REPRESENTATIONS OF RATIONAL CHEREDNIK ALGEBRAS WITH MINIMAL SUPPORT AND TORUS KNOTS

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Abstract. In this paper we obtain several results about representations of rational Cherednik algebras, and discuss their applications. Our first result is the Cohen-Macaulayness property (as modules over the polynomial ring) of Cherednik algebra modules with minimal support. Our second result is an explicit formula for the character of an irreducible minimal support module in type A_{n-1} for $c = \frac{m}{n}$, and an expression of its quasispherical part (i.e., the isotypic part of "hooks") in terms of the HOMFLY polynomial of a torus knot colored by a Young diagram. We use this formula and the work of Calaque, Enriquez and Etingof to give explicit formulas for the characters of the irreducible equivariant D-modules on the nilpotent cone for SL_m. Our third result is the construction of the Koszul-BGG complex for the rational Cherednik algebra, which generalizes the construction of the Koszul-BGG resolution from [BEG] and [Go], and the calculation of its homology in type A. We also show in type A that the differentials in the Koszul-BGG complex are uniquely determined by the condition that they are nonzero homomorphisms of modules over the Cherednik algebra. Finally, our fourth result is the symmetry theorem, which identifies the quasispherical components in the representations with minimal support over the rational Cherednik algebras $H_{\frac{m}{n}}(S_n)$ and $H_{\frac{n}{m}}(S_m)$. In fact, we show that the simple quotients of the corresponding quasispherical subalgebras are isomorphic as filtered algebras. This symmetry was essentially established in [CEE] in the spherical case, and in [Gor2] in the case $GCD(m, n) = 1$, and it has a natural interpretation in terms of invariants of torus knots.

1. Introduction

The goal of this paper is to establish a number of properties of representations of rational Cherednik algebras with minimal support, and connect them to knot invariants. Our motivation came from the connections of representations of Cherednik algebras with quantum invariants of torus knots and Hilbert schemes of plane curve singularities (such as $x^m = y^n$, $GCD(m, n) = 1$), see [GORS].

1.1. Let $\mathfrak{h}$ be a finite dimensional complex vector space, $W \subset GL(\mathfrak{h})$ a finite subgroup, $S \subset W$ the set of reflections, and $c : S \to \mathbb{C}$ a conjugation invariant function. Let $H_c(W, \mathfrak{h})$ be the rational Cherednik algebra attached to $W, \mathfrak{h}$. Let $\mathcal{O}_c = \mathcal{O}_c(W, \mathfrak{h})$ be the category of modules over this algebra which are finitely generated over $\mathbb{C}[\mathfrak{h}] = S\mathfrak{h}^*$ and locally nilpotent under $\mathfrak{h}$. Typical examples of objects of this category are $M_c(\tau)$, the Verma (a.k.a. standard) module over $H_c(W, \mathfrak{h})$ with lowest weight $\tau \in \text{Irrep } W$, and $L_c(\tau)$, the irreducible quotient of $M_c(\tau)$.

Any object $M \in \mathcal{O}_c$, being a finitely generated $\mathbb{C}[\mathfrak{h}]$-module, has support $\text{supp}(M)$ as a module over $\mathbb{C}[\mathfrak{h}]$, which is a closed subvariety of $\mathfrak{h}$.

Definition 1.1. We say that $M \in \mathcal{O}_c$ has minimal support if no subset of $\text{supp}(M)$ of smaller dimension is the support of a nonzero object of $\mathcal{O}_c$.

Our first main result is

Theorem 1.2. If M has minimal support then it is a Cohen-Macaulay module over $\mathbb{C}[\mathfrak{h}]$ of dimension $d = \dim \operatorname{supp}(M)$. In other words, it is free over any polynomial subalgebra $\mathbb{C}[p_1, ..., p_d] \subset \mathbb{C}[\mathfrak{h}]$ (with homogeneous p_i) over which it is finitely generated.

Remark 1.3. Note that the minimal support condition is needed. For example, if $W = S_3$, $c = 1/3$, and $M = L_c(\mathfrak{h})$ is the irreducible module with lowest weight $\mathfrak{h}$, then M is the augmentation ideal in $\mathbb{C}[\mathfrak{h}]$, so it is not Cohen-Macaulay (as it is not free).

1.2. Our second result is the character formula for irreducible minimally supported modules for rational Cherednik algebras of S_n for $c = \frac{m_0}{n_0}$, and its consequences. Let $\mathfrak{h} = \mathfrak{h}_n$ be the reflection representation of S_n, and consider the rational Cherednik algebra $H_c(S_n) := H_c(S_n, \mathfrak{h}_n)$, where $c = \frac{m_0}{n_0}$ and $m_0, n_0 \in \mathbb{Z}_{\geq 1}$ are coprime. Let $n = dn_0 + r$, where $0 \leqslant r < n_0$. Recall from [Wi] that minimally supported modules in the category $\mathcal{O}_c := \mathcal{O}_c(S_n, \mathfrak{h}_n)$ are of the form $L_c(n_0\lambda + \lambda')$, where λ is a partition of d and λ' is a partition of r. Here $n_0\lambda + \lambda'$ is the partition given by $(n_0\lambda + \lambda')_i = n_0\lambda_i + \lambda'_i$.

To state the character formula, define the constants $c^{\nu}_{\lambda,\lambda',n_0}$ by:

$$s_\lambda(x_1^{n_0}, x_2^{n_0}, ...)s_{\lambda'}(x_1, x_2, \ldots) = \sum_\nu c^{\nu}_{\lambda,\lambda',n_0} s_\nu(x_1, x_2, ...),$$

where s_λ are the Schur polynomials. When we write c^{ν}_{λ,n_0}, we mean $c^{\nu}_{\lambda,\varnothing,n_0}$.

Theorem 1.4. In the Grothendieck group $K_0(\mathcal{O}_c)$, we have

$$[L_c(n_0\lambda + \lambda')] = \sum_{\nu:|\nu|=n} c^{\nu}_{\lambda,\lambda',n_0} M_c(\nu).$$

In particular, the character of $L_c(n_0\lambda)$ is given by the formula

$$\operatorname{Tr}_{L_c(n_0\lambda+\lambda')}(\sigma q^{\mathbf{h}}) = \sum_{\nu:|\nu|=n} c^{\nu}_{\lambda,\lambda',n_0} q^{\frac{n-1}{2} - c\kappa(\nu)} \chi_\nu(\sigma) \det{}_{\mathfrak{h}}(1 - q\sigma)^{-1},$$

where $\mathbf{h}$ is the scaling element of the rational Cherednik algebra, $\kappa(\nu)$ is the content of ν (see formula (4)), $\sigma \in S_n$, and χ_ν is the character of the S_n-module attached to the partition ν.

This theorem implies the following explicit formula for the character of the quasispherical part of $L_c(n_0\lambda)$, which provides a connection to the theory of knot invariants. Namely, let $T(m_0, n_0)$ be the torus knot corresponding to the relatively prime integers m_0, n_0, and let $P_\lambda(T(m_0, n_0))(a, q)$ be its colored HOMFLY polynomial; if $a = q^N$ for large enough N, it is computed as the Reshetikhin-Turaev invariant for $U_q(\mathfrak{sl}_N)$, by coloring the knot with the irreducible representation of $\mathfrak{sl}_N$ of highest weight λ. Let

$$\widetilde{P}_\lambda(T(m_0, n_0))(a, q) = a^{\frac{d}{2}(m_0+n_0-m_0n_0)} \frac{q^{-1/2} - q^{1/2}}{1 - a} P_\lambda(T(m_0, n_0))(a, q). \tag{1}$$

We will call this polynomial the *renormalized* colored HOMFLY polynomial.

Using the formula by M. Rosso and V. Jones [RJ] for this polynomial, from Theorem 1.4 we obtain:

Corollary 1.5.

$$\sum_{k=0}^{n-1} (-a)^k \dim_q \operatorname{Hom}_{S_n}(\wedge^k \mathfrak{h}_n, L_c(n_0\lambda)) = q^{-m_0n_0\kappa(\lambda)} \widetilde{P}_\lambda(T(m_0, n_0))(a, q),$$

where $\dim_q(E) := \operatorname{Tr}_E(q^{\mathbf{h}})$.

This shows, in particular, that the sum on the left hand side is symmetric under interchanging m and n, which is not obvious from the representation theoretic viewpoint (and is explained by Theorem 1.10 below). It also shows that $\widetilde{P}_\lambda(T(m_0,n_0))(-a,q)$ (and hence $P_\lambda(T(m_0,n_0))(-a,q)$, under a suitable normalization by a power of $-a$) is a (Laurent) polynomial in a and a power series in q with nonnegative coefficients, which is not straightforward from the knot theory point of view (in fact, the only proof we know uses Cherednik algebras). Moreover, Theorem 1.2 implies that the reduced colored HOMFLY invariant $\widetilde{P}_\lambda(T(m_0,n_0))\cdot\prod_{i=2}^d(1-q^i)$ is a polynomial with nonnegative coefficients.

1.3. The character formula of Theorem 1.4 can be used to solve a problem in Lie theory posed in [CEE, Section 9], namely, to compute the characters of certain equivariant D-modules on the nilpotent cone of the group SL_m.

Let G be a complex simply connected simple algebraic group with Lie algebra $\mathfrak{g}$, $\mathcal{N}\subset\mathfrak{g}^*$ be its nilpotent cone, and $\mathcal{D}_G(\mathcal{N})$ be the category of G-equivariant D-modules on $\mathcal{N}$. This category is known to be semisimple, with simple objects $M_{O,\chi}$ parametrized by nilpotent orbits O and irreducible representations χ of the fundamental group of O. Using Kashiwara's lemma, we can regard objects of this category as equivariant D-modules on $\mathfrak{g}^*$ supported on $\mathcal{N}$, and then they are precisely the Fourier transforms of unipotent character D-modules on $\mathfrak{g}$ (see [CEE, Section 9] and references therein, in particular, [Mir]).

Given $M\in\mathcal{D}_G(\mathcal{N})$, regard it as a D-module on $\mathfrak{g}^*$, and consider its space of global sections, which we will denote also by M for brevity. Then M carries an action of G and a commuting action of the Lie algebra $\mathfrak{sl}_2$ generated by the Laplace operator and the operator of multiplication by the squared norm on $\mathfrak{g}^*$, see [CEE, Section 9]. Moreover, it is shown in [CEE, Subsection 9.4], that for simple M and for any irreducible G-module V, the multiplicity space $\mathrm{Hom}_G(V,M)$ is an $\mathfrak{sl}_2$-module in category $\mathcal{O}$. Thus, one can define the character of M by the formula

$$\mathrm{Ch}_M(q,g)=\mathrm{Tr}_M(gq^{-H})=\sum_{\mu\in P_+}\mathrm{Tr}_{V_\mu}(g)\psi_{M,\mu}(q)$$

with

$$\psi_{M,\mu}(q):=\mathrm{Tr}_{\mathrm{Hom}_G(V_\mu,M)}(q^{-H}), g\in G,$$

where H is the Cartan element of $\mathfrak{sl}_2$, and V_μ is the irreducible representation of G with highest weight μ. This leads naturally to the following interesting problem:

Problem 1.6. Compute the character Ch_M for every simple object $M=M_{O,\chi}$ of $\mathcal{D}_G(\mathcal{N})$.

As far as we know, this problem is open for a general G. In [CEE] it was reduced for $G=SL_m$ to the computation of characters of minimally supported modules for rational Cherednik algebras, and solved for $G=SL_2$ and in the cuspidal case for $G=SL_m$ using this reduction. Thus, using Theorem 1.4, we now obtain the general answer for $G=SL_m$.

Let $s\in[0,m-1]$, and θ_s be the corresponding character of the center of SL_m. Let $d=GCD(m,s)$, $m_0=m/d$ and λ be a partition of d. Let O_μ be the nilpotent orbit corresponding to the partition μ of m. Consider the orbit $O_{m_0\lambda}$. This orbit carries a unique 1-dimensional local system corresponding to the central character θ_s, which we will denote by $\mathcal{L}_s$.

Theorem 1.7. If $M=M_{O_{m_0\lambda},\mathcal{L}_s}$ then

$$\mathrm{Ch}_M(q,g)=(1-q)\lim_{n\to\infty}\sum_{\nu:|\nu|=n}c^\nu_{\lambda,n_0}q^{\frac{n-1}{2}-\frac{m}{n}\kappa(\nu)}s_\nu(x_1,...,x_m,qx_1,...,qx_m,q^2x_1,...),$$

where $n = s + km$ with $k \in \mathbb{Z}_{\geq 0}$, $n_0 = n/d$, and $x_1, ..., x_m$ are the eigenvalues of g.

Here the limit is understood in the sense of stabilization. Namely, define an increasing filtration on M (labeled by $n = s + km$) by setting $M^{(n)}$ to be the isotypic part of M for the representations V_μ of SL_m which occur in $V^{\otimes n}$. Then

$$\mathrm{Ch}_{M^{(n)}}(q,g) = (1-q) \sum_{\nu:|\nu|=n} c^{\nu}_{\lambda,n_0} q^{\frac{n-1}{2} - \frac{m}{n}\kappa(\nu)} s_\nu(x_1, ..., x_m, qx_1, ..., qx_m, q^2x_1, ...),$$

and $\mathrm{Ch}_M = \lim_{n\to\infty} \mathrm{Ch}_{M^{(n)}}$.

1.4. The third result is the construction of the Koszul-BGG complex and the study of its homology. To define this complex, let us say that an irreducible W-subrepresentation $V \subset M_c(\tau)$ is singular if it is annihilated by the action of $\mathfrak{h} \subset H_c(W, \mathfrak{h})$. Then, given a singular subrepresentation $V \subset M_c(\mathbb{C})$ for which $\mathrm{rank}(s-1)|_V = 1$ for every reflection $s \in S$, we consider the Koszul complex $K^\bullet(V)$ (in the sense of commutative algebra) [1]:

$$M_c(\mathbb{C}) \leftarrow M_c(V) \leftarrow M_c(\wedge^2 V) \leftarrow ...$$

Our third main result is the following theorem.

Theorem 1.8. (i) (Proposition 6.1 below) The complex $K^\bullet$ is, in fact, a complex of $H_c(W, \mathfrak{h})$-modules.

(ii) (Theorem 6.3 below) If $W = S_n$, $c = \frac{m}{n}$, $GCD(m, n) = d < n$, and V is the unique singular copy of $\mathfrak{h}$ in degree m (see [D, CE, ES]) then the homology $H_i(K^\bullet)$ vanishes if $i \geq d$, and is the irreducible $H_c(W, \mathfrak{h})$-module $L_c(\lambda_i)$, where $\lambda_i = n_0(d-i, 1^i)$, if $i < d$.

The complex $K^\bullet(V)$ is analogous to the BGG resolution in the representation theory of semisimple Lie algebras, and for this reason it is called the Koszul-BGG complex.

Remark 1.9. In the case when $\dim V = \dim \mathfrak{h}$ and the quotient module $M_c(\mathbb{C})/(V)$ is finite dimensional, the Koszul complex $K^\bullet(V)$ (which is then exact in higher degrees, i.e., a resolution) was considered in [BEG],[Go] for real reflection groups, and in [CE] for complex reflection groups.

We also show in type A that the differentials in the Koszul-BGG complex are uniquely determined up to scaling by the condition that they are nonzero homomorphisms of modules over the Cherednik algebra (Proposition 6.11).

1.5. Finally, our fourth main result concerns symmetry for Cherednik algebras of type A. Let $\mathbf{e}_{i,n}$ be the Young projector in $\mathbb{C}S_n$ corresponding to the "hook" representation $\wedge^i \mathfrak{h}_n$ (which is nonzero iff $0 \leq i \leq n-1$), and let $\overline{\mathbf{e}}_n = \sum_{i=0}^{n-1} \mathbf{e}_{i,n}$ be the idempotent of $\wedge \mathfrak{h}_n$.[2] The subalgebra $\overline{\mathbf{e}}_n H_c(S_n) \overline{\mathbf{e}}_n$ will be called the quasispherical subalgebra.

Note that the algebra $H_c(W, \mathfrak{h})$ has the Bernstein filtration, in which $\deg(\mathfrak{h}) = \deg(\mathfrak{h}^*) = 1$, $\deg(W) = 0$. Also, the module $L_c(\tau)$ is graded by the eigenvalues of the scaling element $\mathbf{h} \in H_c(W, \mathfrak{h})$, and has a descending filtration by the images of the powers of the maximal ideal $\mathfrak{m} \subset \mathbb{C}[\mathfrak{h}]^W$ (this filtration is discussed in [GORS]).

It is shown in [L1] that the algebra $H_c(S_n)$ has a unique maximal two-sided ideal $J_c(n)$. Also, for $m \in \mathbb{Z}_{>0}$ with $GCD(m, n) = d$, it follows from [CEE], [BE] (see also [Wi]) that if λ is a partition of d then the module $L_{\frac{m}{n}}(n_0\lambda)$ has minimal support (its support can

[1] Here we do not use, nor claim, that $\wedge^i V$ are simple W-modules, even though this is true if W is a Coxeter group and V is its reflection representation.

[2] When no confusion is possible, we will often drop the subscript n from the notation for these idempotents.

be explicitly computed from the construction of [CEE], and it follows from [BE] that this support is minimal). This means that the annihilator of $L_{\frac{m}{n}}(n_0\lambda)$ is the maximal ideal $J_{\frac{m}{n}}(n)$.

Our fourth main result is

Theorem 1.10. (i) (Corollary 7.15 below) Let λ be a partition of d. Then for all i there is an isomorphism of vector spaces

$$\phi_{n,m,i} : \mathbf{e}_{i,n} L_{\frac{m}{n}}(n_0\lambda) \cong \mathbf{e}_{i,m} L_{\frac{n}{m}}(m_0\lambda)$$

which preserves the grading and the filtration. In particular, the two-variable characters of these two spaces associated to the grading and the filtration are equal.

(ii) (Theorem 7.11 below) There exists an isomorphism of algebras

$$\mathbf{\Phi}_{n,m} : \overline{\mathbf{e}}_n (H_{\frac{m}{n}}(S_n)/J_{\frac{m}{n}}(n))\overline{\mathbf{e}}_n \to \overline{\mathbf{e}}_m (H_{\frac{n}{m}}(S_m)/J_{\frac{n}{m}}(m))\overline{\mathbf{e}}_m$$

preserving the Bernstein filtration and compatible with $\phi_{n,m,i}$.

Note that this implies that if $i \geq \min(n,m)$ then in both parts of Theorem 1.10, the spaces and the algebras vanish (which is obvious only on one of the two sides).

The proof of Theorem 1.10 is based on comparing two constructions of representations of rational Cherednik algebras of type A from Lie theory (by reduction from equivariant D-modules) - the Gan-Ginzburg construction ([GG]) and the construction from [CEE]. More precisely, we generalize the Gan-Ginzburg construction to the case of hook representations, and then the representations in part (i) of Theorem 1.10 turn out to be realized on the same vector space, yielding a proof of part (i), and the algebras from part (ii) turn out to act on this space by the same operators, yielding a proof of part (ii).

1.6. The organization of the paper is as follows.

Section 2 contains the preliminaries.

In Section 3 we prove Theorem 1.2 (actually, we give two somewhat different proofs), and give some applications.

In Section 4, we prove Theorem 1.4 and Corollary 1.5, providing a link to knot invariants.

In Section 5, we prove Theorem 1.7 on the characters of equivariant D-modules.

In Section 6, we develop the theory of the Koszul-BGG complex, and prove Theorem 1.8. We give two proofs, based on two different approaches.

In Section 7, we generalize the Gan-Ginzburg quantum reduction construction to the "hook" case, and prove Theorem 1.10.

Finally, in Section 8, we study the symmetrized Koszul-BGG complexes, and give a third proof of Theorem 1.8.

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2. Preliminaries and notation

2.1. **Rational Cherednik algebras.** Let $\mathfrak{h}$ be a finite dimensional complex vector space, $W \subset GL(\mathfrak{h})$ a finite subgroup, $S \subset W$ the set of reflections, and $c : S \to \mathbb{C}$ a conjugation invariant function. For $s \in S$, let $\alpha_s \in \mathfrak{h}^*, \alpha_s^\vee \in \mathfrak{h}$ be elements such that $s\alpha_s = \lambda_s \alpha_s$, $\lambda_s \neq 1$, $s\alpha_s^\vee = \lambda_s^{-1}\alpha_s^\vee$, and $(\alpha_s, \alpha_s^\vee) = 2$.

Definition 2.1. The rational Cherednik algebra $H_c(W, \mathfrak{h})$ attached to $W, \mathfrak{h}$ is the quotient of $\mathbb{C}W \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)$ by the relations

$$[x, x'] = [y, y'] = 0, \ [y, x] = (x, y) - \sum_{s \in S} c_s(\alpha_s, y)(\alpha_s^\vee, x)s,$$

where $x, x' \in \mathfrak{h}^*, y, y' \in \mathfrak{h}$.

If W is a reflection group and $\mathfrak{h}$ is its reflection representation, we will also use the abbreviated notation $H_c(W)$ for this algebra.

For a representation τ of W, let $M_c(\tau)$ be the Verma (or standard) module over $H_c(W, \mathfrak{h})$ induced from τ, i.e., $M_c(\tau) = H_c(W, \mathfrak{h}) \otimes_{\mathbb{C}W \ltimes S\mathfrak{h}} \tau$. We have a natural isomorphism $M_c(\tau) \cong S\mathfrak{h}^* \otimes \tau$ of $\mathbb{C}W \ltimes S\mathfrak{h}^*$-modules, and $y \in \mathfrak{h}$ act by Dunkl operators

$$D_y = \partial_y - \sum_{s \in S} \frac{\tilde{c}_s(\alpha_s, y)}{\alpha_s}(1 - s) \otimes s,$$

where $\tilde{c}_s = 2c_s/(1 - \lambda_s)$. The Verma module $M_c(\tau)$ has a unique irreducible quotient $L_c(\tau)$.

Define the category $\mathcal{O}_c = \mathcal{O}_c(W, \mathfrak{h})$ to be the category of $H_c(W, \mathfrak{h})$-modules which are finitely generated over $\mathbb{C}[\mathfrak{h}] = S\mathfrak{h}^*$, and locally nilpotent under $\mathfrak{h}$. Clearly, $M_c(\tau)$ and $L_c(\tau)$ belong to this category.

