# ON SOME ABELIAN *p*-GROUPS AND THEIR CAPABILITY

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Abstract.     A group is said to be capable if it is a central factor group; equivalently, if and only if a group is isomorphic to the inner automorphism group of another group. In this research, the capability of some abelian *p*-groups which are groups of order $p^4$ and $p^5$, where *p* is an odd prime are determined. The capability of the groups is determined by using the classifications of the groups.

*Keywords*     Abelian groups; *p*-groups; capability

## 1.0   INTRODUCTION

Capability of groups was studied by many researchers. Baer [1] in 1938 introduced the capability of groups by determining the capability of finitely generated abelian groups. In the following years, Hall in [2] remarked that:

"The question of what conditions a group *G* must fulfill in order that it may be the central quotient group of another group *H*, $G \cong H/Z(H)$, is an interesting one. But while it is easy to write down a number of necessary conditions, it is not so easy to be sure that they are sufficient."

Later in 1964, Hall and Senior [3] established that a group *G* is called capable if and only if there exists a group *H* such that $G \cong H/Z(H)$.

Beyl *et al*. in [4] established a necessary condition for a group to be capable by using the epicenter of the group. They found that a group is capable if and only if the epicenter of the group is trivial. Meanwhile, Ellis [5] proved that a group is capable if and only if its exterior center is trivial.

Using the nonabelian tensor square, the capability for 2-generator finite *p*-groups of odd order and class 2 was determined by Bacon and Kappe [6] in 2003. Earlier in 2000, Beuerle and Kappe in [7] characterized the capability of infinite metacyclic groups by using nonabelian tensor square of the groups.

All capable *p*-groups with commutator subgroup of order *p* have been obtained by Parvizi and Niroomand in [8]. They proved the capability of these groups by using the Schur multiplier, epicenter, exterior center and abelianization of the groups. By using the same method, Rashid *et al*. (see [9,10]) determined all the capability for groups of composite orders of $p^2q$ and $p^3q$, where *p* and *q* are distinct prime and *p* < *q*.

Throughout the rest of this paper, *p* is denoted for an odd prime. The capability of nonabelian groups of order $p^4$ have been determined by Zainal *et al*. [11] in 2014. In their research, they only focused on the groups with the properties $|Z(G)| = p^2$, $|G'| = p$ and $G' \subseteq Z(G)$. As a continuation of those work, the capability of abelian groups of order $p^4$ and $p^5$ will be characterized in this research.

## 2.0   PRELIMINARIES

In this section, some preliminary results that are used in the computation of the capability of the abelian groups of order $p^4$ and $p^5$ are presented. The classification of these groups has been constructed by Burnside in [12].

The classification of abelian groups of groups of order $p^4$ is stated in the following theorem.

## Theorem 2.1 [12]

Let $G$ be an abelian group of order $p^4$. Then exactly one of the following holds:

$$G \cong \mathbb{Z}_{p^4}. \tag{1.1}$$

$$G \cong \mathbb{Z}_{p^3} \times \mathbb{Z}_p. \tag{1.2}$$

$$G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}. \tag{1.3}$$

$$G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p. \tag{1.4}$$

$$G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p. \tag{1.5}$$

The following theorem presents the classification of abelian groups of order $p^5$.

## Theorem 2.2 [12]

Let $G$ be an abelian group of order $p^5$. Then exactly one of the following holds:

$$G \cong \mathbb{Z}_{p^5}. \tag{1.6}$$

$$G \cong \mathbb{Z}_{p^4} \times \mathbb{Z}_p. \tag{1.7}$$

$$G \cong \mathbb{Z}_{p^3} \times \mathbb{Z}_{p^2}. \tag{1.8}$$

$$G \cong \mathbb{Z}_{p^3} \times \mathbb{Z}_p \times \mathbb{Z}_p. \tag{1.9}$$

$$G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2} \times \mathbb{Z}_p. \tag{1.10}$$

$$G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p. \tag{1.11}$$

$$G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p. \tag{1.12}$$

The following definition, proposition and theorem give preliminary results that will be used in order to prove the main results.

## Definition 2.1 [13]

A group $G$ is called capable if and only if there exists a group $H$ such that $G \cong H/Z(H)$; or equivalently, if and only if $G$ is isomorphic to the inner automorphism group of a group $H$.

**Proposition 2.1[13]**

Every nontrivial cyclic group is not capable.

