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SOLVING ROBIN PROBLEMS IN BOUNDED MULTIPLY CONNECTED REGIONS VIA AN INTEGRAL EQUATIONS AND DIFFERENTIAL EQUATIONS WITH THE GENERALIZED NEUMANN KERNEL

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Abstract. This paper presents a boundary integral equation method for finding the solution of Robin problems in bounded multiply connected regions. The Robin problems are formulated as a Riemann-Hilbert problems which lead to systems of integral equations and the related differential equations are also constructed that give rise to unique solutions are shown. Numerical results on several test regions are presented to illustrate that the approximate solution when using this method for the Robin problems when the boundaries are sufficiently smooth are accurate.

Keywords Robin problem; Riemann-Hilbert problem; Integral equation; Generalized Neumann kernel; Multiply connected region.

1.0 INTRODUCTION

This paper consider Laplace‘s equation $\Delta u = 0$ in bounded multiply connected regions Ω with a linear combination of Dirichlet and Neumann boundary condition on the boundary $\Gamma = \partial\Omega$, generally known as a mixed boundary value problem and commonly called the Robin problem.

Applications of mixed boundary value problem exist in large number in classical mathematical physics, physical geodesy, electro-magnetic, measurement ([1],[2]), and to specific boundary problems such as Dirichlet problem and Neumann problem [3]. The applications of the mixed boundary value problem in potential theory can be found in [4]. It has been shown that the problem of conformal mapping, Dirichlet problem, Neumann problem and a mixed Dirichlet-Neumann problem can all be treated as Riemann-Hilbert (briefly RH) problems as discussed in [5],[6], and [7]. The interplay of RH problems and boundary Fredholm integral equation with the generalized Neumann kernel has been investigated in [8] for simply connected regions with smooth and piecewise smooth boundaries and in [9] for bounded and unbounded multiply connected regions.

Earlier, the known integral equations for RH problem have been employed for solving Dirichlet problem and Neumann problem [5] and mixed Dirichlet-Neumann problem [6]. They are uniquely solvable Fredholm integral equations of the second kind. However, the problems in [5] and [6] are not Robin problem, since the Dirichlet condition and Neumann condition are given separately.

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$$\hat{\phi}(t) := \begin{cases} \hat{\phi}_0(t), & t \in J_0, \\ \hat{\phi}_1(t), & t \in J_1, \\ \vdots & \\ \hat{\phi}_m(t), & t \in J_m, \end{cases} \tag{3}$$

with real Hölder continuous 2π -periodic functions $\hat{\phi}_j$ defined on J_j.

From now on, for complex-valued or real-valued function $\psi \in H$ defined on the boundary Γ and for $t \in J$, we will not distinguish between $\psi(\eta(t))$ and $\psi(t)$. For $t \in J_k$, the values $\psi(t)$ will be denoted by $\psi_k(t)$.

For given functions $\alpha \in H$, $\beta \in H$, $l \in H$, a Robin problem is a boundary value problem for determining a harmonic function $u(x,y)$ harmonic in Ω and continuous on $\Omega \cup \Gamma$ and satisfies the Robin boundary condition [10]

$$\alpha(t)u(\eta(t)) + \beta(t)\frac{\partial u(\eta(t))}{\partial \mathbf{n}} = l(t) \quad \alpha(t), \beta(t) \neq 0, \eta(t) \in \Gamma, \tag{4}$$

where **n**is exterior normal to Γ. If $\frac{\alpha(t)}{\beta(t)} > 0$, then the Robin problem is uniquely solvable (see e.g [11, p. 163] and [12, p. 141]). In this paper, we shall relate the Robin problem with the RH problem. The RH problem consists of finding a function g analytic in Ω, continuous in its closure $\overline{\Omega}$ and has boundary values

$$\text{Re}[Ag] = \gamma, \tag{5}$$

where $\gamma \in H$ and $A(t) \in H$ is a complex continuously differentiable 2π -periodic function with $A \neq 0$ for all t.