The algebra $H_c(W, \mathfrak{h})$ contains the scaling element

$$\mathbf{h} = \sum_i x_i y_i + \frac{\dim \mathfrak{h}}{2} - \sum_{s \in S} \tilde{c}_s s,$$

where $\{y_i\}$ is a basis of $\mathfrak{h}$, and $\{x_i\}$ the dual basis of $\mathfrak{h}^*$. This element has the property that $[\mathbf{h}, x_i] = x_i$, $[\mathbf{h}, y_i] = -y_i$, $[\mathbf{h}, w] = 0$ for all $w \in W$. It is known ([GGOR]) that $\mathbf{h}$ acts locally finitely on every module from category $\mathcal{O}_c$, and acts semisimply in every standard and hence every irreducible module. This implies that any module in $\mathcal{O}_c$ is naturally graded by (generalized) eigenvalues of $\mathbf{h}$, and in particular every standard and irreducible module in this category is $\mathbb{C}^\times$-equivariant (we make $\mathbb{C}^\times$ act trivially on the lowest weight space).

It is known ([GGOR]) that the category $\mathcal{O}_c$ is a highest weight category (with the ordering by real parts of eigenvalues of $\mathbf{h}$). In particular, it has enough projectives, and they admit a filtration in which successive quotients are standard modules. Such a filtration is called a standard filtration.

2.2. **Rational Cherednik algebras in type** A**.** Let $W = S_n$ be the symmetric group in n letters, and $\mathfrak{h} = \mathfrak{h}_n$ be the reflection representation of W (of dimension $n - 1$). Then the reflections are just transpositions, so we have a single conjugacy class. Thus the parameter c is a single complex number. The space $\mathfrak{h}$ is spanned by $y_1, ..., y_n$ permuted by S_n, with $\sum_i y_i = 0$, and $\mathfrak{h}^*$ is spanned by $x_1, ..., x_n$ permuted by S_n with $\sum_i x_i = 0$. The defining relations are:

$$[x_i, x_j] = [y_i, y_j] = 0;$$
$$[y_i, x_j] = -\frac{1}{n} + cs_{ij}, \ i \neq j;$$
$$[y_i, x_i] = 1 - \frac{1}{n} - c\sum_{j \neq i} s_{ij}.$$

2.3. **Idempotents.** We need to fix notation for some idempotents in $\mathbb{C}S_n$. Denote by $\mathbf{e}_{i,n}$ (or shortly $\mathbf{e}_i$ when no confusion is possible) the primitive projector of the representation $\wedge^i\mathfrak{h}_n$ (it is nonzero iff $0 \le i \le n-1$). Denote the symmetrizer $\mathbf{e}_0$ by $\mathbf{e}$ and the antisymmetrizer $\mathbf{e}_{n-1}$ by $\mathbf{e}_-$. Also, set $e_i = \mathbf{e}_i + \mathbf{e}_{i-1}$ (the projector of $\wedge^i\mathbb{C}^n$), and $\overline{\mathbf{e}} = \sum_{i=0}^{n-1} \mathbf{e}_i = \sum_{i\ge 0} e_{2i} = \sum_{i\ge 0} e_{2i+1}$ (the projector of $\wedge\mathfrak{h}_n$).

2.4. **The restriction functors.** The parabolic restriction functors for rational Cherednik algebras were introduced in [BE]. Namely, given a point $b \in \mathfrak{h}$, denote by W_b the stabilizer of b in W. Then one can define the restriction functor $\mathrm{Res}_b : \mathcal{O}_c(W,\mathfrak{h}) \to \mathcal{O}_c(W_b,\mathfrak{h})$, as follows. Given $M \in \mathcal{O}_c(W,\mathfrak{h})$, let $\widehat{M_b}$ be the completion of M at b as a $\mathbb{C}[\mathfrak{h}]$-module. Then $\widehat{M_b}$ is naturally a module over the completion of the algebra $H_c(W_b,\mathfrak{h})$ at zero. By taking the y-nilpotent vectors in $\widehat{M_b}$, we get a module over $H_c(W_b,\mathfrak{h})$, which lies in $\mathcal{O}_c(W_b,\mathfrak{h})$, and is denoted by $\mathrm{Res}_b(M)$. [3]

The functor Res_b is exact. It will be used below in several places.

2.5. **The results of** [Wi]**.** Let us summarize the results of [Wi] (essentially, Theorem 1.8 and Proposition 3.7 in [Wi]) which will be used several times below.

Let m, n, d be as above. Let π_μ be the representation of S_d corresponding to a partition μ of d. Let $X_{d,n/d}(n)$ be the affine variety which is the union of all the S_n-translates of the subspace in $\mathbb{C}^n$ defined by the equations $\sum_i x_i = 0$ and

$$x_1 = \ldots = x_{\frac{n}{d}}, x_{\frac{n}{d}+1} = \ldots = x_{\frac{2n}{d}}, \ldots, x_{(d-1)\frac{n}{d}+1} = \ldots = x_n.$$

Let $X_{d,n/d}(n)^\circ$ be the open subset of $X_{d,n/d}(n)$ where there are d distinct values of x_i. Then $X_{d,n/d}(n)^\circ/S_n$ is isomorphic to the configuration space of d unmarked points on the complex plane with barycenter at the origin, so $\pi_1(X_{d,n/d}(n)^\circ/S_n) = B_d$, the braid group in d strands.

Theorem 2.2. ([Wi]) (i) The minimal support for modules in the category $\mathcal{O}_{\frac{m}{n}}(S_n,\mathfrak{h}_n)$ in $\mathfrak{h}_n$ is the variety $X_{d,n/d}(n)$. The minimally supported irreducible modules are $L_{\frac{m}{n}}(\frac{n}{d}\mu)$, where μ is a partition of d.

(ii) Let Y be the simple finite dimensional module over $H_{\frac{m}{n}}(S_{n/d})$. Given a minimally supported module $M \in \mathcal{O}_{\frac{m}{n}}(S_n,\mathfrak{h}_n)$, let $\mathcal{L}_M$ be the local system on $X_{d,n/d}(n)^\circ/S_n$ whose fiber at a point b is $\mathrm{Hom}_{H_{\frac{m}{n}}(S_{n/d})^{\otimes d}}(Y^{\otimes d}, \mathrm{Res}_b(M))$. Then the local system $\mathcal{L}_M$ corresponds to a representation of B_d which factors through the symmetric group S_d. Moreover, the assignment $M \mapsto \mathcal{L}_M$ is an equivalence of categories between the category $\mathcal{O}_{\frac{m}{n}}(S_n,\mathfrak{h}_n)_{ms}$ of minimally supported modules in $\mathcal{O}_{\frac{m}{n}}(S_n,\mathfrak{h}_n)$ and $\mathrm{Rep}(S_d)$. In particular, the category $\mathcal{O}_{\frac{m}{n}}(S_n,\mathfrak{h}_n)_{ms}$ is semisimple.

(iii) The equivalence of (ii) maps $L_{\frac{m}{n}}(\frac{n}{d}\mu)$ to π_μ.

3. Cohen-Macaulayness of modules of minimal support over rational Cherednik algebras

The goal of this section is to prove Theorem 1.2. We propose two proofs, given in the two subsections below. In the third subsection we give applications in the case of the symmetric group.

[3] Note that this definition is slightly different from the one in [BE], since here, unlike [BE], we don't replace $\mathfrak{h}$ with $\mathfrak{h}/\mathfrak{h}^W$.

3.1. **Proof via homological duality for rational Cherednik algebras.** Here is our first proof of Theorem 1.2. Its idea was suggested to us by R. Bezrukavnikov.

For brevity let $H := H_c(W,\mathfrak{h})$ and $R := \mathbb{C}[\mathfrak{h}]$. Let $n = \dim \mathfrak{h}$.

Proposition 3.1. Let M be a module over H which is free of finite rank over R. Then $\mathrm{Ext}^\bullet_H(M,H)$ lives in dimension n (i.e., $\mathrm{Ext}^i_H(M,H) = 0$ unless $i = n$), and there is a natural isomorphism of R-modules

$$\mathrm{Ext}^n_H(M,H) \cong M^* \otimes \wedge^n \mathfrak{h}^*,$$

where $M^* := \mathrm{Hom}_R(M,R)$.

Proof. Consider the Koszul complex of M as an H-module:

$$M \leftarrow H \otimes_{\mathbb{C}W\ltimes R} M \leftarrow H \otimes_{\mathbb{C}W\ltimes R} (M \otimes \mathfrak{h}) \leftarrow \dots \leftarrow H \otimes_{\mathbb{C}W\ltimes R} (M \otimes \wedge^n \mathfrak{h}) \leftarrow 0, \quad (2)$$

with the differential defined by

$$\partial(h \otimes m \otimes b) = \sum_j (hy_j \otimes m - h \otimes y_j m) \otimes \iota_{x_j}(b), b \in \wedge^i \mathfrak{h},$$

where $\{y_j\}$ is a basis of $\mathfrak{h}$, $\{x_j\}$ the dual basis of $\mathfrak{h}^*$, and ι is the contraction operator.

Lemma 3.2. This differential is well defined.

Proof. If $w \in W$ then

$$\partial(hw \otimes m \otimes b) = \sum_j (hwy_j \otimes m - hw \otimes y_j m) \otimes \iota_{x_j} b =$$

$$\sum_j (hw(y_j) \otimes wm - h \otimes w(y_j)wm) \otimes w\iota_{x_j} b =$$

$$\sum_j (hy_j \otimes wm - h \otimes y_j wm) \otimes \iota_{x_j} wb = \partial(h \otimes wm \otimes wb).$$

On the other hand,

$$\partial(hx_i \otimes m \otimes b) = \sum_j (hx_i y_j \otimes m - hx_i \otimes y_j m) \otimes \iota_{x_j} b =$$

$$\sum_j (hx_i y_j \otimes m - h \otimes x_i y_j m) \otimes \iota_{x_j} b =$$

$$\sum_j (hy_j x_i \otimes m - h \otimes y_j x_i m) \otimes \iota_{x_j} b + \sum_{j,s} c_s(x_i, \alpha_s^\vee)(y_j, \alpha_s)(hs \otimes m - h \otimes sm) \otimes \iota_{x_j} b =$$

$$\partial(h \otimes x_i m \otimes b) + \sum_s c_s(x_i, \alpha_s^\vee)(hs \otimes m - h \otimes sm) \otimes \iota_{\alpha_s} b.$$

Thus, it suffices to show that for each s,

$$(hs \otimes m - h \otimes sm) \otimes \iota_{\alpha_s} b = 0.$$

We have

$$(hs \otimes m - h \otimes sm) \otimes \iota_{\alpha_s} b = h \otimes sm \otimes (s-1)\iota_{\alpha_s} b.$$

So it suffices to show that $(s-1)\iota_{\alpha_s} b = 0$. This is shown in Lemma 6.2 below (in a slightly more general situation). □

The complex (2) is a resolution (i.e., exact in nonzero degrees), since its associated graded under the y-filtration (where M sits in degree 0 and $\deg(y_i) = 1$) is the usual Koszul complex of M as a $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]$-module with $y_i|_M = 0$ (which is a resolution since M is free over R). Moreover, since M is free over R, this is a projective resolution over R, and we can use it to compute the Ext groups of M with other modules (in particular, H). Computing Hom of this resolution to H, and using that

$$\mathrm{Hom}_H(H \otimes_{\mathbb{C}W \ltimes R} M, H) \cong \mathrm{Hom}_{\mathbb{C}W \ltimes R}(M, H) \cong$$

$$\mathrm{Hom}_{\mathbb{C}W \ltimes R}(M, (\mathbb{C}W \ltimes R) \otimes S) \cong M^* \otimes S \cong M^* \otimes_{\mathbb{C}W \ltimes R} H,$$

where $S = S\mathfrak{h}$, we see that the dual is a similar Koszul complex (of right H-modules) with M replaced with $M^* \otimes \wedge^n \mathfrak{h}^*$, and the statement follows. □

Corollary 3.3. If M is in the category $\mathcal{O}_c$, then there is a natural isomorphism of R-modules $\mathrm{Ext}_H^{n+i}(M, H) \cong \mathrm{Ext}_R^i(M, R) \otimes \wedge^n \mathfrak{h}^*$.

Proof. It suffices to show that the corollary holds for projectives in category $\mathcal{O}_c$; then the statement follows in general since we can replace every object by its projective resolution. But projectives admit a standard filtration, so they are free over R, and Proposition 3.1 applies. □

Now we are ready to prove Theorem 1.2. Suppose that $M \in \mathcal{O}_c$ has minimal support.

Let $c^*(s) = c(s^{-1})$. We have an antiisomorphism $\dagger : H_c(W, \mathfrak{h}) \to H_{c^*}(W, \mathfrak{h})$ defined by the formulas: $x \mapsto x$ for $x \in \mathfrak{h}^*$; $y \mapsto -y$ for $y \in \mathfrak{h}$; $s \mapsto s^{-1}$, for $s \in W$. It is shown in [GGOR], Proposition 4.10, that the homological duality functor $M \mapsto \mathrm{Ext}_H^*(M, H)^\dagger$ defines a derived antiequivalence between the categories $\mathcal{O}_c$ and $\mathcal{O}_{c^*}$. Moreover, it is clear from Corollary 3.3 that this antiequivalence preserves supports (in the sense that $\mathrm{Ext}_H^i(M, H)$ is supported on $\mathrm{supp}(M)$ for all i). This implies that the minimal supports are the same in $\mathcal{O}_c$ and $\mathcal{O}_{c^*}$.

Suppose that the support of M has dimension d. Then M is Cohen-Macaulay of dimension d at generic points of its support, so for any $i < n-d$, $\mathrm{Ext}_R^i(M, R)$ is supported in dimension $< d$. On the other hand, by Corollary 3.3, $\mathrm{Ext}_R^i(M, R) \otimes \wedge^n \mathfrak{h}^*$ is a right H_c-module, which can be turned into a left H_{c^*}-module U_i from category $\mathcal{O}_{c^*}$ by the antiisomorphism $\dagger$. Since the support of U_i is a proper subvariety of $\mathrm{supp}(M)$, by the minimality assumption for $\mathrm{supp}(M)$, we must have $U_i = 0$. This implies that M is Cohen-Macaulay.

3.2. **Proof using cohomology with support.** Here is our second proof of Theorem 1.2. Let $M \in \mathcal{O}_c$ have minimal support, and assume that M is not Cohen-Macaulay. Let Y be the non-Cohen-Macaulay locus of M in $\mathfrak{h}$ (which is a Zariski closed subset in $\mathfrak{h}$) and let u be the codimension of Y in $\mathfrak{h}$. Consider the ith cohomology group $H_Y^i(M)$ of M with support in Y. According to [Gr, Expose VIII, Cor. 2.3], $H_Y^i(M)$ is a finitely generated R-module whenever $i < u$. Similarly to the proof of Theorem 6.2.5 in [V] one needs to prove that $H_Y^i(M) = 0$ for $i < u$. Indeed, the vanishing of $H_Y^i(M)$ implies $\mathrm{Ext}_R^i(\mathbb{C}[Y], M) = 0$ for $i < u$ (see [Gr, Expose VII, Prop. 1.2]). Thus, M has depth $\geq u$ near a generic point of Y. This contradicts the condition that Y is the non-Cohen-Macaulay locus for M.

Lemma 3.4. For any closed W-stable subvariety $Z \subset \mathfrak{h}$, any H-module M, and any integer i the space $H_Z^i(M)$ admits a natural action of H extending the R-action.

Proof. Consider the endofunctor Γ_Z on the category of R-modules such that $\Gamma_Z(M)$ is the set of elements of M set-theoretically supported on Z (i.e., killed by some power of the ideal of Z). This functor is representable by the module $R^{\wedge_Z}$, the completion of R along

Z. Since Z is W-invariant, the space $H^{\wedge_Z} := H \otimes_R R^{\wedge_Z}$ has a natural algebra structure and admits algebra embeddings $R^{\wedge_Z} \hookrightarrow H^{\wedge_Z}$ and $H \hookrightarrow H^{\wedge_Z}$ (compare to [BE]). So if M is an H-module, we get, using the Frobenius reciprocity, that $\Gamma_Z(M) = \mathrm{Hom}_H(H^{\wedge_Z}, M)$. Now, $H^i_Z(\bullet) = R^i\Gamma_Z(\bullet) = \mathrm{Ext}^i_R(R^{\wedge_Z}, \bullet)$, so, using the Shapiro lemma, for an H-module M we have $H^i_Z(M) = \mathrm{Ext}^i_H(H^{\wedge_Z}, M)$ (an isomorphism of R-modules). The right hand side is definitely an H-module. □

Now we can finish the proof of the theorem. We may assume that M is irreducible. In this case, it is shown by Ginzburg, [Gi1], that $\mathrm{supp}(M)/W$ is an irreducible subvariety of $\mathfrak{h}/W$. Hence, $\mathrm{supp}(M)$ is an equidimensional variety of the form $W\mathfrak{h}_0$, where $\mathfrak{h}_0$ is a subspace of $\mathfrak{h}$. The subvariety Y is a proper subvariety in the support, W-stable because M is a W-equivariant R-module. From Lemma 3.4 and the preceding discussion we deduce that $H^i_Y(M)$ is an H-module, finitely generated over R for $i < u$. Also the support of that R-module is contained in Y. Since M, by our assumptions, has minimal support, we see that $H^i_Y(M) = 0$, which gives the desired contradiction and completes the proof of the theorem.

Remark 3.5. In fact, our assumption of the minimality of support concerned only the category $\mathcal{O}_c$. But this does not create a problem because $H^i_Y(M)$ automatically lies in the category $\mathcal{O}_c$. Indeed, M is a $\mathbb{C}^\times$-equivariant H-module. So Y is $\mathbb{C}^\times$-stable and we have a natural $\mathbb{C}^\times$-action on $H^{\wedge_Y}$ and hence also on $\mathrm{Ext}^i_H(H^{\wedge_Y}, M) = H^i_Y(M)$. So $H^i_Y(M)$ becomes a $\mathbb{C}^\times$-equivariant H-module. Since this module is finitely generated over R, it lies in the category $\mathcal{O}_c$ (see [BE]).

3.3. **Examples.** Consider now the case of type A, i.e. $W = S_n$. Let $c = \frac{r}{\ell}$, where $r, \ell \in \mathbb{Z}_{\geq 1}$, $GCD(r, \ell) = 1$. In this case, we have the following result.

Proposition 3.6. (see [BE, Example 3.25]). For $i = 0, ..., [n/\ell]$, let $X_{i,\ell}(n)$ be the union of all the S_n-translates of the subspace U_i in $\mathfrak{h}$ defined by the equations

$$x_1 = ... = x_\ell, x_{\ell+1} = ... = x_{2\ell}, ..., x_{(i-1)\ell+1} = ... = x_{i\ell}.$$

Then $X_{i,\ell}(n)$ occur as supports of modules from category $\mathcal{O}_c$, and conversely, the support of any irreducible module in $\mathcal{O}_c$ is $X_{i,\ell}(n)$ for some i.

In particular, since $X_{0,\ell}(n) \supset X_{1,\ell}(n) \supset ... \supset X_{[n/\ell],\ell}(n)$, we see that the only minimal support is $X_{[n/\ell],\ell}(n)$. So we get

Corollary 3.7. Any module $M \in \mathcal{O}_c$ with support $X_{[n/\ell],\ell}(n)$ is Cohen-Macaulay as a module over $\mathbb{C}[x_1, ..., x_n]$ (of dimension $n - 1 - [n/\ell](\ell - 1)$).

In particular, consider the irreducible module $L_c(\mathbb{C})$. We have the following known proposition:

Proposition 3.8. If $c = 1/\ell$ then $L_c(\mathbb{C}) \cong \mathbb{C}[X_{[n/\ell],\ell}(n)]$.

Proof. Let I_c be the ideal of $X_{[n/\ell],\ell}(n)$. Then it is easy to see by completing at the generic point of $X_{[n/\ell],\ell}(n)$ (as in [BE]) that I_c is invariant under the Dunkl operators (i.e., if a polynomial f vanishes on $X_{[n/\ell],\ell}(n)$, then so does the polynomial $D_y f$ for any $y \in \mathfrak{h}$, because it is so at a generic point by the completion argument). Thus, I_c is a submodule in $M_c(\mathbb{C}) = \mathbb{C}[\mathfrak{h}]$. The quotient $M_c(\mathbb{C})/I_c = \mathbb{C}[X_{[n/\ell],\ell}(n)]$ is clearly an irreducible module, since it has minimal support, and its multiplicity at generic points of the support is 1. This proves the proposition. □

This leads to the following corollary in commutative algebra, which appears to be new (note that it is used in the recent paper [BGS]).

Proposition 3.9. For any $\ell \le n$, the variety $X_{[n/\ell],\ell}(n)$ is Cohen-Macaulay.

Proof. This follows from Proposition 3.8 and Corollary 3.7. □

Now consider the situation when $\ell = n/d = n_0$, where $d \in \mathbb{Z}_{\ge 1}$, and $r = m/d$ (so $c = \frac{m}{n}$). Then the minimal support is $X_{d,n/d}(n)$, of dimension $d-1$, and by the results of [Wi] (see Theorem 2.2), the simple modules in $\mathcal{O}_c$ with this support are precisely $L_{\frac{m}{n}}(n_0\lambda)$, where λ is a partition of d (we identify the irreducible S_n-modules with the corresponding partitions). Let p_i be the i-th power sum polynomial. Then $\mathbb{C}[X_{d,n/d}(n)]$ is finite over $\mathbb{C}[p_2, \dots, p_d] = \mathbb{C}[X_{d,n/d}(n)]^{S_n}$ by the Hilbert-Noether theorem. Thus, we get the following result:

Proposition 3.10. For any partition λ of d, $L_{\frac{m}{n}}(n_0\lambda)$ is a free finite rank module over $\mathbb{C}[p_2, \dots, p_d]$.

This proposition is used below in the proof of Theorem 4.19.

3.4. **Cohen-Macaulayness of $X_{k,\ell}(n)$.** Using the above results, one can actually completely answer the question when the variety $X_{k,\ell}(n)$ is Cohen-Macaulay. Namely, we have

Proposition 3.11. [4] The variety $X_{k,\ell}(n)$ for $k > 0, \ell > 1$ is Cohen-Macaulay if and only if either $k = [\frac{n}{\ell}]$ or $\ell = 2$.

