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On the errors-in-variables model with equality and inequality constraints for selected numerical examples

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Abstract It is well known that the errors-in-variables (EIV) model has been treated as a special case of the traditional geodetic model, the nonlinear Gauss–Helmert model (GHM), for more than a century. In this contribution, an adjustment of the EIV model with equality and inequality constraints is investigated based on the nonlinear GHM. In each iteration, the constrained EIV model is linearized to form a quadratic program. Furthermore, the precision description is investigated for the mixed constrained problem. The demonstrated results from the numerical examples show that this approach avoids the large computational expenses of the existing combinatorial solution that normally accompany the number of inequality constraints.

Keywords Total Least-Squares (TLS) · Errors-in-variables model · Equality and inequality constraints · Gauss–Helmert model · Convex quadratic program

1 Introduction

Total Least-Squares (TLS) is a method of fitting that is appropriate when there are errors in both the observation vector and in the design matrix in computational mathematics (Golub and Van Loan 1980) and geodesy (Teunissen 1988; Schaffrin and Wieser 2008; Amiri-Simkooei and Jazaeri 2012; Grafarend and Awange 2012; Xu et al. 2012; Chang 2015; Shi et al. 2015), which is also referred as errors-in-variables (EIV) modelling or orthogonal regression in the statistical community. The TLS/EIV principle was studied by Adcock (1878) and Pearson (1901) already more than one century ago. Kendall and Stuart (1969) described this problem as structural relationship models. In geodetic literature Teunissen (1988) was the first who solved an EIV model in an exact form. The equivalent form of the

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EIV model appears as the condition equation with a random coefficient matrix (Schaffrin and Wieser 2011). From a geodetic point of view, the EIV model is a special case of the nonlinear Gauss Helmert Model (GHM), which generates the standard LS solution after iterative linearization (Neitzel 2010; Schaffrin and Snow 2010; Fang 2011, 2013a; Bányai 2012; Snow 2012). Amiri-Simkooei and Jazaeri (2013) applied data snooping to WTLS and Amiri-Simkooei (2013) applied least squares variance component estimation to WTLS.

A number of geodetic problems may be expressed as an optimization model subject to various specific constraints. Regarding prior information representing constraints of the parameters, Schaffrin and Felus (2009), Mahboub and Sharifi (2013) and Fang (2014a, 2015) investigated the equality constrained EIV model. The EIV model with linear inequality constraints was firstly adjusted by Zhang et al. (2013) with an active set method based on exhaustive tests, under the condition that the weight matrix (including all random errors of the coefficient matrix and the observation vector) is the identity matrix. Later, Fang (2014b) solved the weighted TLS (WTLS) problem with inequality constraints within the standard optimization framework. Recently, Zeng et al. (2015) used the partial EIV model proposed by Xu et al. (2012) to treat the inequality constrained WTLS problem.

Proper incorporation of constraints into the system of equations is a well-documented issue in parameter estimation. When linear equality and inequality constraints are available simultaneously, the equality and inequality constrained TLS (EICTLS) solution should be established. Except being compatible with equality and inequality constraints, the EICTLS solution should be numerically efficient as well as familiar to geodesists. Furthermore, statistical aspects of a (unconstrained or constrained) TLS estimate have not been thoroughly investigated, as most works on TLS focus on theoretical methods and algorithms for numerically finding the TLS estimates (Xu et al. 2012). Therefore, we provide the quality description of the corresponding constrained TLS estimates in the case of finite measurements. The quality description including the covariance matrix of the parameter estimates is given based on the aggregate function proposed by Li (1991), which was used in Peng et al. (2006) and Zeng et al. (2015). Note that the aggregate function based method only provides an approximate quality description, because Roese-Koerner et al. (2012) explained that in an inequality constrained estimate it is not possible to represent the complete stochastic information in form of a variance–covariance matrix as the description of the first two moments of the PDF is no longer sufficient.

