Lolk.

Wednesday, 28th August

Plenary lectures, Chair: M.N. Ouarzazi and A. Nakayama

Time	Room A
9.00-9.40	**W.Q. Tao & Y.L. He** A generalized reconstruction operator for coupling FVM and LBM in multiscale simulation and its applications in simulating transport process in porous medium
9.50-10.30	**M. Kohr** Poisson problems for semilinear Brinkman systems on Lipschitz domains. Applications

Coffee Break

Plenary lectures, Chair: W.Q. Tao and Y.L. He

11.00-11.40	**B. Sri Padmavati** A new approach to discuss Stokes flow past arbitrary shaped porous bodies
11.50-12.30	**I.O. Sert** Enhancement of convective heat transfer with nanofluids - single-phase and two-phase analysis

Coffee Break

Ordinary lectures

Chairs: S.M. Hassanizadeh (Room A) and A. Barletta (Room B)

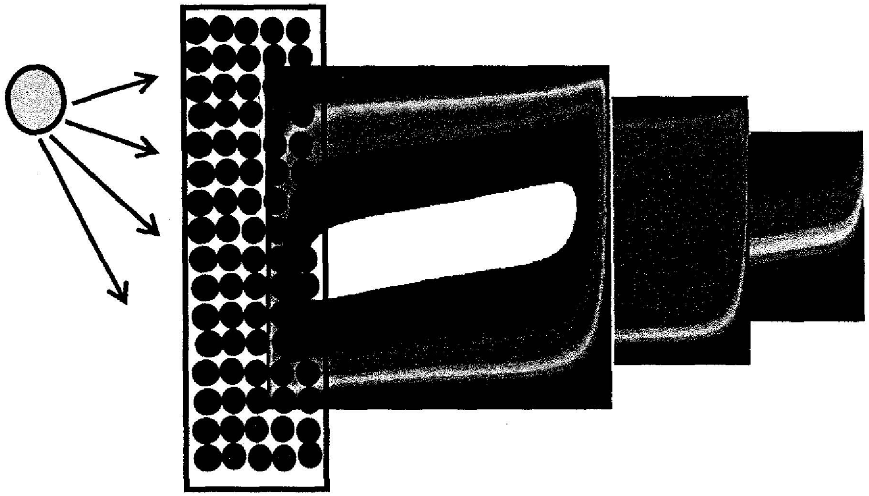
Time	Room A	Room B
13.30-13.50	**G. Lauriat** Effects of velocity slip on permeability and effective thermal conductivity of micro-porous media	**L. Sphaier** Instability of the mixed convection flow in a heated porous channel with an adiabatic upper wall
13.50-14.10	**Z. Mesticou** Influence of the ionic strength on the clogging phenomenon and transport dynamic of microparticles through saturated porous medium	**A.K. Ismail** Temperature profiles and emission characteristics of a liquid-fuel-fired porous burner
14.10-14.30	**M.N. Ouarzazi** Effects of viscous dissipation on convective instability of viscoelastic mixed convection flows in porous media	**T. Groşan** Free convection heat transfer in a square cavity filled with a porous medium saturated by nanofluid
14.30-14.50	**N. Dukhan** The influence of spacing of segmented metal foam on airflow pressure drop	**O. Noah** Experimental evaluation of natural convection heat transfer in packed beds contained in slender cylindrical geometries
14.50-15.10	**A. Satheesh** Behavior in two sided lid driven closed square porous cavity due to double diffusive mixed convection using CFD techniques	**K. Murthy** Influence of MHD forces on double diffusive free convection process induced by boundary layer flow along a vertical surface in a doubly stratified fluid saturated porous medium with Soret and Dufour effects
15.10-15.40	**V. Nustrov** Pressure recovery process in fractured formations	**M.C. Raju** Soret effects due to natural convection in a viscoelastic fluid flow in porous medium with heat and mass transfer

Coffee break

16.00- 17.00: Conclusions, discussions, future planning (Room A)

ICAPM 2013

5th International Conference on Applications of Porous Media



August 25-28, 2013

Cluj–Napoca, ROMANIA

UNIVERSITAS NAPOCENSIS BABEŞ-BOLYAI 1581 ROMANIA

UNIVERSITY OF CALGARY

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Sunday, 25th August

Registration 16.00-18.00
Babeş-Bolyai University
Faculty of Mathematics and Computer Sciences
Str. Mihail Kogălniceanu, Nr. 1
400084 Cluj-Napoca, Romania

Monday, 26th August

Registration 8.00-8.30
Babeş-Bolyai University
Faculty of Mathematics and Computer Sciences
Str. Mihail Kogălniceanu, Nr. 1
400084 Cluj-Napoca, Romania

Opening 8.30 - 9.00, Room A

Plenary lectures, Chairs: R. Nazar and I. Pop

Time	Room A
9.00-9.40	**M. Lanza de Cristoforis** A functional analytic approach to singular perturbation problems: a quasi-linear heat transmission problem in a dilute two-phase composite
10.00-10.40	**A.A. Mohamad** Heat transfer management with porous media
11.00-11.40	**A. Barletta** Thermal instability of a plane porous layer with an inclined temperature gradient: recent results