Proof. Let $r = [\frac{n}{\ell}]$. By Proposition 3.9, it suffices to show that

(1) for $\ell \ge 3$, the variety $X_{k,\ell}(n)$ is not Cohen-Macaulay for $k < r$; and

(2) $X_{k,2}(n)$ is Cohen-Macaulay for all k, n.

Let us prove (1). We first show that $X_{r-1,\ell}(n)$ is not Cohen-Macaulay for n divisible by $\ell \ge 3$ and $n > \ell$. To this end, consider a generic point a of $X_{r,\ell}(n) \subset X_{r-1,\ell}(n)$. We have $a = (a_1, ..., a_1, a_2, ..., a_2, ..., a_r, ..., a_r) \in \mathbb{C}^n$, where the a_i are distinct, and each occurs ℓ times. Consider the formal neigborhood of a in $X_{r-1,\ell}(n)$. When we pass from a to a generic point of this neighborhood, the equalities in exactly one group of ℓ equal coordinates in a have to become inequalities. This means that this neighborhood is a product of a formal polydisc of the appropriate dimension with the formal neighborhood of zero in the union of the subspaces $W_1, ..., W_r$ of dimension $\ell - 1$ inside $W_1 \oplus ... \oplus W_r$. This union is not Cohen-Macaulay by Reisner's theorem ([Re], Theorem 1) if $\ell > 2$.

Now suppose that $X_{k,\ell}(n)$ is not Cohen-Macaulay, and consider a point in $X_{k,\ell}(n+1)$ of the form $(b, ..., b, -nb)$, where $b \ne 0$. The formal neighborhood of this point in $X_{k,\ell}(n+1)$ is the product of the formal disc with the formal neighborhood of zero in $X_{k,\ell}(n)$. Thus, $X_{k,\ell}(n+1)$ is not Cohen-Macaulay, either. This completes the proof of (1).

Now let us prove (2). To this end, note that if $c = 1/\ell$, the defining ideal of $X_{k,\ell}(n)$ is inviariant under Dunkl operators (this can be checked at the generic point of $X_{k,\ell}(n)$ by using restriction functors from [BE], as in the proof of Proposition 3.8), so $\mathbb{C}[X_{k,\ell}(n)]$ is a lowest weight module over $H_c(S_n)$. Therefore, specializing to $\ell = 2$ and arguing as in the proof of Theorem 1.2, we see that if $X_{k,2}(n)$ fails to be Cohen-Macaulay, its non-Cohen-Macaulay locus has to be $X_{s,2}(n)$ for some $s > k$ (as the non-Cohen-Macaulay locus has to be the support of a module over the rational Cherednik algebra at $c = 1/2$, by the proof of Theorem

[4]We thank A. Polishchuk and S. Sam for discussions which led to this result, and S. Sam for computation of the cases $\ell = 3, n = 6, k = 1$ and $\ell = 2, n = 6, 7, k = 2$ in Macaulay-2.

1.2). Consider a generic point of $X_{s,2}(n)$, $v = (a_1, a_1, a_2, a_2, ..., a_s, a_s, b_1, ..., b_{n-2s})$, where $a_i \neq a_j$, $b_i \neq b_j$ for $i \neq j$ and $a_i \neq b_j$ for any i, j. Consider the formal neighborhood of $X_{k,2}(n)$ at the point v. When we pass from v to a generic point of this formal neighborhood, exactly $s-k$ of the equalities inside the s pairs of equal coordinates have to become inequalities. Considering differences of coordinates in the pairs, we see that this formal neighborhood is the product of a formal polydisk of the appropriate dimension with the formal neighborhood of zero in the union of the $s-k$-dimensional coordinate subspaces in $\mathbb{C}^s$. This union is Cohen-Macaulay by Reisner's theorem ([Re], Theorem 1), which is a contradiction. This means that $X_{k,2}(n)$ is always Cohen-Macualay, and (2) is proved. □

4. Characters of minimally supported modules and colored invariants of torus knots

In this section we first prove the character formula for minimally supported modules (Theorem 1.4), an then proceed to apply it to knot invariants. Namely, in [GORS, Theorem 3.6] it was shown that the HOMFLY polynomial of the (m, n) torus knot can be realized as a bigraded character of

$$\mathcal{H}_{\frac{m}{n}} := \mathrm{Hom}_{S_n}(\wedge^\bullet \mathfrak{h}_n, L_{\frac{m}{n}}),$$

where $\mathfrak{h}_n$ is the reflection representation of S_n and $L_{\frac{m}{n}} = L_{\frac{m}{n}}(\mathbb{C})$ is the unique finite-dimensional irreducible representation of the rational Cherednik algebra of type A_{n-1}. The space $L_{\frac{m}{n}}$ has a canonical q-grading, and the second a-grading is defined on $\wedge^\bullet \mathfrak{h}_n$ as exterior degree. This section extends this description to the colored HOMFLY invariants of torus knots. As an application, we prove certain positivity results for these polynomials.

4.1. **Proof of Theorem 1.4.** First let us consider the case when $r = 0$. In this case the computation was basically done in the proof of [SV, Proposition 5.13]. We reproduce the proof for reader's convenience.

The category $\mathcal{O}_c(S_n, \mathfrak{h}_n)$ is equivalent to the category of modules over the q-Schur algebra $\mathcal{S}_q(n)$, where $q := \exp(\pi\sqrt{-1}/n_0)$ (where n_0 is the denominator of c and $n = dn_0$). The equivalence was proved by Rouquier, [R], when $n_0 > 2$ and by the third author, [L4], in general. Under this equivalence the Verma module $\Delta(\nu) := M_c(\pi_\nu)$ goes to the Weyl module $W(\nu)$. Let us represent $\mathcal{S}_q(n)$ as the quotient of $U_q(\mathfrak{gl}_N)$ with some $N \geqslant n$. Then the character of $W(\nu)$ is the same as the character of the irreducible GL_N-module V_ν with highest weight ν. The simple module $L_c(n_0\nu)$ is obtained from V_ν under the pull-back with respect to the quantum Frobenius. So the character of $L_c(n_0\nu)$ is obtained from that of V_ν by replacing each summand e^μ with $e^{n_0\mu}$. This implies Theorem 1.4 in the case when $r = 0$.

Let us proceed to the case when $r > 0$. The proof will follow if we check that $L(n_0\lambda + \lambda') = \mathrm{Ind}_{S_{dn_0}\times S_r}^{S_n} L(n_0\lambda) \boxtimes L(\lambda')$, where on the right hand side we have the Bezrukavnikov-Etingof functor associated to the parabolic subgroup $S_{dn_0} \times S_r \subset S_{dn_0+r}$. First, we will provide an alternative realization of $L(n_0\lambda + \lambda')$ in terms of $L(n_0\lambda)$ and λ'.

Recall that the category $\mathcal{O}_c = \bigoplus_{n=0}^\infty \mathcal{O}_c(n)$ carries a categorical Kac-Moody action of $\hat{\mathfrak{sl}}_{n_0}$, see [Sh]. In particular, we have functors $F_i : \mathcal{O}_c(\bullet) \to \mathcal{O}_c(\bullet + 1), i = 0, 1, \ldots, n_0 - 1$. The functor F_i maps $\Delta(\lambda)$ to a module that admits a filtration with standard quotients, the quotients that occur are $\Delta(\mu)$ with μ being a diagram obtained from λ adding a box with content congruent to i modulo n_0, each $\Delta(\mu)$ with such μ occurs with multiplicity 1. Also the categorical action is highest weight in the sense of [L2].

Choose a Young tableau on λ', let $c_1, \ldots, c_r$ be the residues of boxes in the order they appear in the tableau. We claim that $L(n_0\lambda + \lambda') = F_{c_r} \ldots F_{c_2} F_{c_1} L(n_0\lambda)$. Indeed, let λ_j, for

$j = 1, \ldots, r$, denote the diagram obtained from $n_0\lambda$ by adding the first j boxes. It is enough to prove that $F_{c_j}L(\lambda_{j-1}) = L(\lambda_j)$. As the third author proved in [L2], the crystal of the categorical action on $\mathcal{O}_c$ coincides with the standard crystal of the Fock space. Therefore, the reduced c_j-signature of λ_{j-1} is a single "$-$" and $\tilde{f}_{c_j}\lambda_{j-1} = \lambda_j$. According to [CR, Proposition 5.20], $F_{c_j}L(\lambda_{j-1}) = L(\lambda_j)$.

So it remains to prove that

$$F_{c_r} \ldots F_{c_2}F_{c_1}L(n_0\lambda) = \mathrm{Ind}_{S_{dn_0}\times S_r}^{S_n} L(n_0\lambda) \boxtimes L(\lambda'). \tag{3}$$

By [SV, Proposition 5.15], the functor $\mathrm{Ind}_{S_{dn_0}\times S_\bullet}^{S_{dn_0+\bullet}} L(n_0\lambda)\boxtimes ? : \mathcal{O}_c(\bullet) \to \mathcal{O}_c(dn_0+\bullet)$ commutes with the functors F_i. Let us remark that $L(\lambda') = \Delta(\lambda')$ as the category $\mathcal{O}_c(r)$ is semisimple and so $L(\lambda') = F_{c_r} \ldots F_{c_1}L(\varnothing)$. This completes the proof of (3). □

4.2. **Standard modules revisited.** Let us first recall some facts about representations of S_d. Let λ be a Young diagram, $d = |\lambda|$, and let π_λ denote the irreducible representation of S_d corresponding to λ. It is well known (e.g. [FH, Exercise 4.17(c)], [Hur]) that the central element $\Omega = \sum_{i<j}(i\ j) \in \mathbb{C}S_d$ acts in π_λ by the constant

$$\kappa(\lambda) = \sum_{(i,j)\in\lambda}(i-j) = \frac{1}{2}\sum_j(\lambda_j - 2j + 1)\lambda_j \tag{4}$$

called the **content** of λ. Recall that the *Frobenius character* of a representation π of S_d is defined by the formula

$$\mathrm{ch}\,\pi = \frac{1}{d!}\sum_{\sigma\in S_d}\mathrm{Tr}_\pi(\sigma)p_1^{k_1(\sigma)}\ldots p_r^{k_r(\sigma)},$$

where p_i are power sums and $k_i(\sigma)$ is the number of cycles of length i in σ. The Frobenius character of π_λ is given by the Schur polynomial s_λ. The following lemma is obvious (and well known).

Lemma 4.1. Let $\mathfrak{h}$ be the $(d-1)$-dimensional reflection representation of S_d, $\sigma \in S_d$. Then

$$\det{}_{\mathfrak{h}}(1 - q\sigma) = \frac{1}{1-q}\prod_i(1-q^i)^{k_i(\sigma)} \tag{5}$$

Lemma 4.2. The following equation holds:

$$\sum_{k=0}^{d-1}(-a)^k\dim_q\mathrm{Hom}_{S_d}(\wedge^k\mathfrak{h}, M_c(\lambda)) = q^{\frac{d-1}{2}-c\kappa(\lambda)}\cdot\frac{1-q}{1-a}\cdot\theta_{a,q}(s_\lambda), \tag{6}$$

where $\theta_{a,q}$ is the character of the ring of symmetric functions defined by the formula $\theta_{a,q}(p_i) := \frac{1-a^i}{1-q^i}$.

Here, for simplicity we write $M_c(\lambda)$ for $M_c(\pi_\lambda)$.

Proof. The character of $M_c(\lambda)$ was computed in [BEG, eq. (1.5)]:

$$\mathrm{Tr}_{M_c(\lambda)}(\sigma\cdot q^{\mathbf{h}}) = \frac{q^{\frac{d-1}{2}-c\kappa(\lambda)}\,\mathrm{Tr}_{\pi_\lambda}(\sigma)}{\det{}_{\mathfrak{h}}(1-q\sigma)}.$$

By orthogonality of characters, we have

$$\dim_q\mathrm{Hom}_{S_d}(\wedge^k\mathfrak{h}, M_c(\lambda)) = \frac{1}{d!}\sum_{\sigma\in S_d}\mathrm{Tr}_{M_c(\lambda)}(\sigma\cdot q^{\mathbf{h}})\,\mathrm{Tr}_{\wedge^k\mathfrak{h}}(\sigma).$$

Since $\sum_{k=0}^{d-1}(-a)^k \operatorname{Tr}_{\wedge^k\mathfrak{h}}(\sigma) = \det_{\mathfrak{h}}(1-a\sigma)$, one can rewrite the left hand side of (6) as

$$\frac{1}{d!}\sum_{\sigma\in S_d}\operatorname{Tr}_{M_c(\lambda)}(\sigma\cdot q^{\mathbf{h}})\det{}_{\mathfrak{h}}(1-a\sigma) = q^{\frac{d-1}{2}-c\kappa(\lambda)}\frac{1}{d!}\sum_{\sigma\in S_d}\operatorname{Tr}_{\pi_\lambda}(\sigma)\frac{\det_{\mathfrak{h}}(1-a\sigma)}{\det_{\mathfrak{h}}(1-q\sigma)}.$$

By Lemma 4.1 it is equal to

$$\frac{q^{\frac{d-1}{2}-c\kappa(\lambda)}(1-q)}{1-a}\frac{1}{d!}\sum_{\sigma\in S_d}\operatorname{Tr}_{\pi_\lambda}(\sigma)\prod_i\left(\frac{1-a^i}{1-q^i}\right)^{k_i} =$$

$$\frac{q^{\frac{d-1}{2}-c\kappa(\lambda)}(1-q)}{1-a}\cdot\theta_{a,q}(\operatorname{ch}\pi_\lambda).$$

□

We will need some facts on the colored HOMFLY invariants of knots in the three-sphere. Given a knot K and a Young diagram λ, one can define a rational function $P_\lambda(K)(a,q)$ in variables a and q. We refer the reader to [AM],[LM],[MM], [Resh] for the precise mathematical definitions. The colored $\mathfrak{sl}_N$ invariant $P_{\lambda,N}(K)(q)$ (which can be defined using quantum groups, see e.g. [Resh]) coincides with the specialization of the HOMFLY invariant: $P_{\lambda,N}(K)(q) = P_\lambda(K)(q^N,q)$.

For example, the $\mathfrak{sl}_N$ invariant of the unknot colored by a diagram λ equals to the q-character of the corresponding irreducible representation V_λ, which is equal to

$$P_{N,\lambda}(q) = s_\lambda(q^{\frac{1-N}{2}}, q^{\frac{2-N}{2}},\dots,q^{\frac{N-1}{2}}) = q^{\frac{1-N}{2}|\lambda|}\prod_{(i,j)\in\lambda}\frac{(1-q^{N+i-j})}{(1-q^{h(i,j)})},$$

where $h(i,j)$ is the hook-length for a box $(i,j)\in\lambda$.

Proposition 4.3. The HOMFLY polynomial of the unknot colored by a Young diagram λ equals to

$$P_\lambda(a,q) = \left(\frac{q}{a}\right)^{\frac{|\lambda|}{2}}\prod_{(i,j)\in\lambda}\frac{(1-aq^{i-j})}{(1-q^{h(i,j)})} = \left(\frac{q}{a}\right)^{\frac{|\lambda|}{2}}\theta_{a,q}(s_\lambda). \tag{7}$$

Proof. Note that if $\{x_i\} = \{1,q,\dots,q^{N-1}\}$ then $p_i = \frac{1-q^{iN}}{1-q^i}$, so

$$s_\lambda(1,q,\dots,q^{N-1}) = \theta_{q^N,q}(s_\lambda).$$

Since s_λ is a homogeneous polynomial of degree $|\lambda|$, we get

$$P_{N,\lambda}(q) = s_\lambda(q^{\frac{1-N}{2}}, q^{\frac{2-N}{2}},\dots,q^{\frac{N-1}{2}}) = q^{\frac{1-N}{2}|\lambda|}\theta_{q^N,q}(s_\lambda).$$

If we replace q^N by a, we get

$$P_\lambda(a,q) = \left(\frac{q}{a}\right)^{\frac{|\lambda|}{2}}\theta_{a,q}(s_\lambda).$$

□

Corollary 4.4. The character $\sum_{k=0}^{d-1}(-a)^k\dim_q\operatorname{Hom}_{S_d}\left(\wedge^k\mathfrak{h}, M_c(\lambda)\right)$ of the hook-labeled isotypic components of $M_c(\lambda)$ equals $q^{-m_0n_0\kappa(\lambda)}\widetilde{P}_\lambda(a,q)$.

Proof. Follows from equations (6), (7), and (1). □

Remark 4.5. Corollary 4.4 can be explained in more combinatorial way. We have an isomorphism $\mathrm{Hom}_{S_d}(\wedge^\bullet\mathfrak{h}, M_c(\lambda)) = \mathrm{Hom}_{S_d}(\pi_\lambda, \mathbb{C}[\mathfrak{h}]\otimes\wedge^\bullet\mathfrak{h})$. The space $\mathbb{C}[\mathfrak{h}]\otimes\wedge^\bullet\mathfrak{h}$ is naturally bigraded: the q-grading is the polynomial degree and the a-grading is the degree of an exterior form. It is known that the q-grading defined by eigenvalues of $\mathbf{h}$ differs from the polynomial grading by a constant. The bigraded character of the isotypic component of π_λ in this space was computed in [KP] (see also [Mol]): it is equal to

$$\prod_{(i,j)\in\lambda}\frac{q^{i-1}+aq^{j-1}}{1-q^{h(i,j)}}.$$

It remains to compare this formula with (7).

4.3. **Representations with minimal support and torus knots.** Let Λ be the ring of symmetric polynomials in infinitely many variables. Let us define the *Adams operations* on Λ by the formula

$$\Psi_k(f)(x_1,x_2,\ldots) = f(x_1^k,x_2^k,\ldots).$$

Note that $\Psi_k:\Lambda\to\Lambda$ are ring homomorphisms and $\Psi_k\circ\Psi_m=\Psi_{km}$. We refer the reader to [Gor1] and references therein for more details on Adams operations.

Definition 4.6. Let us define the coefficients c^{μ}_{λ,n_0} by the equation

$$\Psi_{n_0}(s_\lambda) = \sum_{|\mu|=n_0|\lambda|} c^{\mu}_{\lambda,n_0}s_\mu. \tag{8}$$

Theorem 4.7. ([RJ], see also [LZ],[Ste]) The HOMFLY polynomial of the λ-colored (m_0,n_0) torus knot can be computed using the formula

$$P_\lambda(T(m_0,n_0)) = q^{m_0n_0\kappa(\lambda)}a^{\frac{m_0(n_0-1)|\lambda|}{2}}\sum_{\mu}c^{\mu}_{\lambda,n_0}t^{-\frac{m_0}{n_0}\kappa(\mu)}P_\mu(a,q),$$

where $\kappa(\mu)$ is defined by (4).

Proof of Corollary 1.5:

By Theorem 1.4

$$[L_{\frac{m}{n}}(n_0\lambda)] = \sum_{|\mu|=n}c^{\mu}_{\lambda,n_0}[M_c(\mu)]. \tag{9}$$

Consider a linear map $\mathcal{F}:K_0[\mathcal{O}_c(S_n,\mathfrak{h})]\to\mathbb{C}[[a,q]]$ defined by the equation

$$\mathcal{F}([V]) = \sum_{k=0}^{n-1}(-a)^k\dim_q\mathrm{Hom}_{S_n}\left(\wedge^k\mathfrak{h}_n, V\right).$$

By Corollary 4.4, we have

$$\mathcal{F}(M_c(\mu)) = q^{-\frac{m}{n}\kappa(\mu)}a^{\frac{n}{2}}\frac{q^{-1/2}-q^{1/2}}{1-a}P_\mu(a,q),$$

so by (9) we get

$$\mathcal{F}(L_{\frac{m}{n}}(n_0\lambda)) = a^{\frac{n}{2}}\frac{q^{-1/2}-q^{1/2}}{1-a}\sum_{|\mu|=n}c^{\mu}_{\lambda,n_0}q^{-\frac{m}{n}\kappa(\mu)}P_\mu(a,q) =$$

$$q^{-m_0n_0\kappa(\lambda)}a^{\frac{(m_0+n_0-m_0n_0)d}{2}}\frac{q^{-1/2}-q^{1/2}}{1-a}P_\lambda(T(m_0,n_0))(a,q).$$

The last equation follows from Theorem 4.7.□

Corollary 4.8. Consider the space

$$\mathcal{H}_{\frac{m}{n}}(\lambda)=\bigoplus_{k=0}^{n-1}\mathrm{Hom}_{S_n}\left(\wedge^k\mathfrak{h}_n, L_{\frac{m}{n}}(n_0\lambda)\right).$$

It carries a q-grading obtained from the q-grading on $L_{\frac{m}{n}}(n_0\lambda)$, and an a-grading by the exterior degree of $\mathfrak{h}_n$. Then the (a,q)-bigraded characters of $\mathcal{H}_{\frac{m}{n}}(\lambda)$ and $\mathcal{H}_{\frac{n}{m}}(\lambda)$ coincide.

Proof. By Corollary 1.5 these characters compute the λ-colored HOMFLY invariants of the (m_0,n_0) and (n_0,m_0) torus knots respectively. Since the knots are topologically equivalent in S^3, their colored invariants coincide. □

Remark 4.9. Indeed, this coincidence of (a,q)-characters also follows from Theorem 1.10, which shows an isomorphism between $\mathcal{H}_{\frac{m}{n}}(\lambda)$ and $\mathcal{H}_{\frac{n}{m}}(\lambda)$.

Corollary 4.10. Let $\widetilde{P}_\lambda(T(m_0,n_0))(a,q)$ denote, as above, the renormalized λ-colored *unreduced* HOMFLY invariant of the (m_0,n_0) torus knot. Then $\widetilde{P}_\lambda(T(m_0,n_0))(-a,q)$ (and hence $P_\lambda(T(m_0,n_0))(-a,q)$, with an appropriate normalization by a power of $-a$) is a polynomial in a of degree $\min(m_0,n_0)\cdot|\lambda|-1$ and a power series in q with nonnegative coefficients.

4.4. **Invariants of torus links.** Let m,n be two positive integers, $d=GCD(m,n)$. One can consider the (m,n) torus link with d components and compute its quantum invariants.