Although WTLS with inequality constraints and WTLS with equality constraints have been separately investigated in Fang (2014a, b, 2015) and Zeng et al. (2015), their combination is not discussed. For the algorithm design, we applied the Gauss–Newton method instead of the pure Newton method proposed in Fang (2014a, 2015) and the iterative method deduced by Lagrange multipliers (Fang 2014b; Zeng et al. 2015). The iterative Gauss–Newton method is more familiar to geodesists, and its corresponding geometrical interpretation was emphasized in Teunissen (1990). Furthermore, the statistical analysis was investigated in this paper. The influence of variation of the control factor to the parameter estimates is also analyzed, which is missing in the current publications (e.g., Zeng et al. 2015). In the final part, we present some numerical examples taken from other publications in order to check the algorithm and quality description developed in this paper.

2 EIV model with linear equality and inequality constraints

Let the well-known unconstrained EIV model be defined by the following functional and stochastic model:

$$\mathbf{y} + \mathbf{v_y} = (\mathbf{A} + \mathbf{V_A})\boldsymbol{\xi}, \tag{1}$$

$$\mathbf{v} := \begin{bmatrix} vec(\mathbf{V_A}) \\ \mathbf{v_y} \end{bmatrix} = \begin{bmatrix} \mathbf{v_A} \\ \mathbf{v_y} \end{bmatrix} \sim \left(\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \sigma_0^2 \mathbf{Q_{ll}} = \sigma_0^2 \begin{bmatrix} \mathbf{Q_{AA}} & \mathbf{Q_{Ay}} \\ \mathbf{Q_{yA}} & \mathbf{Q_{yy}} \end{bmatrix} = \sigma_0^2 \mathbf{P}^{-1} \right), \tag{2}$$

where $\mathbf{y}$ and $\mathbf{v}_y$ are the observation and the corresponding random correction vector, respectively. $\mathbf{A}$ and $\mathbf{V}_A$ are the full-column rank stochastic coefficient matrix ($n \times u$) and the corresponding random correction matrix, respectively. Vector ξ is the unknown parameter vector with dimension $u \times 1$. Vector $\mathbf{v}$ is the extended random correction vector, in which $\mathbf{v_A} = vec(\mathbf{V_A})$. σ_0^2 is the unknown variance factor. $\mathbf{l}$ is the extended observation vector $\mathbf{l} := vec([\mathbf{A}, \mathbf{y}])$. $\mathbf{Q_{ll}}$ and $\mathbf{P}$ are the cofactor matrix and the weight matrix of extended observation vector, respectively. $\mathbf{Q_{AA}}$ and $\mathbf{Q_{yy}}$ are the cofactor matrices of the vectors $\mathbf{v_A}$ and $\mathbf{v_y}$, respectively. $\mathbf{Q_{Ay}}$ is the cofactor matrix, which refers to the correlations of $\mathbf{v_A}$ and $\mathbf{v_y}$ ($\mathbf{Q}_{\mathbf{Ay}}^T = \mathbf{Q_{yA}}$).

If one incorporates the desired linear equalities and inequalities into the above EIV formulation, the functional part of the constrained EIV model can be expressed as follows:

$$\begin{array}{l} \mathbf{y} + \mathbf{v_y} = (\mathbf{A} + \mathbf{V_A})\boldsymbol{\xi} \\ with \qquad \mathbf{C}\xi \geq \mathbf{c} \\ \qquad\qquad \mathbf{K}\xi = \boldsymbol{\kappa} \end{array}, \tag{3}$$

where $\mathbf{C}$ is the $m \times u$ fixed coefficient matrix of constraints $\mathbf{c}$ is a $m \times 1$ constant vector on the right-hand side of the inequality constraints. $\mathbf{K}$ is a $s \times u$ ($s < u$) fixed coefficient matrix of constraints with full row rank. $\boldsymbol{\kappa}$ is a $s \times 1$ constant vector on the right-hand side of the equality constraints.

3 The EICTLS solution

In this section, we aim on providing the EICTLS solution based on the linearized GHM. In the first part, the linearization of the EIV model is presented to form the linearized GHM, even the transformed GMM with algebraically formulated Jacobian matrices. Shortly afterwards, constraints are incorporated within the linearized GMM to form a quadratic program (QP). Finally, the corresponding EICTLS algorithm is established based on iteratively solving the QP (Nocedal and Wright 2006, p. 448).

