FINDING GLOBAL MINIMUM USING FILLED FUNCTION METHOD

Omar Zahid Bin Ahmad & PM Dr. Rohanin Binti Ahmad

Abstract: Filled function method is an optimization method for finding global minimizers. Filled function method is a combination of a local search in findings local solutions as well as global solution. It is basically a construction and eventually the inclusion of an auxiliary function called the filled function into the algorithm. Optimizing the objective function at an initial point will only yield a local minimizer. By using the auxiliary function, the local minimizer is shifted to a new lower basin of the objective function. The shifted point is the new initial solution for the local search to find the next local minimizer, where the function value is lower. The process continued until the global minimizer is achieved. This research used several test functions to examine the effectiveness of the method in finding global solution. The results show that this method works successfully and further research directions are discussed.

Keywords:Optimization, global optimization, local optimization, global minimum, filled function method

Introduction

The field of optimization has grown rapidly during the past few decades. Many new theoretical, algorithmic, and computational contributions of optimization have been proposed to solve various problems in many areas. Recent developments of optimization methods can be mainly divided into deterministic and heuristic approaches. Deterministic approaches take advantage of the analytical properties of the problem to generate a sequence of points that converge to a global optimal solution. Heuristic approaches have been found to be more flexible and efficient than deterministic approaches; however, the quality of the obtained solution cannot be guaranteed. Moreover, the probability of finding the global solution decreases when the problem size increases. Deterministic approaches (e.g., linear programming, nonlinear programming, and mixed-integer nonlinear programming, etc.) can provide general tools for solving optimization problems to obtain a global or an approximately global optimum. With the increasing reliance on modeling optimization problems in real applications, a number of deterministic methods for optimization problems have been presented. The study focuses on analyzing the recent advances in deterministic optimization approaches (Lin, Tsai and Yu, 2012).

Optimization has attracted a great deal of attention from the research community since many problems arising in many different fields can be posed and solved through mathematical programming techniques. Interest in optimization intensified in the middle of 20^{th} centuries with the linear programming model which was simple, practical, and perhaps the only solvable model using the computing power available at the time.

With the tremendous advances in computing technology nowadays, the situation is changing rapidly. It has been realized that the assumption of linearity is restrictive in modeling a variety of applications (Pardalos and Romeijin, 2002). As a result, researchers seek efficient methods to solve nonlinear programming problems. Initial attempts concentrated on developing local optimization methods, guaranteeing globality under the assumption of convexity. However, many optimization problems of practical relevance are not convex and exhibit multiple local optima, thus demanding the use of global optimization techniques for their solution.

Literature Review

In this section, we focus on one FFM that is widely used and become the reference to many researchers that is P-function, introduced by Renpu Ge in 1990.

$$P(x,r,\rho) \;=\; 1/[r+f(x)]\, exp\left(-\frac{\|x-x_1^*\|^2}{\rho^2}\right). \tag{1}$$

It was originally put forward by RenpuGe as an effective algorithm of finding global minimizer of a multi-modal function.

During the year, the theory is raised mainly to cope with unconstrained optimization problems. The rudimental notion of filled function method is: when a local minimizer is reached, it is hoped that "escaping" this local minimizer is possible. Furthermore, keep searching, and a better local minimizer is reached. By constructing an auxiliary function, a local minimizer is shifted into a local maximizer. We make the point perturbed deterministically, and then, take this perturbed point as the initial point to search, and try to find the auxiliary function's local minimizer which is a better local minimizer of the objective function, or at least as well as the original one (Fang, 2006).

Fang stated that Ge' FFM is a two-phase iterative: the minimizer phase and the filling phase. In the first phase, we can use classic minimizer method, such as quasi-Newton method, the method of steepest descent, etc. to search for a local minimizer x_1^* in objective function. During the second phase, we take the present minimizer as the basic to define a filled function, and using it to find x'. Then we take x' as the starting point and repeat the first phase. This occurs again and again until the best local minimizer could not be found. The filled function method fully grasps the present local information, for it only applies the mature local minimizer method, it is greatly popular with the theoretical and practical operators.

Methodology

An optimization based methods which provide a mechanism to find global solution to the objective function. A local search is used as a tool to minimize the objective function as well as the auxiliary function in order to find local minimizer and new initial point respectively. Which we are already discussed in the literature review. This method provide a 'jump' from one local solution to another local solution until better local called global solution achieved.

This research is concerned with the problem of finding a global minimizer of a twice continuously differentiable function $f(x)$on $\mathbb{R}^n$. Suppose $f(x)$ satisfies the condition

$$f(x) \longrightarrow +\infty \text{ as } \|x\| \rightarrow +\infty \tag{2}$$

Then there exists a closed bounded domain $\Omega \subset \mathbb{R}^n$ whose interior contains all global minimizers of $f(x)$. We assume that Ω is known and our methods only consider points in Ω. We also assume that $f(x)$has only a finite number of minimizers in Ω and therefore, every minimizer is an isolated minimizer.