Theorem 4.11. The uncolored HOMFLY polynomial of the (m,n) torus link is given as the following linear combination of characters of the minimal support representations:

$$P_{\Box}(T(m,n))(a,q)=\sum_{|\lambda|=d}\dim\pi_\lambda\cdot\mathrm{ch}_{a,q}\,\mathcal{H}_{\frac{m}{n}}(\lambda),$$

where, as above,

$$\mathcal{H}_{\frac{m}{n}}(\lambda)=\bigoplus_{i=0}^{n-1}\mathrm{Hom}_{S_n}\left(\wedge^i\mathfrak{h}_n, L_{\frac{m}{n}}\left(\frac{n}{d}\lambda\right)\right).$$

Proof. Let C denote a cycle of length n in S_n. By [RJ, Theorem 8], the HOMFLY polynomial of $T(m,n)$ can be presented as following:

$$P_{\Box}(T(m,n))(a,q)=\sum_{|\mu|=n}q^{-\frac{m}{n}\kappa(\mu)}\,\mathrm{Tr}_{\pi_\mu}(C^m)\cdot P_\mu(a,q).$$

Since C^m is a product of d cycles of length $n_0=n/d$, we have

$$\mathrm{Tr}_{\pi_\mu}(C^m)=\left\langle(p_{n_0})^d,s_\mu\right\rangle=\left\langle\Psi_{n_0}(p_1)^d,s_\mu\right\rangle=$$

$$\left\langle\Psi_{n_0}\sum_{|\lambda|=d}\dim(\pi_\lambda)s_\lambda,s_\mu\right\rangle=\sum_{|\lambda|=d}\dim(\pi_\lambda)\cdot c^{\mu}_{\lambda,n_0}.$$

It remains to apply Theorem 1.4 and Corollary 1.5. □

Corollary 4.12. The series $P_{\Box}(T(m,n))(-a,q)$ (renormalized by a suitable power of $-a$) has nonnegative coefficients.

Remark 4.13. In fact, our argument implies a stronger statement: the function

$$(1+a)^{-1}P_{\square}(T(m,n))(-a,q)\cdot\prod_{i=1}^{d}(1-q^i)$$

is a polynomial with nonnegative coefficients.

Example 4.14. Let us compute the HOMFLY polynomial for the Hopf link, i.e., the $(2,2)$ torus link. We have $m=n=d=2$, $c=1$, the Verma modules $M_{c=1}(\lambda)$ are irreducible and $\dim\pi_\lambda=1$. One can check that

$$\mathrm{ch}_{a,q}(\mathcal{H}_1(2))=q^{-\frac{1}{2}}\frac{1+aq}{1-q^2},\ \mathrm{ch}_{a,q}(\mathcal{H}_1(1,1))=q^{\frac{5}{2}}\frac{1+aq^{-1}}{1-q^2}.$$

Therefore

$$\mathrm{ch}_{a,q}(\mathcal{H}_1(2))+\mathrm{ch}_{a,q}(\mathcal{H}_1(1,1))=q^{-\frac{1}{2}}\frac{1+q^3+aq(1+q)}{1-q^2}.$$

This coincides with the known answer for the HOMFLY polynomial (e.g. [OS, Example 4]). Note that one can cancel the factor $(1+q)$ to get $q^{-\frac{1}{2}}\frac{1-q+q^2+aq}{1-q}$, but this destroys the non-negativity of the coefficients in the numerator.

4.5. **Duality of characters.** Let ω be the involution of the ring of symmetric functions Λ defined by the equation $\omega(p_k)=(-1)^{k-1}p_k$. It is well known that $\omega(s_\lambda)=s_{\lambda^t}$.

Lemma 4.15. Let f be a symmetric function of degree d. Then

$$\omega(\Psi_m(f))=(-1)^{(m-1)d}\Psi_m(\omega(f)).$$

Proof. It is sufficient to check the statement for the power sums $f=p_d$:

$$\omega(\Psi_m(p_d))=(-1)^{md-1}p_{md},\ \Psi_m(\omega(p_d))=(-1)^{d-1}p_{md}=(-1)^{(m-1)d}\Psi_m(\omega(p_d)).$$

□

Corollary 4.16. The coefficients c^{ν}_{λ,n_0} satisfy the equation

$$c^{\nu^t}_{\lambda^t,n_0}=(-1)^{(n_0-1)|\lambda|}c^{\nu}_{\lambda,n_0}. \tag{10}$$

Proof. This follows from Lemma 4.15. □

Theorem 4.17. The characters of $L_c(n_0\lambda)$ and of $L_c(n_0\lambda^t)$ are related (as rational functions in q) by the equation

$$\mathrm{Tr}_{L_c(n_0\lambda^t)}(\sigma q^{\mathbf{h}})=(-1)^{|\lambda|-1}\,\mathrm{Tr}_{L_c(n_0\lambda)}(\sigma q^{-\mathbf{h}}).$$

Proof. By Theorem 1.4 we have

$$\mathrm{Tr}_{L_c(n_0\lambda^t)}(\sigma q^{\mathbf{h}})=\sum_{\nu}c^{\nu^t}_{\lambda^t,n_0}q^{\frac{n-1}{2}-c\kappa(\nu^t)}\chi_{\nu^t}(\sigma)\det{}_{\mathfrak{h}}(1-q\sigma)^{-1}.$$

By (10) we can rewrite this as

$$\sum_{\nu}(-1)^{(n_0-1)|\lambda|}c^{\nu}_{\lambda,n_0}q^{\frac{n-1}{2}+c\kappa(\nu)}(-1)^{\mathrm{sgn}\,\sigma}\chi_{\nu}(\sigma)\det{}_{\mathfrak{h}}(1-q\sigma)^{-1}.$$

On the other hand,

$$\det{}_{\mathfrak{h}}(1-q^{-1}\sigma)=(-1)^{n-1-\mathrm{sgn}(\sigma)}q^{-(n-1)}\det{}_{\mathfrak{h}}(1-q\sigma),$$

hence

$$\mathrm{Tr}_{L_c(n_0\lambda)}(\sigma q^{-\mathbf{h}}) = \sum_{\nu} c^{\nu}_{\lambda,n_0} q^{-\frac{n-1}{2}+c\kappa(\nu)}(-1)^{n-1-\mathrm{sgn}(\sigma)} q^{(n-1)}\chi_\nu(\sigma)\det{}_{\mathfrak{h}}(1-q\sigma)^{-1}.$$

□

Remark 4.18. The statement of Theorem 4.17 should be understood as follows. For nontrivial λ the representation $L_c(n_0\lambda)$ is infinite-dimensional, so its character is an infinite series in q. On the other hand, by Proposition 3.10 this character is a rational function in q of the form

$$\mathrm{ch}\, L_c(n_0\lambda) = \frac{Q_c(n_0\lambda)(q)}{(1-q^2)\cdots(1-q^d)},$$

where $d=|\lambda|$. Theorem 4.17 provides a functional equation for this rational function which is equivalent to the functional equation for its numerator (which is a Laurent polynomial with nonnegative coefficients):

$$Q_c(n_0\lambda^t)(q) = q^{\frac{d(d+1)}{2}-1} Q_c(n_0\lambda)(q^{-1}).$$

4.6. **Reduced colored invariants.** In knot theory one has a notion of the reduced HOMFLY invariant. By definition, it is equal to the normalization of the unreduced λ-colored HOMFLY invariant of a knot K by the unreduced λ-colored HOMFLY invariant of the unknot:

$$P^{red}_\lambda(K) = P_\lambda(K)/P_\lambda(T(1,0)).$$

Motivated by Proposition 3.10, we define *partially reduced* λ-colored HOMFLY invariants by the formula

$$\widehat{P}_\lambda(K) := P_\lambda(K)\cdot\prod_{i=1}^{|\lambda|}(1-q^i).$$

Theorem 4.19. The function $\widehat{P}_\lambda(K)$ has the following properties:

a) $\widehat{P}_\lambda(K)$ is a polynomial in a and q for any knot K.
b) For a torus knot $T(m_0,n_0)$, all the coefficients of the polynomial $\widehat{P}_\lambda(T(m_0,n_0))(-a,q)$ (after renormalizing by a power of $-a$) are nonnegative.
c) The sum of the coefficients of $\widehat{P}_\lambda(T(m_0,n_0))(-a,q)$ equals to

(11)
$$\widehat{P}_\lambda(T(m_0,n_0))(a=-1,q=1) = (\widehat{P}_{(1)}(T(m_0,n_0))(-1,1))^d\cdot\dim\pi_\lambda = (2\dim\mathcal{H}_{\frac{m_0}{n_0}})^d\cdot\dim\pi_\lambda,$$

where $d=|\lambda|$ and π_λ is the irreducible representation of S_d corresponding to λ.

Remark 4.20. In fact, our argument implies a stronger statement that (b): the function $(1+a)^{-1}(1-q)\widehat{P}_\lambda(T(m_0,n_0))(-a,q)$ (after renormalizing by a power of $-a$) is a polynomial with nonnegative coefficients.

Proof. a) We have

$$\widehat{P}_\lambda(K) := P_\lambda(K)\cdot\prod_{i=1}^{d}(1-q^i) = P^{red}_\lambda(K)\cdot P_\lambda(T(1,0))\cdot\prod_{i=1}^{d}(1-q^i).$$

It is known that the function $P_\lambda^{red}(K)$ is a polynomial, and the product of the remaining factors is a polynomial too:

$$P_\lambda(T(1,0))\cdot\prod_{i=1}^{d}(1-q^i)=\prod_{x\in\lambda}(1-aq^{c(x)})\cdot\frac{\prod_{i=1}^d(1-q^i)}{\prod_{x\in\lambda}(1-q^{h(x)})}.$$

Indeed, e.g. by [KP]

$$\dim_q(\pi_\lambda)=\frac{\prod_{i=1}^d(1-q^i)}{\prod_{x\in\lambda}(1-q^{h(x)})}$$

is a polynomial in q with nonnegative coefficients.

b) By Proposition 3.10 the module $L_{\frac{m}{n}}(n_0\lambda)$ is free over $\mathbb{C}[p_2,\ldots,p_d]$:

$$L_{\frac{m}{n}}(n_0\lambda)=N(n_0,m_0,\lambda)\otimes\mathbb{C}[p_2,\ldots,p_d],$$

where $N(n_0,m_0,\lambda)$ is a certain finite-dimensional graded S_n-module. It remains to note that

$$\widehat{P}_\lambda(T(m_0,n_0))=\sum_{j=0}^{n-1}(-a)^j\dim_q\operatorname{Hom}_{S_n}(\wedge^j\mathfrak{h},N(n_0,m_0,\lambda)). \tag{12}$$

c) By (12) the number $\widehat{P}_\lambda(T(m_0,n_0))(-1,1)$ equals

$$\widehat{P}_\lambda(T(m_0,n_0))(-1,1)=\dim\operatorname{Hom}_{S_n}(\wedge^\bullet\mathfrak{h},N(n_0,m_0,\lambda)).$$

Let us use the restriction functor [BE] to compute this dimension. The stabilizer of a generic point b in $X_{d,n/d}(n)$ is isomorphic to $(S_{n_0})^d$, and

$$\operatorname{Res}^{S_{n_0d}}_{(S_{n_0})^d}L_{\frac{m_0}{n_0}}(n_0\lambda)\simeq(L_{\frac{m_0}{n_0}}(\mathbb{C}))^{\otimes d}\otimes E_\lambda$$

for an S_d-module E_λ. It follows from [Wi] that $E_\lambda=\pi_\lambda$. □

Remark 4.21. Equation (11) is similar to the "power growth" phenomenon in the colored Khovanov-Rozansky homology, conjectured by S. Gukov and M. Stosic in [GS, Sec. 4.5.2].

Example 4.22. Consider the $(2,3)$ torus knot colored by $\lambda=(2,1)$. One can check using Theorem 4.7 that (up to an overall monomial factor)

$$P_\lambda^{red}(T(2,3))=1+2q^2-q^3+2q^4+2q^6-q^7+2q^8+q^{10}-a(1+2q^2+3q^4+3q^6+2q^8+q^{10})+$$
$$+a^2(q^2+q^3+q^4+q^6+q^7+q^8)-a^3q^5.$$

We see that $P_\lambda^{red}(T(2,3))(-a,q)$ has two negative coefficients, while all coefficients in

$$\widehat{P}_\lambda(T(2,3))=\frac{(1-a)(1-aq)(1-aq^{-1})(1-q)(1-q^2)(1-q^3)}{(1-q)^2(1-q^3)}P_\lambda^{red}(T(2,3))=$$
$$(1-a)(1-aq)(1-aq^{-1})(1+q)P_\lambda^{red}(T(2,3))$$

have the right sign, and

$$\widehat{P}_\lambda(T(2,3))(-1,1)=8\cdot 54=(2\cdot 3)^{|\lambda|}\cdot 2.$$

Indeed, $\dim\pi_\lambda=2$ and $\widehat{P}_{(1)}(T(2,3))(a,q)=(1-a)(1+q^2-aq)$, so $\widehat{P}_{(1)}(T(2,3))(-1,1)=2\cdot 3$.

5. Characters of equivariant D-modules

5.1. **Proof of Theorem 1.7.** Let us use Theorem 1.4 to prove Theorem 1.7. Let M be an SL_m-equivariant D-module with central character θ_s, $GCD(m,s)=d$, labeled by the Young diagram $d\lambda$. Let $M^{(n)}$ be the isotypic part of M for the representations V_μ of SL_m which occur in $V^{\otimes n}$.

Define the automorphism $\varphi_{\frac{1}{1-q}}$ of the ring of symmetric functions as follows:

$$\varphi_{\frac{1}{1-q}}(p_k) = \frac{p_k}{1-q^k}. \tag{13}$$

Note that

$$\varphi_{\frac{1}{1-q}}(f)(x_1,...,x_m,0,0,...) = f(x_1,...,x_m,qx_1,...,qx_m,q^2x_1,...).$$

By [CEE, Theorem 9.8], $(\mathcal{F}(M)\otimes(\mathbb{C}^m)^n)^{\mathfrak{sl}_m} \simeq L_{\frac{m}{n}}(n_0\lambda)$, where $\mathcal{F}$ denotes the Fourier transform. Therefore, since the Fourier transform changes H to $-H$, by the Schur-Weyl duality, we have $\mathrm{Ch}_{\mathfrak{sl}_m}(M^{(n)}) = \mathrm{ch}_{S_n} L_{\frac{m}{n}}(n_0\lambda)$ (where the right hand side is the character of the $\mathfrak{sl}_m$-module, and the left hand side is the Frobenius character of the corresponding S_n-module).

By the proof of Corollary 1.5, we have

$$\mathrm{ch}_{S_n} L_{\frac{m}{n}}(n_0\lambda) = \sum_\nu c^\nu_{\lambda,n_0} q^{\frac{n-1}{2}-\frac{m}{n}\kappa(\nu)} \mathrm{ch}_{S_n} M_{\frac{m}{n}}(\nu).$$

Since $M_{\frac{m}{n}}(\nu) = \pi_\nu \otimes \mathbb{C}[\mathfrak{h}_n]$, we have $\mathrm{ch}_{S_n} M_{\frac{m}{n}}(\nu) = \varphi_{\frac{1}{1-q}}(s_\nu)$. This implies Theorem 1.7.

5.2. **Character formulas.** Recall that by [CEE, Corollary 8.10] the character of the D-module for SL_m corresponding to the partition (1^m) is given by the equation

$$M_{(1^m)} = q^{\frac{m^2-1}{2}} \sum_\mu \frac{V_\mu P_\mu(q)}{(1-q^2)\cdots(1-q^m)}, \tag{14}$$

where V_μ is the irreducible representation of SL_m labelled by μ (so that μ is a partition with at most m parts), and $P_\mu(q)$ is the q-analogue of the multiplicity of zero weight in V_μ (cf. [K],[Lu],[Gu]). By the Schur-Weyl duality one can get the character of the corresponding representation of the Cherednik algebra:

$$\mathrm{ch}\, L_{\frac{1}{n}}(n(1^m)) = q^{\frac{m^2-1}{2}} \sum_{|\mu|=nm} \frac{s_\mu P_\mu(q)}{(1-q^2)\cdots(1-q^m)},$$

where s_μ denotes the Schur polynomial labeled by μ.

Remark 5.1. The polynomials $P_\mu(q)$ are a special case of the Kostka-Foulkes polynomials. Indeed, the zero weight for SL_m can be represented by the Young diagram $n(1^m) = (n^m)$, and

$$P_\mu(q) = K_{\mu,(n^m)}(q), \quad \sum_{|\mu|=nm} s_\mu P_\mu(q) = Q'_{(n^m)}.$$

Here $Q'_{(n^m)}$ is a transformation of the corresponding Hall-Littlewood polynomial $Q_{(n^m)}$ ([M], [DLT]):

$$Q'_{(n^m)} = \varphi_{\frac{1}{1-q}}(Q_{(n^m)}),$$

where the map $\varphi_{\frac{1}{1-q}}$ is defined by (13). Therefore

$$\operatorname{ch} L_{\frac{1}{n}}(n(1^m)) = \frac{q^{\frac{m^2-1}{2}} Q'_{(n^m)}}{(1-q^2)\cdots(1-q^m)}.$$

This agrees with the observation in [MMS] that the "extended HOMFLY polynomial" of the $(1,n)$ torus knot colored with the diagram (1^m) is given by the Hall-Littlewood polynomial $Q_{(n^m)}$.

We can use Theorem 4.17 to get similar answers for $\lambda = (m)$:

$$M_{(m)} = q^{\frac{m-1}{2}} \sum_{\mu} \frac{V_\mu P_\mu(q^{-1})}{(1-q^2)\cdots(1-q^m)}, \tag{15}$$

$$\operatorname{ch} L_{\frac{1}{n}}(n(m)) = q^{\frac{m-1}{2}} \sum_{|\mu|=nm} \frac{s_\mu P_\mu(q^{-1})}{(1-q^2)\cdots(1-q^m)}.$$

Finally, similarly to [CEE, Theorem 9.18] (and using Lemma 6.13 below) one gets the following equation in the Grothendieck group of representations of SL_m:

$$\sum_{i=0}^{m-1} (-1)^i [M_{(m-i,1^i)}] = \sum_{\mu} \frac{q^{-d(\mu)/2}}{[m]_q} [V_\mu] \dim_q V_\mu, \tag{16}$$

where $[m]_q = (1-q^m)/(1-q)$, $\dim_q(V_\mu)$ is the (non-symmetrized) q-dimension of V_μ, and $d(\mu) = \deg(\dim_q V_\mu) - m + 1$.

5.3. **Equivariant D-modules for SL_2.** We have $\mu = (l_1, l_2)$ with $l_1 + l_2 = 2n$, and $V_\mu \simeq V_{l_1-l_2}$. Since l_1 and l_2 have same parity, zero weight is present in V_μ with multiplicity 1 and $P_\mu(q) = q^{\frac{l_1-l_2}{2}}$. Therefore by (14) and (15) one gets

$$M_{(2)} = \sum_{j=0}^{\infty} V_{2j} \frac{q^{-j+1/2}}{1-q^2}, \quad M_{(1,1)} = \sum_{j=0}^{\infty} V_{2j} \frac{q^{j+3/2}}{1-q^2},$$

$$\operatorname{ch} L_{\frac{1}{n}}(n(2)) = \sum_{l_1+l_2=2n} s_{(l_1,l_2)} \frac{q^{(l_2-l_1+1)/2}}{1-q^2}, \quad \operatorname{ch} L_{\frac{1}{n}}(n(1,1)) = \sum_{l_1+l_2=2n} s_{(l_1,l_2)} \frac{q^{(l_1-l_2+3)/2}}{1-q^2}.$$

Note that

$$[M_{(2)} - M_{(1,1)}] = \sum_{j=0}^{\infty} [V_{2j}] \frac{q^{-j+1/2} - q^{j+3/2}}{1-q^2} =$$

$$\sum_{j=0}^{\infty} \frac{q^{-j+1/2}(1-q^{2j+1})}{1-q^2} [V_{2j}] = \sum_{j=0}^{\infty} \frac{q^{-j+1/2}}{1+q} [V_{2j}] \dim_q V_{2j},$$

what agrees with (16).

5.4. **Equivariant D-modules for SL_3.** We have $\mu = (l_1, l_2, l_3)$ with $l_1 + l_2 + l_3 = 3n$, and $V_\mu \simeq V_{(l_1-l_3, l_2-l_3)} = V_{(\mu_1,\mu_2)}$. The q-dimension of V_μ equals

$$\dim_q V_\mu = \frac{[\mu_1+2]_q[\mu_1-\mu_2+1]_q[\mu_2+1]_q}{[2]_q}.$$

Let $x = \min(\mu_1 - \mu_2 + 1, \mu_2 + 1)$, then one can check that $P_\mu(q) = q^{\mu_1-x+1}[x]_q$. Therefore by (14) and (15) we have

$$M_{(3)} = \sum_\mu V_\mu \frac{q^{-\mu_1+1}[x]_q}{(1-q^2)(1-q^3)},\ M_{(1,1,1)} = \sum_\mu V_\mu \frac{q^{\mu_1-x+5}[x]_q}{(1-q^2)(1-q^3)}. \tag{17}$$

To compute the character of $M_{(2,1)}$, note that

$$\dim_q V_\mu = \frac{[\mu_1+2]_q[x]_q[\mu_1+2-x]_q}{[2]_q},$$

hence by (16) one gets

$$[M_{(3)} - M_{(2,1)} + M_{(1,1,1)}] = \sum_\mu \frac{q^{-\mu_1+1}}{[3]_q}[V_\mu]\dim_q V_\mu =$$

$$\sum_\mu \frac{q^{-\mu_1+1}[x]_q}{(1-q^2)(1-q^3)}(1-q^{\mu_1+2})(1-q^{\mu_1+2-x})[V_\mu] =$$

$$\sum_\mu \frac{[x]_q}{(1-q^2)(1-q^3)}(q^{-\mu_1+1} - q^3 - q^{3-x} + q^{\mu_1-x+5})[V_\mu],$$

therefore

$$M_{(2,1)} = \sum_\mu V_\mu \frac{(q^3+q^{3-x})[x]_q}{(1-q^2)(1-q^3)} = \sum_\mu V_\mu \frac{q^{3-x}[2x]_q}{(1-q^2)(1-q^3)}. \tag{18}$$

The character formulas for the corresponding representations of Cherednik algebras immediately follow from (17) and (18).