3.1 Linearization of the EIV model with algebraically formulated Jacobian matrices

The application of the Gauss–Helmert model (GHM) in the sense of TLS/EIV has been widely studied by e.g., Koch (2014). Neitzel (2010) as well as Amiri-Simkooei and Jazaeri (2012) have theoretically proved that the solution of TLS and WTLS problem can be obtained by a nonlinear GHM, which is supported with the same numerical examples in Amiri-Simkooei and Jazaeri (2012).

In our opinion, the aforementioned EIV model (without constraints) is a special case of the nonlinear GHM model. The nonlinear model

$$\mathbf{f}(\mathbf{l}+\mathbf{v},\boldsymbol{\xi})=(\mathbf{A}+\mathbf{V_A})\boldsymbol{\xi}-\mathbf{y}-\mathbf{v_y}=\mathbf{0} \tag{4}$$

can be linearized at the position $\boldsymbol{\xi}^i$ and $\mathbf{l}^i$ (for ith iteration) through the truncated Taylor series to form the GHM (Fang 2011, p 30)

$$\mathbf{A}^i d\boldsymbol{\xi}+\mathbf{B}^i\mathbf{v}+\mathbf{w}^i=\mathbf{0} \tag{5}$$

with deterministic Jacobian matrices $\mathbf{A}^i=\mathbf{A}+\mathbf{V}^i_{\mathbf{A}}$ and $\mathbf{B}^i=\left[\left(\boldsymbol{\xi}^i\right)^T\otimes\mathbf{I}_n,-\mathbf{I}_n\right]$, the inconsistency vector $\mathbf{w}^i$ and the parameter increment vector $d\boldsymbol{\xi}$.

This linearized model can be further formulated by a GMM as (Fang 2013a, b)

$$\mathbf{A}^i d\boldsymbol{\xi}=\mathbf{l}^i_{\mathbf{B}}+\mathbf{v}^i_{\mathbf{B}}, \tag{6}$$

where $\mathbf{l}^i_{\mathbf{B}}=-\mathbf{B}^i\mathbf{l}=-\mathbf{w}^i$ and $\mathbf{v}^i_{\mathbf{B}}=-\mathbf{B}^i\mathbf{v}\sim\left(\mathbf{0},\mathbf{B}^i\mathbf{Q_{ll}}\left(\mathbf{B}^i\right)^T\right)$.

Thus, the objective function based on the above model, namely the weighted quadratic form of the vector $\mathbf{B}^i\mathbf{v}$, reads

$$\min\left(\mathbf{B}^i\mathbf{v}\right)^T\left(\mathbf{B}^i\mathbf{Q_{ll}}\left(\mathbf{B}^i\right)^T\right)^{-1}\mathbf{B}^i\mathbf{v}, \tag{7}$$

which is equivalent to the minimization of the following function relating to the parameter increment vector:

$$\min\left(\mathbf{A}^i d\boldsymbol{\xi}+\mathbf{w}^i\right)^T\left(\mathbf{B}^i\mathbf{Q_{ll}}\left(\mathbf{B}^i\right)^T\right)^{-1}\left(\mathbf{A}^i d\boldsymbol{\xi}+\mathbf{w}^i\right). \tag{8}$$

3.2 Linearization of the constrained EIV model to form a QP

Now, we consider the constraints also by linearization in the ith iteration

$$\begin{aligned}\mathbf{C}d\boldsymbol{\xi}+\mathbf{C}\xi^i-\mathbf{c}\geq 0\\ \mathbf{K}d\boldsymbol{\xi}+\mathbf{K}\xi^i-\boldsymbol{\kappa}=0.\end{aligned} \tag{9}$$

