The Conjugate Graph and Conjugacy Class Graph of Order at Most 32

Nurul Huda Bilhikmah & Nor Haniza Sarmin

ABSTRACT

A groupis called metacyclic if it has a cyclic normal subgroup such that the quotient groupis also cyclic. The classification of non-Abelianmetacyclic*p*-groups of class two has been found by earlier researcher, which is partitioned into two families of non-isomorphic *p*-groups. The conjugacy classes of these groups are then applied into graph theory. The conjugate graph is a graph whose the vertices are non-central elements of a finite non-Abelian group. Besides, the conjugacy class graph is a graph whose vertices are non-central of a group that is two vertices are connected if their cardinalities are not coprime, in which their greatest common divisor between the vertices is not equal to one. In this study, the conjugacy classes of the metacyclic 2-groups of order at most 32 have been obtained using the definition of conjugacy classes and their group presentations. The conjugate graph and conjugacy class graph of metacyclic 2-groups of order at most 32 are found directly using the definition. These conjugate graph and conjugacy class graph are then used to determine some graph properties such as chromatic number, clique number, dominating number and independent number. The conjugate graph of the groups turned out to be union of complete components of K_2, meanwhile the conjugacy class graph of the groups turned out to be a complete graph.

Keywords: Metacyclic group, conjugacy classes, graph, chromatic number, clique number, dominating number, independent number.

INTRODUCTION

Recent years have seen an increase demand for the application of mathematics. Graph theory has proven to be particularly useful to a large number of rather diverse fields. The exciting and rapidly growing area of graph theory is rich in theoretical results as well as applications to real-world problems. With the increasing importance of the computer, there has been a significant movement away from the traditional calculus courses and toward courses on discrete mathematics, including graph theory.

Many real-life situation can be described by means of a diagram consisting of a set of points with lines joining certain pairs of points. In the recent years, many studies tend to find a relation between group theory and graph theory and that led to introduce numerous type of graphs.

Previously the presentation of metacyclic*p*-groups of nilpotency class two has been found by Beuerle (Beuerle, 2005). Extended to that, this research are trying to find the conjugacy classes of metacyclic2-groups of order at most 32 and their related graphs. The objectives of this research are to find the conjugate graph and conjugacy class graph of metacyclic 2-groups of order at most 32 and their graph's properties. The scope of study will consider three groups in the classification of metacyclic 2-groups of nilpotency class two, their conjugate graph and conjugacy class graph and then their graph properties which include the chromatic number, the clique number, the independent number and the dominating number.

LITERATURE REVIEW

A group G is called metacyclic if it has a cyclic normal subrgroup$H$ such that the quotient group G/H is also cyclic (Reis, 2011). Whereas, in 2013, Erfanian and Tolue introduced a new graph called conjugate graph in which its vertices are the non-central elements of a group. Two vertices of this graph are adjacent if they are conjugate (Erfanian and Tolue, 2013). Another graph which was introduced by Bertram et al. in 1990 is called a conjugacy class graph. This graph is defined as the set of non-central classes of a group. Two distinct vertices are connected by an edge if the greatest common divisor of the size of the conjugacy classes is greater than one (Bertram et al., 1990).

Definition 2.1 (Dummit et al., 2004)

Let a and b be two elements in a finite group G, then a and b are called **conjugate** if there exists an element $g$in G such that $gag^{-1} = b$.

Definition 2.2 (Goodman, 2003)

Let $x \in G$. The **conjugacy class**of x is the set $\mathrm{cl}(x) = \{axa^{-1} | a \in G\}$.In this research, the notation $K(G)$ is used for the number of conjugacy classes in G, while $Z(G)$ is used for the center group G.

Definition 2.4 (Goodman, 2003)

A **presentation** of group is a description of the form $G \cong \langle A|B \rangle$. Where A is the generator and B is the relation.

Definition 2.5 (Bondy and Murty, 1982)

A **graph** is a mathematical structure consisting of two sets namely vertices and edges which are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively.

Definition 2.6 (Bondy and Murty, 1982)

A **subgraph** of a graph Γ is a graph whose vertices and edges are subset of the vertices and edges of Γ.

Definition 2.7 (Godsil and Royle, 2001)

A **complete graph** is a graph where each order pair of distinct vertices are adjacent, and it's denoted by K_n.

Definition 2.8 (Bondy and Murty, 1982)

A non-empty set S of $V(\Gamma)$ is called an **independent set** of Γ. While the **independent number** is the number of vertices in maximum independent set and it's denoted by $\alpha(\Gamma)$.