Coffee Break

Plenary lectures, Chair: A.A. Mohamad and L.A. Sphaier

Time	Room A
13.00-13.40	**S.M. Hassanizadeh** Pore-scale modeling and micro-model experiments for the study of two-phase flow in porous media
14.00-14.40	**I.S. Pop** Non-equilibrium models for two phase flow in porous media: the occurrence of saturation overshoots

Coffee Break

Ordinary lectures

Chairs: G. Lauriat (Room A) and A. Ishak (Room B)

Time	Room A	Room B
15.00-15.20	**A. Tatomir** Characterization of geological reservoirs for storage of carbon dioxide using tracer tests	**Gh. Juncu** Unsteady conjugate forced convection heat transfer from a porous sphere to a surrounding porous media
15.20-15.40	**A. Nakayama** Applications of porous media theory to membrane transport phenomena	**N. Arifin** Boundary layer slip flow over a porous plate in a Darcy-Forchheimer porous medium
15.40-16.00	**A. Carabineanu** A genetic algorithm approach for investigating the free-boundary seepage from a symmetric soil channel with an angular point	**N. Bachok** Flow and heat transfer past a permeable stretching/shrinking surface in a porous medium: Brinkman model
16.00-16.20	**M.A. Sheremet** Numerical simulation of 3D unsteady natural convection in a porous enclosure having finite thickness walls	**F.M. Ali** MHD mixed convection boundary l flow past a vertical flat plate embec in a porous medium with radiation e
16.20-16.40	**M.A. Sheremet** Conjugate natural convection in a partially porous vertical cylinder: A comparison study of different models	**S. Ahmad** Mixed convection boundary layer f at lower stagnation point of a sph through porous medium in presenc heat source/sink
16.40-17.00	**R. Lombarkia** Shape optimization of covers and hoops of packaging metal boxes	**Y.Y. Lok** Mixed convection boundary-layer along a vertical surface embedded porous medium with a convecti boundary condition

Coffee Break

Ordinary lectures

Chairs: I.S. Pop (Room A) and Gh. Juncu (Room B)

Time	Room A	Room B
17.20-17.40	**M.-C. Suárez A.** Modeling of heat and mass transfer in geothermal systems	**M.A.A. Hamad** Magnetohydrodynamics forcec convection boundary layer flow anc transfer over a moving porous flat nanofluid: dual solution
17.40-18.00	**J.C. Umavathi** Heat transfer enhancement for free convection flow of nanofluids in a vertical rectangular duct using Darcy-Forchheimer-Brinkman model	**A. Roşca** Mixed convection boundary layer past a vertical flat plate embedded porous medium filled with water a with a convective boundary condi opposing flow case
18.00-18.20	**D. Filip** Fully developed assisting mixed convection through a vertical porous channel with an anisotropic permeability: case of heat flux	**H. Xu** Boundary-layer similarity flows of Newtonian fluids driven by power shear over a stretching flat plate v suction or injection
18.20-18.40	**D. Cîmpean** Mixed convection flow of a nanofluid between two inclined parallel plates filled with a porous medium – the case of an adiabatic plate	**F. Huţanu** Electrochemical detection of gluc based on Prussian blue modified s printed electrode
18.40-19.00	**Ö. Türküler** Effect of pore to throat size ratio on interfacial convective heat transfer coefficient	**E.D. Kovacs** Formation/accumulation rate estir of target chemicals from mushroc through a novel numerical procec

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Trip to Turda Salt Mine

Alba-Iulia

ICAPM 2013

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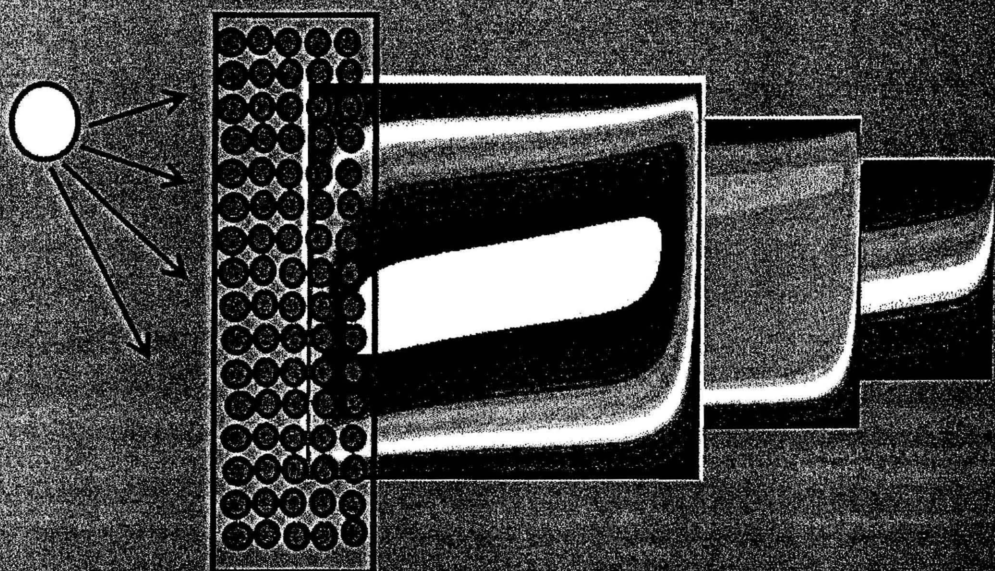