The RH problem can be solved using a boundary integral equation with the generalized Neumann kernel. Define the real kernels M and N as real and imaginary parts [8], [9].

$$M(\tau,t) = \frac{1}{\pi}\text{Re}[\frac{A(\tau)}{A(t)}\frac{\dot{\eta}(t)}{\eta(t)-\eta(\tau)}], \qquad \tau \neq t, \tag{6}$$

$$N(\tau,t) = \frac{1}{\pi}\text{Im}[\frac{A(\tau)}{A(t)}\frac{\dot{\eta}(t)}{\eta(t)-\eta(\tau)}], \qquad \tau \neq t. \tag{7}$$

The kernel $N(\tau,t)$ is called the generalized Neumann kernel formed with A and η. When $A = 1$, it reduces to the classical Neumann kernel

$$N(\tau,t) = \frac{1}{\pi}\text{Im}[\frac{\dot{\eta}(t)}{\eta(t)-\eta(\tau)}], \qquad \tau \neq t. \tag{8}$$

The generalized Neumann kernel (7) is continuous at $t = \tau$ with

$$N(t,t) = \frac{1}{\pi}\text{Im}[\frac{1}{2}(\frac{\ddot{\eta}(t)}{\dot{\eta}(t)}) - \frac{\dot{A}(t)}{A(t)}]. \tag{9}$$

The kernel $M(\tau,t)$ has the representation

$$M(\tau,t) = -\frac{1}{2\pi}\cot\frac{\tau - t}{2} + M_1(\tau,t), \quad t \neq \tau, \tag{10}$$

with the continuous kernel M_1, which takes on the diagonal the values

$$M_1(t,t) = \frac{1}{\pi}\text{Re}[\frac{1}{2}(\frac{\ddot{\eta}(t)}{\dot{\eta}(t)}) - \frac{\dot{A}(t)}{A(t)}]. \tag{11}$$

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For details, see [8] and [9]. Let $\mathbf{N}$ and $\mathbf{M_1}$ be the Fredholm integral operators associated with the continuous kernels N and M_1, i.e.,

$$(\mathbf{N}\mu)(\tau) = \int_J N(\tau,t)\mu(t)dt, \qquad \tau \in J.$$

(12)

$$(\mathbf{M_1}\mu)(\tau) = \int_J M_1(\tau,t)\mu(t)dt, \qquad \tau \in J. \tag{13}$$

Let $\mathbf{M}$ and $\mathbf{K}$ be the singular integral operators

$$(\mathbf{M}\mu)(\tau) = \int_J M(\tau,t)\mu(t)dt, \qquad \tau \in J.$$

(14)

$$(\mathbf{K}\mu)(\tau) = \frac{1}{2\pi}\int_{J_k} \cot(\frac{\tau-t}{2})\mu(t)dt, \qquad \tau \in J_k, \quad k = 0,1,\cdots,m. \tag{15}$$

The integrals (14) and (15) are principal value integrals. The operator $\mathbf{K}$ is known as the conjugation operator. It is also known as the Hilbert transform [8]. It follows from equation (10) that

$$\mathbf{M} = \mathbf{M_1} - \mathbf{K}.$$

(16)

Theorem 1. ([8], [13]) *If g is a solution of the RH problem equation (5) with boundary values*

$$Ag = \gamma + \mathrm{i}\mu, \tag{17}$$

then the imaginary part μ in (17) satisfies the integral equation

$$\mu - \mathbf{N}\mu = -\mathbf{M}\gamma, \tag{18}$$

and the real part γ in (17) satisfies the integral equation

$$\gamma - \mathbf{N}\gamma = \mathbf{M}\mu. \tag{19}$$

The solvability of boundary integral equations with the generalized Neumann kernel is determined by the index (winding number in other terminology) of the function $A(t)$ [9].