Definition 2.9 (Bondy and Murty, 1982)

The **chromatic number** is the smallest number of colors needed to color the vertices of Γ so that no two adjacent vertices share the same color and denoted by χ(Γ).

Definition 2.10 (Erfanian and Tolue, 2012)

Clique is a complete subgraphin Γ. While the **clique number** is the size of the largest clique in Γ and it's denoted by ω(Γ).

Definition 2.11 (Bondy and Murty, 1982)

The **dominating set** X ⊆ V(Γ) is a set where for each voutside X, ∃x ∈ X such that v adjacent to x. The minimum size of X is called the **dominating number** and it's denoted by γ(Γ).

MAIN RESULTS

In this section the conjugacy classes of $\boldsymbol{G_1} \cong \langle \boldsymbol{a}, \boldsymbol{b} : \boldsymbol{a^8} = \boldsymbol{e}, \boldsymbol{b^2} = \boldsymbol{e}, [\boldsymbol{a}, \boldsymbol{b}] = \boldsymbol{a^4} \rangle, \boldsymbol{G_2} \cong \langle \boldsymbol{a}, \boldsymbol{b} : \boldsymbol{a^{16}} = \boldsymbol{e}, \boldsymbol{b^2} = \boldsymbol{e}, [\boldsymbol{a}, \boldsymbol{b}] = \boldsymbol{a^8} \rangle$, and $\boldsymbol{G_3} \cong \langle \boldsymbol{a}, \boldsymbol{b} : \boldsymbol{a^4} = \boldsymbol{e}, \boldsymbol{b^2} = [\boldsymbol{a}, \boldsymbol{b}] = \boldsymbol{a^{-1}} \rangle$are found by directly using the definition of conjugacy class.

Then, theconjugacy classes of G_1 are found and listed as follows:

1. $\text{cl}(e) = \{e\}$,
2. $\text{cl}(a) = \{a, a^5\} = \text{cl}(a^5)$,
3. $\text{cl}(a^2) = \{a^2\}$,
4. $\text{cl}(a^3) = \{a^3, a^7\} = \text{cl}(a^7)$,
5. $\text{cl}(a^4) = \{a^4\}$,
6. $\text{cl}(a^6) = \{a^6\}$,
7. $\text{cl}(b) = \{b, a^4b\} = \text{cl}(a^4b)$,
8. $\text{cl}(ab) = \{ab, a^5b\} = \text{cl}(a^5b)$,
9. $\text{cl}(a^2b) = \{a^2b, a^6b\} = \text{cl}(a^6b)$,
10. $\text{cl}(a^3b) = \{a^3b, a^7b\} = \text{cl}(a^7b)$.

Hence,$K(G_1) = 10$ and $|Z(G_1)| = 4$.

The conjugacy classes of G_2 are listed as follows:

1. $\text{cl}(e) = \{e\}$,
2. $\text{cl}(a) = \{a, a^9\} = \text{cl}(a^9)$,
3. $\text{cl}(a^2) = \{a^2\}$,
4. $\text{cl}(a^3) = \{a^3, a^{11}\} = \text{cl}(a^{11})$,

5. $\mathrm{cl}(a^4) = \{a^4\}$,
6. $\mathrm{cl}(a^5) = \{a^5, a^{13}\} = \mathrm{cl}(a^{13})$,
7. $\mathrm{cl}(a^6) = \{a^6\}$,
8. $\mathrm{cl}(a^7) = \{a^7, a^{15}\} = \mathrm{cl}(a^{15})$,
9. $\mathrm{cl}(a^8) = \{a^8\}$,
10. $\mathrm{cl}(a^{10}) = \{a^{10}\}$,
11. $\mathrm{cl}(a^{12}) = \{a^{12}\}$,
12. $\mathrm{cl}(a^{14}) = \{a^{14}\}$,
13. $\mathrm{cl}(b) = \{b, a^8b\} = \mathrm{cl}(a^8b)$,
14. $\mathrm{cl}(ab) = \{ab, a^9b\} = \mathrm{cl}(a^9b)$,
15. $\mathrm{cl}(a^2b) = \{a^2b, a^{10}b\} = \mathrm{cl}(a^{10}b)$,
16. $\mathrm{cl}(a^3b) = \{a^3b, a^{11}b\} = \mathrm{cl}(a^{11}b)$,
17. $\mathrm{cl}(a^4b) = \{a^4b, a^{12}b\} = \mathrm{cl}(a^{12}b)$,
18. $\mathrm{cl}(a^5b) = \{a^5b, a^{13}b\} = \mathrm{cl}(a^{13}b)$,
19. $\mathrm{cl}(a^6b) = \{a^6b, a^{14}b\} = \mathrm{cl}(a^{14}b)$,
20. $\mathrm{cl}(a^7b) = \{a^7b, a^{15}b\} = \mathrm{cl}(a^{15}b)$.

