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Symbols and the Bifurcation between Procedural and Conceptual Thinking

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Symbols occupy a pivotal position between processes to be carried out and concepts to be thought about. They allow us both to ***do*** *mathematical problems and to* ***think*** *about mathematical relationships. In this presentation we consider the discontinuities that occur in the learning path taken by different students, leading to a divergence between conceptual and procedural thinking. Evidence will be given from several different contexts in the development of symbols through arithmetic, algebra and calculus, then on to the formalism of axiomatic mathematics. This is taken from a number of research studies recently performed for doctoral dissertations at the University of Warwick by students from the USA, Malaysia, Cyprus and Brazil, with data collected in the USA, Malaysia and the United Kingdom. All the studies form part of a broad investigation into why some students succeed yet others fail.*

Introduction: Building a theory

Our purpose in this paper is to build a theory of how individuals use symbols in mathematics and to consider why some are so much more successful than others. To begin with we will consider the wider cognitive growth to see that mathematical symbols work in a very special and powerful way. This involves a compression of knowledge developing the ability to pivot between mental concepts to *think* about problems and time-dependent processes to *do* mathematical operations to produce solutions.

Various theories have been proposed, some building from cognitive studies of children learning elementary mathematics, others based on diverse viewpoints such as the logical structure of propositional thinking or computer metaphors for brain activity. The approach given here is based on how the biological human species builds from activities in the environment to developing highly subtle abstract concepts (Tall, 1995). This begins with the ability to *perceive* things, to *act* on them and to *reflect* upon these actions to build theories (figure 1).

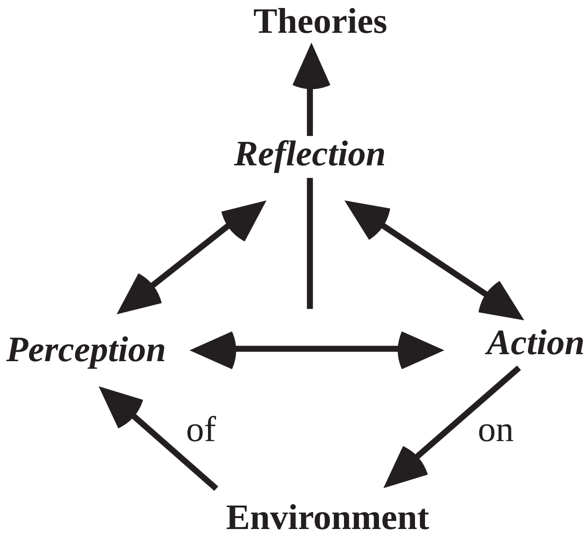

Figure 1: Combining Reflection, Perception & Action

Some authors see various activities occurring in specific sequences. For instance, Dubinsky and his colleagues propose a theory (e.g. Dubinsky, 1991; Cottrill *et al.*, 1996 etc.) in which *actions* become routinized into *processes* that are then encapsulated as *objects*, later to be embedded into cognitive *schemas* (referred to by the acronym APOS). Such a sequence occurs widely in cognitive development, and will often occur in this paper. However, our view is that perception, action and reflection occur in various combinations at a given time and a focus on one more than the others can lead to very different kinds of mathematics.

Perception of the world includes the study of *space and shape*, eventually leading to *geometry*, where verbal formulations support a shift to Euclidean proof. ***Actions on*** the world, such as counting, are represented by symbols and grow into the *symbolic mathematics* of number, arithmetic and thence on to generalised arithmetic and algebra. ***Reflection on perception and action*** in mathematics leads eventually to the desire for a consistent *axiomatic theory* based on formal definitions and deductions (figure 2).

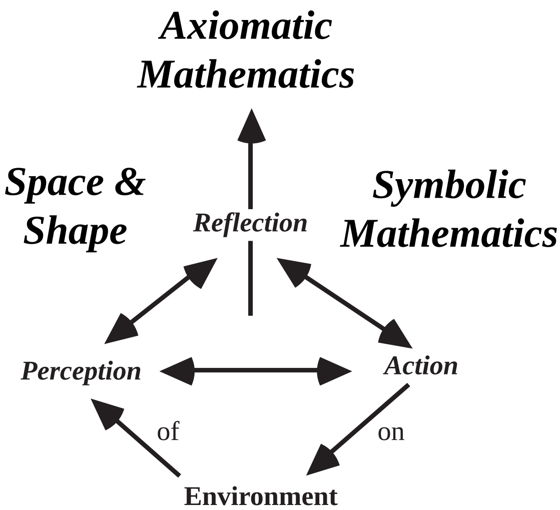

Figure 2: Various types of mathematics

Our hypothesis is that each of the three types of mathematics (space & shape, symbolic mathematics, axiomatic mathematics) is accompanied by a different type of cognitive development (figure 3).

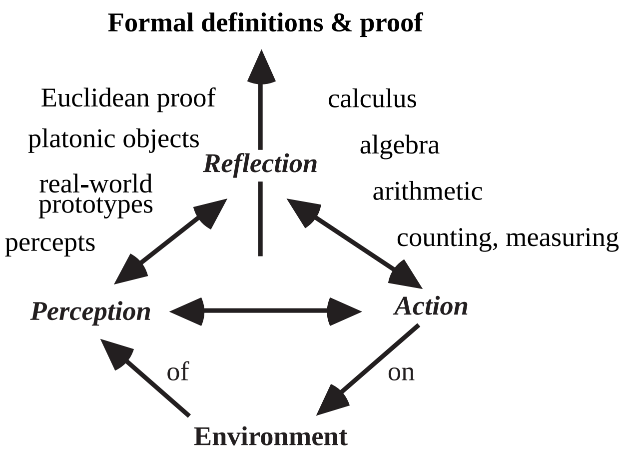

Figure 3: Conceptual development of selected mathematical concepts

Development of Geometric Concepts

Before focusing on the growth of symbolic thinking, we briefly consider the very different cognitive development in geometry. This is rooted in *perceptions* of objects in the world, initially recognised as whole gestalts. Some are specific individual perceptions, such as a child's mother, or the family pet, but more often they are perceived as *prototypes* that apply to a wide range of percepts. For instance, dog, cat, bird are prototypes for various kinds of living creatures. Some creatures are evidently birds (such as a robin), whereas others, such as an ostrich, are also classified as birds even though they fail to fly. It is interesting to note that these classifications do not begin from the bottom up, or from the top down, but in terms of centrally typical levels of recognition. For instance, children usually recognise *dog* before the more specific types of dog such as Alsatian, poodle, or more general notions such as mammal, animal. Likewise in mathematics, the recognition of concepts such as square, rectangle, parallelogram, quadrilateral, polygon, take time to organise into a conceptual hierarchy which is done neither bottom up nor top down.

This development involves various cognitive reconstructions. For instance, in the early stages, squares and rectangles are initially considered by young children as disjoint concepts (a square is *not* a rectangle, because a square has four equal sides whilst a rectangle has only *opposite* sides equal). Disjoint categories of geometric shapes must be reconstructed to give hierarchies of shapes (a square is a rectangle is a parallelogram is a quadrilateral). Further re-constructions are necessary to see a shape not as a physical object, but as a mental object with perfect properties, and then to imagine geometry not just in terms of two and three dimensional euclidean geometry, but as a variety of different geometries (affine, projective, elliptic, hyperbolic, differential, etc.) Such a cognitive development and its succession of cognitive stages have been documented in the work of van Hiele (1986). (Figure 4.)

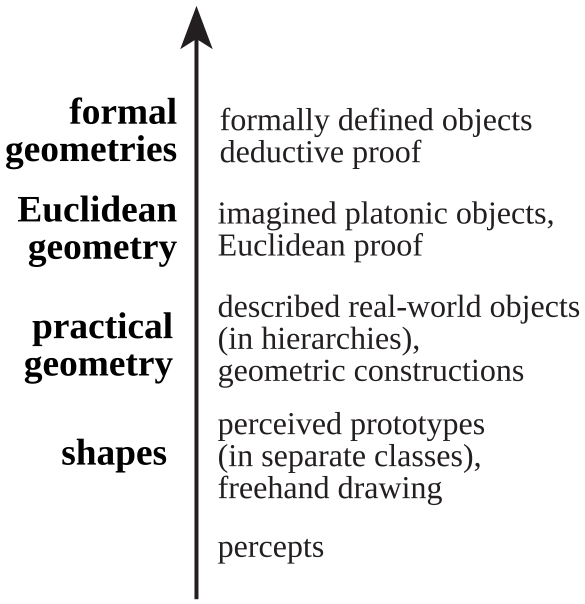

Figure 4: cognitive development of geometrical concepts

Language plays an increasingly subtle part in this geometric development. Prototypical shapes such as a straight line, a triangle, a circle, are described verbally in ways that support the imagination of perfect platonic representations, such as a perfect straight line with no width that may be extended arbitrarily in either direction, or a perfect square, a perfect circle. Thus, paradoxically, perfect geometric entities depend on language to construct their meaning.

