The Method of Lines solution of the Forced Korteweg-de Vries-Burgers Equation (FKdVB)

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Abstract. In this paper, the application of the method of lines (MOL) to the Forced Korteweg-de Vries-Burgers equation with variable coefficient (FKdVB) is presented. The MOL is a powerful technique for solving partial differential equations by typically using finite-difference approximations for the spatial derivatives and ordinary differential equations (ODEs) for the time derivative. The MOL approach of the FKdVB equation led to a system of ODEs. Solution of the system of ODEs was obtained by applying the Fourth-OrderRunge-Kutta (RK4) method. In order to show the accuracy of the presented method, the numerical solutions obtained were compared with its progressive wave solution in terms of maximum absolute error at certain times. It was found that the maximum absolute errors are in theorder of 10^{-6}.

Keywords: FKdVB Equation; The Method of Lines; System of Differential Equation; RungeKutta.
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INTRODUCTION

There are many physical phenomena in engineering and physics which can be expressed by some nonlinear partial differential equations(Demiray, 2003). However, most of them do not have exact analytical solutions. Therefore, these nonlinear equations should be solved using approximate methods (Zedan & Adrous, 2012).

In literature, weakly nonlinear wave propagation in a prestressed fluid-filled stenosed elastic tube filled with a Newtonian fluid with variable viscosity fluid has been studied by (Tay, 2007)by applying the reductive perturbation method and long wave approximation. By using the stretchedcoordinate of initial-value type and extending the field quantities into the asymptotic series of order ε where ε is a small parameter, the governing equations are reduced to the forced Korteweg-de Vries-Burgers (FKdVB) equation with variable coefficients

$$u_\tau + \mu_1 u u_\xi - \mu_2 u_{\xi\xi} + \mu_3 u_{\xi\xi\xi} + \mu_4(\tau) u_\xi = \mu(\tau) \tag{1}$$

where ξ isaspatial variable, τ is a temporal variable, $\mu_1, \mu_2, \mu_3, \mu_4(\tau)$ and $\mu(\tau)$ are the coefficients of nonlinearity, dissipation, dispersion, variable coefficient and forcing term respectively The presence of forcing term $\mu(\tau)$,and variable coefficient term, $\mu_4(\tau)u_\xi$ show the presence of stenosis. The dissipative term $-\mu_2 u_{\xi\xi}$ in FKdVBequation is caused by the effect of variable viscosity. The coefficients of $\mu_1, \mu_2, \mu_3, \mu_4(\tau)$ and $\mu(\tau)$ are defined by (Tay, 2007)as

$$\mu_1=\frac{5}{2\lambda_\theta}+\frac{\beta_2}{\beta_1},\quad \mu_2=\frac{\nu}{2c},\quad \mu_3=\frac{m}{4\lambda_z}+\frac{\lambda_\theta^2}{16}-\frac{\beta_0}{2\beta_1},$$

$$\mu_4(\tau)=\frac{\lambda_\theta\gamma_2}{\beta_1}G(\tau)-\left[\frac{\beta_2}{\beta_1}+\frac{1}{2\lambda_\theta}\right]g(\tau),\quad \mu(\tau)=\frac{1}{2}g'(\tau)-\frac{\lambda_\theta\gamma_1}{2\beta_1}G'(\tau). \qquad (2)$$ where

$$\gamma_0=\frac{1}{\lambda_\theta\lambda_z}\left(\lambda_\theta-\frac{1}{\lambda_\theta^3\lambda_z^2}\right)F(\lambda_\theta,\lambda_z),$$

$$\gamma_1=\frac{1}{\lambda_\theta\lambda_z}\left[\left(1+\frac{3}{\lambda_\theta^4\lambda_z^2}\right)+2\alpha\left(\lambda_\theta-\frac{1}{\lambda_\theta^3\lambda_z^2}\right)^2\right]F(\lambda_\theta,\lambda_z),$$

$$\gamma_2=\frac{1}{2\lambda_\theta\lambda_z}\left[-\frac{12}{\lambda_\theta^5\lambda_z^2}+6\alpha\left(\lambda_\theta-\frac{1}{\lambda_\theta^3\lambda_z^2}\right)\left(1+\frac{3}{\lambda_\theta^4\lambda_z^2}\right)+4\alpha^2\left(\lambda_\theta-\frac{1}{\lambda_\theta^3\lambda_z^2}\right)^3\right]F(\lambda_\theta,\lambda_z),$$