5.5. **The "small" part of the equivariant D-modules with trivial central character.** Consider the special case $s = 0$, so that $d = m$. Let λ be a partition of m, and consider the equivariant D-module M_λ attached to the nilpotent orbit corresponding to λ. Also, let μ be another partition of m, and V_μ be the corresponding "small" representation of SL_m (in the sense of A. Broer, [Br]), i.e., one occurring in $(\mathbb{C}^m)^{\otimes m}$. Consider the isotypic component of V_μ in M_λ, and let us compute its character. [5] We may take $n = m$, so $n_0 = 1$, the Verma modules are irreducible, and thus the formula of Theorem 1.7 is greatly simplified:

$$\mathrm{Ch}_{M_\lambda^{(m)}}(q,g) = (1-q)q^{\frac{m-1}{2}-\kappa(\lambda)}s_\lambda(x_1,...,x_m,qx_1,...,qx_m,q^2x_1,...).$$

Thus, to compute the character of the multiplicity space for V_μ, we need to find the coefficient of $s_\mu(x_1,...,x_m)$ in the decomposition of $s_\lambda(x_1,...,x_m,qx_1,...,qx_m,q^2x_1,...)$ with respect to Schur functions.

[5] The dual representations to these V_μ are also small representations in the sense of Broer, but we don't have to consider them, since each M_λ is clearly self-dual as a graded SL_m-module (being stable under outer automorphisms of SL_m), so the characters of the multiplicity spaces for V_μ and V_μ^* are the same.

Let π_λ be the representation of S_m corresponding to λ, and $E_{\lambda,\mu}(q)$ be the character of the multiplicity space of π_λ in $\pi_\mu \otimes S\mathbb{C}^m$ (with grading defined by $\deg(\mathbb{C}^m) = 1$). It is easy to see that the desired coefficient equals $E_{\lambda,\mu}(q)$. Thus, we get

$$\mathrm{Ch}_{\mathrm{Hom}_{SL_m}(V_\mu, M_\lambda)}(q) = (1-q)q^{\frac{m-1}{2}-\kappa(\lambda)}E_{\lambda,\mu}(q).$$

In particular, consider the special case $V_\mu = \mathbb{C}$ (i.e., $\mu = (1^m)$), which gives the character of the invariants $M_\lambda^{SL_m}$. We have that $E_{\lambda,(1^m)}(q)$ is the character of the multiplicity space of π_{λ^t} in $S\mathbb{C}^m$, where λ^t is the dual partition to λ. This character is well known to equal a power of q times the reciprocal of the hook polynomial ([M]):

$$E_{\lambda,(1^m)}(q) = q^{\sum(i-1)\lambda_i^t}\prod_{x\in\lambda}(1-q^{h(x)})^{-1}.$$

This implies that

$$\mathrm{Ch}_{M_\lambda^{SL_m}}(q) = q^{\frac{m-1}{2}+\sum_{i\geq 1}(i-1)\lambda_i}(1-q)\prod_{x\in\lambda}(1-q^{h(x)})^{-1}.$$

Thus, we see that

$$\sum_\lambda \mathrm{Ch}_{M_\lambda^{SL_m}}(q)\pi_\lambda$$

is $q^{\frac{m-1}{2}}\det_{\mathfrak{h}_m}(1-q\sigma)^{-1}$, where $\det_{\mathfrak{h}_m}(1-q\sigma)^{-1}$ is the graded character of $S\mathfrak{h}_m$ as an S_m-module.

6. The Koszul-BGG complex for rational Cherednik algebras

6.1. **The definition of the Koszul-BGG complex.** We keep the notation of Section 2. Let $V \subset S\mathfrak{h}^* = M_c(\mathbb{C})$ be a representation of W where $\mathrm{rank}(1-s) \leq 1$ for all $s \in S$ (this includes, for instance, the Galois twists of the reflection representation for complex reflection groups). Assume that V is singular, i.e., the Dunkl operators act on V by zero. We will attach to V a complex of $H_c(W,\mathfrak{h})$-modules from category $\mathcal{O}_c$, called the Koszul-BGG complex.

For $s \in S$ let $0 \neq \beta_s^* \in V^*$ be such that s acts trivially on $\mathrm{Ker}\,\beta_s^*$, and let $s\beta_s^* = \mu_s\beta_s^*$. Let $\beta_s \in V$ be such that $s\beta_s = \mu_s^{-1}\beta_s$ and $(\beta_s, \beta_s^*) = 1$.

We have the Koszul complex $K^\bullet(V)$, where $K^i(V) = S\mathfrak{h}^* \otimes \wedge^i V = M_c(\wedge^i V)$.

Proposition 6.1. The complex $K^\bullet(V)$:

$$M_c(\mathbb{C}) \leftarrow M_c(V) \leftarrow M_c(\wedge^2 V) \leftarrow ...$$

is a complex of $H_c(W,\mathfrak{h})$-modules.

Proof. By definition, the Koszul complex is a complex of $\mathbb{C}W \ltimes S\mathfrak{h}^*$-modules. So we need to show that the Koszul differential d commutes with the Dunkl operators. Let $f \in S\mathfrak{h}^*$, $u \in \wedge^m V$. We have

$$d(f\otimes u) = \sum_j v_j f \otimes \iota_{v_j^*}u,$$

where $\{v_j\}$ is a basis of V and $\{v_j^*\}$ the dual basis of V^*. Thus,

$$[D_y, d](f\otimes v) = \sum_j \partial_y(v_j)f\otimes \iota_{v_j^*}u - \sum_{s\in S}\frac{\tilde{c}_s(\alpha_s, y)}{\alpha_s}(1-s)(v_j)sf\otimes s\iota_{v_j^*}u.$$

Since $D_y v_j = 0$, this equals to

$$\sum_{s\in S}\frac{\tilde{c}_s(\alpha_s,y)}{\alpha_s}(1-s)(v_j)sf\otimes(1-s)\iota_{v_j^*}u,$$

so our job is to show that the expression

$$T(u):=\sum_{s\in S}\frac{\tilde{c}_s(\alpha_s,y)}{\alpha_s}(1-s)(v_j)s\otimes(1-s)\iota_{v_j^*}u.$$

vanishes. To this end, we note that

$$\sum_j(1-s)(v_j)\otimes v_j^*=\sum_j(1-\mu_s)(\beta_s^*,v_j)\beta_s\otimes v_j^*=(1-\mu_s)\beta_s\otimes\beta_s^*.$$

This implies that

$$T(u)=\sum_{s\in S}\tilde{c}_s(1-\mu_s)(\alpha_s,y)\frac{\beta_s}{\alpha_s}s\otimes(1-s)\iota_{\beta_s^*}u.$$

So the result follows from the following lemma.

Lemma 6.2. For any $u\in\wedge^m V$ and $s\in S$, $(1-s)\iota_{\beta_s^*}u=0$.

Proof. Let $u=u_1\wedge\ldots\wedge u_m$. If $m=1$, there is nothing to prove, so assume that $m\geq 2$. Then

$$(1-s)\iota_{\beta_s^*}u=$$

$$\mathrm{Alt}_m\sum_{j=1}^{m-1}(u_m,\beta_s^*)u_1\otimes\ldots\otimes u_{j-1}\otimes(1-s)(u_j)\otimes su_{j+1}\otimes\ldots\otimes su_{m-1}=$$

$$\mathrm{Alt}_m\sum_{j=1}^{m-1}(u_m,\beta_s^*)u_1\otimes\ldots\otimes u_{j-1}\otimes(1-\mu_s)(u_j,\beta_s^*)\beta_s\otimes su_{j+1}\otimes\ldots\otimes su_{m-1}.$$

This is zero, since we have skew-symmetrization with respect to j and m, which now occur symmetrically. □

The proposition is proved. □

In the special case when W is a real irreducible reflection group, $c=m/h$, where h is the Coxeter number of W, $m\in\mathbb{Z}_{\geq 1}$, $GCD(m,h)=1$, and $V=\mathfrak{h}$ is the reflection representation, this complex was studied in [BEG] and [Go]. In this case, this complex is actually a resolution of a finite dimensional $H_c(W,\mathfrak{h})$-module of dimension m^r, where r is the rank of W. Later it was studied in [CE] in the case when the representation $S\mathfrak{h}^*/(V)$ is finite dimensional (it follows from the fact that the expression in Theorem 2.3(iii) in [CE] is a polynomial that in this case S acts by reflections in V). This resolution is analogous to the BGG resolution in Lie theory, so it was called the BGG resolution. Thus we will call the complex $K^\bullet$ the **Koszul-BGG complex**.

6.2. **The Koszul-BGG complex for $W = S_n$.** It follows from the paper [FS] that if W is an irreducible real reflection group, and $c = m/h$, where $m = d - 1 + \ell h$, $\ell \in \mathbb{Z}_{\geq 0}$, and d is a degree of W, then there is a singular copy of $V = \mathfrak{h}$ in degree m of $S\mathfrak{h}^*$, so the complex $K^\bullet(V)$ is nontrivial. A similar result for cyclotomic wreath product groups $G(l, r, n)$ follows from the paper [DO] (see also [CE] and [L3] for the case $r = 1$).

In particular, if $W = S_n$, and $\mathfrak{h}$ is the reflection representation, the Koszul-BGG complex has a nonzero differential for any $c = \frac{m}{n}$, where m is not divisible by n. In this case, it was shown by Dunkl, [D] (see also [CE]) that the singular representation V is spanned by partial derivatives of the polynomial

$$F_{m,n}(x_1, ..., x_n) := Res_{u=\infty}((u - x_1)...(u - x_n))^{\frac{m}{n}} du. \tag{19}$$

Note that this works also if m is divisible by n, except that in this case $F_{m,n} = 0$, so the differential in the corresponding complex is zero. Thus, we have defined a complex for every $n \geq 1$ and $m \geq 1$. Let us denote this complex by $K^\bullet_{m,n}$.

Now let m, n be positive integers, and $d = GCD(m, n)$. Write $m = m_0 d$, $n = n_0 d$. Our main result about the Koszul-BGG complex for S_n is the following theorem.

Theorem 6.3. (i) The homology $H_i(K^\bullet_{m,n})$ is nonzero if and only if $0 \leq i \leq d - 1$.

(ii) If $0 \leq i \leq d-1$ then $H_i(K^\bullet_{m,n})$ is the irreducible representation $L_{\frac{m}{n}}(\lambda_i)$ of the rational Cherednik algebra $H_c(S_n)$, where $\lambda_i = n_0(d - i, 1^i)$.

Two proofs of this theorem are contained in the next three subsections, and a third one in Subsection 8.3; these proofs are based on different ideas, so we present all three of them.

6.3. **Proof of Theorem 6.3.**

Lemma 6.4. Let V be a finite dimensional subspace of $R := \mathbb{C}[x_1, ..., x_N]$. Assume that the zero set $Z(V)$ of V in $\mathbb{C}^N$ has dimension $k < N$. Then there are polynomials $f_1, ..., f_{N-k} \in V$, which form a regular sequence.

Proof. We prove by induction in i (for $i \leq N - k$) that one can choose a regular sequence $f_1, ..., f_i \in V$. The base of induction is obvious. To make the step of induction, suppose that for some $i < N - k$, the polynomials $f_1, ..., f_i$ have been chosen. Then the zero set Z_i of $f_1, ..., f_i$ has pure codimension i. Since the zero set Z of V has codimension $> i$, none of the components of Z_i is contained in Z, so a generic element f_{i+1} of V does not vanish identically on any of these components; this completes the step of induction. □

The vanishing of H_i for $i \geq d$ follows from the standard properties of the Koszul complex. Namely, we know that the module $H_0 = L_{\frac{m}{n}}(\mathbb{C})$ has minimal support (by 2.2; see also [BE]), so this support is of dimension $d - 1$. By Lemma 6.4, this means that there exists a basis $f_1, ..., f_{n-1}$ of the space V spanned by the partial derivatives of $F_{m,n}$ such that $f_1, ..., f_{n-d}$ is a regular sequence. Define a grading on $K^\bullet_{m,n}$ by the number of f_i in the wedge part with $i > n - d$. Then the differential preserves the filtration defined by this grading, and the associated graded complex is of the form $K^\bullet(f_1, ..., f_{n-d}) \otimes \wedge^\bullet(f_{n-d+1}, ..., f_{n-1})$ (with the Koszul differential of the first factor). The first factor is acyclic in positive degrees, so this complex has no homology in degrees $\geq d$. Hence the same is true for the filtered complex.

Thus, we just need to prove part (ii).

To this end, note that the support of H_0 is the union $X_{d,n/d}(n)$ of all the images of the subspace defined by the equations $x_i = x_j$ when $i - j = 0$ modulo d under permutations. By Theorem 2.2, this is the minimal support of modules in category $\mathcal{O}$ for $H_{\frac{m}{n}}(S_n)$. By the theory of Koszul complexes, this implies that all the homology modules H_i are supported on $X_{d,n/d}(n)$, i.e. have minimal support. By the results of Wilcox, [Wi, Theorem 1.8, Proposition 3.7] (see Theorem 2.2), the category of such modules is equivalent to the category of representations of S_d, by considering restriction Res_b to the open stratum of $X_{d,n/d}(n)$ and looking at the monodromy of the resulting local system. Namely, this equivalence sends a representation of S_d corresponding to the Young diagram μ to the representation $L_{\frac{m}{n}}(n_0\mu)$ over the rational Cherednik algebra $H_{\frac{m}{n}}(S_n)$ with minimal support (see Theorem 2.2). Moreover, this equivalence is compatible with restrictions to points of $X_{d,n/d}(n)/S_n$ (i.e. restriction from Cherednik algebra to its parabolic subalgebras corresponds under this equivalence to the restriction from the symmetric group S_d to its parabolic subgroups).

Let $\wedge_d^i$ be the i-th exterior power of the reflection representation of S_d. We will need the following simple lemma.

Lemma 6.5. Suppose that π is a representation of S_d such that for any $0<k<d$,

$$\pi|_{S_k\times S_{d-k}} = \oplus_{r-1\le i+j\le r} \wedge_k^i \otimes \wedge_{d-k}^j,$$

and $\pi^{S_d}=0$. Then $\pi=\wedge_d^r$.

Proof. Clearly, $\wedge_d^r$ satisfies the condition. Hence, the character of the difference $\pi-\wedge_d^r$ has zero restriction to the subgroups $S_k\times S_{d-k}$, i.e., vanishes on all non-cyclic permutations in S_d. Thus, it is an integer multiple of the virtual character $\chi(g)=\sum_\lambda \mathrm{Tr}_{\pi_\lambda}(g)\pi_\lambda$, where g is a cyclic permutation in S_d. This virtual character involves a copy of the trivial representation. So $\pi=\wedge_d^r$. □

Now we prove part (ii) of the theorem. Our job is to show that $H_i=L_{\frac{m}{n}}(n_0(d-i,1^i))$ for $0<i<d$ (we already know that $H_0=L_{\frac{m}{n}}((n)))$.

We will use the following proposition. Let $0<k<d$. Let b be a point in $\mathfrak{h}$ with coordinates $x_i=(d-k)z$ for $i\le n_0k$, and $x_i=-kz$ for $i>n_0k$, for some $z\ne 0$.

Proposition 6.6. We have an isomorphism of complexes of $\mathbb{C}(S_{n_0k}\times S_{n_0(d-k)})\ltimes\mathbb{C}[\mathfrak{h}]$-modules

$$\mathrm{Res}_b(K^\bullet_{m,n})\cong K^\bullet_{m_0k,n_0k}\otimes K^\bullet_{m_0(d-k),n_0(d-k)}\otimes\Omega^\bullet, \tag{20}$$

where $\Omega^\bullet$ is the two-step complex $\mathbb{C}[t]\leftarrow\mathbb{C}[t]$ with the zero differential.

Proof. First of all, if $f_1,...,f_r\in R=\mathbb{C}[x_1,...,x_n]$ and $K^\bullet(f_1,...,f_r,R)$ is the corresponding Koszul complex, then by definition, the completion $\hat{K}^\bullet_b(f_1,...,f_r,R)$ of $K^\bullet(f_1,...,f_r,R)$ at any point $b\in\mathbb{C}^n$ is naturally isomorphic to $K^\bullet(f_1,...,f_r,\hat{R}_b)$, where $\hat{R}_b$ is the completion of R at b.

Next, suppose $\bar{f}_i\in R$, $1\le i\le r$, are linearly independent quasihomogeneous polynomials of the same degree D (i.e., homogeneous polynomials of degree D for a grading in which $\deg(x_j)=d_j$ for some positive integers d_j), and assume that $f_i=\bar{f}_i+$higher degree terms $\in \hat{R}=\mathbb{C}[[x_1,...,x_n]]$ are deformations of these polynomials. Also let $g_p\in\hat{R}$, $p=1,...,s$, be elements whose lowest degree is $>D$.

Lemma 6.7. Assume that $\bar{f}_i$ generate the same ideal in $\hat{R}$ as f_i,g_p. Then

$$K^\bullet(f_1,...,f_r,g_1,...,g_s,\hat{R})\cong K^\bullet(\bar{f}_1,...,\bar{f}_r,\hat{R})\otimes\wedge(\xi_1,...,\xi_s),$$

as complexes of $\hat{R}$-modules, where $\partial\xi_i=0$.

Proof. We can choose elements $a_{ij} \in \hat{R}$ such that

$$f_i = \sum_j a_{ij} \bar{f}_j.$$

Then, since $\bar{f}_j$ have the same homogeneity degree, we have

$$\bar{f}_i = \sum_j a_{ij}(0) \bar{f}_j.$$

This implies that $a_{ij}(0) = \delta_{ij}$ and hence (a_{ij}) is invertible. Also, we have the matrix (c_{pj}), $c_{pj} \in \hat{R}$ such that $g_p = \sum_j c_{pj} \bar{f}_j$.

We claim that the matrices $(a_{ij}), (c_{pj})$ define the desired isomorphism

$$\theta : K(f_1, ..., f_r, g_1, ..., g_s, \hat{R}) \cong K(\bar{f}_1, ..., \bar{f}_r, \hat{R}) \otimes \wedge(\xi_1, ..., \xi_s).$$

Namely, let $\eta_1, ..., \eta_r$ be the odd generators of $K(\bar{f}_1, ..., \bar{f}_r, \hat{R})$ over $\hat{R}$, and let $\eta'_1, ..., \eta'_r, \xi'_1, ...\xi'_s$ be the odd generators of $K(f_1, ..., f_r, g_1, ..., g_s, \hat{R})$ over $\hat{R}$ (so that

$$K(\bar{f}_1, ..., \bar{f}_r, \hat{R}) = \hat{R} \otimes \wedge(\eta_1, ..., \eta_r),\ K(f_1, ..., f_r, g_1, ..., g_s, \hat{R}) = \hat{R} \otimes \wedge(\eta'_1, ..., \eta'_r, \xi'_1, ..., \xi'_s),$$

and $\partial\eta_i = \bar{f}_i$, $\partial\eta'_i = f_i$, $\partial\xi'_p = g_p$). Then θ is defined by the formula

$$\theta(\eta'_i) = \sum_j a_{ij}\eta_j, \theta(\xi'_p) = \xi_p + \sum_j c_{pj}\eta_j.$$

This proves the lemma. □

Now, consider the singular polynomials f_i, $i = 1, ..., n$, generating the Koszul complex $K^\bullet_{m,n}$. As explained above, $f_i = \partial_i F_{m,n}$, where

$$F_{m,n} = \frac{1}{2\pi \mathrm{i}} \int_\gamma ((u - x_1)...(u - x_n))^{\frac{m}{n}} du,$$

where the integration is over a large enough circle γ in the counterclockwise direction. This polynomial has degree $m + 1$. Let us consider the completion at the point b, and introduce new variables:

$$t = \frac{1}{n_0} \sum_{1 \le i \le n_0 k} x_i - k(d-k)z;\ \ x'_i = x_i - \frac{t}{k} - (d-k)z, i \le n_0 k;$$

$$x''_i = x_i + \frac{t}{d-k} + kz, i > n_0 k$$

(thus, $\sum_i x'_i = \sum_j x''_j = 0$).

Lemma 6.8. We have

$$F_{m,n}(x) = C' F_{m_0 k, n_0 k}(x') + C'' F_{m_0(d-k), n_0(d-k)}(x'') + \text{higher terms}, C', C'' \in \mathbb{C}^\times,$$

where higher terms are of two kinds:

(1) degree $s' \ge n_0 k + 1$ in x' and degree s'' in x'', t with $s' + s'' - (n_0 k + 1) > 0$;

(2) degree $s'' \ge n_0(d-k) + 1$ in x'' and degree s' in x', t with $s' + s'' - (n_0(d-k) + 1) > 0$.

Proof. Consider a point x close to b. Then x_i cluster around $z(d-k)$ for $i \le n_0k$, and around $-zk$ for $i > n_0k$. So by the Cauchy integral formula the contour integral defining $F_{m,n}$ can be represented as a sum of integrals over two contours going around each of the two clusters (note that the integrand is single-valued on these contours). Shifting the integration variable in each of the integrals to make the contours go around the origin, we get

$$F_{m,n} = \frac{1}{2\pi \mathrm{i}}(zd)^{m_0(d-k)} \oint \prod_{i=1}^{n_0k}(v-x_i')^{m_0/n_0} \prod_{i=n_0k+1}^{n}\left(1+\frac{v-x_i''}{zd}+\frac{t}{zk(d-k)}\right)^{m_0/n_0} dv+$$

$$\frac{1}{2\pi \mathrm{i}}(-zd)^{m_0k} \oint \prod_{i=1}^{n_0k}\left(1-\frac{v-x_i'}{zd}+\frac{t}{zk(d-k)}\right)^{m_0/n_0} \prod_{i=n_0k+1}^{n}(v-x_i'')^{m_0/n_0} dv.$$

This implies the lemma, with $C' = (zd)^{m_0(d-k)}$ and $C'' = (-zd)^{m_0k}$ (the two terms in the formula come from the two resulting integrals, and the form of the higher terms is clear from the form of these integrals; we just expand the expressions of the form $(1+u)^{m_0/n_0}$ appearing in the integrals in a Taylor series with respect to u). □

Now let $\deg(x_i') = d-k$, $\deg(x_i'') = k$. Then the polynomials $\bar{f}_i = \frac{\partial}{\partial x_i'}F_{m_0k,n_0k}(x')$ for $i \le n_0k$, and $\bar{f}_i = \frac{\partial}{\partial x_i''}F_{m_0(d-k),n_0(d-k)}(x'')$ for $i > n_0k$ are quasihomogeneous of degree $n_0k(d-k)$. Note that $\sum_{i\le n_0k}\bar{f}_i = \sum_{i>n_0k}\bar{f}_i = 0$.