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Universitatea Babeş-Bolyai
Presa Universitară Clujeană
Director: Codruţa Săcelean
Str. Hasdeu nr. 51
400371 Cluj-Napoca, România
Tel./fax: (+40)-264-597.401
E-mail: editura@editura.ubbcluj.ro
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Contents

Mixed Convection Boundary-Layer Flow along a Vertical Surface Embedded in a Porous Medium with a Convective Boundary Condition

Y.Y. Lok[a], J.H. Merkin[b], I. Pop[c]

[a]Mathematics Division, School of Distance Education, Universiti Sains Malaysia, 11800 USM, Pulau Pinang, Malaysia
[b]Department of Applied Mathematics, University of Leeds, LS2 9JT, UK
[c]Department of Mathematics, Babeş-Bolyai University, R-400084 Cluj-Napoca, Romania
*Correspondence author: Fax: +604 657 6000 Email: lokyy@usm.my, lyianyian@yahoo.com

Abstract

An analysis of the mixed convection boundary-layer flow on one face of a semi-infinite vertical surface embedded in a fluid-saturated porous medium is presented. It is assumed that the other face of the surface is in contact with a hot or cooled fluid maintaining the surface at a constant temperature T_f. Using an appropriate similarity transformation, the governing system of partial differential equations is transformed into a system of ordinary differential equations, which are then solved numerically. The dependence of the reduced Nusselt number on the convective (Biot) number and the buoyancy or mixed convection parameter is investigated. The results indicate that dual solutions exist for opposing flow, whereas for the assisting flow the solution is unique. Limiting asymptotic forms are also derived.

Nomenclature

f	modified streamfunction
g	acceleration due to gravity
h_f	heat transfer coefficient
k	surface thermal conductivity
K	porosity of porous medium
T	fluid temperature
T_f, T_∞	surface/ambient temperature
ΔT	temperature difference ($= T_f - T_\infty$)
u	streamwise velocity
U_∞	outer flow
v	transverse velocity
x	streamwise coordinate
y	transverse coordinate
α_m	thermal diffusivity
β	coefficient of thermal expansion
$, \epsilon$	dimensionless parameters defined in (9)
θ	dimensionless temperature difference
η	similarity variable
ν	kinematic viscosity of the fluid
ψ	streamfunction

Introduction

In the study of convective heat transfer it is customary to treat the problem as either purel forced convection or free convection. However, the combination of both forced and fre convection arises in many transport processes in nature and in engineering devices, such a in atmospheric boundary layers, heat exchangers, solar collectors, nuclear reactors, elec tronic equipment, etc., in which the effects of a forced flow on a buoyantly-induced flov are significant. A great deal of work has already been performed on the study of convectiv flows in fluid-saturated porous media. The problem of mixed convection in porous medi has important applications in such fields as geothermal energy extraction, oil recover modelling, food processing, thermal insulating systems, in manufacturing processes, envi ronment, heat storage systems, etc. Many of these applications can be found in the recen books by Pop and Ingham [1], Bejan et al. [2], Ingham et al. [3], Ingham and Pop [4], Vafa [5, 6], Nield and Bejan [7] and Vadasz [8].

It appears that Cheng [9] was the first to study the problem of mixed convection adja cent to inclined surfaces embedded in porous media using the boundary-layer approxima tion. Similarity solutions were obtained for the situation where the free stream velocity anc the surface temperature distribution vary according to the same power function of the dis tance along the surface. Further, Merkin [10, 11] examined the effect of opposing buoyancy forces on the boundary-layer flow on a semi-infinite vertical flat surface at a constant (isothermal) temperature in a uniform free stream, while Aly et al. [12] considered the surface temperature to vary as x^{λ}, where x is the coordinate measuring distance from the leading edge along the surface and λ is a fixed constant. It was shown in these papers that, for opposing flow, the numerical solutions break down and the boundary layer may separate from the surface, giving rise to rather unusual heat transfer characteristics. The governing similarity equations can also admit multiple (dual) solutions. The steady boundary-layer flow near the stagnation point on an impermeable vertical surface with slip that is embedded in a fluid-saturated porous medium has been investigated by Harris et al. [13] using the Darcy-Brinkman fluid model. It was found that dual solutions exist for assisting flows, as well as those usually reported in the literature for opposing flows. The temporal stability of their steady flow solutions for different values of the mixed convection parameter has been performed using a linear stability analysis.

Most mixed convection studies in porous media assume an isothermal or variable surface condition, but not a convective boundary condition. The idea of using a convective (or conjugate) boundary conditions was first introduced by Merkin [14] for the problem of free convection past a vertical flat plate immersed in a viscous (Newtonian) fluid. More recently, Aziz [15] used the convective boundary condition to study the classical problem of forced convection boundary-layer flow over a flat surface. Since then, a number of boundary-layer flows have been reconsidered now applying convective boundary conditions, see Aziz [15], Bataller [16], Makinde and Olanrewaju [17], Merkin and Pop [18] and Makinde and Aziz [19], for examples.