$$\beta_0=\frac{1}{\lambda_\theta}\left(\lambda_z-\frac{1}{\lambda_\theta^2\lambda_z^3}\right)F(\lambda_\theta,\lambda_z),\ \beta_1=\gamma_1-\frac{\gamma_0}{\lambda_\theta},\quad \beta_2=\gamma_2-\frac{\beta_1}{\lambda_\theta}, \qquad (3)$$

given that $F(\lambda_\theta,\lambda_z)=\exp\left[\alpha\left(\lambda_\theta^2+\lambda_z^2+\frac{1}{\lambda_\theta^2\lambda_z^2}-3\right)\right]$, $\alpha=1.948,\ \lambda_\theta=\lambda_z=1.6,\ \nu=1, c=15.391,\quad m=0.1,\ G(\tau)=0$ and $g(\tau)=\text{sech}(0.01\tau)$. Here α refers tomaterial constant, λ_θ is the initial circumferential stretch ratio, λ_z is the initial axial stretch ratio, ν is kinematic viscosity, *m* is mass of artery and*c* is thescale parameter.

The application of the MOL to the FKdVBequation (1)will be presented in this paper. The MOL approach of the FKdVB equation led to a system of ODEs. Solution of the system was obtained by applying the RK4 method. The solution of the FKdVBequation that is obtained using theMOL with progressive wave solution conducted by (Tay, 2007)will be compared in terms of its maximum absolute error at certain time τ.

METHODOLOGY

The MOL

The MOL is a powerful method used to solve partial differential equations (PDEs). It involves making an approximation to the spatial derivatives and reducing the problem into a system of ODEs (Hall & Watt, 1976; Loeb & Schiesser, 1974; Schiesser, 1994). In addition, this system of ODEs can be solved by using time integrator. The most important advantage of the MOL approach is that it has not only the simplicity of the explicit methods (Dehgan, 2006) but also the superiority (stability advantage) of the implicit ones unless a poor numerical method for solution of ODEs is employed. It is possible to achieve higher-order approximations in the discretization of spatial derivatives without significant increases in the computational complexity. This method has wide applicability to physical and chemical systems modeled by PDEs such as delay differential equations (Koto, 2004), two-dimensional sine-Gordon equation (Bratsos, 2007), the Nwogu one-dimensional extended Boussinesq equation(Hamdi et.al, 2005),the fourth-order Boussinesq equation, the fifth-order Kaup–Kupershmidt equation and an extended Fifth-Order Korteweg-de Vries (KdV5) equation(Saucez et al, 2004).

In this paper, the spatial derivatives are firstly discretized using central finite difference formulae as follows:

$$u_\xi\approx\frac{u_{j+1}-u_{j-1}}{2\Delta\xi},$$

$$u_{\xi\xi}\approx\frac{u_{j+1}-2u_j+u_{j-1}}{(\Delta\xi)^2},$$

$$u_{\xi\xi\xi}\approx\frac{u_{j+2}-2u_{j+1}+2u_{j-1}-u_{j-2}}{2(\Delta\xi)^3}, \qquad (4)$$

where ξ is the spatial variable, τ is the temporal variable, *j* is the index denoting the spatial position along ξ-axis and $\Delta\xi$ is the step size along the spatial axis. The ξ - interval is divided into M points with $j = 1,2,\ldots,M-2,M$. Therefore, the MOL approximation of the equation (1) is given by

$$\frac{\partial u_j}{\partial \tau} = -\frac{\mu_1}{2\Delta\xi} u_j \left(u_{j+1} - u_{j-1}\right) + \frac{\mu_2}{(\Delta\xi)^2}\left(u_{j+1} - 2u_j + u_{j-1}\right)$$
$$-\frac{\mu_3}{2(\Delta\xi)^3}\left(u_{j+2} - 2u_{j+1} + 2u_{j-1} - u_{j-2}\right) - \frac{\mu_4(\tau)}{2\Delta\xi}\left(u_{j+1} - u_{j-1}\right)$$
$$+\mu(\tau) \equiv f(u_j). \tag{5}$$