Theorem 2. ([14] Cauchy Integral Formula) *Let f be a function that is analytic everywhere in Ω and on a simple closed contour Γ. Then the Cauchy integral formula is given by*

$$\frac{1}{2\pi i}\int_J \frac{f(\eta)}{\eta - z}d\eta = \begin{cases} f(z), & \text{if } z \in \Omega, \\ 0, & \text{if } z \notin \Omega. \end{cases} \tag{20}$$

1.2 Reduction Robin Problems in Bounded Multiply Connected Regions to RH problem

It can be shown that the Robin problem (4) cab be reformulated as [15]

$$\alpha(t)\operatorname{Re}[f(\eta(t))] - \beta(t)\operatorname{Re}[\frac{\mathrm{i}\dot{\eta}(t)f'(\eta(t))}{|\dot{\eta}(t)|}] = l(t). \tag{21}$$

Multiply both sides of (21) by $|\dot{\eta}(t)|$ we get,

$$\operatorname{Re}[-\mathrm{i}\beta(t)\left\{\frac{d}{dt}(f(\eta(t))) + \mathrm{i}e(t)f(\eta(t))\right\}] = l(t)|\dot{\eta}(t)|, \tag{22}$$

where

$$e(t) = \frac{\alpha(t)}{\beta(t)}|\dot{\eta}(t)|, \qquad t \in J. \tag{23}$$

Hence

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$$-\mathrm{i}\beta(t)[\frac{d}{dt}(f(\eta(t)))+\mathrm{i}e(t)f(\eta(t))]=l(t)\,|\,\dot{\eta}(t)\,|+\mathrm{i}\mu(t), \qquad t\in J, \tag{24}$$

where $\mu(t)\in H$ is an unknown function. By means of integrating factor, we obtain

$$-\mathrm{i}\beta(t)A(t)[\frac{d}{dt}(f(\eta(t)))+\mathrm{i}e(t)f(\eta(t))]=l(t)\,|\,\dot{\eta}(t)\,|\,e^{\mathrm{i}\zeta(t)}+\mathrm{i}\mu(t)e^{\mathrm{i}\zeta(t)}, \qquad t\in J, \tag{25}$$

$$A(t)=e^{\mathrm{i}\int_0^t e(\tau)d\tau}=e^{\mathrm{i}\zeta(t)}, \tag{26}$$

$$\zeta(t)=\int_0^t e(\tau)d\tau. \tag{27}$$

Then equations (25) becomes

$$\frac{d}{dt}[A(t)(-\mathrm{i})(f(\eta(t)))]=\frac{l(t)\,|\,\dot{\eta}(t)\,|\,e^{\mathrm{i}\zeta(t)}+\mathrm{i}\mu(t)e^{\mathrm{i}\zeta(t)}}{\beta(t)}.$$

(28)

Letting $g=-\mathrm{i}f$, which is analytic on Ω, we obtain

$$\frac{d}{dt}[A(t)g(\eta(t))]=\frac{l(t)\,|\,\dot{\eta}(t)\,|\,e^{\mathrm{i}\zeta(t)}+\mathrm{i}\mu(t)e^{\mathrm{i}\zeta(t)}}{\beta(t)}. \tag{29}$$

Hence, it follows from (29) that

$$\begin{aligned} A(t)g(\eta(t)) &= \int_0^t \frac{l(\tau)\,|\,\dot{\eta}(\tau)\,|\cos\zeta(\tau)}{\beta(\tau)}d\tau+\mathrm{i}\int_0^t \frac{l(\tau)\,|\,\dot{\eta}(\tau)\,|\sin\zeta(\tau)}{\beta(\tau)}d\tau \\ &+\mathrm{i}\int_0^t \frac{\mu(\tau)\cos\zeta(\tau)}{\beta(\tau)}d\tau-\int_0^t \frac{\mu(\tau)\sin\zeta(\tau)}{\beta(\tau)}d\tau+c_1+\mathrm{i}c_2 \\ &=\gamma_1(t)+\mathrm{i}\gamma_2(t)+\mathrm{i}\mu_1(t)-\mu_2(t)+c_1+\mathrm{i}c_2 \\ &=(\gamma_1(t)-\mu_2(t)+c_1)+\mathrm{i}(\gamma_2(t)+\mu_1(t)+c_2), \qquad t\in J, \end{aligned} \tag{30}$$