Euclidean proof builds on this use of language to give verbal argument to support deductions based on visual concepts. Later still, the question arises as to whether the verbal proof is complete in itself, or whether it depends on implicit properties of the geometric objects. (For instance, in Euclidean geometry the notion of "inside" is not defined and yet theorems refer to the diagonals of a rhombus meeting "inside the figure.) This led to the need to formulate verbal axioms and definitions to build a system where all the properties of that system could be formally deduced from explicit assumptions.

Symbolism as a mental pivot between process and concept

The main focus of this paper—the development of symbols in arithmetic, algebra, calculus and undergraduate mathematics—is in our view very different from that of geometric objects. These symbols give *Homo sapiens* an incredibly simple way of dealing with quantities for calculation, problem solving and prediction. They simply act as a *pivot* between the symbol thought of as a concept (such as number) to a process (such as counting). This gives an instant shift from *thinking* about symbols as manipulable entities to *doing* mathematics. There are many instances of symbols allowing the switch between *process* and *concept*. (Table 1).

symbol	***process***	***concept***
4	counting	number
3+2	addition	sum
–3	subtract 3 (3 steps left)	negative 3
3/4	sharing/division	fraction
$3+2x$	evaluation	expression
$v=s/t$	ratio	rate
$y=f(x)$	assignment	function
dy/dx	differentiation	derivative
$\int f(x)\,dx$	integration	integral
$\left.\begin{matrix}\lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right)\\ \sum_{n=1}^{\infty} 1/n^2\end{matrix}\right\}$	tending to limit	value of limit
$(x_1, x_2, \ldots, x_n)$	Vector shift	point in n-space
$\sigma \in S_n$	permuting$\{1,2,\ldots,n\}$	element of S_n

Table 1: Symbols as process and concept

The dual use of symbol as process and concept often begins by becoming familiar with the process as a step-by-step procedure, then routinizing it so that it can be carried out without conscious attention to details sometimes in increasingly sophisticated ways. Counting, for instance, is a complex process of saying a sequence of number words at the same time as pointing in turn at objects in a collection once and once only. As a child counts a number of apples, (s)he might say "there are one, two, *three* apples." As this becomes more routine, the counting may be performed silently, "there are [one, two,] three apples", then compressed into "there are … *three* apples" or just "there are three apples." In this way the *process* of counting is compressed into the *concept* of number. The symbol 3 then evokes either a (counting) process or a (number) concept. Likewise the symbol 3+2 can evoke either the process of addition or the concept of sum.

Gray & Tall (1994) refer to the combination of symbol representing both a process and the output of that process as a *procept* (figure 5).

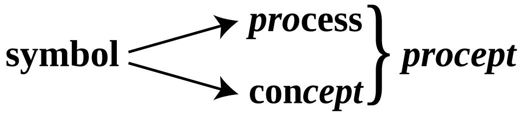

Figure 5: the symbol as pivot between process and concept forming a procept

The procept notion has been given increasingly subtle meaning since its first formulation (Gray & Tall, 1991). It is now seen mainly as a *cognitive* construct, in which the symbol can act as a *pivot*, switching from a focus on process to compute or manipulate, to a concept that may be *thought* about as a manipulable entity. We believe that procepts are at the root of human ability to manipulate mathematical ideas in arithmetic, algebra and other theories involving manipulable symbols. They allow the biological brain to switch effortlessly from *doing* a process to *thinking* about a concept in a minimal way.

Focus of attention during actions on objects

When acting on objects, perception and action are intimately connected. The addition of 2 and 3 to give 5 can be seen in terms of the combination of 2 objects and 3 objects to give five objects (figure 6).

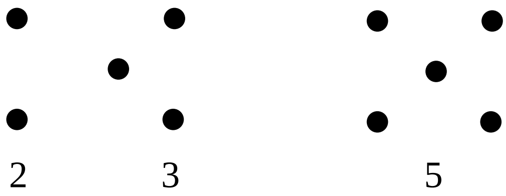

Figure 6: 2+3 is 5

In this way simple arithmetic can simply be "seen" in terms of operations on arrays of objects. It can lead to personal methods of carrying out imaginary processes to conceptualise the operation of addition using recognisable images for numbers.

For instance, the addition 5+3 might be seen initially as:

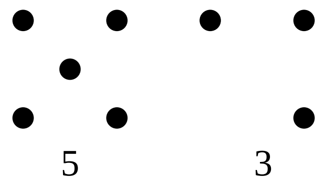

Figure 7: Adding 5 things and 3 things

A ‘slow learner’ explained to Eddie Gray that he visualized the middle dot of the 5 moving to the vacant space of the three to give two groups of four, which is eight.

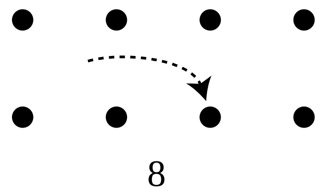

Figure 8: A mental picture of the addition

Such experiences can prove of great value in building up meaning for number relationships. However, a focus on *objects* is limited to what can be visualized. The handling of larger numbers requires more efficient methods of representation to utilize the combination of large memory store and small focus of attention.

Some children persist in seeing arithmetic in terms of mental images that prevent them from flying into the higher realms of arithmetic. Pitta & Gray (1997) investigated children selected as ‘low achievers’ and ‘high achievers’. They found that the low achievers carrying out simple arithmetic of whole numbers tended to focus attention on imagined objects that may possess shape and in many instances colour. Frequently they “saw” mental objects having characteristics of a number track (a line of cubes stuck together), although the common object that formed the basis of each unit of the track was often derived from an experience with fingers. Some reported seeing full images of fingers, others reported images that were ‘finger like’. This resulted in the children carrying out mental operations that were the analogues of counting fingers or other perceptual items. Pitta and Gray hypothesised that these mental representations were essential to their calculations and occupied much of their focus of attention. When items became more difficult, the children reverted to the use of real items.

The ‘high achievers’ on the other hand seemed to focus more often on the symbolism itself. They often either responded automatically or reported that they “talked things over in their heads.” When they did describe mental images, they often described them as coming and going very quickly. “I saw ‘3+4’ flash through my mind and I told you the answer”, “I saw a flash of answer and told you.” On occasion there were traces of intermediate activity, for instance, when given 9+7 one eleven year old gave the answer 16 accompanied by the statement. “10 and 6 flashed through my mind.” This provides vivid evidence of powerful mental connections enabling the individual to move almost instantaneously from one focus of attention to another.

This reveals a difference in focus of attention between the more successful and less successful. The less successful focus on details which may or may not be appropriate to the arithmetic task in hand—the colour of the objects, their shape and position—whilst the more successful use their focus of attention in a more productive way. Pitta (1998) placed five red cubes before some seven-year-old children at the extremes of mathematical ability and asked them to indicate what they thought about when they saw the cubes and what they thought would be worth remembering about them. The higher achievers all mentioned the word ‘five’ and thought that ‘five cubes’ was worth remembering. The lower achievers talked about the pattern, the colour, or the possible rearrangements of the cubes.

There is an amusing Simpsons cartoon which illustrates an extreme case of this phenomenon. Bart Simpson is taking an intelligence test, involving calculations for trains travelling certain distances in certain times. Instead of thinking about the numbers and calculating the relationships between them, Bart has a nightmare in which he sees himself on the train without a ticket, trying desperately to avoid being found by the ticket collector. For him the problem he is trying to solve is not one involving mathematics, but episodic images of problems related to his fantasies about the situation.

The divergence between higher and lower achievers in performing simple arithmetic was termed the ‘proceptual divide’ by Gray & Tall (1994). They observed that the low achievers sought the security of counting procedures focusing on physical or mentally perceived objects whilst the high achievers used more efficient and flexible methods, using known facts or relationships between symbols where possible and reverting to efficient counting methods otherwise. It is the growing ability to use symbols as a pivot between process and concept that gives the power to numerical symbols as procepts.

Compression of symbol usage through procedure, process and procept

Dubinsky (1991) focuses on the development from cognitive process to mental object and the subsequent “de-encapsulation” back to process once the mental object has been constructed. Sfard (1991) also sees the move from operational mathematics of doing processes to forming mental objects whose properties may then be a focus of a more structural kind of mathematics. Our approach (Gray & Tall, 1994) looks at the nature of the mathematical activities concerned. The word *procedure* is used to mean a specific sequence of steps carried out a step at a time. The term *process* is used in a more general sense to include any number of procedures which essentially “have the same effect.” For instance, the process of differentiating the function $(1+x^2)/x^2$ can be done by various procedures such as the quotient rule, the product rule (for $1+x^2$ and $1/x^2$), or other strategies such as simplification to $x^{-2}+1$ prior to differentiation.