**Theorem 2.3[14]**

Let $G$ be a finitely generated capable group. Then every central element $z$ in $G$ has order dividing $\exp\left(\left(G/\langle z\rangle\right)^{ab}\right)$.

**Theorem 2.4[4]**

$G$ is capable if and only if the natural map $M(G) \to M\left(G/\langle x\rangle\right)$ has a nontrivial kernel for all $1 \neq x \in Z(G)$.

## 3.0    RESULTS AND DISCUSSION

In this section, the capability of abelian groups of order $p^4$ is stated in Theorem 3.1. Meanwhile, the capability of abelian groups of order $p^5$ is stated in Theorem 3.2.

**Theorem 3.1**

Let $G$ be an abelian group of order $p^4$. Then $G$ is capable if $G$ is a group of Type (1.3) and Type (1.5).

**Proof:**

Since the group of Type (1.1) is nontrivial, then by Proposition 2.1, $G$ is not capable.

For the group of Type (1.2), the presentation of $G$ can be rewritten as $\left\langle a,b \mid a^{p^3} = b^p = 1, [a,b]=1\right\rangle$. The derived subgroup for this group is trivial $\left(G' = \langle[a,b]\rangle = 1\right)$ and the center of $G$, $Z(G) = G$. Let $z \in Z(G)$ such that $z$ is of order $p^3$ and thus $\left|G/\langle z\rangle\right| = p$. Since $G$ is abelian, then $G/\langle z\rangle$ is abelian and $G/\langle z\rangle$

is isomorphic to $\mathbb{Z}_p$. Since the order of $z$ does not divide $\exp\left(\left(G/\langle z \rangle\right)^{ab}\right)$, where $\exp\left(\left(G/\langle z \rangle\right)^{ab}\right) = p$. Thus, by Theorem 2.3, $G$ is not capable.

Next to prove the group of Type (1.3), i.e $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$ is capable. The Schur multiplier for this group is $M(G) = \mathbb{Z}_{p^2}$ and the $Z(G) = G$. Let $z \in Z(G)$ and $|z| = p$, it is found that $\left|G/\langle z \rangle\right| = p^3$. By choosing $G/\langle z \rangle \cong \left(\mathbb{Z}_p\right)^3$, it is found that $M\left(G/\langle z \rangle\right) \cong \left(\mathbb{Z}_p\right)^3$. Therefore, $M(G) \ncong M\left(G/Z(G)\right)/\left(G' \cap Z(G)\right)$ that is, the natural map $M(G) \to M\left(G/Z(G)\right)$ is not injective. Thus, the kernel of this map cannot be trivial. Therefore by Theorem 2.4, $G$ is capable.

For the group of Type (1.4), which is $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p$, let $|z| = p^2$, where $z \in Z(G)$ and $\left|G/\langle z \rangle\right| = p^2$. Since $G$ is abelian, therefore $G/\langle z \rangle$ is abelian. Since the order of $z$ does not divide $\exp\left(\left(G/\langle z \rangle\right)^{ab}\right)$, where $G/\langle z \rangle \cong \left(\mathbb{Z}_p\right)^2$ and $\exp\left(\left(G/\langle z \rangle\right)^{ab}\right) = p$. Thus, by Theorem 2.3, $G$ is not capable.

Lastly, for the group of Type (1.5), which is $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, where $z \in Z(G) = G$. By choosing $|z| = p$ and with similar proof with group of Type (1.3), it is clear that $\left|G/\langle z \rangle\right| = p^3$. It is found that $M\left(G/\langle z \rangle\right) \cong \left(\mathbb{Z}_p\right)^3$. Since $M(G) = \left(\mathbb{Z}_p\right)^6$, then $M(G) \ncong M\left(G/Z(G)\right)/\left(G' \cap Z(G)\right)$ that is, the natural map $M(G) \to M\left(G/Z(G)\right)$ is not injective. Thus, the kernel of this map cannot be trivial. Therefore by Theorem 2.4, $G$ is capable.

**Theorem 3.2**
Let $G$ be an abelian group of order $p^5$. Then $G$ is capable if $G$ is group of Type (1.10) and Type (1.12).

**Proof:**

Let $G$ be a nontrivial cyclic group, then by Proposition 2.1, $G$ is not capable. Hence, the group of Type (1.6) is not capable.