Combining Eqs. (8) and (9) and after rearranging, we form the constrained objective function as follows:

$$\begin{aligned}&\min f(d\boldsymbol{\xi})=d\boldsymbol{\xi}^T\mathbf{N}^i d\boldsymbol{\xi}+2d\boldsymbol{\xi}^T\mathbf{n}^i\\ &\quad \textit{subject to}\\ &\quad \mathbf{C}d\boldsymbol{\xi}+\left(\mathbf{C}\xi^i-\mathbf{c}\right)\geq 0\\ &\quad \mathbf{K}d\boldsymbol{\xi}+\left(\mathbf{K}\xi^i-\boldsymbol{\kappa}\right)=0\end{aligned} \tag{10}$$

with the known matrix $\mathbf{N}^i$ and the known vector $\mathbf{n}^i$

$$\begin{aligned}\mathbf{N}^i&=\left(\mathbf{A}^i\right)^T\left(\mathbf{B}^i\mathbf{Q_{ll}}\left(\mathbf{B}^i\right)^T\right)^{-1}\mathbf{A}^i\\ \mathbf{n}^i&=\left(\mathbf{A}^i\right)^T\left(\mathbf{B}^i\mathbf{Q_{ll}}\left(\mathbf{B}^i\right)^T\right)^{-1}\mathbf{w}^i.\end{aligned} \tag{11}$$

It is obvious that Eq. (10) refers to standard QP. Because of the positive definiteness of the (well-conditioned) matrix $\mathbf{N}$, the solution of Eq. (10) can be readily obtained.

3.3 Quality description of EICTLS estimates

One of the possible methods to approximately assess the quality of parameter estimates subject to an inequality constrained problem is converting all inequality into an aggregate function as proposed by Peng et al. (2006).

In this paper it is presented that a real-valued and smooth optimization problem with (nonlinear) inequality constraints can be converted by the optimization problem with an aggregate constraint. Therefore, we can formulate linear inequality constraints by a reformulation with a surrogate equality constraint $c_{ag}(\boldsymbol{\xi})$:

$$c_{ag}(\boldsymbol{\xi}) = (1/p)\ln\sum_{i=1}^{s} e^{p\cdot(\mathbf{C}_{\bullet i}\boldsymbol{\xi}-c_i)} = 0, \tag{12}$$

where $\mathbf{C}_{\bullet i}$ is the ith row of the matrix $\mathbf{C}$ and c_i is the ith element of the vector of $\mathbf{c}$. p is a scale factor, which is usually chosen as a large number, for example $p = 10^6$ in Peng et al. (2006). ln stands for the natural logarithm. Note that at least one active constraint is assumed within the given inequality constraints.

It is an easy exercise to deduce the analytical form of the elements of the gradient vector of the aggregate function $c_{ag}(\boldsymbol{\xi})$ as follows

$$\mathbf{C}_{\mathrm{ag}} = \frac{\partial c_{ag}(\boldsymbol{\xi})}{\partial \xi_j} = \frac{\sum_{i=1}^{s}\left(e^{p\cdot(\mathbf{C}_{\bullet i}\boldsymbol{\xi}-c_i)}\cdot c_{ij}\right)}{\sum_{i=1}^{s} e^{p\cdot(\mathbf{C}_{\bullet i}\boldsymbol{\xi}-c_i)}}, \tag{13}$$

where c_{ij} is the corresponding element within the matrix $\mathbf{C}$.

Therefore, if the iterations terminates, we can give an appropriate formulation for the approximated precision of the parameter estimates according to Koch (1999, p 172) as follows

$$\begin{aligned}
D(\boldsymbol{\xi}) &= \frac{\left(\mathbf{A}^i d\boldsymbol{\xi} + \mathbf{w}^i\right)^T\left(\mathbf{B}^i\mathbf{Q}_{\mathbf{ll}}\left(\mathbf{B}^i\right)^T\right)^{-1}\left(\mathbf{A}^i d\boldsymbol{\xi} + \mathbf{w}^i\right)}{n-u+s}\left(\mathbf{D} - \mathbf{DH}^T\left(\mathbf{HDH}^T\right)^{-1}\mathbf{HD}\right)\\
\mathbf{D} &= \left(\left(\mathbf{A}^i\right)^T\left(\mathbf{B}^i\mathbf{Q}_{\mathbf{ll}}\left(\mathbf{B}^i\right)^T\right)^{-1}\mathbf{A}^i\right)^{-1}\\
\mathbf{H} &= \begin{bmatrix}\mathbf{C}_{ag}^T\\ \mathbf{K}\end{bmatrix},
\end{aligned} \tag{14}$$

where s is the number of the active constraints. The operator D stands for the dispersion matrix.