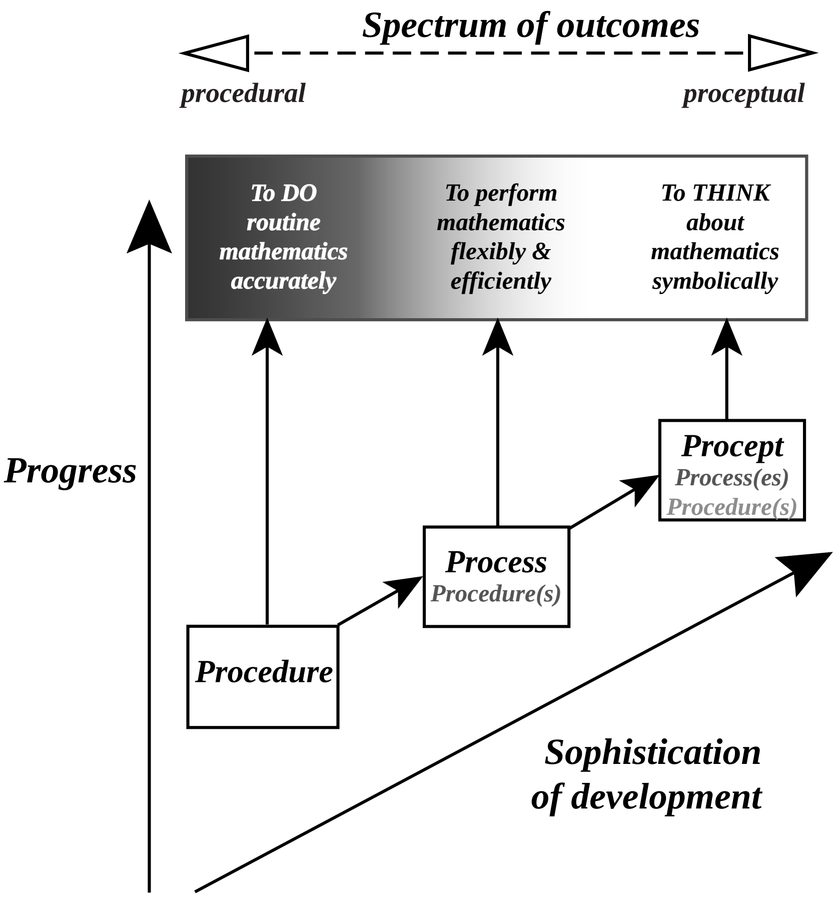

Figure 9: A spectrum of performance in the carrying out of mathematical processes

Knowing a specific procedure allows the individual to *do* a specific computation or manipulation. Having one or more alternatives available allows greater flexibility and efficiency to choose the most suitable route for a given purpose. But also being able to think about the symbolism as an entity allows it to be manipulated itself, to think about mathematics in a compressed and manipulable way, moving easily between process and concept. This gives a spectrum of performance (figure 9) in which it is possible, at certain stages, for students with different capacities all to succeed with a given routine problem, yet the possible development for the future is very different. Those who are procedurally oriented are limited to a particular procedure, with attention focused on the steps themselves, whilst those who see symbolism as process or concept have a more efficient use of cognitive processing. Long-term, as students meet new tasks the same kind of spectrum occurs, with more and more tending to be coerced into procedural thinking. This means that those who are (or who become) focused mainly on the procedural have a considerably greater burden to face in learning new mathematics than those who are able (in addition) to focus on the essential qualities of the symbolism as both process and concept.

Procedure, Process and Procept in Algebra

The processing of expressions in algebra is highly prone to the procedural-process-procept spectrum. DeMarois (1998) asked a class of college (pre-) algebra students to write down the output of the two function boxes in figure 10 in algebraic form and asked if they were the same.

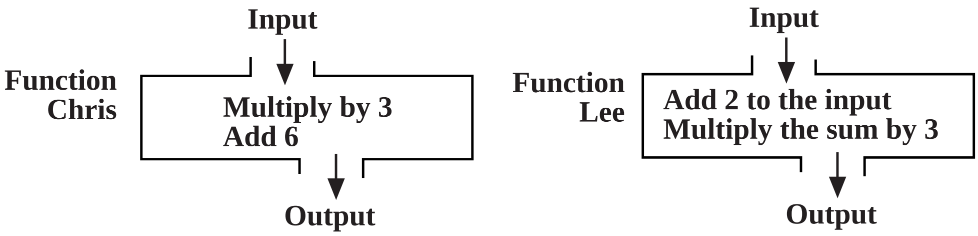

Figure 10: What are the outputs of these two function boxes and are they the same?

Three students were chosen, so that Student 1 was one of the highest achievers, Student 2 was in the middle and Student 3 was struggling. Their responses were as follows (table 2):

	Function Chris	Function Lee	*Are functions equal?*
Student 1	$3x+6$	$3(x+2)$	Yes, if I distribute the 3 in Lee, I get the same function as Chris.
Student 2	$x3+6$	$(x+2)3$	Yeah, but different processes.
Student 3	$3x+6$	$x+2(3\times)$	No, you come up with the same answer, but they are different processes.

Table 2: A spectrum of responses to functions as procept, process and procedure

Student 1 flexibly uses the notation in the usual way, manipulating the algebra, but thinking that the expression remains the same throughout the manipulation. We classify this as the *procept* level. Student 2 uses non-standard (but clearly meaningful) algebraic symbolism, asserting the functions to be "equal" but being highly conscious that there are "different processes'. We classify this as a *process* level response. Student 3 wrote his answer for the function Lee with the x in "$x+2$" and the "×" in "3×" both as a cross. A probable interpretation of this expression is as meaning "$x+2$, three times." This response is classified as being *procedural*. The three responses therefore exemplify the spectrum of figure 5. (DeMarois & Tall, 1999.)

Discontinuities in the development of symbols

There is a general perception amongst educators that curriculum design requires the construction of a sequence of lessons in which each builds smoothly and inexorably on the previous ones. This does not happen in mathematics. Working in a given context leads to beliefs that may need reconstructing at a later stage. For instance, in using numbers for counting, the "next" number after 3 is 4, so how can there be any numbers "in between"? For some individuals this causes great difficulties with fractions. Likewise, "you can't have less than nothing" when working with whole numbers and fractions, which requires a further reconstruction when introducing negative numbers. Handling the product of two negative numbers requires even more conceptual reconstruction. Many just "accept" the result and begin the slippery slope to learning by rote to pass examinations.

The routinizing of mathematical procedures can cause tensions to arise which manifest themselves in a variety of ways. Not only may old rules remain unchanged and be used inappropriately, new rules may supplant old rules and be used incorrectly when earlier work is recalled. For instance, rules of thumb in arithmetic can be mistranslated in algebra, and those in algebra can be mistranslated in arithmetic. An example of the former is when a fraction is seen to involve "dividing the top number by the bottom number" so that $\frac{12}{6}$ is correctly computed by dividing 12 by 6. However, in the algebraic expression $\frac{a^{12}}{a^{6}}$, "dividing the top number by the bottom number" may be incorrectly given as a^2. The reverse also occurs with new rules in algebra interfering with operations in arithmetic. For instance, the rule to compute $3a^2 \times 4a^3$ to give $12a^5$ by "multiplying numbers and adding powers" may be mis-applied in arithmetic to compute $3^2 \times 4^3$ as 12^5. Both of these errors (and a variety of others) prove to be made by a significant proportion of students (Anderson, 1997).

Even when a mathematical idea is learnt in a meaningful way in one context, this may cause conceptual difficulties when the old meanings no longer hold in a new context. For instance, the power 2^3, meaningfully means "three twos multiplied together." From this meaning, the properties of powers $2^3 \times 2^4 = 2^7$ easily follows because the left side has three lots of two times four lots of two, giving seven lots of two. But from this meaning, what does $2^{1/2}$ mean? How can one have "half a lot of twos multiplied together"? This leads to deep confusions when students who prefer to learn meaningfully are asked to use the power law in a context where – for them – it has no meaning.

Flexible and Procedural Links in Algebra

Students attempting to learn algebra occupy a wide spectrum of development. There are those who understand the symbolism flexibly as process and concept and find it almost trivial to manipulate the symbols in a meaningful way. Others have already fallen foul of difficult reconstructions and are hanging on to a collection of half-remembered procedures to apply to a limited range of specific problems. The cognitive links that each individual forms between various aspects of symbolism and visual representations of graphs are widely different. The way in which they solve problems can give fascinating insight into their available cognitive structures. For example, the *x*- and *y*- intercepts for a given straight line equation may be found by several methods: reading the points off a graph, setting $y=0$ solving for x, and then $x=0$, solving for y. A student who has understood how to manipulate expressions as mental concepts may have a variety of methods at their finger-tips and choose the most efficient method to suit a given situation. However, a student who seeks the security of procedures may not always be so fortunate. In a study by Crowley of students taking a preparatory course for college algebra (Crowley & Tall, 1999), they were asked:

Find the *x*- and *y*- intercepts of $3y+x-12=0$.