Hence,$K(G_2) = 20$ and $|Z(G_2)| = 8$.

Lastly, the conjugacy classes of G_3 are listed as follows:

1. $\mathrm{cl}(e) = \{e\}$,
2. $\mathrm{cl}(a) = \{a, a^3\} = \mathrm{cl}(a^3)$,
3. $\mathrm{cl}(a^2) = \{a^2\}$,
4. $\mathrm{cl}(b) = \{b, a^2b\} = \mathrm{cl}(a^2b)$,
5. $\mathrm{cl}(ab) = \{ab, a^3b\} = \mathrm{cl}(a^3b)$.

Hence,$K(G_3) = 5$and $|Z(G_3)| = 2$.

Then after all the conjugacy classes are found, the next step is to find the conjugate graph by using the definition of conjugate graph, the conjugate graphs of three groups are given as belows:

The Conjugate Graph of G_1:

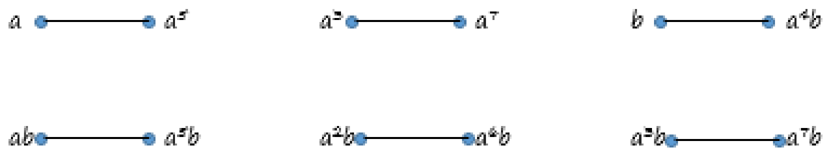

Figure 3.1 : The Conjugate Graph of G_1.
$\boldsymbol{\Gamma(G_1) = K_2 \cup K_2 \cup K_2 \cup K_2 \cup K_2 \cup K_2}$.

The properties:

a) The independent number of conjugate graph of G_1, $\alpha(G_1) = 6$.

b) The chromatic number of conjugate graph of G_1, $\chi(G_1) = 2$.

c) The clique number of conjugate graph ofG_1, $\omega(G_1) = 2$.

d) The dominating number of conjugate graph ofG_1, $\gamma(G_1) = 1$.

The Conjugate Graph of G_2:

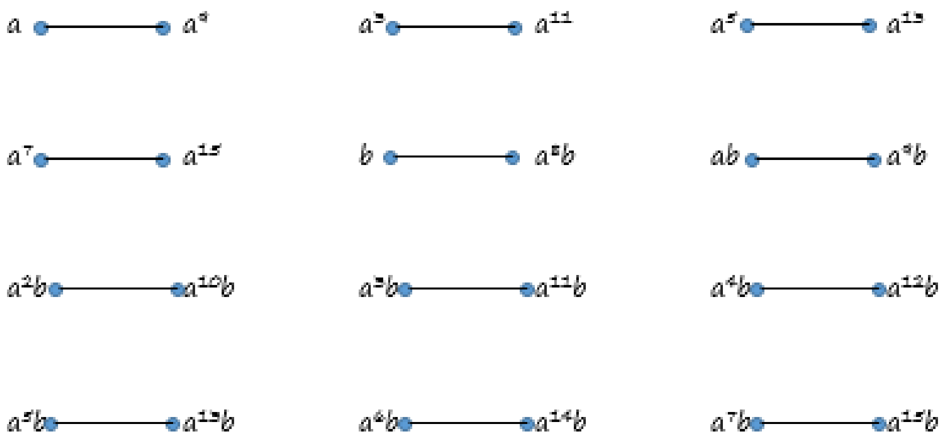

Figure 3.2 : The Conjugate Graph of G_2.
$\boldsymbol{\Gamma(G_2) = K_2 \cup K_2 \cup K_2 \cup K_2 \cup K_2 \cup K_2 \cup K_2 \cup K_2 \cup K_2 \cup K_2 \cup K_2 \cup K_2}$.

The properties:

a) The independent number of conjugate graph of G_2, $\alpha(G_2) = 12$.

b) The chromatic number of conjugate graph of G_2, $\chi(G_2) = 2$.

c) The clique number of conjugate graph of G_2, $\omega(G_2) = 2$.

d) The dominating number of conjugate graph of G_2, $\gamma(G_2) = 1$.

