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"Billion is a large number for us if we are dealing
with years or tons of water, but this is a small
number when it comes to hydrogen molecules
or even a water drop in the ocean."
Émile Borel

GEOMETRIC CHARACTERISTICS OF FOUR-DIMENSIONAL HYPERCOMPLEX SYSTEMS

Systems of natural, integer, real and even complex numbers have long been known. The latter are widely applied not only in mathematics, but also in all areas of modern science. But this is not the end of the search for new systems of numbers. At present, the issue of the discovery of more general types of numbers is very topical, as new problems in mathematics arise that cannot be solved only with the help of these systems. For instance, it turns out that in order to describe the rotation in $\mathbb{R}^3$ 4 numbers are necessary, so we need vectors of four-dimensional space, which are quaternions. Therefore, that consideration of the basic geometric properties of four-dimensional hypercomplex systems, namely quaternions, biquaternions, coquaternions, is quite perspective and topical.

The Concept of Quaternion. A quaternion is a hypercomplex number that has 4 imaginary units. Components for imaginary units are chosen from algebra, which are fields. [2]

The Hamilton formula $ijk = -1$ gives the multiplication table of quaternions [1]:

	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	-1	I
k	k	j	-i	-1

Fig. 1. **Quaternion Multiplication Table** [2]

Thus, a quaternion is a vector of a four-dimensional real space with basis 1, i, j, k (which are called basic quaternions): $a + bi + cj + dk$. Number a is called the real part (the scalar), and the three-dimensional vector $v = bi + cj + dk$ – the imaginary part of the quaternion. The word “vector” appeared precisely in this theory. In times of William Hamilton there were no vectors, so the scientist had to invent all the terminology for this theory.

The “numbers” $1, i, j, k$ are called basic quaternions. [1]

For our purposes, the "hypercomplex" form of recording quaternions is more convenient. Let the symbols i, j, k be unit vectors, which form a fixed (the same for all quaternions) orthogonal basis in three-dimensional space. Then quaternions can be represented as “the sum of scalar and vector”.

$$Q = q_0 + q_1 i + q_2 j + q_3 k = q_0 + \vec{q},$$

where $\vec{q}$ is a vector with coordinates $q_\mu (\mu = 1,2,3)$ in the basis i, j, k. Accordingly, we call q_0 the scalar part (the scalar) of the Q quaternion, and $\vec{q}$ – its vector part (the vector).

Basic properties of quaternions:

1. $\lambda Q = Q\lambda = \lambda q_0 + \lambda \vec{q}, \lambda \in \mathbb{R}$
2. $P + Q = (p_0 + q_0) + (\vec{p} + \vec{q})$
3. $P \cdot Q = (p_0 + \vec{p})(q_0 + \vec{q}) = p_0 q_0 - (\vec{p}\vec{q}) + p_0 \vec{q} + q_0 \vec{p} + [\vec{p} \times \vec{q}]$.

Here $(\vec{p}\vec{q})$ – scalar product of vectors, $[\vec{p} \times \vec{q}]$ – vector product.

4. Let $Q = q_0 + \vec{q}$. The quaternion $\bar{Q} = q_0 - \vec{q}$ is called (quaternionically) conjugate to Q.

Obviously, $\bar{\bar{Q}} = Q, \overline{(PQ)} = \bar{P} \cdot \bar{Q}$

5. The product of quaternion is conjugated to it. It is always an essential scalar

$$Q\bar{Q} = \bar{Q}Q = (q_0 + \vec{q})(q_0 - \vec{q}) = q_0^2 + q_1^2 + q_2^2 + q_3^2 \geq 0,$$

which is called quaternions square module. Thus, the module is defined as a quaternion

$$|Q| = \sqrt{Q\bar{Q}} = \sqrt{q_0^2 + |\vec{q}|^2} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.$$

Operation of division for quaternions is defined as the product of inversed quaternions.

Thus, the paper deals with studying of four-dimension hypercomplex system and some of its geometric characteristics.

LITERATURE

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