For those who realise that intercepts are found by putting the other variable equal to zero and solving, there is a symmetry between the two cases. Putting x=0 gives 3y=12, so y is 4 and putting y=0 gives x=12. Some students can just look at the equation and see each answer in a single glance. Kristi, however, had a strategy for most of the problems in this course, which was "put it into y=mx+b form and then work form there. When asked, "What would you do here?", she replied:

> *Divide everything by 3. … in my mind I'm visually moving everything, and dividing x by 3 is … . one third x plus …, so the y-intercept is 4.*

She found the y-intercept easily, but then had to cope with the x-intercept with the equation now in the form $y = -\frac{1}{3}x + 4$. This presented difficulties with fractions and negatives, so putting y=0 was abandoned and she took out her calculator to draw the graph and use the INTERSECT facility to find the x-intercept. (Figure 11.)

Kristi seems typical of a kind of student who has difficulty manipulating symbols meaningfully, but has overall aims of following certain procedures to move her closer to her ultimate goal. It is not that she lacks processing power, for she often seems to be working *harder* than her more flexible colleagues. She seems to expend so much mental power on the details of her particular approach that she has difficulty maintaining control of the whole enterprise.

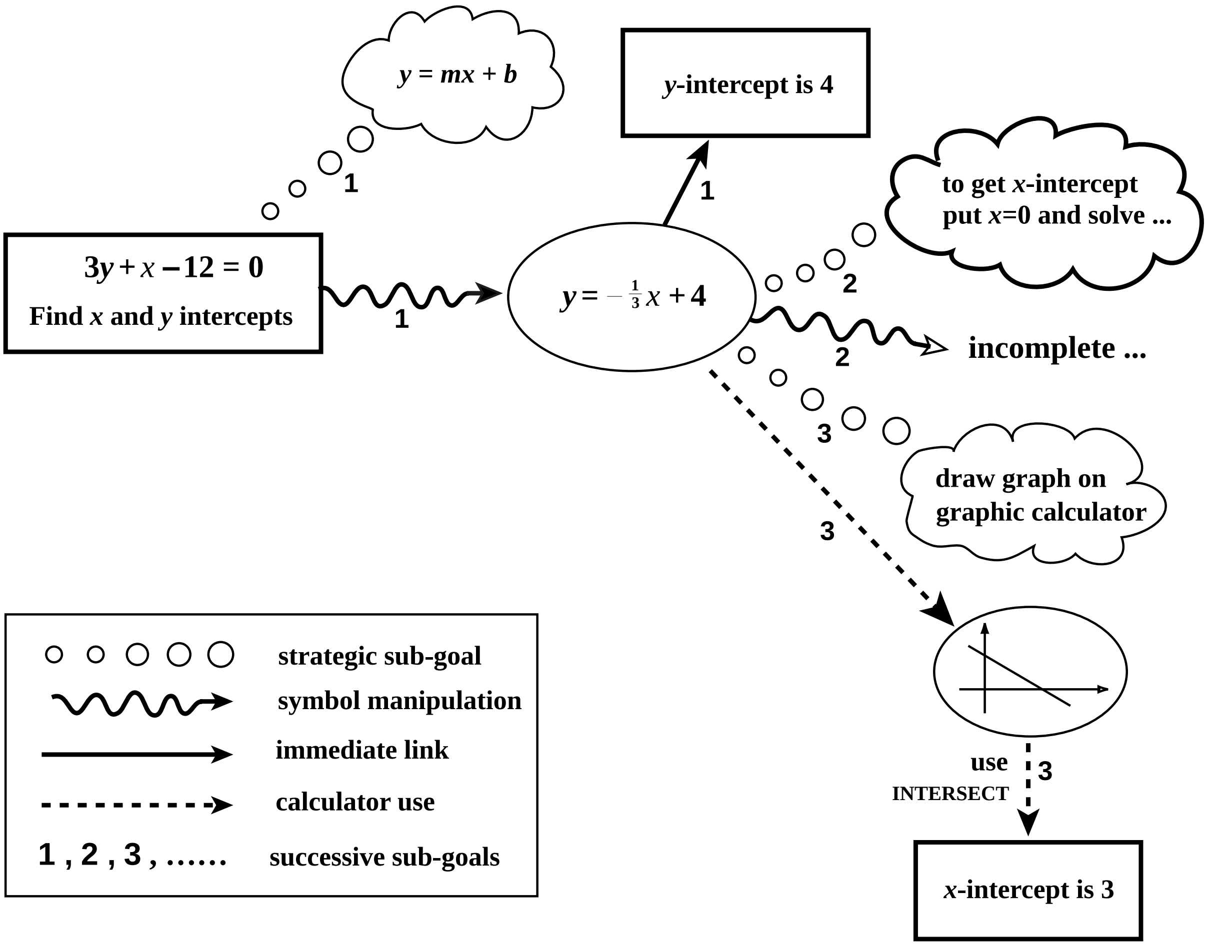

Figure 11: Kristi's strategies for finding x- and y- intercepts of 3y+x–12=0.

Student concept maps

A further method of investigating the differing cognitive constructions of students is through concept maps that the students are asked to draw at intervals during their course. McGowen (1998) did this with a class of college students studying a preliminary algebra course based on the function concept. Students were asked to build up a concept map using moveable "post-it" labels before making a permanent record after 4, 9 and 15 weeks of the course. Figures 12 and 13 show the first two maps of student SK who struggled to make sense of the course.

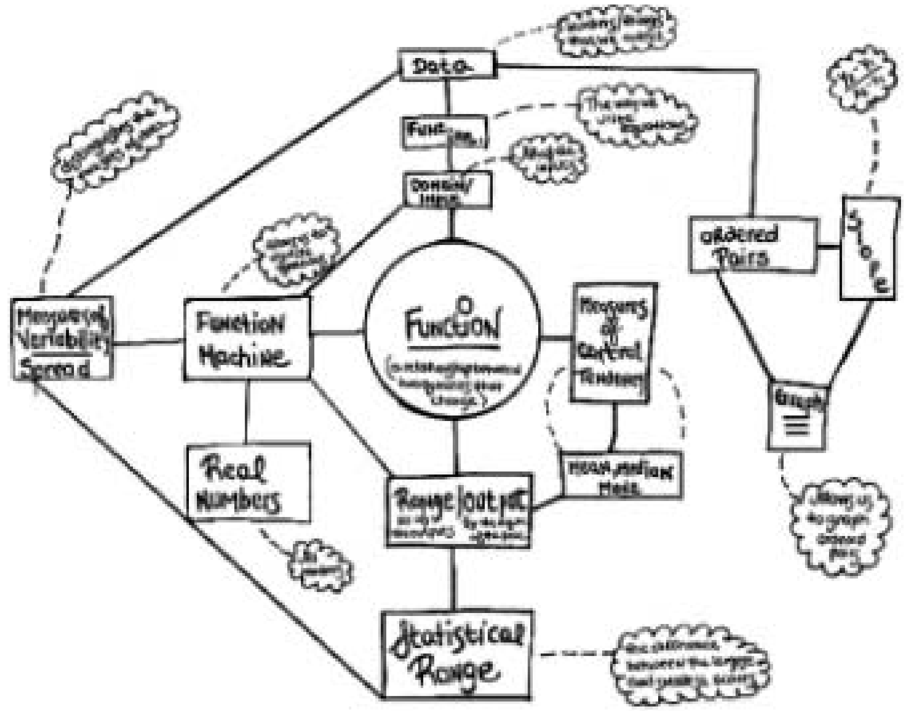

Figure 12: First concept map of SK in week 4

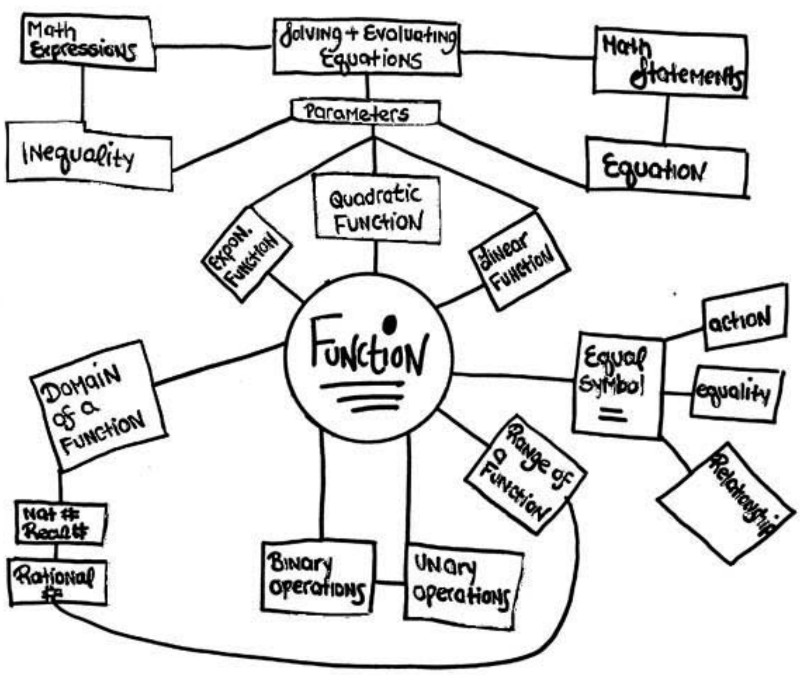

Figure 13: Second concept map of SK in week 9

On inspection it may be seen that the links between boxes do not always relate to conceptual relationships; for instance at the top of figure 12, “data” is linked to concepts involving (linear) functions such as ordered pairs, graphs, slope, with no direct link between the latter and function machine. Furthermore, on comparing the two concept maps, it soon becomes apparent that, apart from the box for “function” in the centre, almost all of the items in the second concept map are new and those that are not are moved into new positions.