In the present paper, the effect of steady mixed convection flow over a semi-infinite flat surface embedded in a fluid-saturated porous medium is studied, in the case when the surface is heated or cooled convectively. Using pseudo-similarity variables, the basic continuity, Darcy and energy equations are reduced to a coupled system of ordinary differential equations. Conditions are identified for the existence of a true similarity solution. The similarity equations are solved numerically and the results discussed. It is worth, however, mentioning that the present problem extends that of Merkin [10, 11] to the case of convective boundary condition.

Model

We consider the steady boundary-layer flow along a vertical flat surface embedded in a fluid-saturated porous material of constant ambient temperature T_∞. We assume that the constant velocity of the outer (potential) flow is U_∞ and that the surface is heated by convection from a hot or cooled fluid at the temperature T_f, providing a heat transfer coefficient $h_f = h_f(x)$, where $T_f > T_\infty$ corresponds to a heated plate (assisting flow) and $T_f < T_\infty$ corresponds to a cooled surface (opposing flow), respectively. Under these assumptions as well as the usual Boussinesq and boundary-layer approximations, the basic equations are, see Ingham and Pop [4], Vafai [5], Nield and Bejan [7] for example,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$\frac{\partial u}{\partial y} = \frac{gK\beta}{\nu}\frac{\partial T}{\partial y} \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_m \frac{\partial^2 T}{\partial y^2} \tag{3}$$

subject to the boundary conditions

$$\begin{gathered} v = 0, \quad -k\frac{\partial T}{\partial y} = h_f(T_f - T) \quad \text{on} \quad y = 0 \\ u \to U_\infty, \quad T \to T_\infty \quad \text{as} \quad y \to \infty \end{gathered} \tag{4}$$

where x and y are the Cartesian coordinates measured along the surface and normal to it; u and v are respectively the velocity components in the x and y directions; T is the fluid temperature; g is the acceleration due to gravity; K is the permeability of the porous medium; α_m is the thermal diffusivity of the porous medium; k is the thermal conductivity of the surface, β is the coefficient of thermal expansion and ν is the kinematic viscosity.

Following Merkin [10, 11] and Aziz [15] for example, equations (1–3) with the boundary conditions (4) can be transformed into ordinary differential equations by the similarity transformation

$$\psi = (2\alpha_m U_\infty x)^{1/2} f(\eta), \quad T - T_\infty = \Delta T \theta(\eta), \quad \eta = y\left(\frac{U_\infty}{2\alpha_m x}\right)^{1/2} \tag{5}$$

where $\Delta T = T_f - T_\infty$ and where ψ is the stream function defined by $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$. Using (5), equations (2) and (3) reduce to the ordinary differential equations

$$f'' = \epsilon\theta', \quad \theta'' + f\theta' = 0 \tag{6}$$

In order to have a similarity solution for equations (6), the boundary conditions (4) require that heat transfer coefficient h_f must be proportional to $x^{-1/2}$. Thus we assume

$$h_f = kC_0 x^{-1/2} \tag{7}$$

where C_0 is a constant. Boundary conditions (4) now become

$$f = 0, \quad \theta' = -\gamma(1-\theta) \quad \text{on} \quad \eta = 0,$$
$$f' \to 1, \quad \theta \to 0 \quad \text{as} \quad \eta \to \infty \tag{8}$$

where primes denotes differentiation with respect to η. The dimensionless quantities γ and ϵ are respectively the convective and the buoyancy (or mixed convection) parameters and are defined by

$$\gamma = C_0\left(\frac{2\alpha_m}{U_\infty}\right)^{1/2}, \quad \epsilon = \frac{g\beta K \Delta T}{\nu U_\infty} \tag{9}$$

We note that having $\epsilon > 0$ corresponds to a heated plate $T_f > T_\infty$ or assisting flow; having $\epsilon < 0$ corresponds to a cooled plate ($T_f < T_\infty$) or opposing flow, with $\epsilon = 0$ corresponding to forced convection.

We can combine equations (6), noting that $f' = 1 + \epsilon\theta$, to reduce the problem to the single equation

$$f''' + ff'' = 0 \tag{10}$$

with boundary conditions (4) giving

$$f = 0, \quad f'' = -\gamma(1 + \epsilon - f') \quad \text{on} \quad \eta = 0,$$
$$f' \to 1 \quad \text{as} \quad \eta \to \infty \tag{11}$$

It is the problem given by (10, 11) that we now consider in detail.