where c_1, c_2, are unknown piecewise real constants in *H*, and

$$\gamma_1(t):=\int_0^t \frac{l(\tau)\,|\,\dot{\eta}(\tau)\,|\cos\zeta(\tau)}{\beta(\tau)}d\tau, \qquad t\in J, \tag{31}$$

$$\gamma_2(t):=\int_0^t \frac{l(\tau)\,|\,\dot{\eta}(\tau)\,|\sin\zeta(\tau)}{\beta(\tau)}d\tau, \qquad t\in J, \tag{32}$$

are known functions in *H*, and

$$\mu_1(t):=\int_0^t \frac{\mu(\tau)\cos\zeta(\tau)}{\beta(\tau)}d\tau, \qquad t\in J, \tag{33}$$

$$\mu_2(t):=\int_0^t \frac{\mu(\tau)\sin\zeta(\tau)}{\beta(\tau)}d\tau, \qquad t\in J, \tag{34}$$

are unknown functions in *H*.

Then we can write (30) briefly as

$$A(t)g(t)=\gamma(t)+\mathrm{i}\mu(t), \qquad t\in J, \tag{35}$$

where $g(t)=g(\eta(t))$,

$$\gamma(t)=\gamma_1(t)-\mu_2(t)+c_1, \tag{36}$$

$$\mu(t)=\gamma_2(t)+\mu_1(t)+c_2. \tag{37}$$

The real part of (35) yields the RH problem. The function $A(t)=e^{\mathrm{i}\zeta(t)}$ is in general not periodic. To apply the result of Theorem 1, $A(t)$ must be periodic. The function $A(t)$ is periodic if we assume $\zeta(2\pi)-\zeta(0)=2\pi$. Thus, Theorem 1 implies that (35) can be reformulated as

$$(\mathbf{I}-\mathbf{N})(\gamma_2(t)+\mu_1(t)+c_2)=-\mathbf{M}(\gamma_1(t)-\mu_2(t)+c_1), \qquad t\in J, \tag{38}$$

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$$(\mathbf{I}-\mathbf{N})(\gamma_1(t)-\mu_2(t)+c_1)=\mathbf{M}(\gamma_2(t)+\mu_1(t)+c_2), \qquad t\in J. \tag{39}$$

It can be shown that the above two integral equations are linearly independent [15].

Theorem 3. *Let* $\frac{\alpha(t)}{\beta(t)}>0$. *Then the following system of integral equations are linearly independent*

$$(\mathbf{I}-\mathbf{N})x=-\mathbf{M}y, \tag{40}$$

$$(\mathbf{I}-\mathbf{N})y=\mathbf{M}x \tag{41}$$

for $x,y\in H$.

Equations (38) and (39) imply

$$(\mathbf{I}-\mathbf{N})\mu_1(t)-\mathbf{M}\mu_2(t)+\mathbf{M}c_1+(\mathbf{I}-\mathbf{N})c_2=-\mathbf{M}\gamma_1(t)-(\mathbf{I}-\mathbf{N})\gamma_2(t), \qquad t\in J, \tag{42}$$

$$-\mathbf{M}\mu_1(t)-(\mathbf{I}-\mathbf{N})\mu_2(t)+(\mathbf{I}-\mathbf{N})c_1-\mathbf{M}c_2=-(\mathbf{I}-\mathbf{N})\gamma_1(t)+\mathbf{M}\gamma_2(t), \qquad t\in J. \tag{43}$$