Equation (13) on the precision of the WTLS estimate (without the constraint) comes from Amiri-Simkooei and Jazaeri (2012, 2013), Amiri-Simkooei (2013), Amiri-Simkooei et al. (2014) in which the WTLS was formulated by the standard least squares theory. The term $\mathbf{DH}^T\left(\mathbf{HDH}^T\right)^{-1}\mathbf{HD}$ comes from introducing the constraints to the WTLS.

3.4 Design of the EICTLS algorithm

In summation, the EICTLS algorithm can be formulated in the following

Algorithm 1 *The EICTLS solution*

Begin with the initial values of the parameter vector $\hat{\xi}^1 = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{y}$ and $\mathbf{A}^1 = \mathbf{A}$.

Compute for $i \in N$

$$\hat{\mathbf{B}}^{i+1} = \left[\left(\hat{\xi}^i \right)^T \otimes \mathbf{I}_n, -\mathbf{I}_n \right]$$
$$\mathbf{N}^{i+1} = \left(\mathbf{A}^{i+1} \right)^T \left(\mathbf{B}^{i+1} \mathbf{Q}_{\mathbf{ll}} \left(\mathbf{B}^{i+1} \right)^T \right)^{-1} \mathbf{A}^{i+1}$$
$$\mathbf{n}^{i+1} = \left(\mathbf{A}^{i+1} \right)^T \left(\mathbf{B}^{i+1} \mathbf{Q}_{\mathbf{ll}} \left(\mathbf{B}^{i+1} \right)^T \right)^{-1} \left(\mathbf{A}\xi^i - \mathbf{y} \right)$$

Solve the QP:

$$\begin{aligned} &\min f(d\boldsymbol{\xi}) = d\boldsymbol{\xi}^T \mathbf{N}^{i+1} d\boldsymbol{\xi} + 2 d\boldsymbol{\xi}^T \mathbf{n}^{i+1} \\ &\textit{subject to} \\ &\mathbf{C} d\boldsymbol{\xi} + \left(\mathbf{C}\xi^i - \mathbf{c} \right) \geq 0 \\ &\mathbf{K} d\boldsymbol{\xi} + \left(\mathbf{K}\xi^i - \boldsymbol{\kappa} \right) = 0 \end{aligned}$$

to estimate the parameter increment vector $d\boldsymbol{\xi}^i$ and the parameter vector $\hat{\xi}^{i+1} = \hat{\xi}^i + d\boldsymbol{\xi}^i$.

Reconstruction of the matrix $\mathbf{A}^{i+1}$

$$\hat{\mathbf{v}}_{\mathbf{A}}^{i+1} = \left[\mathbf{Q}_{\mathbf{AA}} \quad \mathbf{Q}_{\mathbf{Ay}} \right] \left(\mathbf{B}^{i+1} \right)^T \left(\mathbf{B}^{i+1} \mathbf{Q}_{\mathbf{ll}} \left(\mathbf{B}^{i+1} \right)^T \right)^{-1} \left(\mathbf{y} - \mathbf{A}\xi^{i+1} - \mathbf{A}^{i+1} d\hat{\xi}^{i+1} \right)$$

$\mathbf{A}^{i+1} = \mathbf{A} + \mathbf{V}_{\mathbf{A}}^{i+1} = \mathbf{A} + Invec_{n \times u}\left(\hat{\mathbf{v}}_{\mathbf{A}}^{i+1} \right)$ (Invec denotes the reshape operator according to the given dimension).

End when $\left\| d\boldsymbol{\xi}^i \right\| < \varepsilon$ (ε is a sufficiently small positive threshold), $\hat{\xi}_{EICTLS} := \hat{\xi}^i$.