1. Filled Function Method Procedure

In general, the application of the algorithm of filled function methods is as the following phases:

Phase 1: Finding the local minimizer

We need to find a local minimizer x_k^* starting from an initial point and use any local minimization method such as Newton method.

Phase 2: Finding a new solution in a lower basin

After the local minimum x_k^* in the Phase 1 is obtained, then the filled function is constructed at that point and minimize the filled function in order to identify a point $x_{k+1}$with $f(x_{k+1}) < f(x_k^*)$. If x_{k+1} is formed, x_{k+1} is certainly in a lower basin than B_k^*.Use x_{k+1} as an initial point in Phase 1 again.

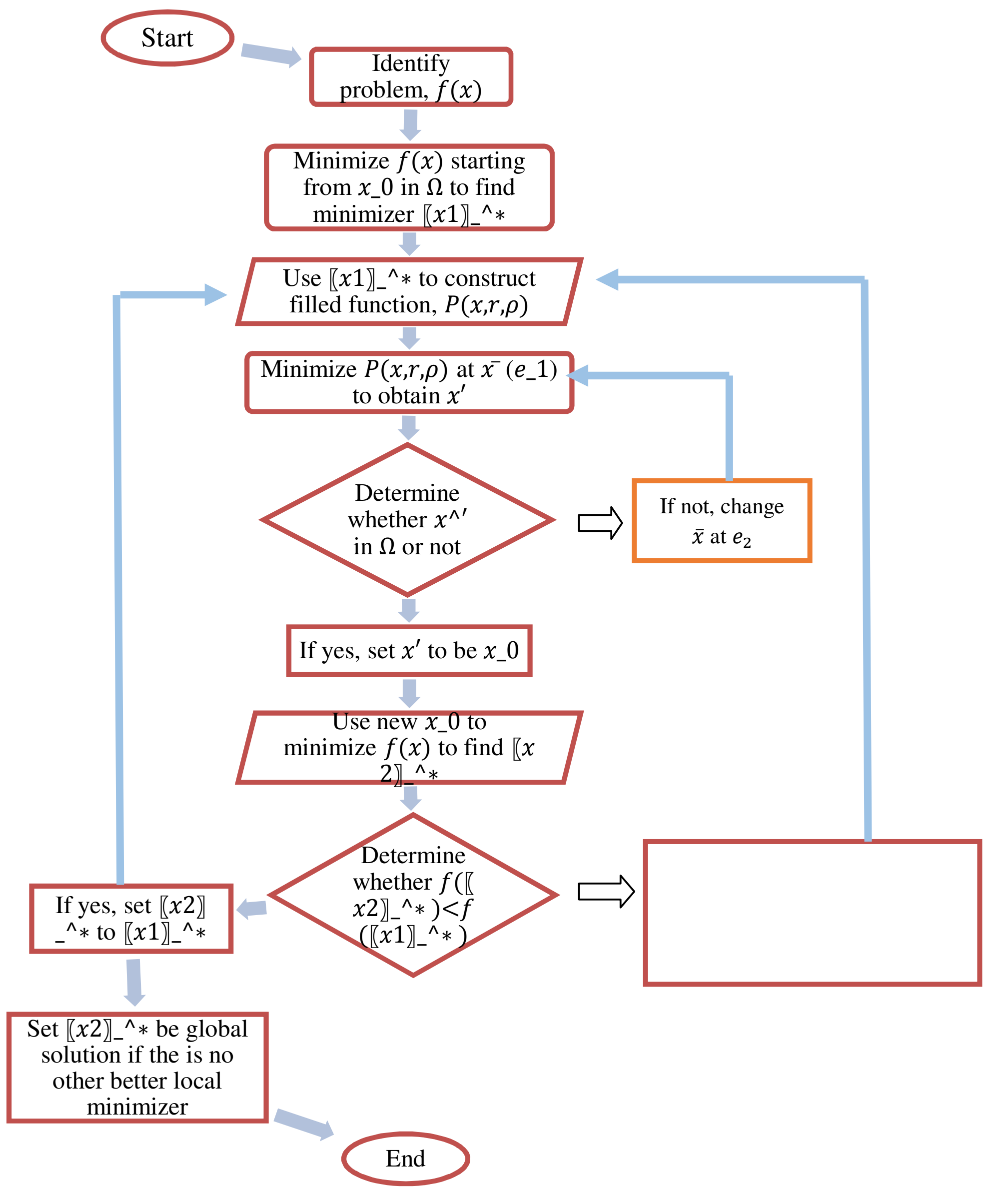

Figure 1: Procedure flow of the algorithm of filled function method

Figure 1 shows the flow chart on the procedure flow that we are concerning in this research study. The basic idea from Ge (1990) is as follows. At the beginning one may use any local minimization method. For instance, fminsearch function (Built is used to find a minimizer x_1^* of $f(x)$in the domain Ω. Then one attempts to find another minimizer of $f(x)$, x_2^*say, which satisfies the inequality