Applying the definitions of the integral operators $\mathbf{N}$ and $\mathbf{M}$, we get

$$\begin{aligned}&\mu_1(\tau)-\int_J N(\eta(\tau),\eta(t))\mu_1(t)dt-\int_J M(\eta(\tau),\eta(t))\mu_2(t)dt\\&+\int_J M(\eta(\tau),\eta(t))c_1dt+(c_2-\int_J N(\eta(\tau),\eta(t))c_2dt)\\&=-\int_J M(\eta(\tau),\eta(t))\gamma_1(t)dt-(\gamma_2(\tau)-\int_J N(\eta(\tau),\eta(t))\gamma_2(t)dt), \qquad \tau\in J,\end{aligned} \tag{44}$$

$$\begin{aligned}&-\int_J M(\eta(\tau),\eta(t))\mu_1(t)dt-\mu_2(\tau)+\int_J N(\eta(\tau),\eta(t))\mu_2(t)dt\\&+(c_1-\int_J N(\eta(\tau),\eta(t))c_1dt)-\int_J M(\eta(\tau),\eta(t))c_2dt\\&=-(\gamma_1(\tau)-\int_J N(\eta(\tau),\eta(t))\gamma_1(t)dt)+\int_J M(\eta(\tau),\eta(t))\gamma_2(t)dt, \qquad \tau\in J.\end{aligned} \tag{45}$$

The system of integral equations (44) and (45) is in two unknown real functions $\mu_1(t),\mu_2(t)$ and two unknown real constants c_1,c_2. Furthermore, by the definitions of the functions $\mu_1(t)$, $\mu_2(t)$ given in (33), (34), we have a condition in the form of a differential equation

$$\sin(\zeta(t))\mu_1'(t)-\cos(\zeta(t))\mu_2'(t)=0, \qquad t\in J, \tag{46}$$

and the conditions

$$\mu_1(0)=0, \qquad \text{and} \qquad \mu_2(0)=0. \tag{47}$$

For bounded region Ω, since $\operatorname{Im} f(0)=0$, then $\operatorname{Re}[g(0)]=0$. Applying the Cauchy integral formula for bounded region Ω in Theorem 2 together with (35), (36) and (37), yields

$$\begin{aligned}g(0)&=\frac{1}{2\pi\mathrm{i}}\int_\Gamma\frac{g(\eta)}{\eta}d\eta=\frac{1}{2\pi\mathrm{i}}\int_\Gamma\frac{g(\eta(t))}{\eta(t)}\dot{\eta}(t)dt\\&=\frac{1}{2\pi\mathrm{i}}\int_J\frac{[\gamma_1(t)-\mu_2(t)+c_1+\mathrm{i}(\gamma_2(t)+\mu_1(t)+c_2)]\dot{\eta}(t)}{A(t)\eta(t)}dt, \qquad t\in J.\end{aligned} \tag{48}$$

Thus $\operatorname{Re}[g(0)]=0$ implies that

$$\int_J[(\gamma_1(t)-\mu_2(t)+c_1)\operatorname{Im}\frac{\dot{\eta}(t)}{A(t)\eta(t)}+(\gamma_2(t)+\mu_1(t)+c_2)\operatorname{Re}\frac{\dot{\eta}(t)}{A(t)\eta(t)}]dt=0, \quad t\in J, \tag{49}$$

which gives rise to the following condition for $\mu_1(t),\mu_2(t),c_1$ and c_2:

$$\int_J \mathrm{Re}\frac{\dot{\eta}(t)}{A(t)\eta(t)}\mu_1(t)dt - \int_J \mathrm{Im}\frac{\dot{\eta}(t)}{A(t)\eta(t)}\mu_2(t)dt + \int_J \mathrm{Im}\frac{\dot{\eta}(t)}{A(t)\eta_0(t)}c_1 dt$$
$$+\int_J \mathrm{Re}\frac{\dot{\eta}(t)}{A(t)\eta(t)}c_2 dt = -\int_J \mathrm{Im}\frac{\dot{\eta}(t)}{A(t)\eta(t)}\gamma_1(t)dt - \int_J \mathrm{Re}\frac{\dot{\eta}(t)}{A(t)\eta_0(t)}\gamma_2(t)dt. \quad (50)$$