Compute the precision of the parameter estimates:

$$D(\boldsymbol{\xi}) = \frac{\left(\mathbf{A}^i d\boldsymbol{\xi} + \mathbf{w}^i \right)^T \left(\mathbf{B}^i \mathbf{Q}_{\mathbf{ll}} \left(\mathbf{B}^i \right)^T \right)^{-1} \left(\mathbf{A}^i d\boldsymbol{\xi} + \mathbf{w}^i \right)}{n - u + s} \left(\mathbf{D} - \mathbf{D}\mathbf{H}^T \left(\mathbf{H}\mathbf{D}\mathbf{H}^T \right)^{-1} \mathbf{H}\mathbf{D} \right)$$
$$\mathbf{D} = \left(\left(\mathbf{A}^i \right)^T \left(\mathbf{B}^i \mathbf{Q}_{\mathbf{ll}} \left(\mathbf{B}^i \right)^T \right)^{-1} \mathbf{A}^i \right)^{-1} \text{ and } \mathbf{H} = \begin{bmatrix} \mathbf{C}_{ag}^T \\ \mathbf{K} \end{bmatrix}$$

After terminating the iterative process, a desired local solution is obtained if one correctly updates the model matrices in accordance to Pope (1972). This solution pursues the minimization of the objective function in the WTLS case, and simultaneously fulfills the linear equality and inequality constraints. Of course, the algorithm can also solve the WTLS problem only with equality constraints or inequality constraints.

The iterative QP process differs essentially from the sequential QP presented in Fang (2014b) as we solve a convex problem within each iteration instead of the nonconvex one. The convex problem may be more familiar with geodesists (Teunissen 1990; Roese-Koerner and Schuh 2014). The nonconvex problem is usually hard to handle because the properties of its local and global solution are different (Fang 2013b).

4 Applications

The main purpose of this section is to substantiate the presented EICTLS algorithm through geodetic applications. In the first example, data of a geodetic application presented in Schaffrin and Felus (2009) shows a simplified "geodetic resection" problem. In the second application, we use the data presented in Peng et al. (2006) and Zhang et al. (2013) to compare results.

4.1 Application 1

This simplified "geodetic resection" problem is represented by the data matrix **A**, an observation vector **y** and a identity matrix with the given size as the cofactor matrix:

$$\mathbf{A} = \begin{bmatrix} -0.5 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \ \mathbf{y} = \begin{bmatrix} 6 \\ 3 \\ 4 \\ 10 \end{bmatrix}, \ \mathbf{Q_{ll}} = \mathbf{I}_{16}.$$

We assume that different sets of constraints are available in the TLS problem. Four cases (with equality or inequality constraints) are shown in the second and third row of Table 1. We implement our EICTLS algorithm in each of the four cases, and present the results (parameter estimates and total sum of squared residuals, TSSR) in the following part of this table.

The results of Case 1 which contain one inequality constraint correspond with the TLS solution without any constraints as presented in Schaffrin and Felus (2009). This indicates that the inequality constraint refers to an inactive constraint, which does not influence the parameter estimates at all. If we vary the sign of the inequality constraint from '>' to '<', this change leads to the fact that Cases 2 and 3 provide identical results. The phenomenon illustrates that the inequality constraint in Case 2 is an active constraint, and functions as the equality constraint in this TLS problem. However, the identical results reflect that our method provides an approximated quality description, because as an inequality constraint is less restrictive than an equality constraint it should also reduce the standard deviation less.