Results

We start by noting that, for $\gamma \gg 1$, the problem reduces to equation (10) but now subject to the boundary conditions

$$f = 0, \quad f' = 1 + \epsilon \quad \text{on} \quad \eta = 0,$$
$$f' \to 1 \quad \text{as} \quad \eta \to \infty \tag{12}$$

treated by Merkin [10, 11]. The main point to note about the results given in [10, 11] is the existence of a saddle-node bifurcation at $\epsilon_c \approx -1.3541$, with dual solutions for $\epsilon_c < \epsilon < -1$, no solutions for $\epsilon < \epsilon_c$ and only a single solution for $\epsilon > -1$. This leads us to expect the existence of a critical point $\epsilon_c = \epsilon_c(\gamma)$ in the present problem.

We also note that, when $\epsilon = 0$, we have the forced convection limit with then $f = \eta$. By perturbing about this limit it is straightforward to show that

$$f'(0) \sim 1 + \frac{\gamma\sqrt{\pi}}{\sqrt{2} + \gamma\sqrt{\pi}}\epsilon + \cdots, \quad f''(0) \sim -\frac{\gamma\sqrt{2}}{\sqrt{2} + \gamma\sqrt{\pi}}\epsilon + \cdots \quad \text{for } \epsilon \ll 1 \tag{13}$$

Opposing flow, $\epsilon < 0$: We plot both $f'(0)$ and $f''(0)$ against ϵ in Figure 1 for representative values of γ. We see that there is a critical point at $\epsilon_c = \epsilon_c(\gamma)$ with the values of $|\epsilon_c|$ increasing as γ is decreased. There are dual solutions for $\epsilon_c < \epsilon < -1$ with the lower branch solutions terminating in a singularity as $\epsilon \to -1$ from below in a similar manner to that described in [10, 11]. We also note that all solutions have $f'(0) = 1$, $f''(0) = 0$ at $\epsilon = 0$ in agreement with (13).

In Figure 2 we plot $f'(0)$ and $f''(0)$ against γ for representative values of ϵ, both positive and negative. We see that, for $\epsilon < \epsilon_c$, there is a critical value of γ with this critical value decreasing as $|\epsilon|$ is increased, consistent with the results shown in Figure 1. Also, for values of $\epsilon > \epsilon_c$, the solution continues to large values of γ with, for a given value of γ, both $f'(0)$ and $f''(0)$ increasing as ϵ is increased. As perhaps expected from (13), $f'(0) < 1$, $f''(0) > 0$ for $\epsilon_c < \epsilon < 0$ and $f'(0) > 1$, $f''(0) < 0$ for $\epsilon > 0$. We see that, for $\epsilon = -1.5$ and -2.0, there is an upper bound on γ for the existence of a solution, consistent with the results shown in Figure 3.

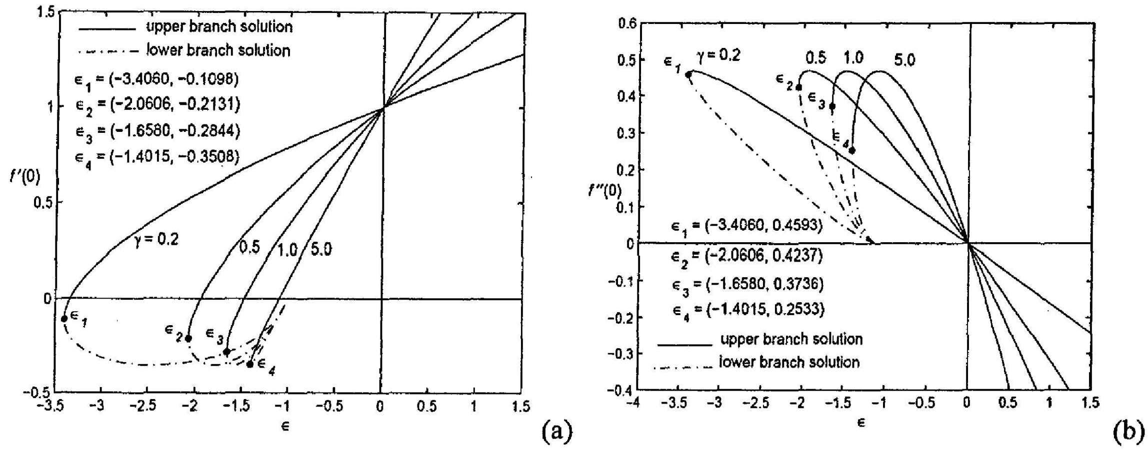

Figure 1. Plots of (a) $f'(0)$ and (b) $f''(0)$ against ϵ for some values of γ obtained from the numerical solution of equations (10, 11). The values of $f'(0)$ and $f''(0)$ at the critical points are noted on the figure.