Next, we consider the group of Type (1.7), the derived subgroup for this group is trivial and the center of $G$, $Z(G) = G$. By choosing $z \in Z(G)$ and the order of $z$ is $p^4$. The order of $G/\langle z \rangle$ is equal to $p$ and $G/\langle z \rangle$ is isomorphic to cyclic group of order $p$. Thus, $\exp\left(\left(G/\langle z \rangle\right)^{ab}\right) = p$. Hence, the order of $z$ does not divide $\exp\left(\left(G/\langle z \rangle\right)^{ab}\right)$. Therefore, by Theorem 2.3, $G \cong \mathbb{Z}_{p^4} \times \mathbb{Z}_p$ is not capable.

For the group of Type (1.8), the proof is similar with the previous group. Let $z \in Z(G)$ and $|z| = p^3$, thus, the order of $G/\langle z \rangle$ is equal to $p^2$. Since $G$ is abelian, then $G/\langle z \rangle$ is abelian and it is isomorphic to $\mathbb{Z}_{p^2}$. So that, $\exp\left(\left(G/\langle z \rangle\right)^{ab}\right) = p^2$. Again in this case, $|z| \nmid \exp\left(\left(G/\langle z \rangle\right)^{ab}\right)$. Thus, by Theorem 2.3, $G$ is not capable.

Next, we prove that the group of Type (1.9) is not capable. For this group, we choose an element $z \in Z(G)$ and the order is equal to $p^3$. Thus, it implies that $\left|G/\langle z \rangle\right| = p^2$. The group $G/\langle z \rangle$ is abelian since $G$ is abelian. Thus $G/\langle z \rangle \cong \mathbb{Z}_{p^2}$. So that, we found that $\exp\left(\left(G/\langle z \rangle\right)^{ab}\right) = p^2$, implies that $|z| \nmid \exp\left(\left(G/\langle z \rangle\right)^{ab}\right)$. Therefore, by Theorem 2.3, $G$ is not capable.

Next, we prove that group of Type (1.10) is capable. Let $z \in Z(G)$ and the order of $z$ is equal to $p^2$. We have $G/\langle z \rangle$ of order $p^3$ and $G/\langle z \rangle$ is abelian, that is, $G/\langle z \rangle = \left(\mathbb{Z}_p\right)^3$. It is found that $M\left(G/\langle z \rangle\right) \cong \left(\mathbb{Z}_p\right)^3$. Since $M(G) = \mathbb{Z}_{p^2} \times \left(\mathbb{Z}_p\right)^2$, then $M(G) \ncong M\left(G/Z(G)\right)/\left(G' \cap Z(G)\right)$ that is, the natural map

$M(G) \to M(G/Z(G))$ is not injective. Thus, the kernel of this map cannot be trivial. Therefore by Theorem 2.4, $G$ is capable.

For the group of Type (1.11), the derived subgroup for this group is trivial and the center of $G$, $Z(G) = G$. Let $z \in Z(G)$ and $|z| = p$ and $|G/\langle z \rangle| = p^3$. Since $G$ is abelian, then $G/\langle z \rangle$ is abelian. Hence, $G/\langle z \rangle \cong \left( \mathbb{Z}_p \right)^3$, thus $\exp\left( \left( G/\langle z \rangle \right)^{ab} \right) = p$. Therefore, $|z| \nmid \exp\left( \left( G/\langle z \rangle \right)^{ab} \right)$. Therefore, by Theorem 2.3, $G$ is not capable.

Lastly, for the group of Type (1.12), i.e $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, where $z \in Z(G) = G$ and $|z| = p$. Note that, $M(G) = \left( \mathbb{Z}_p \right)^{10}$. it is found that, $|G/\langle z \rangle| = p^4$. By choosing $G/\langle z \rangle \cong \left( \mathbb{Z}_{p^2} \right)^2$ and $M(G/\langle z \rangle) \cong \mathbb{Z}_{p^2}$. Thus, $M(G) \ncong M(G/Z(G)) / (G' \cap Z(G))$ where the natural map $M(G) \to M(G/Z(G))$ is not injective. Thus, the kernel of this map cannot be trivial. Therefore by Theorem 2.4, $G$ is capable.

## 4.0    CONCLUSIONS

In this paper , the capability of the abelian groups of order $p^4$ and $p^5$, where $p$ is an odd prime has been determined. For the abelian groups of order $p^4$, $G$ is capable if $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$ or $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$. Meanwhile, for the abelian groups of order $p^5$, $G$ is capable if $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2} \times \mathbb{Z}_p$ or $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$. The proofs for these cases are also provided in this paper.

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