These changes between successive concept map can be represented in what are termed “schematic diagrams” (figure 14). The two columns of three pictures show the changes in two students, SK on the left and MC on the right. The first picture in each column is the outline of the first concept map, then successive diagrams represent the *changes* from the previous diagram. The sequence of schematic diagrams for student SK reveal the almost total change in the second diagram, and a further major change on the third diagram, this time bringing back some items from the first concept map, but in different positions.

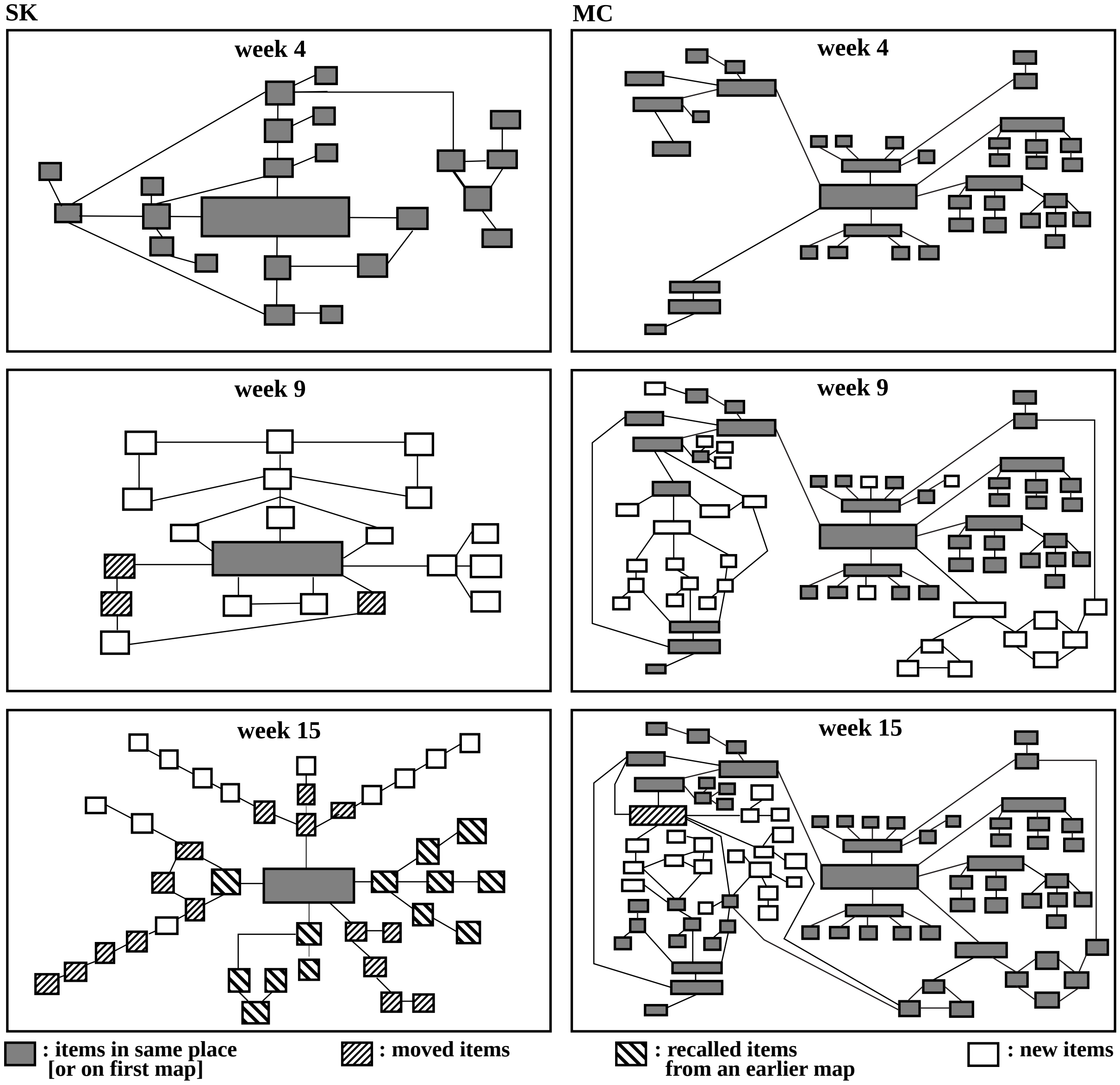

Figure 14: Schematic concept maps for students SK and MC

Student MC, on the other hand, begins with a more complex map, and each successive map builds directly on the previous one. This emphasises the manner in which MC builds a growing complex of related concepts, whilst SK (who is far less successful) builds a new map every time. SK appears to lack the stable connections that give MC the conceptual power, leaving SK mainly with procedures or processes to carry out. Whilst SK stays procedural, MC builds a more conceptual structure. (McGowen & Tall, 1999.)

Different kinds of procepts

Another theme developing through the cognitive building of arithmetic, algebra and calculus is the way in which different procepts operate in different ways leading to the need for cognitive reconstruction. In whole number arithmetic the symbols have a built-in computational process which children learn to compute a specific answer. The basic arithmetic symbols all have a dual meaning as process and concept, but there are subtle nuances that are different. For instance the sum of two whole numbers is another whole number so that the process of addition outputs a number of the same kind. But division of whole numbers can give an entirely new entity—a fraction. These violate previous experiences of (counting) numbers. For instance, although five is the "next" number after "four", fractions introduce many "numbers" between four and five and, more generally, no number has a "next" number. Similarly the expansion from counting numbers to positive and negative integers leads to old beliefs like "you can't have less than nothing" being violated in the new context, causing discontinuity in cognitive growth that may cause great difficulties for some.

The shift from arithmetic to algebra leads to a new kind of procept where the expression $2+3x$ has only a *potential* process of evaluation (when the numerical value of x is known). Thus the student again has to reconstruct experiences to give meaning to these new kinds of procept. In particular, many students have difficulty conceiving of expressions as manipulable mental objects, being unable to accept such expressions as "answers" to problems. For many of these students, the equals sign in an equation such as 3+2=5 is seen as a left-to-right process where the left hand side "makes" the right-hand side through computation. Students with such an interpretation may be able to solve an equation such as

$$3x + 1 = 16$$

by reasoning that if "$3x+1$" makes 16, if the final 1 wasn't added, then the $3x$ part of the expression only makes 15, and if 3 times x is 15, then x must be 5.

An equation of the form

$$3x + 1 = 4x - 4$$

is an altogether stranger beast. Both sides "make" something, but by different calculations that cannot be undone by reversing each of them. At the very best, if both sides are seen as mental objects representing a number, then "doing the same thing to each side" will continue to give the "same things" but these are different from the "same things" on the previous line. Faced with problems of meaning, many students concentrate on learnt procedures to "get an answer", such as “change sides, change signs”, “move the numbers over to the right”, “move the *x*s to the left”, “divide both sides by the coefficient of *x*.” Students may be able to “do” mathematics procedurally, yet not understand it relationally. The symbols have little meaning other than carrying out learned rules impressed on them by their teacher.

The shift from algebra to calculus poses even new problems. The limit symbols which occur such as

$$\lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right),\quad \sum_{n=1}^{\infty} 1/n^2,$$

all have *potentially infinite* processes. They seem to “go on forever”, perhaps never reaching the output limit concept. Again the difficulties of moving from finite algebra to potentially infinite limit processes have been widely documented (see Cornu, 1991 for a review). The limits are often sensed, not as fixed values, but as variable quantities that are “arbitrarily small”, or “arbitrarily close” or “arbitrarily large.”

Faced with such difficulties, it is no wonder that so many are relieved to find that the rules of differentiation such as

$$\frac{d}{dx}(\sin x \cos x)$$

can be performed by a *finite* manipulative process. This returns the student to a kind of security reminiscent of the operational procepts in arithmetic. The rules for computing derivatives again give a definite answer, albeit in the form that an operation on a formula gives another formula. Few students cope with the limit concept and many seek the procedural security of the rules of differentiation. Thus, in the calculus, procedural students are happier with the rules of differentiation and may make no formal sense of the limit concept. Likewise, in the theory of limits of sequences and series, students often prefer the achievable computation offered by the tests for convergence of series (such as comparison test, ratio test, or the alternating sign test). All of these have a familiar operational notion of a built-in finite computation to give an answer.