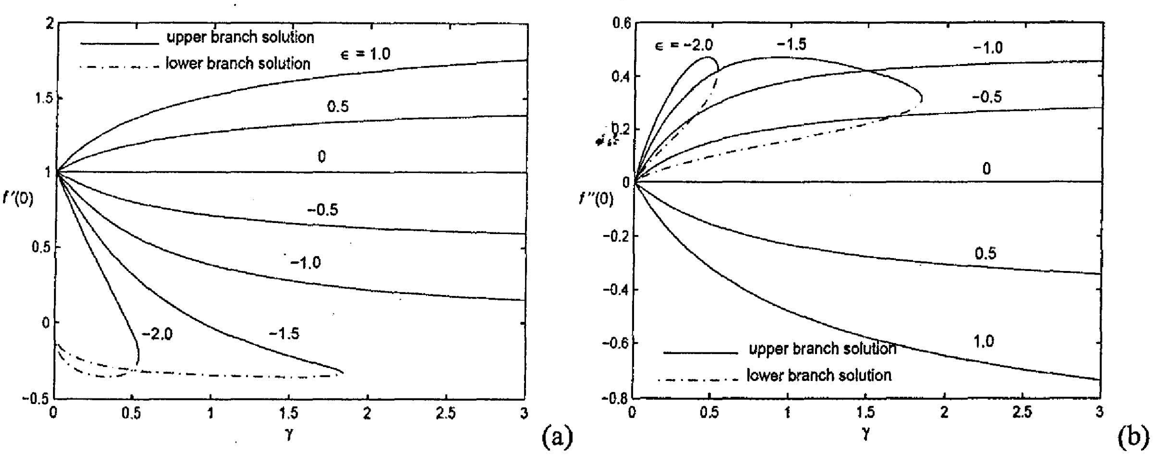

Figure 2. Plots of (a) $f'(0)$ and (b) $f''(0)$ against γ for the values of ϵ noted on the figure obtained from the numerical solution of equations (10, 11)

We can calculate the critical values ϵ_c numerically following the approach described in [20] for example. In Figure 3 we plot ϵ_c against γ with this figure showing that, consistent with Figure 1, $|\epsilon_c|$ increases as γ is decreased, appearing to become unbounded as $\gamma \to 0$. Also, ϵ_c approaches the large γ limit of -1.3541 mentioned above and shown by a broken line as γ is increased.

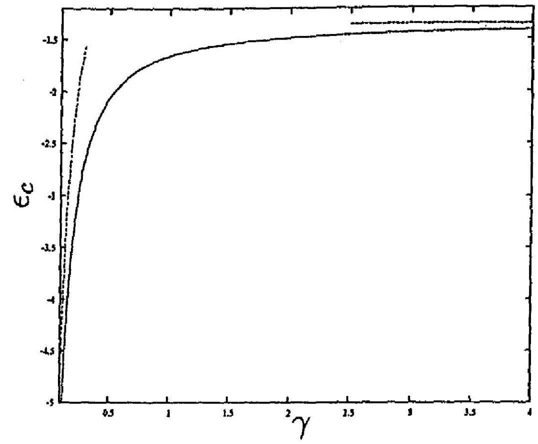

Figure 3. The critical values ϵ_c of equations (10, 11) plotted against γ. The asymptotic limits of $\epsilon_c = -1.3541$ for large γ and expression (18) for γ small are shown by broken lines

We now consider the behaviour of the solution for γ small. For ϵ of $O(1)$, boundary condition (11) gives $f''(0) = 0$ and the solution is simply $f = \eta$. If we then put

$$f = \eta + \gamma\phi_0 + \cdots \tag{14}$$

we find that ϕ_0 satisfies, at leading order,

$$\begin{gathered} \Phi_0''' + \eta\Phi_0'' = 0 \\ \Phi_0(0) = 0 \quad \Phi_0''(0) = -\epsilon \quad \Phi_0' \to 0 \quad \text{as} \quad \eta \to \infty \end{gathered} \tag{15}$$

Equation (15) has the solution

$$\Phi_0' = \epsilon \int_\eta^\infty e^{-s^2/2}\, ds \quad \text{giving} \quad \Phi_0'(0) = \epsilon\sqrt{\frac{\pi}{2}} \tag{16}$$

which is the form given in (13) when expanded for small γ.

This approach breaks down when ϵ is large, of $O(\gamma^{-1})$. We now put $\epsilon = \delta\gamma^{-1}$ and assume that δ is of $O(1)$. In this case the problem reduces to, at leading order, equation (10) but now subject to the boundary conditions

$$f(0) = 0, \quad f''(0) = -\delta, \quad f' \to 1 \quad \text{as} \quad \eta \to \infty \tag{17}$$

This problem has a solution similar to that described in [10, 11] in that there is a critical value δ_c of δ with $\delta_c = -0.46960$ and dual solutions for $\delta_c < \delta < 0$, no solutions for $\delta < \delta_c$ and only a single solution for $\delta > 0$. Thus we have

$$\epsilon_c \sim -0.46960\gamma^{-1} + \cdots \quad \text{as} \quad \gamma \to 0 \tag{18}$$

We also show expression (18) in Figure 3 by a broken line, showing reasonable agreement with the numerical values, given that we expect an $O(1)$ correction to this expression for small γ.

Aiding flow, $\epsilon > 0$: Our numerical solutions indicate that there is only one solution for $\epsilon > 0$ (in fact for $\epsilon > -1$) having $f'(0) > 1$ and $f''(0) < 0$ for all values of γ and $\epsilon > 0$ tried. This can be seen in Figures 1 and 2 where we plot $f'(0)$ and $f''(0)$ against ϵ (Figure 1) and against γ (Figure 2). $f'(0)$ increases and $f''(0)$ decreases as ϵ is increased, with both $f'(0)$ and $|f''(0)|$ becoming large for large ϵ.