The Conjugate Graph of G_3:

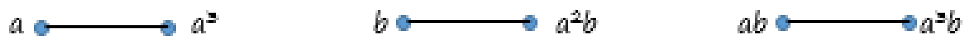

Figure 3.3 : The Conjugate Graph of G_3.
$\Gamma(G_3) = K_2 \cup K_2 \cup K_2$.

The properties:

a) The independent number of conjugate graph of G_3, $\alpha(G_3) = 3$.

b) The chromatic number of conjugate graph of G_3, $\chi(G_3) = 2$.

c) The clique number of conjugate graph of G_3, $\omega(G_3) = 2$.

d) The dominating number of conjugate graph of G_3, $\gamma(G_3) = 1$.

Next, by directly using the definition of conjugacy class graph, the conjugacy class graph of all three groups and their graph properties are drawn in the followings:

The Conjugacy Class Graph of G_1:

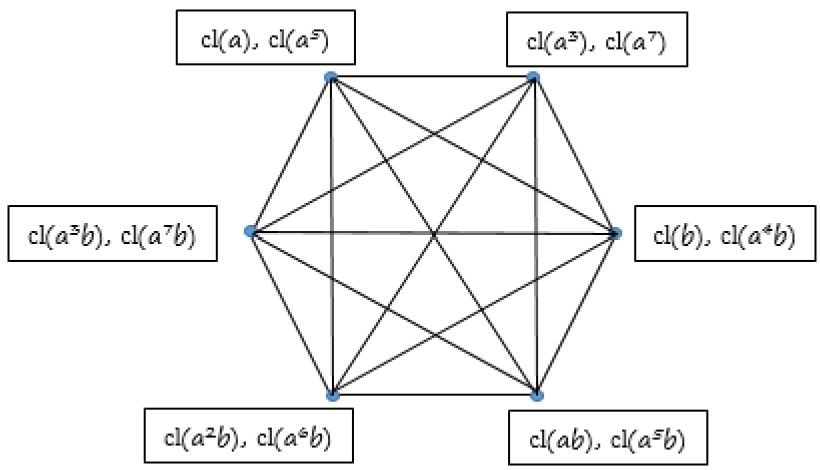

Figure 3.4 : The Conjugacy Class Graph of G_1, K_6.

The properties:

a) The independent number of conjugacy class graph of G_1, $\alpha(G_1) = 1$.

b) The chromatic number of conjugacy class graph of G_1, $\chi(G_1) = 6$.

c) The clique number of conjugacy class graph of G_1, $\omega(G_1) = 6$.

d) The dominating number of conjugacy class graph of G_1, $\gamma(G_1) = 1$.

The Conjugacy Class Graph of G_2:

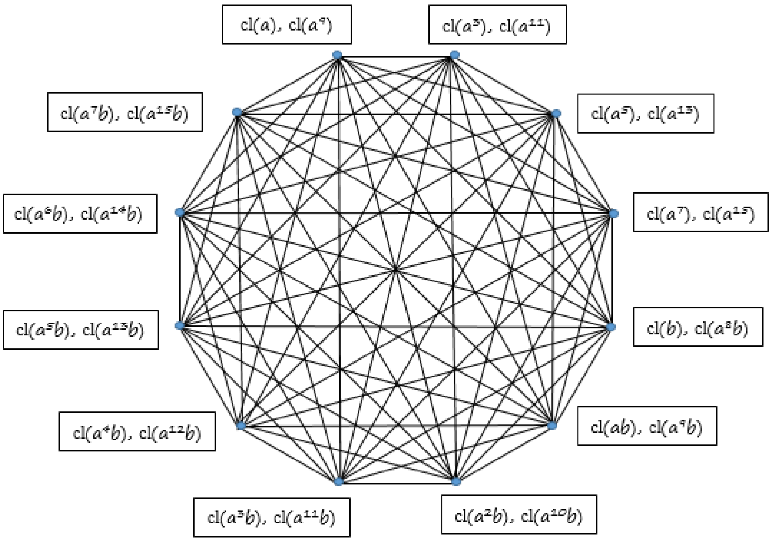

Figure 3.5 : The Conjugacy Class Graph of G_2, K_{12}.

The properties:

a) The independent number of conjugacy class graph of G_2, $\alpha(G_2) = 1$.

b) The chromatic number of conjugacy class graph of G_2, $\chi(G_2) = 12$.

c) The clique number of conjugacy class graph of G_2, $\omega(G_2) = 12$.

d) The dominating number of conjugacy class graph of G_2, $\gamma(G_2) = 1$.