Equation (5) is written as an ODE since there is only one independent variable, which is τ. Also, equation (5) represents a system of M equations of ODEs. The initial condition for equation (5) after discretization is given by

$$u(\xi_j, \tau = 0) = u_0(\xi_j), \qquad j = 1, 2, \ldots, M-1, M. \tag{6}$$

For the time integration, the RK4 method is applied. Thus, the numerical solution at time τ_{i+1} is

$$u_{i+1,j} = u_{i,j} + \frac{1}{6}\left(a_{i,j} + 2b_{i,j} + 2c_{i,j} + d_{i,j}\right), \tag{7}$$

where

$$a_{i,j} = \Delta\tau\, f(u_{i,j}),$$
$$b_{i,j} = \Delta\tau\, f\left(u_{i,j} + \frac{1}{2}a_{i,j}\right),$$
$$c_{i,j} = \Delta\tau\, f\left(u_{i,j} + \frac{1}{2}b_{i,j}\right),$$
$$d_{i,j} = \Delta\tau\, f\left(u_{i,j} + c_{i,j}\right). \tag{8}$$

Here $\Delta\tau$ is the step size of the temporal coordinates.

Progressive Wave Solution

The progressive wave solution of the FKdVB equation as given by (Tay, 2007) is

$$u = \frac{a}{\mu_1} + \frac{3\mu_2^2}{25\mu_1\mu_3}\left(\operatorname{sech}^2\zeta - 2\tanh\zeta\right) + \frac{1}{2}\left[g(\tau) - \frac{\lambda_\theta\lambda_1}{\beta_1}G(\tau)\right], \tag{9}$$

where a is a constant. The phase function ζ can be expressed as

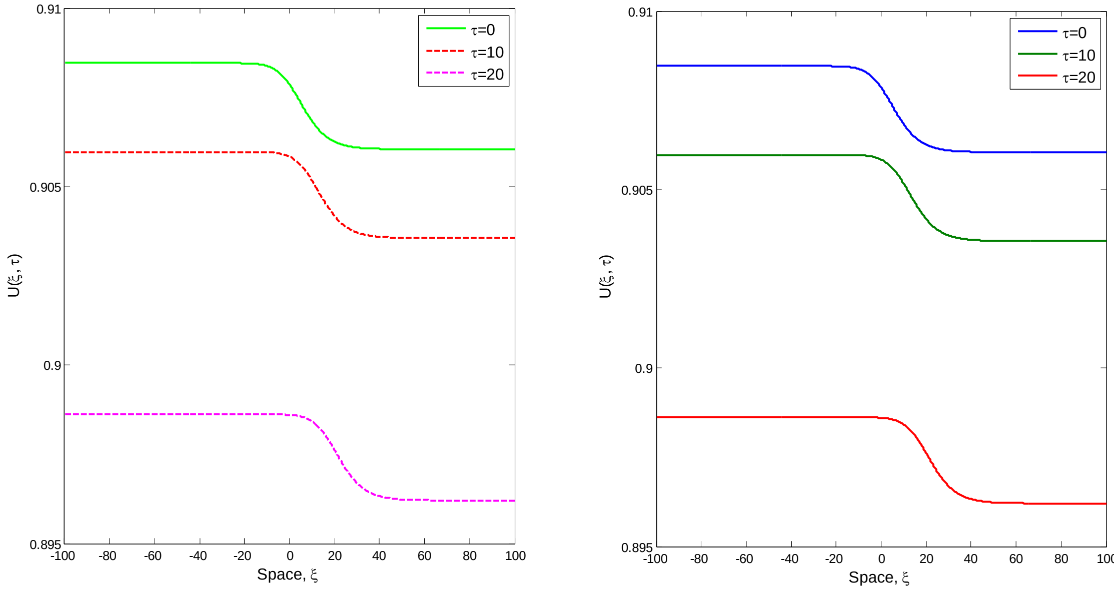
$$\zeta = \frac{\mu_2}{10\mu_3}\left\{\xi - a\tau - \int_0^\tau \left[\left(\frac{3}{4\lambda_\theta} - \frac{\beta_2}{2\beta_1}\right)g(s) + \frac{\lambda_\theta}{\beta_1}\left(\gamma_2 - \frac{\mu_1\gamma_1}{2}\right)G(s)\right]ds\right\}. \tag{10}$$