So, it suffices to show that f_i generate the same ideal in $\mathbb{C}[[x_1',\dots,x_{n_0k}',x_{n_0k+1}'',\dots,x_n'',t]]$ as $\bar{f}_i$, $i = 1,...,n$. Then by virtue of Lemma 6.8 the Proposition will follow by applying Lemma 6.7 to f_i for $i \ne n_0k, n$ and $g_1 = \sum_{i\le n_0k} f_i$ (as power series in x', x'', t).

To this end, note that f_i, $1 \le i \le n$, generate an ideal I that is invariant under the Dunkl operators D_j. Let us expand the Dunkl operators at b, with respect to the coordinates x_i', x_i'', t. These "formal" Dunkl operators are non-homogeneous in the variables x_i', x_i'', t, and we have $D_i = \bar{D}_i + R_i$, where $\bar{D}_i$ are the homogeneous parts (of degree -1 in the grading where the degrees of the x_i', x_i'', t are 1), and R_i are the regular parts. Clearly, I is invariant under R_i, so it is invariant under $\bar{D}_i$, which are the Dunkl operators of the parabolic subgroup $W_b = S_{n_0k} \times S_{n_0(d-k)}$ stabilizing the point b. Thus, I corresponds to a proper submodule J in the polynomial module $M_c(\mathbb{C})$ over the parabolic Cherednik algebra $H_c(S_{n_0k} \times S_{n_0(d-k)}, \mathfrak{h}' \oplus \mathfrak{h}'')$, where $\mathfrak{h}', \mathfrak{h}''$ are the reflection representations of S_{n_0k} and $S_{n_0(d-k)}$, respectively (here we use the fact that the restriction of the polynomial module is the polynomial module over the parabolic subalgebra, which follows from the definition of the restriction functors in [BE]). Since $M_c(\mathbb{C})/J$ has $d-1$-dimensional (i.e., minimal) support, J must be the image of $M_c(\mathfrak{h}') \boxtimes M_c(\mathbb{C}') \oplus M_c(\mathbb{C}'') \boxtimes M_c(\mathfrak{h}'')$ in $M_c(\mathbb{C}') \boxtimes M_c(\mathbb{C}'')$, where $\mathbb{C}', \mathbb{C}''$ are the trivial representations of S_{n_0k} and $S_{n_0(d-k)}$, respectively. This means that I is generated by the elements $\bar{f}_i$.

Thus, by Lemma 6.7, we have the required isomorphism of complexes of $\mathbb{C}[\mathfrak{h}]$-modules. Moreover, since all the constructions in the proof are equivariant under the group $S_{n_0k} \times S_{n_0(d-k)}$, it follows from the proof of Lemma 6.7 that this is actually an isomorphism of $\mathbb{C}(S_{n_0k} \times S_{n_0(d-k)}) \ltimes \mathbb{C}[\mathfrak{h}]$-modules, as desired.

The proposition is proved. □

Corollary 6.9. The two complexes in Proposition 6.6 have isomorphic cohomology groups, as $H_{\frac{m}{n}}(S_{n_0k} \times S_{n_0(d-k)}, \mathfrak{h})$-modules.

Proof. By Proposition 6.6, the cohomology groups of the two complexes are isomorphic as $\mathbb{C}(S_{n_0k} \times S_{n_0(d-k)}) \ltimes \mathbb{C}[\mathfrak{h}]$-modules. Also, we know that these cohomology groups are semisimple modules over $H_{\frac{m}{n}}(S_{n_0k}\times S_{n_0(d-k)},\mathfrak{h})$, by the result of [Wi] (see Theorem 2.2), since they have minimal support, and the category of minimally supported modules is semisimple. Hence, the corollary is a consequence of the following lemma. □

Lemma 6.10. A semisimple object in $\mathcal{O}_c(W,\mathfrak{h})$ is uniquely determined, up to an isomorphism, by its structure of a $\mathbb{C}W\ltimes\mathbb{C}[\mathfrak{h}]$-module.

Proof. Let $\mathfrak{m}$ be the augmentation ideal of $\mathbb{C}[\mathfrak{h}]$ (generated by $\mathfrak{h}^*$). If $N=\oplus_{\tau\in\mathrm{Irrep}\ W}m_\tau L_c(\tau)$, then $N/\mathfrak{m}N=\oplus_{\tau\in\mathrm{Irrep}(W)}m_\tau\tau$ as a W-module. So N is determined by its structure of a $\mathbb{C}W\ltimes\mathbb{C}[\mathfrak{h}]$-module, as desired. □

Now note that for $r>0$, $H_r(K^\bullet_{m,n})$ cannot contain $L_c(\mathbb{C})$, since $L_c(\mathbb{C})$ is not a composition factor of $M_c(\wedge^i\mathfrak{h})$ for $i>0$ (because it has smaller lowest eigenvalue of $\mathbf{h}$ than any eigenvalue of $\mathbf{h}$ in $M_c(\wedge^i\mathfrak{h})$, $i>0$). So, varying k and using Lemma 6.5, we conclude that the statement follows by induction in d from the known case $d=1$. Namely, if under the correspondence of [Wi], [Theorem 1.8 and Proposition 3.7] (see Theorem 2.2), $H_r(K^\bullet_{m,n})$ for some $r>0$ corresponds to some (possibly reducible) representation $\pi\in\mathrm{Rep}\ S_d$, then we have $\pi^{S_d}=0$ (as $L_c(\mathbb{C})$ does not occur) and $\pi|_{S_k\times S_{d-k}}=\oplus_{r-1\le i+j\le r}\wedge^i_k\otimes\wedge^j_{d-k}$, by Corollary 6.9. So the result follows from Lemma 6.5.

6.4. **Uniqueness for the Koszul-BGG complex.** It turns out that $K^\bullet_{m,n}$, where m is not divisible by n, is the only complex with nonzero differentials with terms $M_c(\wedge^i\mathfrak{h})$ that one can write.

Proposition 6.11. The space $\mathrm{Hom}(M_c(\wedge^{i+1}\mathfrak{h}),M_c(\wedge^i\mathfrak{h}))$ is one-dimensional for all $i\le n-1$.

Proof. Let $\mathcal{O}$ denote the direct sum $\bigoplus_{i=0}^\infty\mathcal{O}_i$, where $\mathcal{O}_i$ is the category $\mathcal{O}$ for the algebra $H_c(S_i)$. According to Shan, [Sh], there is a categorical $\widehat{\mathfrak{sl}}_{n_0}$-action on $\mathcal{O}$. This gives rise to n_0 categorical $\mathfrak{sl}_2$-actions, one for each simple root in $\widehat{\mathfrak{sl}}_{n_0}$. These categorical actions are highest weight in the sense of [L3].

Now let λ,λ' be the hooks corresponding to $\wedge^{i+1}\mathfrak{h},\wedge^i\mathfrak{h}$. Since $|\lambda|$ is divisible by n_0, one obtains λ' from λ by moving an a-box (where a is a suitable residue mod n). It follows that λ and λ' lie in the same family (in the terminology of [L3, Section 3]) for the subalgebra $\mathfrak{sl}_2\subset\hat{\mathfrak{sl}}_{n_0}$ corresponding to the residue a. Since $\lambda<\lambda'$ we conclude from [L3, Proposition 7.4, Remark 7.8] that $\dim\mathrm{Hom}_{\mathcal{O}}(M_c(\wedge^{i+1}\mathfrak{h}),M_c(\wedge^i\mathfrak{h}))=1$. □

Let $\mathrm{Sh}_c:\mathcal{O}_c\to\mathcal{O}_{c+1}$ be the shift functor, which is a right exact functor defined by

$$\mathrm{Sh}_c(V)=H_c(S_n)\mathbf{e}\otimes_{\mathbf{e}H_c(S_n)\mathbf{e}=\mathbf{e}_-H_{c+1}(S_n)\mathbf{e}_-}\mathbf{e}_-V,$$

where $\mathbf{e}=\mathbf{e}_0$ is the symmetrizer, and $\mathbf{e}_-=\mathbf{e}_{n-1}\in\mathbb{C}S_n$ is the antisymmetrizer (see [BEG]). It is shown in [BE] that Sh_c is an equivalence of categories for $c>0$.

Corollary 6.12. One has $\mathrm{Sh}_{\frac{m}{n}}(K^\bullet_{m,n})\cong K^\bullet_{m+n,n}$.

Proof. This follows from Proposition 6.11 and the fact that the shift functor maps Verma modules to Verma modules, see [GL]. Namely, Lemma 4.3.2 in [GL] says that if a category is equipped with two highest weight structures such that the classes of the standard objects coincide in K_0, then the structures coincide. But the shift functor is clearly the identity on K_0 (with the basis of standard modules), as it is flat with respect to c for $c>0$, and is obviously the identity for generic c (by looking at the eigenvalues of the scaling element $\mathbf{h}$). □

6.5. **Another proof of Theorem 6.3.**

Lemma 6.13. Theorem 6.3 holds on the level of the Grothendieck groups, i.e.,

$$\bigoplus_{i=0}^{n-1}(-1)^i[M_{\frac{m}{n}}(\wedge^i\mathfrak{h}_n)] = \bigoplus_{i=0}^{d-1}(-1)^i[L_{\frac{m}{n}}(\lambda_i)].$$

Proof. Let $\lambda_i(k) = (k-i, 1^i)$. By Theorem 1.4 one has

$$[L_{\frac{m}{n}}(n_0\lambda_i(d))] = \sum_{|\mu|=n} c^{\mu}_{\lambda_i(d),n_0}[M_{\frac{m}{n}}(\mu)],$$

where the coefficients $c^{\mu}_{\lambda_i(d),n_0}$ are defined by the equation $\Psi_{n_0}(s_{\lambda_i(d)}) = \sum_\mu c^{\mu}_{\lambda_i(d),n_0}s_\mu$. The statement now follows from a symmetric function identity

$$\sum_{i=0}^{d-1}(-1)^i\Psi_{n_0}(s_{\lambda_i(d)}) = \Psi_{n_0}\left(\sum_{i=0}^{d-1}(-1)^i s_{\lambda_i(d)}\right) = \Psi_{n_0}(p_d) = p_n = \sum_{i=0}^{n-1}(-1)^i s_{\lambda_i(n)}.$$

Here we used the equation $p_k = \sum_{i=0}^{k-1}(-1)^i s_{\lambda_i(k)}$ twice: for $k=d$ and $k=n$. □

Let us recall again that all categories $\mathcal{O}_c$ with $c = \frac{r}{n_0}$, where $GCD(r, n_0) = 1$ and $r > 0$, are equivalent as highest weight categories. From Proposition 6.11 it follows that the equivalences preserve the Koszul-BGG complexes. So it is enough to prove the theorem for $c = \frac{1}{n_0}$.

Lemma 6.14. The multiplicity of $H_i(K^\bullet_{d,n})$ in a generic point of the support of $L_{\frac{1}{n_0}}(n)$ equals $\binom{d-1}{i}$.

Proof. This follows from Lemma 6.4. Namely, for $c = 1/n_0$, the zero set Z of $f_1, ..., f_n$ is generically reduced,[6] so for a suitable generic point $z \in Z$, the differentials $df_1(z), ..., df_{n-d}(z)$ are linearly independent. This implies that in the formal neighborhood of z, there exist functions c_{ij}, $j \leq n-d$, $i > n-d$, such that $f_i = \sum_{j=1}^{n-d} c_{ij}f_j$ for $i > n-d$. This implies (similarly to the proof of Lemma 6.7) that the completion of $K^\bullet_{d,n}$ at z is the tensor product of the Koszul complex defined by $f_1, ..., f_{n-d}$ with the exterior algebra $\wedge(\xi_1, ..., \xi_{d-1})$ in $d-1$ generators. This implies the statement, as the dimension of the degree i component of $\wedge(\xi_1, ..., \xi_{d-1})$ is $\binom{d-1}{i}$. □

Proof of Theorem 6.3. The proof is by induction on i. Assume that $H_i(K^\bullet_{d,n}) = L_{\frac{m}{n}}(n_0\lambda_i(d))$ for all $i < k$, where k is a fixed number from 0 to d (for $k = 0$ the assumption is vacuous). We want to prove that $H_k(K^\bullet_{d,n}) = L_{\frac{m}{n}}(n_0\lambda_k(d))$. First of all, for $i < k$, $L_{\frac{m}{n}}(n_0\lambda_k(d))$ is not a composition factor of $H_i(K^\bullet_{d,n})$, by the inductive assumption. We also claim that $L_{\frac{m}{n}}(n_0\lambda(d))$ does not appear in $H_i(K^\bullet_{d,n})$ for $i > k$. Indeed, assume the converse. Then $L_{\frac{m}{n}}(n_0\lambda(d))$ has to be a composition factor of $M_{\frac{m}{n}}(\lambda_i(n))$. However, $\lambda_i(n) \not\geq n_0\lambda_k(d)$ in the dominance ordering as $\lambda_i(n)$ has more rows than $n_0\lambda_k(d)$. So $L_{\frac{m}{n}}(n_0\lambda_k(d))$ cannot appear as a composition factor of $M_{\frac{m}{n}}(\lambda_i(n))$. But $L_{\frac{m}{n}}(n_0\lambda_i(d))$ has to appear in some $H_i(K^\bullet_{d,n})$ thanks to Lemma 6.13 and so we must have $i = k$. Also the generic ranks of $L_{\frac{m}{n}}(n_0\lambda_k(d))$ and $H_k(K^\bullet_{d,n})$ coincide by Lemma 6.14. It follows that $L_{\frac{m}{n}}(n_0\lambda_k(d)) = H_k(K^\bullet_{d,n})$. □

[6] In fact, it is reduced, but we use only generic reducedness.

7. Symmetry for rational Cherednik algebras of type A

7.1. **The statement.** We start by recalling the type A rational Cherednik algebra. We will need a universal version, which is slightly different from the one defined in the preliminaries section. Let $n \geq 2$ be an integer and $\mathfrak{h}$ be the $n-1$ dimensional vector space viewed as the subspace $\{(x_1, \ldots, x_n) | \sum_{i=1}^n x_i = 0\} \subset \mathbb{C}^n$. Then $\mathfrak{h}$ can be thought of as the Cartan subalgebra in the Lie algebra $\mathfrak{sl}_n$. Let $\Delta_+ \subset \mathfrak{h}^*$ be the root system. The corresponding Weyl group is S_n. Fix independent variables h, c. The (universal) rational Cherednik algebra $\mathbf{H}$ is the quotient of the semi-direct product $\mathbb{C}S_n \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)[h, c]$ by the relations

$$[x, x'] = 0, [y, y'] = 0, [y, x] = h\langle y, x\rangle - c \sum_{\alpha \in \Delta_+} \langle \alpha, y\rangle \langle x, \alpha^\vee \rangle s_\alpha,$$

with $x, x' \in \mathfrak{h}^*, y, y' \in \mathfrak{h}$. Here $\alpha^\vee, s_\alpha$ are, respectively, the coroot and the reflection corresponding to a root α.

We remark that the algebra $\mathbf{H}$ is bigraded: h, c lie in bidegree $(1,1)$, $\mathfrak{h}$ is in bidegree $(0,1)$, $\mathfrak{h}^*$ is in bidegree $(1,0)$, and S_n is in bidegree $(0,0)$.

Recall that we introduced the idempotents $\mathbf{e}_i$ in $\mathbb{C}S_n$ corresponding to the irreducible representations $\wedge^i \mathfrak{h}$ of S_n, and $e_i := \mathbf{e}_i + \mathbf{e}_{i-1}$, where $\mathbf{e}_{-1}, \mathbf{e}_n = 0$. The element $e_i \in \mathbb{C}S_n$ should be thought of as the idempotent corresponding to the S_n-module $\wedge^i \mathbb{C}^n = \wedge^i \mathfrak{h} \oplus \wedge^{i-1} \mathfrak{h}$. Recall that $\overline{\mathbf{e}} = \sum_k e_{2k} = \sum_k e_{2k+1} = \sum_j \mathbf{e}_j$. Our goal is to understand the structure of the *quasispherical* subalgebra $\overline{\mathbf{e}}\mathbf{H}\overline{\mathbf{e}} \subset \mathbf{H}$. The latter is not a unital subalgebra, instead $\overline{\mathbf{e}}$ is the unit in $\overline{\mathbf{e}}\mathbf{H}\overline{\mathbf{e}}$. We will identify the algebra $\overline{\mathbf{e}}\mathbf{H}\overline{\mathbf{e}}$ with a certain quantum Hamiltonian reduction generalizing the description of $e_0\mathbf{H}e_0$ obtained by Gan and Ginzburg, [GG].

Set $\mathbf{V}_n := \mathfrak{sl}_n(\mathbb{C}) \oplus \mathbb{C}^n$. We will call it $\mathbf{V}$ if no confusion is possible. The group $G := \mathrm{GL}_n(\mathbb{C})$ acts naturally on $\mathbf{V}$. Also G acts on the symplectic vector space $T^*\mathbf{V}$ with moment map $\mu : T^*\mathbf{V} \to \mathfrak{g} := \mathrm{Lie}(G)$ given by $\mu(A, B, i, j) = [A, B] + i \otimes j, A, B \in \mathfrak{sl}_n(\mathbb{C}), i \in \mathbb{C}^n, j \in (\mathbb{C}^n)^*$, where we identify $\mathfrak{sl}_n(\mathbb{C})$ with its dual via the trace pairing. The space $T^*\mathbf{V}$ also carries an action of the two-dimensional torus $(\mathbb{C}^\times)^2$ commuting with G:

$$(t_1, t_2).(A, B, i, j) = (t_1^{-1}A, t_2^{-1}B, t_1^{-1}i, t_2^{-1}j).$$

Consider the subtori $T_1 := \{(t, t^{-1}) \in (\mathbb{C}^\times)^2\}, T_2 := \{(t, t) \in (\mathbb{C}^\times)^2\}$.

The symplectic vector space $T^*\mathbf{V}$ admits a natural quantization, the algebra $\mathcal{D}_h(\mathbf{V})$ of homogenized differential operators. The latter can be obtained from the algebra $\mathcal{D}(\mathbf{V})$ of differential operators by using the Rees construction. Namely, the algebra $\mathcal{D}(\mathbf{V})$ is filtered by the subspaces $\mathcal{D}_{\leqslant i}(\mathbf{V})$ of differential operators of order $\leqslant i$. Then $\mathcal{D}_h(\mathbf{V}) := \bigoplus_i \mathcal{D}_{\leqslant i}(\mathbf{V})h^i \subset \mathcal{D}(\mathbf{V})[h]$. The algebra $\mathcal{D}_h(\mathbf{V})$ is bigraded: its component of bidegree (i, j), by definition, consists of all elements in $h^j\mathcal{D}_{\leqslant j}(\mathbf{V})$ that have degree $i + j$ with respect to the grading induced by the $\mathbb{C}^\times$-action on $\mathbf{V}$ by $(t, v) \mapsto t^{-1}v$. In particular, $\mathbf{V}^* \subset \mathcal{D}_{\leqslant 0}(\mathbf{V})$ has bidegree $(1,0)$, while $\mathbf{V} \subset \mathrm{Vect}(\mathbf{V}) \subset h\mathcal{D}_{\leqslant 1}(\mathbf{V})$ has bidegree $(0,1)$ and h has bidegree $(1,1)$. Also the action of G on $\mathbf{V}$ gives rise to the quantum comoment map $\Phi_h : \mathfrak{g} \to \mathcal{D}_h(\mathbf{V})$, under which an element ξ maps to the corresponding vector field in $h\mathcal{D}_{\leqslant 1}(\mathbf{V})$ induced by the action of $\mathfrak{g}$. We remark that the quantum comoment map has image in bidegree $(1,1)$.

More generally, let U be a G-module. Consider the tensor product $\mathcal{D}_h(\mathbf{V}) \otimes \mathrm{End}(U)$ that inherits the bigrading from $\mathcal{D}_h(\mathbf{V})$ with $\mathrm{End}(U)$ being of bidegree $(0,0)$. There is the quantum comoment map $\Phi_h^U(\xi) := \Phi_h(\xi) \otimes 1 + h \otimes \xi_U : \mathfrak{g} \to \mathcal{D}_h(\mathbf{V}) \otimes \mathrm{End}(U)$, where ξ_U just stands for the image of ξ in $\mathrm{End}(U)$.

Let $\mathfrak{z}$ denote the center of $\mathfrak{g}$. It is naturally identified with $\mathbb{C}$ via $z \mapsto z\,\mathrm{id}_{\mathbb{C}^n}$. Let β denote the basis element in $\mathfrak{z}$ corresponding to 1.

The quantum Hamiltonian reduction we are going to consider will be defined first at the level of sheaves. A sheaf of interest will be on a formal deformation $\widetilde{X}$ of the Hilbert scheme $X = \mathrm{Hilb}_n$, to be recalled first.

The variety X can be produced by Hamiltonian reduction as follows. Consider the character $\theta := \det$ of $G = \mathrm{GL}_n(\mathbb{C})$ and let $\overline{\mathbf{V}}^{ss}$ be the open subset of θ-semistable points in $\overline{\mathbf{V}} := T^*\mathbf{V}$. Then $\overline{\mathbf{V}}^{ss} \cap \mu^{-1}(0) = \{(A, B, i, 0) | [A, B] = 0, \mathbb{C}[A, B]i = \mathbb{C}^n\}$. By definition, X is the Hamiltonian reduction of $\overline{\mathbf{V}}^{ss}$ by the action of G, i.e., $X = (\mu^{-1}(0) \cap \overline{\mathbf{V}}^{ss})/G$. This is a smooth symplectic variety equipped with a $(\mathbb{C}^\times)^2$-action and also with a morphism $X \to \mu^{-1}(0)//G \cong (\mathfrak{h} \oplus \mathfrak{h}^*)/S_n$ that is a resolution of singularities.

In fact, we will need to work over a larger scheme. Namely, consider the Hamiltonian reduction $[\overline{\mathbf{V}}^{ss} \cap \mu^{-1}(\mathfrak{z}^*)]/G$. This is a scheme over $\mathfrak{z}^*$. Its non-zero fiber is the so called Calogero-Moser space and the G-action over such fiber is known to be free. Let $\widetilde{X}$ be the completion of this scheme at the zero fiber, this is a formal scheme over the formal neighborhood $(\mathfrak{z}^*)^{\wedge_0}$. The scheme $\widetilde{X}$ comes equipped with a fiberwise symplectic form, say $\widetilde{\omega}$.