The solution in Case 4 differs from that in Case 3, which means that the newly added inequality constraint is active. This table also indicates that the TSSR becomes larger when additional active constraints are available. It is obvious that the variances of the parameter estimates are very large in the case of no active constraints (Case 1) in comparison with

Table 1 Numerical results of Application 1

	Case 1	Case 2	Case 3	Case 4
Equality	Not available	Not available	$[-2 \quad 0 \quad 3]\boldsymbol{\xi} - 16 = 0$	$[-2 \quad 0 \quad 3]\boldsymbol{\xi} - 16 = 0$
Inequality	$[-2 \quad 0 \quad 3]\boldsymbol{\xi} \leq 16$	$[-2 \quad 0 \quad 3]\boldsymbol{\xi} \geq 16$	Not available	$[1 \quad -2 \quad 15]\boldsymbol{\xi} \geq 100$
$\hat{\xi}_1$	4.68316	2.36823	2.36823	2.85629
$\hat{\xi}_2$	6.24535	5.69850	5.69850	5.70962
$\hat{\xi}_3$	5.13041	6.91215	6.91215	7.23753
TSSR	0.18400	0.21284	0.21284	0.22079
$D\left(\hat{\xi}_1\right)$	27.30558	2.91866	2.91866	0.12621
$D\left(\hat{\xi}_2\right)$	10.54932	5.09582	5.09582	3.81813
$D\left(\hat{\xi}_3\right)$	14.62061	1.29718	1.29718	0.05609

Case 2 and 3. By adding an active constraint (Case 4), the variance of the estimated parameters are further reduced. The variances of the second parameter change less compared with the first and third parameters. This may be the cause because $\hat{\xi}_2$ is not involved in the active constraint. It should be emphasized that in this paper we discuss only the variances of the parameter estimates and the bias detection will be investigated in a further work.

4.2 Application 2

In the second example, Table 2 not only presents the coefficient matrix **A** and the traditional observation vector **y**, but also includes the inequality $\mathbf{B}_0\boldsymbol{\xi} \leq \mathbf{d}_0$ and box constraints $-0.1 \leq \xi_i \leq 2.0, \quad i = 1, 2, 3, 4$.

The inequality constraints can be formulated together as:

$$\text{Inequality constraints:} \quad \begin{bmatrix} -\mathbf{B}_0^T & \mathbf{I}_4 \otimes \begin{bmatrix} 1 & -1 \end{bmatrix} \end{bmatrix}^T \boldsymbol{\xi} \geq \begin{bmatrix} -\mathbf{d}_0^T & \mathbf{1}_4 \otimes \begin{bmatrix} -0.1 & -2 \end{bmatrix} \end{bmatrix}^T$$

In addition, we add an equality constraint to the aforementioned 11 inequalities:

$$\text{Equality constraints:} \quad \begin{bmatrix} -2 & 1 & 3 & 0 \end{bmatrix} \boldsymbol{\xi} = 2$$

In order to study the role of the controlling factor p in the aggregate function, we implemented Algorithm 1 for different values of p, varying from 10 to 10^{15} (see Fig. 1). The positive differences are presented by the results of the WTLS objective function subject to the aggregate function minus the inequality constrained WTLS (ICWTLS) cost function in Fang (2014a). Figure 1 shows that the positive difference becomes smaller when the factor p increases. This phenomenon clarifies that an equivalence relationship can be considered to be correct in this numerical computation by using an aggregate function, provided that p is sufficiently large. Note that it is not possible to use a larger or infinite factor here due to numerical reasons.

After implementing the TLS solution only with inequality constraints as well as with both equality and inequality constraints, we present the results in Table 3. The results of the inequality constraints in the second column in this table correspond with the results presented in Zhang et al. (2013) exactly, but our method does not need exhaustive searching, which leads to large computational expenses. The results presented in the third

Table 2 Data from Peng et al. (2006) and Zhang et al. (2013)

A				**y**
0.9501	0.7620	0.6153	0.4057	0.0578
0.2311	0.4564	0.7919	0.9354	0.3528
0.6068	0.0185	0.9218	0.9169	0.8131
0.4859	0.8214	0.7382	0.4102	0.0098
0.8912	0.4447	0.1762	0.8936	0.1388
$\mathbf{B}_0$				$\mathbf{d}_0$
0.2027	0.2721	0.7467	0.4659	0.5251
0.1987	0.1988	0.4450	0.4186	0.2026
0.6037	0.0152	0.9318	0.8462	0.6721
Box constraints: $-0.1 \leq \xi_i \leq 2.0$, $i = 1, 2, 3, 4$				

