005593		

Table 7 Transformation residuals in the common case

Station name	Iteration based on function value (m)			Iteration based on derivative (m)		
	X	Y	Z	X	Y	Z
Solitude	0.0960	0.1373	0.1532	0.0959	0.1373	0.1530
Buoch Zeil	0.0594	-0.0477	0.0224	0.0596	-0.0476	0.0226
Hohenneuffen	-0.0400	-0.0881	0.0002	-0.0398	-0.0881	0.0004
Kuehlenberg	0.0232	-0.0208	-0.0707	0.0228	-0.0209	-0.0711
Ex Mergelaec	-0.0898	0.0176	0.0064	-0.0899	0.0177	0.0063
Ex Hof Asperg	-0.0096	0.0096	-0.0420	-0.0098	0.0096	-0.0421
Ex Kaisersbach	-0.0294	0.0065	0.0080	-0.0291	0.0066	0.0083
σ_m (m)	0.1830			0.1830		

convergent and the results are listed in Tables 4, 5, 6 and 7 respectively. σ_m in Tables 5 and 7 is mean error and computed by

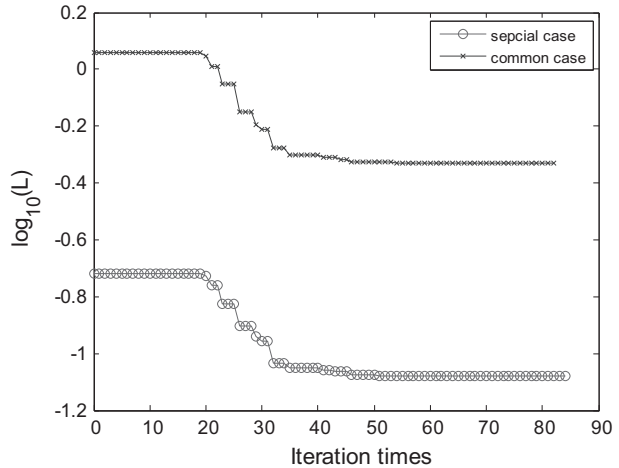
$$\sigma_m = \sqrt{\frac{L}{3n - 7}}, \quad (44)$$

and L is the Lagrangian extremum by Eq. (1) or (18).

Comparing the Table 4 and the result of Grafarend and Awange (2003) namely I-LESS procrustes algorithm, it is seen that the result of iterative approach based on derivatives is identical to that of I-LESS procrustes algorithm. It is worthy to be noted that there is a little mistake in Grafarend and Awange (2003) that the second element and third element in the first row of rotation matrix wrongly exchange position. In addition the result of iterative approach dependent on function value is slightly different from that of I-LESS procrustes algorithm, and the bias of translation parameter is cm level. However, from Table 5, it is seen that the transformation residual is in sub-mm level.

It is seen from Table 6 that the result of iterative approach dependent on function value is slightly different from that of iterative approach based on derivatives, and the bias of translation parameter is dm level. However, from Table 7 it is seen that the transformation residual is in sub-mm level.

Fig. 1 Iterative process based on function value



In order to investigate the performance of iterative approach dependent on function value namely pattern search method, the iterative process is drawn in Fig. 1. It shows that the convergent process is highly similar although the weight matrix is different for the two cases. And the approach is stable and feasible.

6 Conclusion

Through lots of derivation of this paper, it is found that analytical solution of 3D datum transformation does not exist in the case that the weight matrix is non-isotropic and even correlated in different coordinate direction. Thus the iterative approach of 3D datum transformation is investigated. The paper presents two kinds of iterative approach, i.e. iterative approach dependent on the objective function value and iterative approach dependent on the derivative of function. In order to improve the speed and reliability, the form of rotation matrix represented by Rodrigues matrix rather than rotation angles or unit quaternion is adopted for both two iterative approaches.

The numerical experiment and analysis shows the presented two kinds of iterative approach are both correct and efficient. Further it is thought that iterative approach dependent on function value is convenient for its objective function can be established easily and fast, however it needs a great deal of iterative computation, therefore it is slower than the iterative approach based on derivatives, and its result is slightly worse than the iterative approach based on derivatives. On the other hand, the iterative approach based on derivatives has lots of work to deduce derivatives, but once the model is established, the iterative computation is faster and more reliable than the iterative approach dependent on function value. From the view of computing speed and reliability, the iterative approach based on derivatives is preferred.

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