We will define a sheaf $\mathcal{D}_h^U$ of $\mathbb{C}[[\mathfrak{z}^*, h]]$-algebras on $\widetilde{X}$ as follows. We sheafify the h-adic completion of $D_h(\mathbf{V})$ to $\overline{\mathbf{V}}$. Abusing notation, we denote the resulting sheaf again by $D_h(\mathbf{V})$. Then set

$$\mathcal{D}_h^U := \left[(\mathrm{End}(U) \otimes D_h(\mathbf{V}))|_{\overline{\mathbf{V}}^{ss}} / (\mathrm{End}(U) \otimes D_h(\mathbf{V}))|_{\overline{\mathbf{V}}^{ss}} \Phi_h^U([\mathfrak{g}, \mathfrak{g}])\right]^G. \tag{21}$$

The group $(\mathbb{C}^\times)^2$ naturally acts on $\mathcal{D}_h^U$, where we have $(t_1, t_2).h = t_1 t_2 h$. Let $\mathcal{A}_h(\mathbf{V}, U)$ stand for the subalgebra of T_2-finite elements in $\Gamma(\widetilde{X}, \mathcal{D}_h^U)$. This is an algebra over $\mathbb{C}[\beta, h]$ equipped with an action of $(\mathbb{C}^\times)^2$ by algebra automorphisms.

Theorem 7.1. There is a $(\mathbb{C}^\times)^2$-equivariant $\mathbb{C}[h]$-linear isomorphism

$$\Upsilon : \mathcal{A}_h(\mathbf{V}, \wedge^{n-2\bullet}\mathbb{C}^n) \xrightarrow{\sim} \overline{\mathbf{e}}\mathbf{H}\overline{\mathbf{e}}$$

that maps β to $c + h$. [7] This isomorphism induces an isomorphisms

$$\Upsilon_j : \mathcal{A}_h(\mathbf{V}, \wedge^{n-2j}\mathbb{C}^n) \xrightarrow{\sim} e_{n-2j}\mathbf{H}e_{n-2j}$$

for $j \geq 0$.

Theorem 7.1 is proved in the next three subsections.

7.2. **Procesi bundle.** In the proof we use a remarkable bundle on X, the Procesi bundle $\mathcal{P}$ originally constructed by M. Haiman, [Hai]; an alternative construction was produced by Ginzburg, [Gi2].

The Hamiltonian reduction construction equips X and $\widetilde{X}$ with natural vector bundles $\mathcal{T}, \widetilde{\mathcal{T}}$ of rank n. Namely, we can consider the G-equivariant vector bundle on $T^*\mathbf{V}$ that is trivial as a vector bundle, and such that G acts on a fiber as on the tautological n-dimensional representation. We also equip this bundle with the $(\mathbb{C}^\times)^2$-action that is trivial on the fiber. The bundle $\mathcal{T}$ is the descent of the restriction of this bundle to $\mu^{-1}(0) \cap \overline{\mathbf{V}}^{ss}$. The bundle $\mathcal{T}$ is $(\mathbb{C}^\times)^2$-equivariant. The bundle $\widetilde{\mathcal{T}}$ on $\widetilde{X}$ is defined in a similar way.

There is another bundle on X, the Procesi bundle $\mathcal{P}$. It is a $(\mathbb{C}^\times)^2$-equivariant bundle with a fiberwise action of S_n having the following properties:

(i) $\mathrm{End}_{\mathcal{O}_X}(\mathcal{P}) = \mathbb{C}S_n \ltimes \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]$ (a $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{S_n}$- and S_n- and $(\mathbb{C}^\times)^2$-linear isomorphism).
(ii) $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{P}, \mathcal{P}) = 0$ for $i > 0$.

[7] Here $\wedge^{n-2\bullet}\mathbb{C}^n := \oplus_j \wedge^{n-2j}\mathbb{C}^n$.

(iii) $e_0\mathcal{P} = \mathcal{O}_X$.
(iv) $e_1\mathcal{P} = \mathcal{T}$.

Ginzburg generalized (iv): the vector bundle $e_i\mathcal{P} = \operatorname{Hom}_{S_n}(\wedge^i\mathbb{C}^n, \mathcal{P})$ is naturally isomorphic to $\wedge^i\mathcal{T}$. This follows from [Gi2, Theorem 1.6.1] and is the main ingredient in the proof of Theorem 7.1. (Note that this property fails if we replace $\wedge^i$ in both places by a more general Schur functor!)

Because of (ii), the bundle $\mathcal{P}$ uniquely extends to a $(\mathbb{C}^\times)^2$-equivariant bundle $\widetilde{\mathcal{P}}$ on $\widetilde{X}$. Moreover, since $\wedge^i\mathcal{T}$ is a direct summand of $\mathcal{P}$, we get $\operatorname{Ext}^1_{\mathcal{O}}(\wedge^i\mathcal{T}, \wedge^i\mathcal{T}) = 0$. So $\wedge^i\mathcal{T}$ is a unique $(\mathbb{C}^\times)^2$-equivariant extension of $\wedge^i\widetilde{\mathcal{T}}$. So we see that $\widetilde{\mathcal{P}}$ has the following properties:

$\widetilde{\text{(i)}}$ $\operatorname{End}_{\mathcal{O}_{\widetilde{X}}}(\widetilde{\mathcal{P}})/(\mathfrak{z}) = \mathbb{C}S_n \ltimes \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]$.
$\widetilde{\text{(ii)}}$ $\operatorname{Ext}^i_{\mathcal{O}_{\widetilde{X}}}(\widetilde{\mathcal{P}}, \widetilde{\mathcal{P}}) = 0$ for $i > 0$.
$\widetilde{\text{(iii)}}$ $e_0\widetilde{\mathcal{P}} = \mathcal{O}_{\widetilde{X}}$.
$\widetilde{\text{(iv)}}$ $e_1\widetilde{\mathcal{P}} = \widetilde{\mathcal{T}}$. More generally, the multiplicity space of the S_n-module $\wedge^i\mathbb{C}^n$ in $\widetilde{\mathcal{P}}$ is isomorphic to $\wedge^i\widetilde{\mathcal{T}}$.

7.3. **Quantization.** Set $\mathcal{D}_h := \mathcal{D}_h^{\mathbb{C}}$, where $\mathbb{C}$ stands for the trivial G-module. This is a quantization of the structure sheaf $\mathcal{O}_{\widetilde{X}}$. We remark that $\mathcal{D}_h$ is almost the same as the canonical quantization of $\widetilde{X}$ studied in [BK] and [L6]. The only difference is that the structure of the $\mathbb{C}[[\mathfrak{z}^*, h]]$-algebra is changed, since here we have used a non-symmetrized quantum comoment map to define $\mathcal{D}_h$.

To a G-module U we can assign a bundle $\widetilde{\mathcal{U}}$ on $\widetilde{X}$ as before. One can construct a quantization $\widetilde{\mathcal{U}}_h$ of $\widetilde{\mathcal{U}}$ as follows:

$$\widetilde{\mathcal{U}}_h := [(U \otimes D_h(\mathbf{V}))|_{\overline{\mathbf{V}}^{ss}}/(U \otimes D_h(\mathbf{V}))|_{\mathbf{V}^{ss}}\Phi_h([\mathfrak{g}, \mathfrak{g}])]^G.$$

It is clear from the construction that $\widetilde{\mathcal{U}}_h$ is a $(\mathbb{C}^\times)^2$-equivariant right $\mathcal{D}_h$-module. Let us relate $\widetilde{\mathcal{U}}_h$ to $\mathcal{D}_h^U$.

Lemma 7.2. There is a natural identification of the sheaves of algebras $\mathcal{E}nd_{\mathcal{D}_h}(\widetilde{\mathcal{U}}_h)$ and $\mathcal{D}_h^U$.

Proof. There is a natural action of $\mathcal{D}_h^U$ on $\widetilde{\mathcal{U}}_h$ from the left commuting with a right action of $\mathcal{D}_h$. This gives rise to a homomorphism $\mathcal{D}_h^U \to \mathcal{E}nd_{\mathcal{D}_h}(\widetilde{\mathcal{U}}_h)$. The endomorphism sheaf is flat modulo h because $\widetilde{\mathcal{U}}_h$ is a locally free right $\mathcal{D}_h$-module. The sheaf $\mathcal{D}_h^U$ is complete in the h-adic topology. This is because the sheaf $\operatorname{End}(U) \otimes D_h(\mathbf{V})|_{\overline{\mathbf{V}}^{ss}}$ is Noetherian and so every left ideal is finitely generated and hence closed with respect to the h-adic topology, compare to [L5, Lemma 2.4.4]. So it is enough to check that the homomorphism is an isomorphism modulo h. Equivalently, we need to show that the endomorphism sheaf of $\widetilde{\mathcal{U}}$ is the sheaf induced by $\operatorname{End}(U)$. But this is clear. □

Also, thanks to $\widetilde{\text{(ii)}}$ we have a unique quantization $\widetilde{\mathcal{P}}_h$ of $\widetilde{\mathcal{P}}$, where $\widetilde{\mathcal{P}}_h$ is again a $(\mathbb{C}^\times)^2$-equivariant right $\mathcal{D}_h$-module. We still have a natural action of S_n on $\widetilde{\mathcal{P}}_h$. Consider the endomorphism algebra $\operatorname{End}_{\mathcal{D}_h}(\widetilde{\mathcal{P}}_h)$. This is a $\mathbb{C}[[\mathfrak{z}^*, h]]$-algebra equipped with a $(\mathbb{C}^\times)^2$-action by automorphisms. Consider the subalgebra $\operatorname{End}_{\mathcal{D}_h}(\widetilde{\mathcal{P}}_h)_{T_2-fin}$ of T_2-finite elements in $\operatorname{End}_{\mathcal{D}_h}(\widetilde{\mathcal{P}}_h)$. The results of [L6, Section 6] relate the latter algebra to $\mathbf{H}$. Summarizing these results, we obtain the following proposition.

Proposition 7.3. We have an S_n-linear, $(\mathbb{C}^\times)^2$-equivariant isomorphism of $\mathbb{C}[h]$-algebras $\Upsilon : \mathbf{H} \to \operatorname{End}_{\mathcal{D}_h}(\widetilde{\mathcal{P}}_h)_{T_2-fin}$. It maps $c \in \mathbf{H}$ to $-\beta$ or to $\beta - h$.

We will see in the next subsection that actually $\Upsilon(c) = \beta - h$ (i.e. $\beta \mapsto c + h$, as desired).

Now let $\tilde{e} \in \mathbb{C}S_n$ be an idempotent. Then Υ induces an isomorphism

$$\tilde{e}\mathbf{H}\tilde{e} \xrightarrow{\sim} \tilde{e}\,\mathrm{End}_{\mathcal{D}_h}(\widetilde{\mathcal{P}}_h)_{T_2-fin}\tilde{e} = \mathrm{End}_{\mathcal{D}_h}(\tilde{e}\widetilde{\mathcal{P}}_h)_{T_2-fin}.$$

7.4. **Proof of Theorem 7.1.** Let us remark that if a G-module U satisfies $\widetilde{\mathcal{U}} = \tilde{e}\widetilde{\mathcal{P}}$, then $\widetilde{\mathcal{U}}_h = \tilde{e}\widetilde{\mathcal{P}}_h$. This is because $\mathrm{Ext}^i_{\mathcal{O}_{\widetilde{X}}}(\tilde{e}\widetilde{\mathcal{P}}, \tilde{e}\widetilde{\mathcal{P}}) = 0$ for $i > 0$ and so $\widetilde{\mathcal{P}}$ admits a unique quantization. So by applying $\mathbf{e}_-$, we have a $(\mathbb{C}^\times)^2$-equivariant $\mathbb{C}[[\mathfrak{z}^*, h]]$-linear isomorphism $\Gamma(\widetilde{X}, \mathcal{D}_h^U) \to \mathrm{End}_{\mathcal{D}_h}(\tilde{e}\widetilde{\mathcal{P}}_h)$ and hence a $(\mathbb{C}^\times)^2$-equivariant $\mathbb{C}[\beta, h]$-linear isomorphism $\mathcal{A}_h(\mathbf{V}, U) \xrightarrow{\sim} \mathrm{End}_{\mathcal{D}_h}(\tilde{e}\widetilde{\mathcal{P}}_h)_{T_2-fin}$. So, to prove Theorem 7.1, it remains to take $U = \wedge^{n-2\bullet}\mathbb{C}^n$ (where we use Ginzburg's result on $\wedge^i\mathcal{T}$) and verify that in Proposition 7.3, we have $\Upsilon(c) = \beta - h$.

Assume the converse, $\Upsilon(c) = -\beta$. Consider the determinant representation $\wedge^n\mathbb{C}^n$. Then $\widetilde{\mathcal{U}} = \mathbf{e}_-\widetilde{\mathcal{P}}$, where $\mathbf{e}_-$ is the idempotent corresponding to the sign representation. So we have an isomorphism $\mathbf{e}_-\mathbf{H}\mathbf{e}_- \cong \mathcal{A}_h(\mathbf{V}, \wedge^n\mathbb{C}^n)$. Consider the specialization of this isomorphism at $h = 1$ and $c = p$. Since $\Upsilon(c) = -\beta$, we get $\mathbf{e}_-\mathbf{H}_{1,p}\mathbf{e}_- \cong \mathcal{A}_{1,-p}(\mathbf{V}, \wedge^n\mathbb{C}^n)$. It is known ([BEG]) that there is an isomorphism $\sigma_1 : \mathbf{e}\mathbf{H}_{1,p-1}\mathbf{e} \xrightarrow{\sim} \mathbf{e}_-\mathbf{H}_{1,p}\mathbf{e}_-$. Also, by the definition of quantum hamiltonian reductions, there is an isomorphism $\sigma_2 : \mathcal{A}_{1,-1-p}(\mathbf{V}, \mathbb{C}) \xrightarrow{\sim} \mathcal{A}_{1,-p}(\mathbf{V}; \wedge^n\mathbb{C}^n)$. So, we have an isomorphism $\sigma_2^{-1} \circ \mathbf{e}_-\Upsilon \circ \sigma_1 : \mathbf{e}\mathbf{H}_{1,p-1}\mathbf{e} \to \mathcal{A}_{1,-1-p}(\mathbf{V}, \mathbb{C})$. On the other hand, we have $\mathbf{e}\Upsilon : \mathbf{e}\mathbf{H}_{1,p+1}\mathbf{e} \to \mathcal{A}_{1,-1-p}(\mathbf{V}, \mathbb{C})$. This gives rise to an isomorphism $\mathbf{e}\mathbf{H}_{1,p+1}\mathbf{e} \cong \mathbf{e}\mathbf{H}_{1,p-1}\mathbf{e}$ for all p. It is clear, however, that such an isomorphism cannot exist (for example, from considering dimensions of irreducible finite dimensional representations, see [BEG]).

7.5. **Local vs. global quantum Hamiltonian reductions.** We also can form the global hamiltonian reduction

$$A_h(\mathbf{V}, U) := [(\mathrm{End}(U) \otimes D_h(\mathbf{V}))/(\mathrm{End}(U) \otimes D_h(\mathbf{V}))\Phi_h^U([\mathfrak{g}, \mathfrak{g}])]^G,$$

where the algebra $D_h(\mathbf{V})$ is not completed. This definition yields a $(\mathbb{C}^\times)^2$-equivariant $\mathbb{C}[\beta, h]$-linear algebra homomorphism $\varphi : A_h(\mathbf{V}, U) \to \mathcal{A}_h(\mathbf{V}, U)$.

We do not know whether this homomorphism is an isomorphism with one exception: $U = \mathbb{C}$. In this case modulo (h, β), the homomorphism ϕ is the map $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{S_n} \to \mathbb{C}[X]$. This is an isomorphism. Since the algebras in consideration are graded and flat over $\mathbb{C}[h, \beta]$, the homomorphism φ is an isomorphism.

Also let us point out that our construction is independent of U in the following sense. Let e', e'' be two commuting idempotents in $\mathbb{C}S_n$ such that $e'e'' = 0$. Assume that U', U'' be G-modules such that $\mathcal{U}' \cong e'\mathcal{P}, \mathcal{U}'' \cong e''\mathcal{P}$. Then we have homomorphisms

$$A_h(\mathbf{V}, U') \to \mathcal{A}_h(\mathbf{V}, U') \xrightarrow{\sim} e'\mathbf{H}e', A_h(\mathbf{V}, U'') \to \mathcal{A}_h(\mathbf{V}, U'') \xrightarrow{\sim} e''\mathbf{H}e'',$$

$$A_h(\mathbf{V}, U' \oplus U'') \to \mathcal{A}_h(\mathbf{V}, U' \oplus U'') \xrightarrow{\sim} (e' + e'')\mathbf{H}(e' + e'').$$

The latter homomorphisms map idempotents corresponding to U', U'' to the analogous idempotents. The induced homomorphisms between $A_h(\mathbf{V}, U^?), \mathcal{A}_h(\mathbf{V}, U^?), e^?\mathbf{H}e^?$ coincide with the homomorphisms above.

Specializing to $h = 1, \beta = c$ we get a homomorphism $A_c(\mathbf{V}, U) \to e_U\mathbf{H}_c e_U$. We note that for $U = \mathbb{C}$ the homomorphism $A_c(\mathbf{V}, U) \to \mathcal{A}_c(\mathbf{V}, U) = \mathbf{e}H_c\mathbf{e}$ is an isomorphism.

7.6. **The CEE construction.** Let V be an m-dimensional vector space, and M a D-module on $\mathfrak{g} = \mathfrak{sl}(V) = \mathfrak{sl}_m$. Let $G = SL(V)$ (note that we now consider $\mathfrak{sl}_m$ instead of $\mathfrak{gl}_m$, and use m instead of n for the size of matrices).

Consider the vector space

$$F_n(M) := (M \otimes V^{*\otimes n})^G$$

Remark 7.4. Let M_f be the locally finite part of M under the $\mathfrak{sl}(V)$-action by conjugation. It is clear that $F_n(M) = F_n(M_f)$, so we may assume that $M = M_f$, i.e. that M is G-equivariant. In this case, $M = \oplus_{s=0}^{m-1} M(s)$, where $M(s)$ is the subspace on which the center of $SL(V)$ acts as it does in $V^{\otimes s}$. It is easy to see that $F_n(M) = F_n(M(\bar{n}))$, where $\bar{n}$ is the remainder under division of n by m.

Following [CEE], Subsection 9.6 [8] and replacing V with V^* (using the isomorphism $\mathfrak{sl}(V) \cong \mathfrak{sl}(V^*)$ given by $A \mapsto -A^*$), we obtain the following proposition

Proposition 7.5. The space $F_n(M)$ carries a natural action of the rational Cherednik algebra $H_{\frac{m}{n}}(n)$.

If $M = M(s)$, we say that M has central character s. Let μ be a partition of $d = GCD(m, n)$, and M_μ be the irreducible G-equivariant D-module on $\mathfrak{sl}(V)$ with central character $\bar{n}$ supported on the nilpotent orbit O_μ corresponding to μ, as in [CEE]. Then, as shown in [CEE], $F_n(\mathcal{F}(M_\mu)) = L_{\frac{m}{n}}(n\mu/d)$. Thus, $\overline{\mathbf{e}}F_n(\mathcal{F}(M_\mu)) = \overline{\mathbf{e}}L_{\frac{m}{n}}(n\mu/d)$.

Thus, applying $\overline{\mathbf{e}}$ to the statement of Proposition 7.5, we immediately obtain the following corollary.

Corollary 7.6. The space

$$\overline{\mathbf{e}}F_n(M) = (M \otimes (\oplus_{j \text{ even}} S^{n-j}V^* \otimes \wedge^j V^*))^G$$

carries a natural action of the algebra $\overline{\mathbf{e}}H_{\frac{m}{n}}(n)\overline{\mathbf{e}}$. If $M = \mathcal{F}(M_\mu)$, where μ is a partition of d, with central character $\bar{n}$, then this space is naturally isomorphic to $\overline{\mathbf{e}}L_{\frac{m}{n}}(n\mu/d)$.

7.7. **Matching of representations with minimal support for spherical Cherednik algebras.** Let m, n be positive integers with $GCD(m, n) = d$. By $H_c(n)$ we will mean the rational Cherednik algebra $H_c(S_n, \mathbb{C}^{n-1})$. It is known [L1] that if c has denominator d, then the proper two-sided ideals in $H_c(S_n)$ form a chain $0 = I_0 \subset I_1 \subset ... \subset I_{[n/d]}$; so $I_{[n/d]}$ is a maximal ideal. We will denote it by $I(c)$. Let $\mathbf{e} = \mathbf{e}_0$ be the symmetrizing idempotent for S_n, and $I_{\mathbf{e}}(c) = \mathbf{e}I(c)\mathbf{e} \subset \mathbf{e}H_c(n)\mathbf{e}$.

Recall that the spherical Cherednik algebra $\mathbf{e}H_c(n)\mathbf{e}$ contains the x-subalgebra generated by the power sums $p_j(x_1, ..., x_n)$, the y-subalgebra generated by the power sums $p_j(y_1, ..., y_n)$, and the $\mathfrak{sl}_2$-subalgebra generated by $\sum x_i^2$ and $\sum y_i^2$. The following proposition is proved in [CEE, Proposition 9.5]:

Proposition 7.7. There is an isomorphism $\phi : \mathbf{e}H_{\frac{m}{n}}(n)\mathbf{e}/I_{\mathbf{e}}(\frac{m}{n}) \to \mathbf{e}H_{\frac{n}{m}}(m)\mathbf{e}/I_{\mathbf{e}}(\frac{n}{m})$, which preserves the Bernstein filtration, the x-subalgebra, the y-subalgebra, and the $\mathfrak{sl}_2$-subalgebra.