RESULTS AND DISCUSSION

To test MOL on the FKdVB equation, we need the initial condition as follows:

$$u(\xi,0) = \frac{a}{\mu_1} + \frac{3\mu_2^2}{25\mu_1\mu_3}\operatorname{sech}^2\left(\frac{\mu_2}{10\mu_3}\xi\right) - \frac{6\mu_2^2}{25\mu_1\mu_3}\tanh\left(\frac{\mu_2}{10\mu_3}\xi\right) + 0.5. \tag{11}$$

Figure 1 (a) gives the MOL solution of the FKdVB equation (1) with spatial parameters at certain time τ, while Figure 1 (b) represents the progressive wave solution of the FKdVB equation (1) with spatial parameters at certain time τ. The solution of the FKdVB equation (1) with space ξ shows a decreasing shock profile propagating to the right with a decrease in wave amplitude as time τ increases.



(a) the MOL solution (b) the progressive wave solution

FIGURE 1. Solutions of the FKdVB equation versus space ξ for different time τ at $\Delta\xi = 0.1$

We then computed the absolute error between the progressive wave and MOL solutions for each discretized spatial point at certain time τ and later find the maximum absolute error. The maximum absolute errors between the progressive wave and MOL solutions are calculated based on the formula:

$$L_\infty = _{\max} \left| U_{progressive} - U_{MOL} \right| . \tag{12}$$

Table a) gives the maximum absolute error between the progressive wave solution and MOL solution. It shows the maximum absolute errors are in orderof 10^{-3} .

TABLE a). Maximum absolute error of the FKdVB equation for different time τ at $\Delta\tau = \partial\xi^{-3}$..

Time, τ	**0**	**10**	**20**
L_∞	0	0.76506×10^{-6}	1×10^{-6}

CONCLUSION

The MOL was employed to solve the FKdVBequation. It involved replacing the spatial derivatives in the PDE with finite-difference approximations and by doing that, the spatial derivatives are longer stated explicitly in terms of spatial independent variables. This leads to a system of ODEs. The system is then solved by using the RK4 method. This paper describes the effect of computational effort with respect to the accuracy of results. The MOL solution of the FKdVB equation (1) is plotted versus its progressive wave solution. From the observation, it was found that there were no differences for both MOL and progressive wave solutions. The maximum absolute errors between both MOL and progressive wave solutions at certain time τ are computed. Results revealed that the maximum absolute errors are in the order of 10^{-6} for $\Delta\xi = 0.1$ and $\Delta\tau = 1\times10^{-3}$. Hence, it can be concluded thatthe FKdVB equationcan be solved successfully using the MOL.

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REFERENCES

1. A. G. Bratsos, *Journal of Computational and Applied Mathematics*, 206, 251-277 (2007).
2. M. Dehgan, *Mathematics and Computers in Simulation*, 71, 16-30 (2006).
3. H. Demiray, *Applied Mathematics and Computation*, 145, 179-184 (2003).
4. G. Hall andJ. M. Watt, Modern Numerical Methods for Ordinary Differential Equations. Clarendon Press, Oxford, 1976.
5. S. Hamdi et al.,*Journal of Computational and Applied Mathematics*, 183, 327–342(2005).
6. T. Koto, *Computers & Mathematics with Applications*, 48, 45-59 (2004).
7. A. M. Loeb andW. E. Schiesser, “Clarendon Press” in *Stiff Differential Systems*, edited by R. A. Willoughby., The IBM Research Symposia Series, Springer, New York, 1974.
8. P. Saucez et al.,*Journal of Computational and Applied Mathematics*,413-423 (2004).
9. W. E. Schiesser, *Computers & Mathematics with Applications*, 28, 147–154 (1994).
10. K. G. Tay, “Non-linear wave’s propagation in an elastic tube”,Ph.D. Thesis, UTM,2007.
11. H. A. Zedan, andE. E. Adrous, *Abstract and Applied Analysis*, 19 pages.(2012).