At the formal level, there are still procepts (for instance, the elements of a transformation group can be thought of both as processes and concepts). However, the procept notion is now reduced to a more minor role. For instance, the notion of a group itself is not a procept—it is an altogether bigger structure given by a definition that specifies properties it must have. The processes to construct formal meaning are now *logical* processes and the concepts are *formally* constructed. A further discontinuity, this time of major proportions intervenes, signalling the step from “elementary mathematics” of calculation and manipulation to “advanced mathematics” of defining and proving (figure 15).

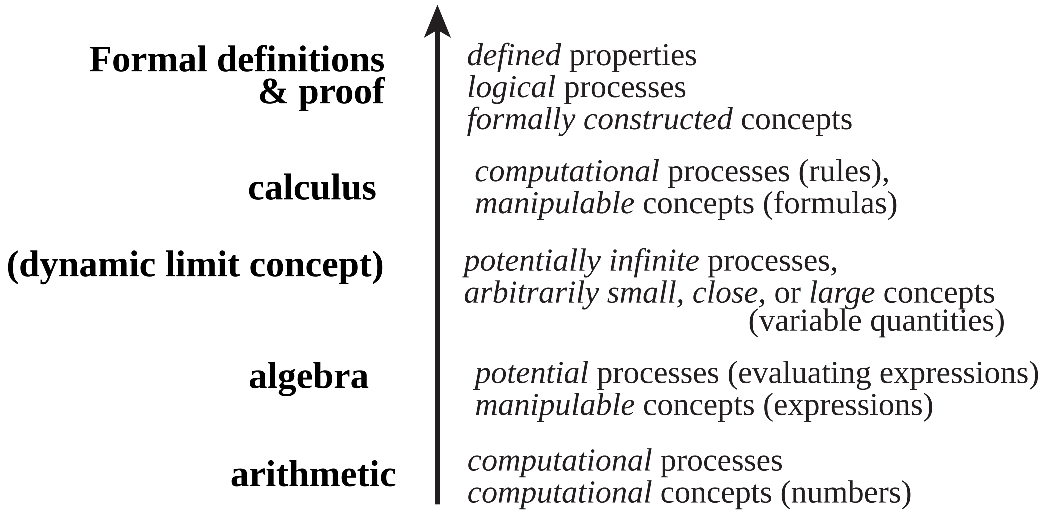

Figure 15: Different types of process & concept in mathematics

Conceptual Preparation for Calculus Procedures

As some students develop from single procedures to use alternative, more efficient, solutions, it is interesting to see how flexible they become in solving calculus problems that essentially only require the selection and operation of an appropriate procedure. The rules of calculus, such as the derivative of a product or quotient may benefit from a little conceptual preparation before carrying out the algorithm. For instance, the problem:

Determine the derivative of $\dfrac{1+x^2}{x^2}$

becomes quite complicated if it is treated immediately as a quotient:

$$y = \frac{1+x^2}{x^2},$$

$$\frac{dy}{dx} = \frac{(2x)(x^2)-(2x)(1+x^2)}{(x^2)^2} = \frac{2x^3-2x-2x^3}{x^4} = -\frac{2x}{x^4} = -\frac{2}{x^3}.$$

However, if the expression is first simplified as $x^{-2}+1$, then its derivative is immediately seen to be $-2x^{-3}$, affording a considerable reduction in processing. More successful students may be able to “see” the symbol $\dfrac{1+x^2}{x^2}$ as two fractions like this:

$$\frac{1}{x^2} + \frac{x^2}{x^2}.$$

By seeing $\dfrac{1}{x^2}$ as x^{-2}, and $\dfrac{x^2}{x^2}$ as +1, the solution can be written in a single step.

Maselan Bin Ali (1996), chose 36 students in three groups of 12, who were high (grade A), medium (grade B) and low (grade C) achievers respectively. The students in the various grades performed as in table 3.

Students' grade	Conceptual preparation	Post-algorithmic simplification	No further simplification
A	10	2	0
B	6	6	0
C	4	7	1
Total	20	15	1

Table 3: Student responses to a differentiation problem

The difference between the A and C grade students is significant at the 5% level using a χ^2 test (with Yates correction). The more successful students are more likely to use conceptual preparation to minimise their work in carrying out the algorithm.

Students were asked how many different ways they could do this example, (e.g. by product rule, quotient rule, simplification first, or implicitly differentiating $yx^2 = x^2+1$). The number of students offering different (correct) methods are given in table 4.

Of those giving two or three methods, the number of A students (9 out of 12) is significantly better than the number of C students (3 out of 12) at the 5% level using a χ^2 test (with Yates correction). However, the spectrum in this example is not an all-or-nothing phenomenon. Some A students use single procedures just as some C students show some flexibility. The A students, therefore have a greater tendency to be (at least) process-oriented than the more procedurally-oriented C students (Ali & Tall, 1996).

Students' grade	0 or 1 methods **[procedure]**	2 or 3 methods **[process]**
A	3	9
B	7	5
C	9	3
Total	19	17

Table 4: Flexibility of student solution processes

Procedural and Conceptual Knowledge

In the previous section we saw a distinction between those who learn to cope with a step-by-step procedure and those who develop flexibility to use a variety of methods and choose the most efficient. This is only part of a range of performance. Beyond this, as we saw earlier, there is also the duality of the use of symbolism as process and concept allowing the student to go even further and *think* about the symbols as mental concepts that can be manipulated. Flexible thinking, however, can go beyond the spectrum from procedural via process to procept using only symbols. The flexible use of a wider range of linkages between symbols and visual representations is called *versatile* thinking (Tall & Thomas, 1991). Essentially this involves the relationships between perceptual aspects of objects and manipulative aspects of symbols represented in figure 1. Expand this by reflection to higher level theories of figure 1 and we get a full presentation of *conceptual* knowledge in the sense of Hiebert & Lefevre (1986).

Using conceptual knowledge flexibly requires the development of a wider range of problem-solving skills. Yusof (1995) found that students taking a course focusing on developing broader problem-solving skills changed their attitudes to mathematics as a collection of skills to be rote-learned towards a creative problem-solving attitude, more willing to attack problems they had not been seen before. In this study, it happened that half the students claimed that mathematics made sense to them and half claimed it did not. These responses were not related to success at all, each group containing the same spectrum of success in examinations. The research indicated that a wide range of students gained some advantage from problem solving with the possible exception of the least successful of those who claimed that mathematics did not make sense to them. The high attainers for whom "mathematics makes sense" tended to approach problems in an open, creative way but that some lower attaining students for whom "mathematics does not make sense" treated problem-solving techniques as a new sequence of routine procedures (Yusof & Tall, 1996, 1999). Once again, we find a bifurcation between those who succeed and those who fail now extended to the full range of conceptual/procedural spectrum of thinking.

The Transition: perceptual/proceptual to formal

The move from elementary to advanced mathematics requires a significant reconstruction in thinking. The perceived shapes in space are the result of direct perception; these lead on to platonic images in euclidean geometry which perfectly represent imagined properties of the geometric figures. Meanwhile, arithmetic begins by counting actual objects and the resulting number properties (such as associativity, commutativity and distributivity) are all directly experienced by the individual.

The transition to advanced mathematical thinking makes a complete shift in focus from the existence of perceived objects and symbols representing actions on the objects to new theories based on specified *properties* of formally defined mathematical structures. Geometric experiences can be used to focus on certain properties (points, lines, intersections, curves, continuity, etc) to formulate new axiomatic systems such as non-euclidean geometry, topology and analysis. Properties of arithmetic and algebraic symbols are formulated and generalised to give axioms for groups, rings, fields, vector spaces, and so on. These newly developed theories are still built by the human brain and to a greater or lesser extent they continue to link with various kinds of mental imagery. Such imagery is useful, even essential for suggesting what kinds of definitions will be most useful and what theorems might be proved. However, the essential quality that makes advanced mathematical thinking different from elementary mathematics is the introduction of formal definitions and proof. (Figure 16.)