In summary to solve the Robin problem (4) on bounded multiply connected regions, we solve for $\mu_1(t)$, $\mu_2(t)$, c_1 and c_2 from (44),(45), (47) and (50). Then we compute $g(\eta(t))$ from (30). Using the relation $g=-\mathrm{i}f$ we get $f(\eta(t))=\mathrm{i}g(\eta(t))$ and hence $u(\eta(t))=\mathrm{Re}[f(\eta(t))]$. The interior values of f can be computed via the Cauchy integral formula (20) which yields $u(z)=\mathrm{Re}[f(z)]$ for $z\in\Omega$.

1.3 Numerical Implementation

Since the functions $A(t)$ and $\eta(t)$ are $2\pi-$periodic, the integral equations in (42) and (43) can be best discretized on an equidistant grid by the Nyström method with trapezoidal rule using n equidistant nodes [16]. Since $\mathbf{M}=\mathbf{M_1}-\mathbf{K}$, the integrals involving the singular kernel $K(\tau,t)$ are discretized using Wittich method [17]. Hence, we obtain $2mn$ equations in $2mn+2m$ variables $(m=1,2,\ldots)$

$$\mu_{01}(t_1),\mu_{01}(t_2),\ldots,\mu_{01}(t_n),\ldots,\mu_{0m}(t_1),\mu_{0m}(t_2),\ldots,\mu_{0m}(t_n),$$
$$\mu_{11}(t_1),\mu_{11}(t_2),\ldots,\mu_{11}(t_n),\ldots,\mu_{1m}(t_1),\mu_{1m}(t_2),\ldots,\mu_{1m}(t_n),c_{01},\ldots,c_{0m},c_{11},\ldots,c_{1m}. \quad (51)$$

The condition (46) is discretized using a five-point central difference method [18] to obtain mn equations. We now have $3mn$ equations in $2mn+2m$ in unknowns (82). For the Robin problem on bounded multiply connected region, discretizing condition (50) gives one more equation. We also have $2m$ conditions from (47). Finally, we obtained $(3mn+2m+1)$ by $(2mn+2m)$ over-determined linear system which is full ranked.

In this case, the obtained over-determined systems are solved using the MATLAB's \ operator that makes use the QR factorization with column pivoting method.

From the computed solutions $\mu_1(t),\mu_2(t),\gamma_1(t),\gamma_2(t),c_1$ and c_2, the approximate boundary values of the analytic function $f_n(\eta(t))$ are calculated using the formula

$$f_n(\eta(t))=\frac{-(\gamma_2(t)+\mu_1(t)+c_2)+\mathrm{i}(\gamma_1(t)-\mu_2(t)+c_1)}{e^{\mathrm{i}\varsigma(t)}}. \quad (52)$$

The approximate interior values of the function $f(z)$ are calculated via the Cauchy integral formula in the form of

$$f(z)=\frac{\frac{1}{2\pi\mathrm{i}}\int_J \frac{f(\eta)}{\eta-z}d\eta}{\frac{1}{2\pi\mathrm{i}}\int_J \frac{d\eta}{\eta-z}}. \quad (53)$$

The formula in (53) has the advantage that the denominators in this formula compensate for the error in the numerators [19], [20]. The integrals in (53) are approximated by the trapezoidal rule.

1.4 Numerical Example

We consider an example of solving Robin problem with Robin boundary condition (4) in bounded multiply connected regions.