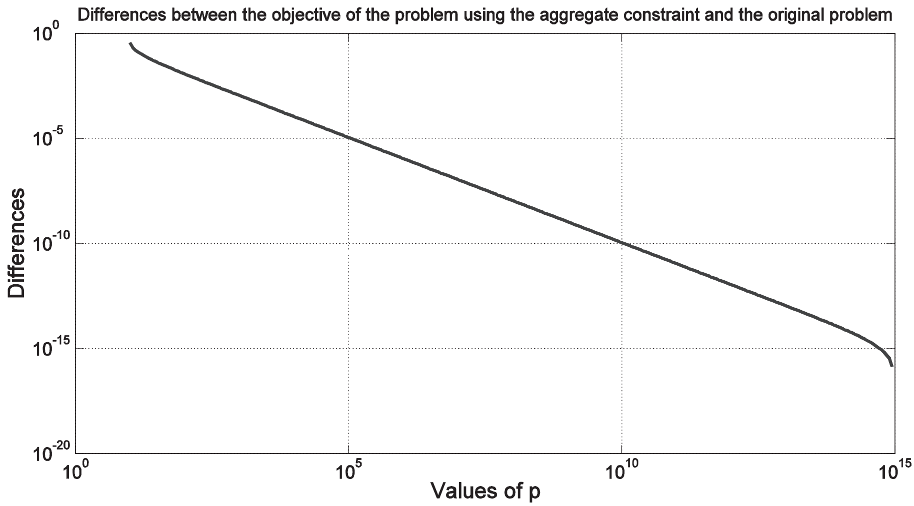

Fig. 1 Difference between the estimated objective function of the ICWTLS problem and its converted problem with an aggregate constraint

Table 3 The results of the inequality constrained TLS and EICTLS

	Inequality constrained TLS	EICTLS
$\hat{\xi}_1$	−0.100000	−0.100000
$\hat{\xi}_2$	−0.100000	−0.100000
$\hat{\xi}_3$	0.168547	0.633333
$\hat{\xi}_4$	0.399777	−0.094322
TSSR	0.139737	0.210737

column are tested to verify that the estimates fulfill the equality and inequality constraints. Due to the newly added equality constraint, the TSSR is significantly larger than when it is only obtained by the inequality constraints available.

We implemented Algorithm 1 for the precision of the EICTLS problem when the factor is chosen by $p = 10^6$ as recommended in Peng et al. (2006). The results are presented in Table 4. We can find that the newly added constraint can improve the precision of the parameter estimates.

5 Conclusions and outlook

The authors have described algorithms to solve TLS problems with constraints. The proposed algorithm was derived from the GHM and can uniquely solve the problem of TLS with both quality and inequality constraints.

The proposed EICTLS algorithm can solve the constrained TLS problem not only in an equally weighted case, but in a weighted or even structured case as well, providing the cofactor matrix is perfectly described. Furthermore, the algorithm avoids a major drawback

Table 4 The precision of the results of the inequality constrained TLS and EICTLS

	Inequality constrained TLS	EICTLS
$D\left(\hat{\xi}_1\right)$	0.09069	0.06845
$D\left(\hat{\xi}_2\right)$	0.10011	0.07127
$D\left(\hat{\xi}_3\right)$	0.08969	0.06866
$D\left(\hat{\xi}_4\right)$	0.09747	0.09309

of the combinatorial method used in Zhang et al. (2013) when applied to the inequality constrained problem. In addition, the precision of the parameter estimates can be given.

The linearized GHM model always provides a global solution within the certain iterations, but it does not indicate that it refers to the global solution of the original EICTLS optimization problem. Whether the estimates of the proposed EICTLS algorithm have got global characteristics might depend on the initial position of the parameters, relative ratio of the noise size between the coefficient matrix and the observation vector as well convexity of the restricted regions defined by the equality and inequality constraints (Fang 2013b). Furthermore, the more rigorous statistical analysis including bias detection needs to be further investigated, which depends on the precision of the observations and on the geometrical properties of the nonlinear manifold—curvature for instance (Teunissen 1984, 1985, 1990).

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