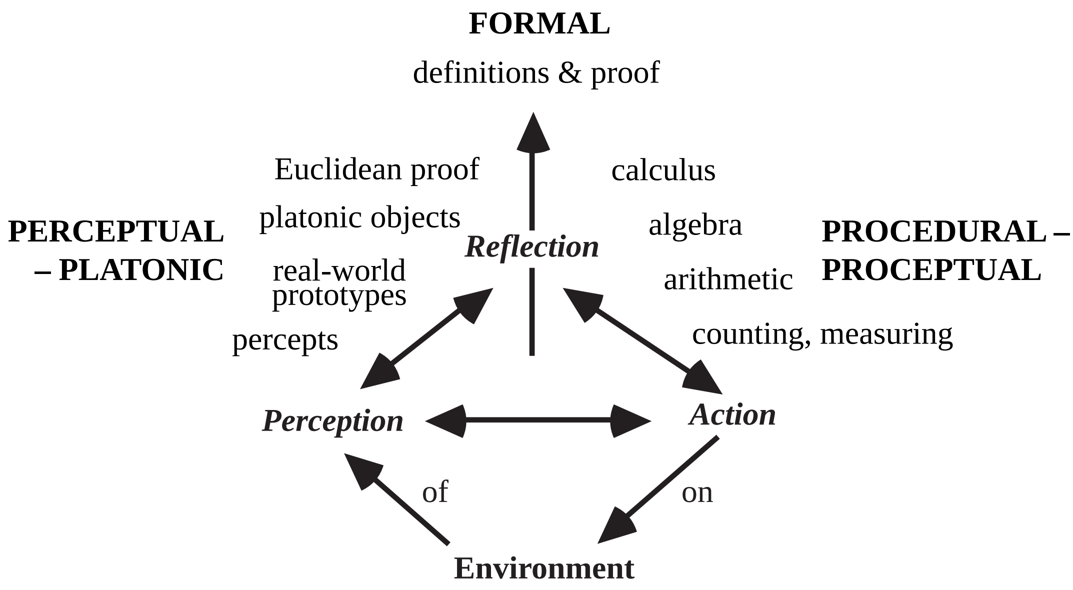

Figure 16: From perceptual & procedural to formal mathematics

The cognitive shift to formal mathematics presents a serious new discontinuity. In almost all previous experience, students have encountered objects that possess properties and symbols that can be manipulated. In both cases, the meaning of the objects and symbols comes from the experience of playing with them and finding out their properties. In formal mathematics this development is *reversed*. The student is now presented with *definitions* in words and symbols that give rise to new mathematical entities through deduction, building up their properties through a sequence of theorems and proofs.

Students often have *enormous* difficulty coping with this new view of definitions. Their current rich conceptual structure, where they already "know" a great deal of mathematics, is not entirely consistent with a formal theory where everything must now be deduced from definitions by logical inference. The fact is that *mathematicians do not use only logic*. They have an interplay between imagery (to suggest) and deduction (to prove). Likewise there are different tendencies noticeable among students. Marcia Pinto (1998) studied a spectrum of students working through a beginning analysis course to see how they handle definition and deduction. She found two widely differing strategies:

- ***giving meaning*** to a definition from a range of personal images, percepts, processes, examples, non-examples, etc.,
- ***extracting meaning*** from the definition by formal deduction in proving theorems.

A hypothetical analysis might suggest that these two strategies are best used in sequence. First one *gives* meaning, by constructing examples and non-examples and building a range of possibilities that might be deduced from the definitions. Then one moves to the logical *extraction* of the hypothesised results by formulating them as theorems and proving them. Amongst mathematicians there are some that tend more to one strategy than the other. Geometers, topologists and those building theories based on conceptual imagery often prefer giving meaning using visuo-spatial insights. Others build on existing mathematical theories, putting them together in new ways and defining new theoretical constructs. Pinto found that, although students may use either strategy at different times, many preferred one strategy to such an extent that the other is hardly used. She observed that *students can be* ***successful*** *with either route*.

The two strategies has different places where difficulties occur. The student who prefers to *give* meaning is involved in continual reconstruction of ideas as (s)he expands mental images to take account of new phenomena. The student who *extracts* meaning must first routinize the definition to be able to write it down from memory, and then use it to build up a repertoire of properties proved from the definitions.

Some students succeed in their chosen strategies, but many fail. Those who attempt to "give meaning" from perceptual images, so they can "see what happens", often find they are trying to prove something that is "obvious" for which the far more obscure proof has no meaning. These can have a "sense" of what is going on but fail to do any more than rote-learn proofs for exams. Alternatively those who attempt to "extract meaning" from a definition that they often cannot remember, let alone understand, are in even greater difficulties. They may not lack mental pictures, but these are not generative in the same way as those who "give meaning." Instead they often represent a single instance (such as a monotonically increasing sequence that does not 'reach' a limit). They are therefore inflexible and intimate properties that are not implied by the formal definition. Such students have only confused images and weak grasp of formalism, so that little progress is possible beyond minimal rote learning. (Pinto, 1998, Pinto & Tall, 1999.)

Examples of successful students giving and extracting meaning

Two students, who we will call Chris and Ross, used widely differing strategies (Pinto, 1998, chapter 8). Chris is a "giver of meaning" who did not attempt to memorise definitions. Instead he refined his mental pictures until the definition becomes evident to him. (Figure 17.)

> "I don't memorise that [the definition of limit]. I think of this [picture] every time I work it out, and then you just get used to it. I can nearly write that straight down."

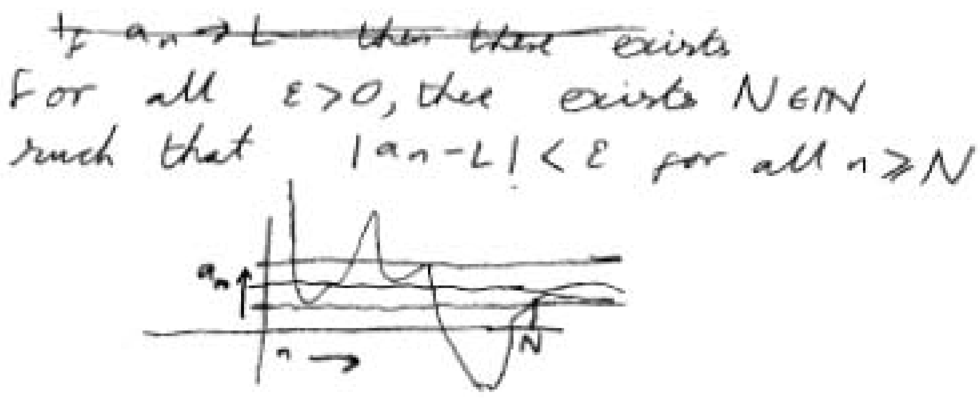

Figure 17: A picture of convergence

As he drew the picture, he gestured with his hands to show that first he imagined how close he required the values to be (either side of the limit), then how far he would need to go along to get all successive values of the sequence inside the required range. As he explained his ideas, he realised that he had represented a sequence graphically as a continuous curve.

> ”I think of it graphically ... you got a graph there and the function there, and I think that it’s got the limit there ... and then ε once like that, and you can draw along and then all the ... points after N are inside of those bounds. ... When I first thought of this, it was hard to understand, so I thought of it like that’s the n going across there and that’s a_n. ... Err this shouldn’t really be a graph, it should be points.”
>
> (Chris, first interview)

Ross, on the other hand, took a formal approach, extracting meaning from the formal definition. He explained that he learns the definition:

> “Just memorising it, well it’s mostly that we have written it down quite a few times in lectures and then whenever I do a question I try to write down the definition and just by writing it down over and over again it gets imprinted and then I remember it.”
>
> (Ross, first interview. ibid. p. 166)

They continued to use these different approaches throughout the course. For example, Ross the extractor of meaning soon came to terms with the manipulation of the symbols and wrote down what it meant for a sequence not to converge by writing out the definition of convergence. (Figure 18.) He negated this by interchanging the quantifiers (after first taking care of the fact that the definition has an implicit quantifier $\exists L$ at the beginning). (Figure 19.)

Figure 18: The definition of convergence (ibid. p. 175)

Figure 19: The negation of the definition of convergence (ibid. p. 175)

Chris, on the other hand, wrote the definition of non-convergence in a single thought experiment without using the interchange of quantifiers.

Both students made minor errors in their working on occasions, but were able to correct themselves by relating specific concepts to a wider range of coherent ideas. Both were successful in building up a formal theory.

Less successful students

The students who were less successful had a range of difficulties (Pinto, 1998). One major problem for many students involved concept imagery that was so dominant that it prevented them from even beginning to come to terms with deduction from the definition. Laura evoked many personal ideas of the limit concept, in a manner well-known in the literature (see Cornu, 1991):

> “The limit is the number where the sequence gets to, but never quite reaches.”
>
> “Let a_n be the sequence and L is the limit which it tends to. Then when some initial values are placed into the formula of the sequence the answers will never reach the value of L (negative or positive).”
>
> “... oh, yes, I put ‘never reach’, and it *can* reach, and that will be the limit of it. ...”

"... But it won't never get bigger than the limit. The limit is like the top number it can possibly reach. And I put never reach."

(Laura, various quotations, first interview)

Her concept imagery allowed her to give meaning to the statements of theorems, but she was unable to write down the definition in any formal sense. Thus formal definitions and formal proof seemed to be complicated fussing over what, to her, was "obvious". She was totally unable to "give meaning" to the formal definition in any sense that then allowed her to use it for logical deduction.