Example *Consider a bounded 4-multiply connected region* Ω *bounded by*

$$\Gamma_0 : \eta_0(t) = 1.5\cos t + \mathrm{i}\ \sin t,$$
$$\Gamma_1 : \eta_1(t) = -0.5 + 0.3\mathrm{i} + 0.3\,e^{-\mathrm{i}t},$$
$$\Gamma_2 : \eta_2(t) = 0.5 + 0.3\mathrm{i} + 0.3\,e^{-\mathrm{i}t},$$
$$\Gamma_3 : \eta_3(t) = -0.35\mathrm{i} + 0.3\,e^{-\mathrm{i}t}, \qquad 0 \le t \le 2\pi.$$

For $\alpha(t)$, $\beta(t)$, $u(t)$ in (4), we choose

$$\alpha(t) := \begin{cases} 2 - 0.4\cos t, & t \in J_o, \\ 1 - 0.4\cos t + 0.04\cos^2 t, & t \in J_1, \\ 0.96 + 0.04\sin^2 t, & t \in J_2. \\ 10 - 2\cos t, & t \in J_3. \end{cases} \quad \text{and} \quad \beta(t) := \begin{cases} \sqrt{5\sin^2 t + 4}, & t \in J_o, \\ 3 - 0.6\cos t, & t \in J_1, \\ 0.3 + 0.06\cos t, & t \in J_2. \\ 3, & t \in J_3. \end{cases}$$

The function $l(t)$

$$l(t) = \begin{cases} (2 - 0.4\cos t)(2.25\cos^2 t - \sin^2 t - 2) + (1.5\cos 2t), & t \in J_0, \\ (10 - 4\cos t + 0.4\cos^2 t)(-1.84 - 0.3\cos t + 0.09\cos 2t + 0.18\sin t) + \\ (3 - 0.6\cos t)(\cos t - 0.6\sin t - 0.6\cos 2t) & t \in J_1, \\ (0.96 + 0.04\sin^2 t)(-1.84 + 0.3\cos t + 0.09\cos 2t + 0.18\sin t) + \\ (0.3 - 0.06\cos t)(-\cos t - 0.6\sin t - 0.6\cos 2t) & t \in J_2, \\ (1 - 0.2\cos t)(-2.1225 + 0.09\cos 2t + 0.21\sin t) + \\ 0.21\sin t - 0.18\cos 2t, & t \in J_3, \end{cases}$$

in (4) is obtained by choosing the exact solution $u(z) = \mathrm{Re}[f(z)]$

$$u(t) = \begin{cases} 2.25\cos^2 t - \sin^2 t - 2, & t \in J_o, \\ -1.84 - 0.3\cos t + 0.09\cos 2t + 0.18\sin t, & t \in J_1, \\ -1.84 + 0.3\cos t + 0.09\cos 2t + 0.18\sin t, & t \in J_2, \\ -2.1225 + 0.09\cos 2t + 0.21\sin t, & t \in J_3. \end{cases}$$

where $f(z) = z^2 - 2$. This yields the exact values

$$c_1 = \begin{cases} 0, & t \in J_o, \\ -0.12, & t \in J_1, \\ 0.48, & t \in J_2, \\ -0.21, & t \in J_3, \end{cases} \quad \text{and} \quad c_2 = \begin{cases} -0.25, & t \in J_o, \\ 2.05, & t \in J_1, \\ 1.45, & t \in J_2, \\ 2.0325, & t \in J_3. \end{cases}$$

For this example, $A(t) = e^{\mathrm{i}(t - 0.2\sin t)}, t \in J_j = [0, 2\pi]$. The integrals in (36) and (37) are calculated by the Gauss-Legendre rule with 256 nodes.

Table 1 lists the maximum error norm $\| u(\eta(t)) - u_n(\eta(t)) \|_\infty$, where n is the number of nodes and $u_n(\eta(t))$ is the numerical approximation of $u(\eta(t))$ based on our method. The error $\| f(z) - f_n(z) \|$ at some selected points are listed in Table 2. The absolute error $| u(z) - u_n(z) |$ for selected points in the entire domain are plotted in Figure 2. Figure 3 shows the surface plot of $u_n(z)$ with n=1024.