$$f(x_2^*) \leq f(x_1^*). \tag{3}$$

The idea of finding x_2^* from x_1^* is to fill the basin of $f(x)$at x_1^* and other higher basins of $f(x)$ than B_1^* so that x_1^* is a maximizer of the filled function and the basin B_1^* is a part of a hill of the filled function. Furthermore, the filled function has no minimizers or saddle points in higher basins of $f(x)$then B_1^*, but it does have a "minimizer" along the direction $x - x_1^*$ in a lower basin of *F(x)* than B_1^* if one exists. Thus, one can use an initial point near x_1^* to minimize the filled function. The minimization sequence $\{x_k\}$leads away from x_1^* and tends to a point x' which is in a lower basin $B_2^*$of $f(x)$. Using x' as an initial point to minimize *F(x),* one can obtain a lower minimizer $x_2^*$of $f(x)$, and so on.

Results and Discussion

In this section, the algorithm is tested on some optimization problems.

(a) $f(x) = -2\sin^2 x - \sin x - 2\sqrt{x}$
s.t $0 \leq x \leq 6$

Table 1: Results of minimizing P-function at x_1^*.

i	x_1^*	$f(x_1^*)$	e_i	$\bar{x}$	x'
1	4.8665	-5.3768	3	6.9878	7.2235
2			-3	2.8866	1.9638

From the result tabulated in Table 4.1, the first iteration of minimizing the P-function, the algorithm took the first preset direction, e_1 give $x' = 7.2235$ from the first local minimizer $x_1^* = 4.8665$. This value, x' is outside of the Ω, so this value is rejected and continue the algorithm with the next preset direction, e_2. Eventually, the algorithm arrived at $x' = 1.9638$ which in the Ω and then, take it as an initial point to minimize $f(x)$, get local minimizer $x_2^* = 1.7251$ and the function value $f(x_2^*) = -5.5677$. Compare the value of $f(x_2^*)$ with $f(x_1^*)$. In this case, $f(x_2^*) < f(x_1^*)$infact $f(x_2^*) \approx f(x^*)$, hence the method arrives at the global minimizer.

From this test, we can see clearly the flow of the algorithm form Phase 1 (minimizing the objective function) to Phase 2 (minimizing the P-function). The process repeat until the global minimizer found. Thus, the initial testing for the algorithm is quite successful. Next, the algorithm is tested with higher number of local minima problem.

(b) $f(x) = \sin x + \sin 2x - \cos 4x$

$\text{s.t} - 2 \leq x \leq 4.$

Table 2 : Result of minimizing problem (a) using the algorithm.

k	i	x_k^*	$f(x_k^*)$	e_i	$\bar{x}$	x'	x_{k+1}^*	$f(x_{k+1}^*)$
1	1	3.079333	-1.03113	-0.5	2.72578	-0.15206	-0.19637	-1.28488
	2			-1	-	-	-	-
2	1	-0.19637	-1.28488	-0.5	-0.54993	-3.5825	-	-
	2			-1	-0.69135	-1.18632	-1.45232	-2.11752

The result shows that the algorithm able to find another new local minimizer in lower basin from the previous local minimizer. This confirmed that the algorithm continues to run until there is no local minimizer found anymore.

Noted that, the use of preset directions, e_i are different from problem (a) with problem (b). It is define differently according to the problem as well as the parameters $\rho$and $r + f(x)$. Though the value of the parameter need to be chose and adjust until the global solution come out from the algorithm. The preset direction also taking as important part as it take role as movement of the P-function in finding the x'.

Table 3 : Result of minimizing Problem (4.2) at different value of parameter ρ.

k	x_k^*	$f(x_k^*)$	x'	x_{k+1}^*	$f(x_{k+1}^*)$
1	3.07933	-1.03113	1.38627	1.70618	-0.13354
2	1.70618	-0.13354	1.04766	1.70618	-0.13354
	-	-	0.64132	-0.19634	-1.28488
3	-0.19634	-1.28488	-1.21458	-1.45227	-2.11752

From table above, at different value of ρ, that is $\rho = 0.5$, the computing of the algorithm will arrive at local minimizer of *b* basin. The result is then, $f(x_2^*) \geq f(x_1^*)$ which means the algorithm had through Step 7 as in the algorithm in Section 3.5. The value of ρ is increased so that the ratio $\rho^2/[r + f(x_1^*)]$ becomes small.