This leads us to consider the asymptotic solution for ϵ large, assuming that γ is of $O(1)$. We put

$$f = (\epsilon\gamma)^{1/3}\bar{f}, \quad \bar{\eta} = (\epsilon\gamma)^{1/3}\eta \tag{19}$$

which leaves equation (10) unaltered except that primes now denote differentiation with respect to $\bar{\eta}$. Boundary condition (11) becomes

$$\bar{f} = 0, \quad \bar{f}'' = -1 + \gamma^{2/3}\epsilon^{-1/3}\bar{f}' - \epsilon^{-1} \quad \text{on} \quad \bar{\eta} = 0, \tag{20}$$

$$\bar{f}' \to (\epsilon\gamma)^{-2/3} \quad \text{as} \quad \bar{\eta} \to \infty \tag{21}$$

Expression (20) suggests an expansion of the form

$$\bar{f} = f_0 + \epsilon^{-1/3}f_1 + \epsilon^{-2/3}f_2 + \cdots \tag{22}$$

The leading-order problem

$$\begin{gathered} f_0''' + f_0 f_0'' = 0, \quad f_0(0) = 0, \\ f_0''(0) = -1, \quad f_0' \to 0 \quad \text{as} \quad \bar{\eta} \to \infty \end{gathered} \tag{23}$$

has arisen previously, see [21] for example, and has $f_0'(0) = 1.36427$.

For the problem at $O(\epsilon^{-1/3})$ we put $f_1 = \gamma^{2/3}\tilde{f}_1$ giving

$$\begin{gathered} \tilde{f}_1''' + f_0\tilde{f}_1'' + f_0''\tilde{f}_1 = 0 \quad \tilde{f}_1(0) = 0, \\ \tilde{f}_1''(0) = f_0'(0) \quad \tilde{f}_1' \to 0 \quad \text{as} \quad \bar{\eta} \to \infty \end{gathered} \tag{24}$$

A numerical integration of (24) gives $\tilde{f}_1'(0) = -1.24081$.

At $O(\epsilon^{-2/3})$ we write $f_2 = \gamma^{4/3}\tilde{f}_2 + \gamma^{-2/3}g_2$, so that

$$\begin{gathered} \tilde{f}_2''' + f_0\tilde{f}_2'' + f_0''\tilde{f}_2 = -\tilde{f}_1\tilde{f}_1'' \quad \tilde{f}_2(0) = 0, \\ \tilde{f}_2''(0) = \tilde{f}_1'(0) \quad \tilde{f}_2' \to 0 \quad \text{as} \quad \bar{\eta} \to \infty \end{gathered} \tag{25}$$

and

$$\begin{gathered} g_2''' + f_0 g_2'' + f_0'' g_2 = 0, \quad g_2(0) = 0, \\ g_2''(0) = 0, \quad g_2' \to 1 \quad \text{as} \quad \bar{\eta} \to \infty \end{gathered} \tag{26}$$

The numerical integration of (25, 26) gives $\tilde{f}_2'(0) = 0.84640$, $g_2'(0) = 0.43531$. Thus we have that

$$\begin{aligned} f'(0) \sim (\gamma\epsilon)^{2/3}\big(1.36427 &- 1.24081\gamma^{2/3}\epsilon^{-1/3} \\ &+ \left(0.84640\gamma^{4/3} + 0.43531\gamma^{-2/3}\right)\epsilon^{-2/3} + \cdots\big) \end{aligned} \tag{27}$$

and

$$f''(0) \sim \epsilon\gamma\left(-1 + 1.36427\gamma^{2/3}\epsilon^{-1/3} - 1.214081\gamma^{4/3}\epsilon^{-2/3} + \cdots\right) \tag{28}$$

as $\epsilon \to \infty$. Expressions (27, 28) show that $f''(0)$ is negative and decreasing and that $f'(0)$ is positive and increasing with ϵ, being respectively of $O(\epsilon)$ and of $O(\epsilon^{2/3})$ for ϵ large.

Finally we note that this asymptotic expansion breaks down when γ is small, of $O(\epsilon^{-1})$ In this case we again put $\delta = \gamma\epsilon$ and the problem, at least to leading order, reduces to tha give above by (10) and (17).

Conclusions

We have considered the mixed convection boundary-layer flow on a vertical surface that is heated convectively. We reduced the problem to similarity form, equations (10, 11), though to do so we required a spatially dependent surface heat transfer coefficient h_f which had to take a specific functional form, expression (7). The problem was found to involve two dimensionless parameters, namely ϵ, a mixed convection parameter that could be either positive or negative, and a surface heat transfer parameter γ. We considered both opposing, $\epsilon < 0$, and aiding, $\epsilon > 0$ flows. In the former case we found a critical value ϵ_c of ϵ, dependent on γ , with solutions to our equations (10, 11) being possible only for $\epsilon \geq \epsilon_c$. We also found dual solutions for $-1 > \epsilon > \epsilon_c$, see Figure 1. The values of ϵ_c were determined numerically and were found to be negative for all γ to increase smoothly with γ, Figure 3. The asymptotic limits of large and small γ were discussed. For large γ the flow is essentially that given by a prescribed wall temperature with a critical value ϵ_c previously determined [10, 11]. Whereas for small γ the flow is essentially that resulting from a prescribed wall heat flux with there being a smooth transition between these two flow regimes. Our analysis showed that the range of existence of solutions increased as γ was decreased with ϵ_c being of $O(\gamma^{-1})$ for γ small, expression (18).