Table 1. The error $\| u(\eta(t)) - u_n(\eta(t)) \|_\infty$ on boundary Γ.

n	$\| u(\eta(t)) - u_n(\eta(t)) \|_\infty$

32	9.00 (-3)
64	6.33 (-4)
128	4.41 (-5)
256	2.86 (-6)
512	1.91 (-7)
1024	1.28 (-8)

Table 2. Absolute error $|f(z)-f_n(z)|$ at some selected points on Ω.

n	-1-0.2i	-0.5-0.6i	0.2+0.6i	1.2-0.3i
32	1.0 (-3)	1.3 (-3)	1.9 (-3)	2.5 (-4)
64	8.80 (-5)	6.89 (-5)	1.17 (-4)	1.13 (-5)
128	5.53(-6)	4.38 (-6)	7.33 (-6)	6.98 (-6)
256	3.16(-7)	3.00 (-7)	4.69 (-7)	4.68 (-7)
512	1.77 (-8)	2.07 (-8)	3.03 (-8)	3.16 (-8)
1024	9.86 (-9)	1.41 (-9)	1.97 (-9)	2.08 (-9)

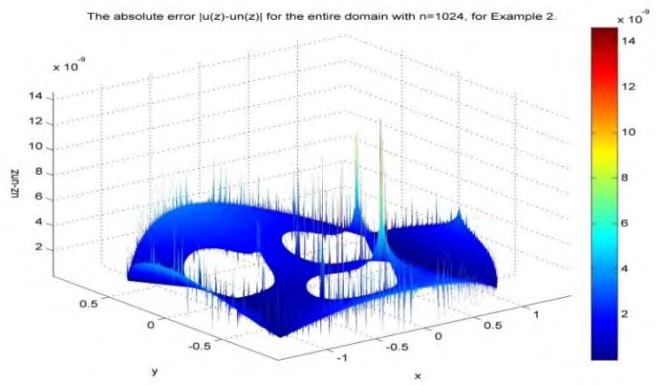

Figure 2 The absolute error $|u(z) - u_n(z)|$ for the entire domain with n= 1024.

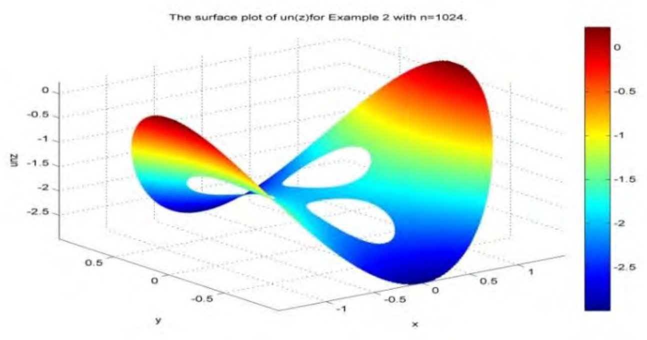

Figure 3 The surface plot of $u_n(z)$ with n= 1024.

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THE n^{th} COMMUTATIVITY DEGREE OF NONABELIAN GROUPS OF ORDER 16 WITH A CYCLIC SUBGROUP OF INDEX FOUR

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Abstract. The commutativity degree of a group is the probability that two randomly selected elements in the group commute. Furthermore, the n^{th} commutativity degree of a group is defined as the probability that the n^{th} power of a random element commutes with another random element of the same group. The commutativity degree of nonabelian groups of order 16 with a cyclic subgroup of index four has been determined in 2015. In this paper, the n^{th} commutativity degree of the nonabelian groups of order 16 with a cyclic subgroup of index four are determined.

Keywords. commutativity degree; n^{th} commutativity degree; cyclic subgroup of index four.

1.0 INTRODUCTION

Probabilistic theory is the branch of mathematics concerned with probability and the analysis of random phenomena. Recently, probabilistic theory shown up to be useful in the solution of some difficult tasks in group theory. A group G is called abelian if any two elements in G with binary operation $*$ commute that is, $a*b=b*a$ for all a,b in G. However, not all groups are abelian, in which they are called nonabelian groups.

Miler [1] was the first person introduced the concept of commutativity degree of a finite group in his paper in 1944. This commutativity degree can be written as

$$P(G)=\frac{|\{(x,y)\in G\times G \mid xy=yx\}|}{|G|^2}.$$