According to Ge (1990), the reasons for obtaining $f(x_2^*) \geq f(x_1^*)$are either the ratio $\rho^2/[r + f(x_1^*)]$ is not small enough or $\|x_k - x_1^*\|$is too large so that the computer cannot recognize the change of $P(x, r, \rho)$. Therefore, the both ρ^2 and $[r + f(x_1^*)]$need to be increase to make the ratio $\rho^2/[r + f(x_1^*)]$smaller than the previous one as in Step 7 (refer to FFM algorithm in Section 3.5). Do not only decrease ρ^2 to decrease the ratio $\rho^2/[r + f(x_1^*)]$ because too small ρ^2 makes the filled function descend very quickly when x is close to x_1^* but very slowly when x is a little bit further from x_1^* and therefore some difficulty arises in the minimization process of $P(x, r, \rho)$.

(c) $\text{Griewank}_n(x) = \sum_{i=1}^{n} \frac{{x_i}^2}{4000} - \prod_{i=1}^{n} \cos\left(\frac{x_i}{\sqrt{i}}\right) + 1$

$$-600 \leq x_i \leq 600 \text{ for } i = 1, \dots, n.$$

The global minimum value is 0 and the global is located in the origin, but the function also has a very large number of local minima, exponentially increasing with n.

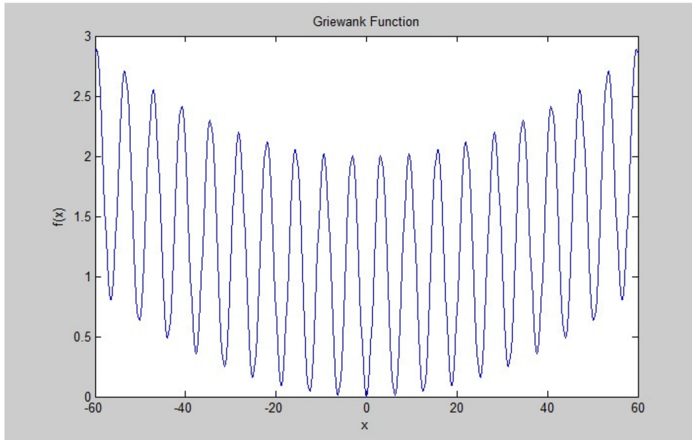

Figure 3 : Graph of Griewank function

Table 4 : Result of minimizing the Griewank function

k	x_k^*	$f(x_k^*)$	$\bar{x}$	x'	x_{k+1}^*	$f(x_{k+1}^*)$
1	50.24036	0.631339	46.70483	46.55751	43.96031	0.483369
2	43.96031	0.483369	40.42478	40.27747	37.68027	0.355128
3	37.68027	0.355128	33.99742	33.99742	31.40023	0.246617
4	31.40023	0.246617	27.86469	27.71738	25.12018	0.157835
5	25.12018	0.157835	21.58465	21.43733	18.84014	0.088782
6	18.84014	0.088782	15.30460	15.15729	12.56009	0.039459
7	12.56009	0.039459	9.02456	8.877243	6.280046	0.009865
8	6.280046	0.009865	2.74451	2.597197	0.0000	0.0000

For the simulation, first, the initial points is chose within the box-constraint$-600 \leq x_i \leq 600$, and the parameters of the FFM that are $r + f(x_1^*) = 3$ and $\rho = 1$ be fixed as well as the number of iteration for the algorithm to run that is Ns=100 and $e_i = \{-5, -10\}$. As the first simulation, let $x_0 = 50$. The result is as the table below.From the result tabulated in Table 4, the global minimizer is 0, thus the algorithm successful to arrive at the global solution

Conclusion

Filled function method (FFM) by Ge's provided a better mechanism in finding global solution for global optimization problem. Whereas, the conventional optimization method such as Newton's Method incapable finding the global but local optima only. Though, the FFM combined with the conventional optimization method in the algorithm for this to work together with its auxiliary function that is P-function.

However, Ge's FFM had several drawbacks in his algorithm which become a field to explore for other researchers. There are two major drawbacks that many researchers eventually take into consideration. First, there are two adjustable parameters, r and ρ, which greatly affect the performance of the algorithms and need to be appropriately adjusted. However, how to adjust the parameters is a very difficult task. Second, the filled functions are likely to be ill-conditioned in practice since their function values increase exponentially due to an exponential function. As the adjustable parameter becomes larger and larger, which is required by the FFM itself, the rapidly increasing exponential function value may result in an overflow in the computation (Wei, Wang and Shang, 2013).

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