Other students made the attempt to work with the definition, but made serious errors. Rolf, for example, learned the definition by rote, but was not as successful as he thought:

"Umm ... I wrote it many times because we use it all the time, every time we are asked a question we have to use and that's how I remembered it. *I don't think I will ever forget it now*. We have done it so many times." (Rolf, first interview)

However, the definition he gave was unsatisfactory (figure 20) and he was then unable to build up a coherent theory.

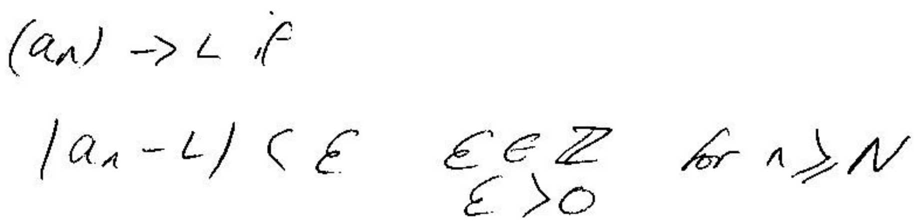

Figure 20: Rolf's attempted definition of convergence

Success and failure with the move to formal mathematics

We therefore see that students attempt to move to formal mathematics in a variety of ways. All begin with imagery before encountering the formalism. For some this imagery is so coercive that they fail to make any significant shift to formalism at all. Those who do make an attempt to make sense of the formalism have two essentially complementary activities. First the consideration of examples and non-examples to begin to build up meaning for the definition, then the deduction of other properties from the definition. What is interesting is that so many students seem to focus much more strongly on one of these activities. Some *give* meaning to the definition by manipulating their mental imagery. Successful students with this strategy can build a rich formal theory with interconnections to other suggestive imagery. Other students focus more on deduction from the definition with little or no intervening imagery, *extracting* meaning from the definition. This strategy can also build up a formal theory, in this case more confined within itself and less linked to other aspects of the student's cognitive structure. It is interesting to note that professional mathematicians work in various ways that are consistent with this spectrum of student development.

These observations differ from the theory of Dubinsky *et al.* (1988) who presuppose that processes of operation on quantified definitions lead to encapsulated formal concepts. Such a route is clearly taken by Ross, who works with the formal definition and regards visualisation as a secondary, often flawed, insight into the mathematics. Chris, however, works with his visuo-spatial imagery, giving this meaning linguistically, so that he can operate using both global gestalt insight and a sequential deductive thinking. In essence we would see aspects of perception and action (on the environment) at the root of our theory (figure 1) being extended to formal mathematics. Whilst Ross prefers to follow the sequential processes of deduction, Chris works with his internal perceptions of the concepts. Ross may be considered as encapsulating formal concepts from logical processes, but Chris manipulates his mental perceptions to construct the formalism. The mental objects *already exist* for him; they do not need to be encapsulated from processes, only moulded to fit with the theory. We therefore see that the spectrum of approach to formalism found in professional mathematicians is already present in students. It represents the full spectrum of facilities available to *Homo sapiens*, extending from perceptions of and actions on the environment to perceptions of mental objects and actions on mental objects.

Reflections: Considering the broader picture

We have seen that the development through symbolic and axiomatic mathematics has a range of discontinuities involving changes in meaning of concepts as symbols are used to compress different kinds of processes into new types of concept. At the formal level these are compounded by coping with definitions which students respond to in a variety of ways. Figure 21 shows an outline of the development, with a number of discontinuities marked.

These are by no means the only discontinuities that may occur, for reconstruction of mental concepts in new contexts is a vital part of mathematical learning. As mathematicians, we may not be aware of the precise nature of students’ difficulties. This suggests that mathematics *cannot* be structured as a

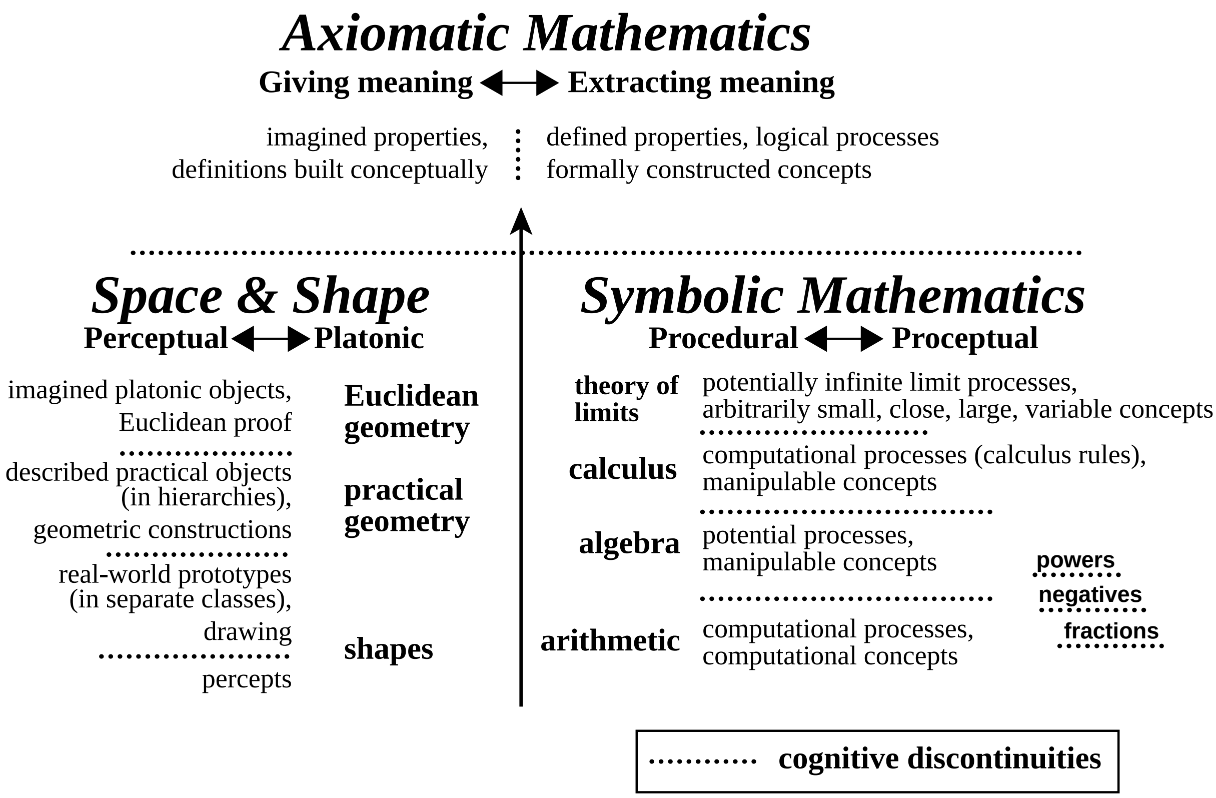

Figure 21: Cognitive growth in selected topics in mathematics, with associated *discontinuities*

simple curriculum steadily expanding the concepts building on old foundations in established ways. It requires constant re-thinking of concepts which proves possible, even invigorating, for some, but forces others into rote-learnt rules to cope in new contexts where the old ideas no longer hold true. In our examples of student development in proof, Chris positively enjoyed the struggle of making sense of ideas that confused him. He had a long experience of the satisfying pleasure of success and now sought the excitement of the struggle to maintain his high state of mental awareness. Laura, on the other hand, had learnt to fail. She took no pleasure out of failing again and could not begin to make sense of formal proof in a context which she would never meet again in her life as a primary school teacher.

The whole of the curriculum, from elementary school through university mathematics is a fascinating journey of reconstruction and conquest which appeals to those who develop a taste for the struggle to overcome defeats and taste new victories. It is a minefield for others, who may genuinely attempt to understand mathematics at one level, yet are thrown off course by a discontinuity in learning that renders new ideas incomprehensible.

Is there a moral to this tale? Certainly we do not claim that *all* students can learn mathematics if it is "presented right" and they are willing to "work hard". The "American dream" that *anyone* can do *anything* we suggest is causing a nightmare for a vast number of students studying mathematics. The evidence shows that some failing students are faced with a much greater cognitive load to achieve a lesser, more pragmatic, procedural goal. On the other hand, we do not claim that it is *impossible* for a specific student to conquer specific difficulties. There will be individuals who fail at one time yet succeed despite the initial complexity of their view of the task in hand.

All of this must be seen in the wider context of the processes of development involved in learning mathematics through arithmetic, algebra, calculus and beyond. This is a challenge that leads to a bifurcation between those who succeed in compressing knowledge into a flexible form and those who tend to seek security in learned procedures. Whilst the flexible knowledge compressors have a more powerful system at their disposal, the procedural learners may be able to solve routine problems but have a cognitive structure which makes it more difficult to build up further sophisticated knowledge.

Given the manner in which students' development diverges into a spectrum of qualitatively differently thinking, we do not believe there is a single way of *teaching* mathematics without taking into account different ways of student *learning*. The human interface between teaching and learning is a constant source of renewal and frustration that will continue to encourage imaginative teachers to seek pragmatic solutions that respect individual student needs in complex learning situations.

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