The Conjugacy Class Graph of G_3:

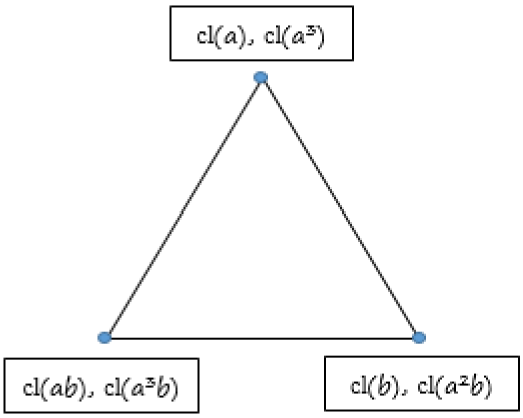

Figure 3.6 : The Conjugacy Class Graph of G_3, K_3.

The properties:

a) The independent number of conjugacy class graph of G_3, $\alpha(G_3) = 1$.

b) The chromatic number of conjugacy class graph of G_3, $\chi(G_3) = 3$.

c) The clique number of conjugacy class graph of G_3, $\omega(G_3) = 3$.

d) The dominating number of conjugacy class graph of G_3, $\gamma(G_3) = 1$.

CONCLUSION

In this research the conjugacy classes of metacyclic 2-groups of order at most 32 are computed. It is proven that the groupG_1, of order 16 with, $G_1 \cong \langle a, b : a^8 = e, b^2 = e, [a, b] = a^4 \rangle$, has ten conjugacy classes, i.e$K(G_1) = 10$ and four elements in the center of group, i.e$|Z(G_1)| = 4$.ForgroupG_2, group of order 32 with $C \cong \langle a, b : a^{16} = e, b^2 = e, [a, b] = a^8 \rangle$, the number of conjugacy class is 20, i.e$K(G_2) = 20$ and the number of elements in the center of group,$|Z(G_2)| = 8$. The third group G_3, namely quaternion of order 8 with $G_3 \cong \langle a, b : a^4 = e, b^2 = [a, b] = a^{-2} \rangle$, has five conjugacy classes,$K(G_3) = 5$ and two elements in the center of the group, $|Z(G_3)| = 2$.

It is shown too that the conjugate graphs of all three groups is a union of complete components of K_2. The conjugacy class graph of group G_1 is K_6, when group G_2 produced K_{12} as it conjugacy class graph, and the conjugacy class graph of group G_3 is K_3

ACKNOWLEDGEMENT

First of all I will like to thank Almighty Allah SWT for giving me His blessing before, during and up to the end of this research work.

In preparing this research, I was in contact with many people, lecturer, researcher and seniors. They have contributed towards my understanding and thoughts. In particular I wish to express my sincere appreciation to my supervisor, Prof. Dr. Nor HanizaSarmin, for encouragement, guidance, positive criticisms and above all supervising this work.

I am also very thankful to Dr. Sanaa Mohamed Saleh Omer for her guidance, advices and motivation. Without their continued support and interest, this thesis would not have been the same as presented here.

Moreover, I would like to thank all my family members especially my parents for always believing in me and for all the sacrifices they made in completing my studies, may Allah reward them abundantly. And then to my sisters and brothers for always keep supporting me. Last but not least, I would like to thank all my coursemates, SSCE 2011/2015 for lending me their shoulders.

REFERENCES

Beuerle, J. R. *An elementary Classification of Finite Metacyclic p-groups of Class at Least Three.* Algebra Colloq., 2005. 12(4): 553-562.

Bondy, J., Murty, G. *Graph Theory with Application*, 5th edn. North Holand, Boston New York. 1982.

Dummit, David S. and Richard M. Foote. *Abstract Algebra*. Third Edition John Wiley and Sons Inc. 2004.

Erfanian, A and Tolue, B. *Conjugate Graphs of Finite Groups. Discrete Mathematics, Algorithms and Applications*. 2012. 4(2): 35-43.

Godsil, C., Royle, G. *Algebraic Graph Theory.* 5th edn. Springer,Boston New York. 2001.

Goodman, F. M. *Algebra Abstract and Concrete. Stressing Symmetry.* Pearson Education, Inc. Upper Saddle River. 2003.

King, B. W. *Presentation of Metacyclic Groups*. Bull Aust Math Soc. 1973. 8(1): 103-131.

Reis, C. *Abstract Algebra: An Introduction to Groups, Rings and Fields*. World Scientific. 2011.

Tolue, B., and Erfanian, *A. Relative Non-Commuting Graph of a Finite Group*. Journal of Algebra and Its Applications. 2013. 12(2): 1-11.