We also considered aiding flows, Figures 1 and 2, where we saw that the solution could be continued to large values of ϵ. We derived an asymptotic solution valid for ϵ large with the leading order problem being the free convection limit corresponding to that for a prescribed wall heat flux [18], perhaps as might be expected. This solution is independent of the surface heat flux parameter γ, though the higher order perturbations to it did depend on γ, see expressions (27, 28) for example.

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References

[1] Pop, I., and Ingham, D.B., 2001, *Transport Phenomena in Porous Media*, Elsevier, Oxford.

[2] Bejan, A., Dincer, I., Lorente, S., Miguel, A.F., and Reis, A.H., 2004, *Porous and Complex Flow Structures in Modern Technologies*, Springer, New York.

[3] Ingham, D.B., Bejan, A., Mamut, E., and Pop, I. (eds.), 2004, *Emerging Technologies and Techniques in Porous Media*, Kluwer, Dordrecht.

[4] Ingham, D.B., and Pop, I. (eds.), 2005, *Transport Phenomena in Porous Media, Vol. III*, Elsevier, Oxford.

[5] Vafai, K. (ed.), 2005, *Handbook of Porous Media* (2nd edition), Taylor and Francis, New York.

[6] Vafai, K., 2010, *Porous Media: Applications in Biological Systems and Biotechnology*, CRC Press, Tokyo.

[7] Nield, D.A., and Bejan, A., 2006, *Convection in Porous Media* (3rd edition), Springer, New York.

[8] Vadasz, P., 2008, *Emerging Topics in Heat and Mass Transfer in Porous Media*, Springer, New York.

[9] Cheng, P., 1977, Combined Free and Forced Convection Flow about Inclined Surfaces in Porous Media, Int. J. Heat Mass Transfer, **20**, pp. 807–814.

[10] Merkin, J.H., 1980, Mixed Convection Boundary Layer Flow on a Vertical Surface in a Saturated Porous Medium, J. Engng. Math., **14**, pp. 301–313.

[11] Merkin, J.H., 1985, On Dual Solutions Occurring in Mixed Convection in a Porous Medium, J. Engng. Math., **20**, pp. 171–179.

[12] Aly, E.H., Elliott, L., and Ingham, D.B., 2003, Mixed Convection Boundary-Layer Flow over a Vertical Surface Embedded in a Porous Medium, Euro. J. Mech. B-Fluids, **22**, pp. 529–543.

[13] Harris, S.D., Ingham, D.B., and Pop, I., 2009, Mixed Convection Boundary-Layer Flow near the Stagnation Point on a Vertical Surface in a Porous Medium: Brinkman Model with Slip, Transport in Porous Media, **77**, pp. 267–285.

[14] Merkin, J.H., 1994, Natural Convection Boundary Layer Flow on a Vertical Surface with Newtonian Heating, Int. J. Heat and Fluid Flow, **15**, pp. 392–398.

[15] Aziz, A., 2009, Similarity Solution for Laminar Thermal Boundary Layer over a Flat Plate with a Convective Surface Boundary Condition, Commun. Nonlinear Sci. Numer. Simulat., **14**, pp. 1064–1068.

[16] Bataller, R.C., 2008, Radiation Effects for the Blasius and Sakiadis Flows with a Convective Surface Boundary Condition, Appl. Math. Comput., **206**, pp. 832–840.

[17] Makinde, O.D., and Olanrewaju, P.O., 2010, Buoyancy Effects on Thermal Boundary Layer over a Vertical Plate with a Convective Surface Boundary Condition, ASME J. Fluids Eng., **132**, pp. 044502 (1–4).

[18] Merkin, J.H., and Pop, I., 2011, The Forced Convection Flow of a Uniform Stream over a Flat Surface with a Convective Surface Boundary Condition, Commun. Nonlinear Sci. Numer. Simulat., **16**, pp. 3602–3609.

[19] Makinde, O.D., and Aziz, A., 2011, Mixed Convection from a Convectively Heated Vertical Plate to a Fluid with Internal Heat Generation, ASME J. Heat Transfer, **133**, pp. 12250 (1–6).

[20] Merkin, J.H., and Mahmood, T., 1989, Mixed Convection Boundary Layer Similarity Solutions: Prescribed Wall Heat Flux, J. Applied Mathematics and Physics (ZAMP), **40**, pp. 51–68.

[21] Merkin, J.H., and Pop, I., 2010, Natural Convection Boundary-Layer Flow in a Porous Medium with Temperature Dependent Boundary Conditions, Transport in Porous Media, **85**, pp. 397–414.