Consider the isomorphism ϕ in more detail. Let $x_1, ..., x_n$ be the x-variables for the first spherical Cherednik algebra, and $x'_1, ..., x'_m$ be the x-variables for the second one. Let $p_{r,n} = x_1^r + ... + x_n^r$. By [CEE, Proof of Proposition 9.7, line 3], we have $\phi(p_{r,n}(x)) = \frac{n}{m}p_{r,m}(x')$. Denote the x-subalgebras of these two spherical Cherednik algebras (i.e., the subalgebras generated by $p_{r,n}$ and $p_{r,m}$ respectively) by $A_{m,n}$ and $A_{n,m}$, respectively, and define the

[8] Note that in [CEE], the parameter m is denoted by N.

affine schemes $\mathcal{X}_{m,n} = \mathrm{Spec}\ A_{m,n}$ and $\mathcal{X}_{n,m} = \mathrm{Spec}\ A_{n,m}$. Then we have an isomorphism $\phi : A_{m,n} \to A_{n,m}$, and hence an isomorphism of schemes $\phi^* : \mathcal{X}_{n,m} \to \mathcal{X}_{m,n}$. This isomorphism induces an isomorphism of the corresponding reduced schemes (i.e., affine varieties) $\phi^* : \overline{\mathcal{X}}_{n,m} \to \overline{\mathcal{X}}_{m,n}$. These varieties, by the results of [CEE, Section 9], are just the minimal supports of category $\mathcal{O}$ modules over the corresponding spherical Cherednik algebras. This means that $\overline{\mathcal{X}}_{m,n} = X_{d,n/d}(n)/S_n$ is the image (under taking the quotient by permutations) of the locus where $x_i = x_j$ when $i-j$ is divisible by d ($i,j \in [1,n]$), and $\overline{\mathcal{X}}_{n,m} = X_{d,m/d}(m)/S_m$ is the image of the locus where $x_i' = x_j'$ when $i-j$ is divisible by d ($i,j \in [1,m]$). Set $z_i = x_i$ and $z_i' = x_i'$ for $i = 1,...,d$. Then $p_{r,n}(x) = \frac{n}{d}p_{r,d}(z)$ on $\overline{\mathcal{X}}_{m,n}$, and $p_{r,m}(x') = \frac{m}{d}p_{r,d}(z')$ on $\overline{\mathcal{X}}_{n,m}$. Thus, $\overline{\mathcal{X}}_{m,n}$ and $\overline{\mathcal{X}}_{n,m}$ are d-dimensional affine spaces with coordinates $p_{r,d}(z)$, $p_{r,d}(z')$, respectively ($r = 1,...,d$), and we have $\phi(p_{r,d}(z)) = p_{r,d}(z')$. Hence $\phi(\Delta^2(z)) = \Delta^2(z')$, where Δ is the Vandermonde determinant. Let $Z \subset \mathcal{X}_{n,m}$ be the zero locus of $\Delta(z)$ (i.e. the image of the locus where $z_i = z_j$ for some $i \neq j$), and $Z' \subset \mathcal{X}_{m,n}$ be the zero locus of $\Delta(z')$ (i.e., the image of the locus where $z_i' = z_j'$ for some $i \neq j$). We see that $\phi^*(Z') = Z$.

Now consider the category $\mathcal{O}$ of modules over $\mathbf{e}H_{\frac{m}{n}}(n)\mathbf{e}/I_{\mathbf{e}}(\frac{m}{n})$ (i.e., the category of modules which are finitely generated over $\mathbb{C}[x_1,...,x_n]^{S_n}$ and locally nilpotent under the action of the augmentation ideal of $\mathbb{C}[y_1,...,y_n]^{S_n}$). This category is equivalent to the category of minimal support modules in the category $\mathcal{O}$ for $H_{\frac{m}{n}}(n)$: namely, an equivalence is given by $M \mapsto \mathbf{e}M$. So by the results of [Wi, Theorem 1.8 and Proposition 3.7] (see Theorem 2.2), it is a semisimple category with the simple objects $\mathbf{e}L_{\frac{m}{n}}(\frac{n}{d}\mu)$, where μ is a partition of d.

We will need the following proposition on how the isomorphism ϕ acts on these modules.

Proposition 7.8. For any partition μ of d, the pushforward map ϕ_* under the isomorphism ϕ of Proposition 7.7 sends the module $\mathbf{e}L_{\frac{m}{n}}(\frac{n}{d}\mu)$ to the module $\mathbf{e}L_{\frac{n}{m}}(\frac{m}{d}\mu)$.

Proof. It is clear that ϕ_* sends the module $\mathbf{e}L_{\frac{m}{n}}(n\mu/d)$ to the module $\mathbf{e}L_{\frac{n}{m}}(m\sigma(\mu)/d)$, where σ is a certain permutation of the set of partitions of d, and our job is to show that $\sigma = \mathrm{id}$. Let us localize our algebras and modules with respect to the loci Z and Z' (i.e., to the complements of these loci), and denote the corresponding localizations by the subscript "loc". Since $\phi^*(Z') = Z$ (as shown above), we have an isomorphism $\phi_{\mathrm{loc}} : (\mathbf{e}H_{\frac{m}{n}}(n)\mathbf{e}/I_{\mathbf{e}}(\frac{m}{n}))_{\mathrm{loc}} \to (\mathbf{e}H_{\frac{n}{m}}\mathbf{e}/I_{\mathbf{e}}(\frac{n}{m}))_{\mathrm{loc}}$, which maps the module $\mathbf{e}L_{\frac{m}{n}}(n\mu/d)_{\mathrm{loc}}$ to the module $\mathbf{e}L_{\frac{n}{m}}(m\sigma(\mu)/d)_{\mathrm{loc}}$.

On the other hand, we see from the results of [Wi] (see Section 4 of [Wi], in particular Theorem 4.4) that the algebra $(\mathbf{e}H_{\frac{m}{n}}(n)\mathbf{e}/I_{\mathbf{e}}(\frac{m}{n}))_{\mathrm{loc}}$ can be naturally identified with the algebra $(D(\mathbb{C}^d \setminus \text{diagonals}) \otimes \mathrm{End}(Y(m,n)^{\otimes d}))^{S_d}$, where $Y(m,n)$ is the spherical part of the irreducible finite dimensional representation of $H_{\frac{m}{n}}(n_0)$. But it is shown in [CEE] (see Section 9, in particular, Proposition 9.5) that there is a natural identification of graded spaces $\gamma : Y(m,n) \cong Y(n,m)$, and upon this identification the map ϕ_{loc} becomes the identity map. But it follows from [Wi] (see Section 4 of [Wi] and Theorem 2.2 above) that the module $\mathbf{e}L_{\frac{m}{n}}(\frac{n}{d}\mu)$ corresponds under the above identification to the local system on $(\mathbb{C}^d/S_d) \setminus Z$ which is attached to the representation $\pi_\mu \otimes Y^{\otimes d}$ of S_d (where $Y = Y(m,n) = Y(n,m)$ and π_μ is the irreducible representation of S_d attached to the partition μ). Thus, we see that $\pi_\mu \cong \pi_{\sigma(\mu)}$, which implies that $\mu = \sigma(\mu)$, as desired. □

7.8. **The generalized Gan-Ginzburg construction.** Recall the setting of quantum hamiltonian reduction introduced above (but now for numerical values of parameters, and n replaced with m). Let $\mathfrak{g} = \mathfrak{sl}_m$, $V = \mathbb{C}^m$, $\mathbf{V}_m = \mathfrak{g} \times V$. We will denote $\mathbf{V}_m$ by $\mathbf{V}$ for brevity. Let $0 \le i \le m$, and consider the algebra $\overline{A} := D(\mathbf{V}) \otimes \mathrm{End}(\wedge^{m-2\bullet}V)$.

For $a \in \mathfrak{gl}_m$, let X_a be the vector field on $\mathbf{V}$ corresponding to the action of a, and let us consider the homomorphism (the quantum moment map) $\mu : \mathfrak{gl}_m \to \overline{A}$ defined by $\mu(a) := X_a \otimes 1 + 1 \otimes a$. Let $c \in \mathbb{C}$ and $\chi_c : \mathfrak{gl}_m \to \mathbb{C}$ be the character defined by $\chi_c(a) = c\,\mathrm{Tr}(a)$. Let $A_c(\mathbf{V}, \wedge^{m-2\bullet}V) := (\overline{A}/\overline{A}(\mu - \chi_c)(\mathfrak{gl}_m))^{\mathfrak{gl}_m}$ be the (global) quantum hamiltonian reduction.

Proposition 7.9. Let M be a $D(\mathfrak{g})$-module. Then the algebra $A_c(\mathbf{V}, \wedge^{m-2\bullet}V)$ acts naturally on the space $(M \otimes SV^* \otimes \wedge^{m-2\bullet}V \otimes \chi_{-c})^{\mathfrak{gl}_m}$.

Proof. This follows directly from the definition of the quantum Hamiltonian reduction. □

Note that the space $(M \otimes SV^* \otimes \wedge^{m-j}V \otimes \chi_{-c})^{\mathfrak{gl}_m}$ is nonzero if and only if $-mc+m-j$ is a nonnegative integer ℓ, and in this case this space is $(M \otimes S^\ell V^* \otimes \wedge^{m-j}V)^{\mathfrak{g}}$. Thus, replacing c by $1-c$ and $\wedge^{m-j}V$ by $\wedge^j V^*$ (which are isomorphic $\mathfrak{sl}_m$-modules), we obtain the following corollary.

Corollary 7.10. For a nonnegative integer ℓ, on the space $(M \otimes (\oplus_{j \text{ even}} S^{mc-j}V^* \otimes \wedge^j V^*))^{\mathfrak{g}}$, there is a natural action of the algebra $A_{1-c}(\mathbf{V}, \wedge^{m-2\bullet}V)$.

We are now ready to state and prove the main theorem of this section. Recall from Subsection 2.3 that $\overline{\mathbf{e}} = \sum_{i=0}^{n-1} \mathbf{e}_i = \sum_{i \geq 0} e_{2i} = \sum_{i \geq 0} e_{2i+1}$ is an idempotent for S_n. We have: $\overline{\mathbf{e}}\mathbb{C}S_n = \wedge\mathfrak{h}_n$, where $\mathfrak{h}_n$ is the reflection representation of S_n.

Theorem 7.11. We have a natural isomorphism of algebras

$$\overline{\mathbf{e}}(H_{\frac{n}{m}}(m)/I(\frac{n}{m}))\overline{\mathbf{e}} \cong \overline{\mathbf{e}}(H_{\frac{m}{n}}(n)/I(\frac{m}{n}))\overline{\mathbf{e}}$$

preserving the filtration and the grading, and mapping $\mathbf{e}_j$ to $\mathbf{e}_j$ and e_j to e_j.

A proof of Theorem 7.11 is given in the next subsection.

Corollary 7.12. We have natural isomorphisms of algebras

$$e_j(H_{\frac{n}{m}}(m)/I(\frac{n}{m}))e_j \cong e_j(H_{\frac{m}{n}}(n)/I(\frac{m}{n}))e_j,$$

$$\mathbf{e}_j(H_{\frac{n}{m}}(m)/I(\frac{n}{m}))\mathbf{e}_j \cong \mathbf{e}_j(H_{\frac{m}{n}}(n)/I(\frac{m}{n}))\mathbf{e}_j$$

Proof. The Corollary follows from Theorem 7.11 by applying e_j, respectively $\mathbf{e}_j$ on both sides. □

Remark 7.13. If $n > m$ then for $m < j \leq n$, $e_j \in I(\frac{n}{m})$, so for any i,

$$e_i(H_{\frac{m}{n}}(n)/I(\frac{m}{n}))e_j = e_j(H_{\frac{m}{n}}(n)/I(\frac{m}{n}))e_i = 0.$$

7.9. **Proof of Theorem 7.11.** We will show that there is a homomorphism

$$\Phi_{m,n} : \overline{\mathbf{e}}(H_{\frac{m}{n}}(n)/I(\frac{m}{n}))\overline{\mathbf{e}} \to \overline{\mathbf{e}}(H_{\frac{n}{m}}(m)/I(\frac{n}{m}))\overline{\mathbf{e}}$$

preserving the Bernstein filtration. This homomorphism must be injective since the algebra $\overline{\mathbf{e}}(H_{\frac{m}{n}}(n)/I(\frac{m}{n}))\overline{\mathbf{e}}$ is simple [9]. This implies that we have a self-inclusion $\Phi_{n,m} \circ \Phi_{m,n}$ of $\overline{\mathbf{e}}(H_{\frac{m}{n}}(n)/I(\frac{m}{n}))\overline{\mathbf{e}}$ preserving the Bernstein filtration. Since the Bernstein filtration has finite dimensional quotients, this self-inclusion must be an isomorphism, which implies the theorem.

[9]Indeed, this algebra is Morita equivalent to the algebra $H_{\frac{m}{n}}(n)/I(\frac{m}{n})$, which is simple, and simplicity is a Morita invariant property.

To construct $\Phi_{m,n}$, recall that by Proposition 7.6, we have a map

$$\tau : \overline{\mathbf{e}} H_{\frac{m}{n}}(n) \overline{\mathbf{e}} \to \mathrm{End}\left((M \otimes \oplus_{j \text{ even}} S^{n-j} V^* \otimes \wedge^j V^*)^{\mathfrak{g}} \right),$$

which is obtained by applying $\overline{\mathbf{e}}$ on both sides to the map $\widetilde{\tau} : H_{\frac{m}{n}}(n) \to \mathrm{End}((M \otimes (V^*)^{\otimes n})^{\mathfrak{g}})$ provided by [CEE], see Proposition 7.5 above (on the right hand side, e_j symmetrizes with respect to the first $n-j$ indices and antisymmetrizes with respect to the last j indices, and $\overline{\mathbf{e}} = \sum_{j \text{ even}} e_j$).[10] Moreover, we know from [CEE], Section 9, that this map kills the ideal $\overline{\mathbf{e}} I(\frac{m}{n}) \overline{\mathbf{e}}$, so it defines a map

$$\bar{\tau} : \overline{\mathbf{e}}(H_{\frac{m}{n}}(n)/I(\frac{m}{n}))\overline{\mathbf{e}} \to \mathrm{End}((M \otimes (\oplus_{j \text{ even}} S^{n-j} V^* \otimes \wedge^j V^*))^{\mathfrak{g}}).$$

Now, according to [CEE], the action $\bar{\tau}$ is given by global differential operators with values in $U = \oplus_{j \text{ even}} \wedge^{m-j} V^*$. Let $\xi : A_{1-\frac{n}{m}}(\mathbf{V}, U) \to \mathrm{End}((M \otimes (\oplus_{j \text{ even}} S^{n-j} V^* \otimes \wedge^j V^*))^{\mathfrak{g}})$ (where $\mathbf{V} = \mathfrak{sl}(V) \oplus V$) be the action of the global hamiltonian reduction from Corollary 7.10. Let $K = \mathrm{Ker}(\xi)$, and $\bar{\xi}$ be the corresponding injective map

$$\bar{\xi} : A_{1-\frac{n}{m}}(\mathbf{V}, U)/K \to \mathrm{End}((M \otimes (\oplus_{j \text{ even}} S^{n-j} V^* \otimes \wedge^j V^*))^{\mathfrak{g}}).$$

Since the action $\bar{\tau}$ is given by global differential operators, it must factor through $\bar{\xi}$, i.e., there exists a unique homomorphism $\bar{\theta} : \overline{\mathbf{e}}(H_{\frac{m}{n}}(n)/I(\frac{m}{n}))\overline{\mathbf{e}} \to A_{1-\frac{n}{m}}(\mathbf{V}, U)/K$ such that $\bar{\tau} = \bar{\xi} \circ \bar{\theta}$.

Now recall that for any c we have an algebra homomorphism from the global hamiltonian reduction to the global sections of the local hamiltonian reduction, $\pi : A_c(\mathbf{V}, U) \to \mathcal{A}_c(\mathbf{V}, U)$ (see Subsection 7.5). This descends to $\bar{\pi} : A_c(\mathbf{V}, U)/K \to \mathcal{A}_c(\mathbf{V}, U)/\langle \pi(K) \rangle$ (where $\langle S \rangle$ denotes the ideal generated by S). Also, since $e_i H_c e_i \cong e_{m-i} H_{-c} e_{m-i}$, by Theorem 7.1 we have an isomorphism $\varphi : \mathcal{A}_{1-\frac{n}{m}}(\mathbf{V}, U) \to \overline{\mathbf{e}}(H_{\frac{n}{m}}(m))\overline{\mathbf{e}}$, which induces isomorphisms $\varphi_j : \mathcal{A}_{1-\frac{n}{m}}(\mathbf{V}, \wedge^{m-2j}\mathbb{C}^m) \to e_j H_{\frac{n}{m}}(m) e_j$. This isomorphism descends to an isomorphism

$$\varphi' : \mathcal{A}_{1-\frac{n}{m}}(\mathbf{V}, U)/\langle \pi(K) \rangle \to \overline{\mathbf{e}} H_{\frac{n}{m}}(m) \overline{\mathbf{e}}/\varphi(\langle \pi(K) \rangle).$$

Since $\overline{\mathbf{e}} I(\frac{n}{m})\overline{\mathbf{e}}$ is a maximal ideal, and since ideals in the Cherednik algebra form a chain (see the beginning of Subsection 7.7), we see that $\overline{\mathbf{e}} I(\frac{n}{m})\overline{\mathbf{e}} \supset \varphi(\langle \pi(K) \rangle)$, and hence we have a projection $\gamma : \overline{\mathbf{e}} H_{\frac{n}{m}}(m)\overline{\mathbf{e}}/\phi(\langle \pi(K) \rangle) \to \overline{\mathbf{e}}(H_{\frac{n}{m}}(m)/I(\frac{n}{m}))\overline{\mathbf{e}}$. So the map φ' gives rise to a map $\bar{\varphi} : \mathcal{A}_{1-\frac{n}{m}}(\mathbf{V}, U)/\langle \pi(K) \rangle \to \overline{\mathbf{e}}(H_{\frac{n}{m}}(m)/I(\frac{n}{m}))\overline{\mathbf{e}}$.

So altogether we have a map

$$\Phi_{m,n} := \bar{\varphi} \circ \bar{\pi} \circ \bar{\theta} : \overline{\mathbf{e}}(H_{\frac{m}{n}}(n)/I(\frac{m}{n}))\overline{\mathbf{e}} \to \overline{\mathbf{e}}(H_{\frac{n}{m}}(m)/I(\frac{n}{m}))\overline{\mathbf{e}},$$

as desired.

It remains to show that the map $\Phi_{m,n}$ preserves the Bernstein filtration. To show this, note that the map $\bar{\theta}$ preserves the Bernstein filtration by the CEE construction, the map $\bar{\varphi}$ preserves the Bernstein filtration by the generalized Gan-Ginzburg construction, and the map $\bar{\pi}$ preserves the Bernstein filtration because the corresponding map of the Rees algebras is $(\mathbb{C}^\times)^2$-equivariant. This implies the required statement.

[10]Here M can be taken to be any D-module for which the corresponding spaces of invariants are nonzero; for example, one can take $M = \mathcal{F}(M_\lambda)$ for λ being a partition of d, as in Section 5.

7.10. **Correspondence between modules over quasi-spherical subalgebras.**

Proposition 7.14. For $j = 0$, the isomorphisms of Corollary 7.12 coincide with the isomorphism constructed above in Proposition 7.7.

Proof. It suffices to show that these isomorphisms coincide on the elements $\sum_i x_i^k$ and $\sum_i y_i^2$, since by the results of [BEG], such elements generate the corresponding algebras.

To this end, note that it follows from [CEE], Section 9, that in the proof of Theorem 7.11 one has $\bar{\theta}(\sum_{i=1}^n x_i^r) = \frac{n}{m}\operatorname{Tr}(X^r)$ where $X \in \mathfrak{sl}_m$. Also, it is clear that $\bar{\pi}(\operatorname{Tr}(X^r)) = \operatorname{Tr}(X^r)$, and $\bar{\phi}(\operatorname{Tr}(X^r)) = \sum_{i=1}^m x_i^r$. Thus, $\Phi_{m,n}(\sum_{i=1}^n x_i^r) = \frac{n}{m}\sum_{i=1}^m x_i^r$. Similarly, it follows from [CEE], Section 9, that one has $\bar{\theta}(\sum_{i=1}^n y_i^2) = \frac{n}{m}\Delta_{\mathfrak{g}}$ (the Laplacian of $\mathfrak{g}$). Also, it is clear that $\bar{\pi}(\Delta_{\mathfrak{g}}) = \Delta_{\mathfrak{g}}$, and it is known from [EG] that $\bar{\phi}(\Delta_{\mathfrak{g}}) = \sum_{i=1}^m y_i^2$. Thus, $\Phi_{m,n}(\sum_{i=1}^n y_i^2) = \frac{n}{m}\sum_{i=1}^m y_i^2$, as desired. □

Corollary 7.15. The isomorphism of Theorem 7.11 maps $\overline{\mathbf{e}}L_{\frac{n}{m}}(\frac{m}{d}\mu)$ to $\overline{\mathbf{e}}L_{\frac{m}{n}}(\frac{n}{d}\mu)$, and the isomorphisms of Corollary 7.12 map $e_jL_{\frac{n}{m}}(\frac{m}{d}\mu)$ to $e_jL_{\frac{m}{n}}(\frac{n}{d}\mu)$, and $\mathbf{e}_jL_{\frac{n}{m}}(\frac{m}{d}\mu)$ to $\mathbf{e}_jL_{\frac{m}{n}}(\frac{n}{d}\mu)$. Thus, we have natural isomorphisms of vector spaces preserving the gradings and the filtrations: $\overline{\mathbf{e}}L_{\frac{n}{m}}(\frac{m}{d}\mu) \cong \overline{\mathbf{e}}L_{\frac{m}{n}}(\frac{n}{d}\mu)$, $e_jL_{\frac{n}{m}}(\frac{m}{d}\mu) \cong e_jL_{\frac{m}{n}}(\frac{n}{d}\mu)$, $\mathbf{e}_jL_{\frac{n}{m}}(\frac{m}{d}\mu) \cong \mathbf{e}_jL_{\frac{m}{n}}(\frac{n}{d}\mu)$.

Proof. The algebra $\overline{\mathbf{e}}H_c(n)\overline{\mathbf{e}}$ is Morita equivalent to both $\mathbf{e}H_c(n)\mathbf{e}$ and $H_c(n)$, and the symmetrizer $\mathbf{e}$ can be regarded as an idempotent in $\overline{\mathbf{e}}H_c(n)\overline{\mathbf{e}}$. So, we see by Proposition 7.8 and Proposition 7.14 that the pullback of $\overline{\mathbf{e}}L_{\frac{m}{n}}(\frac{n}{d}\mu)$ to $\overline{\mathbf{e}}H_{1/c}(m)\overline{\mathbf{e}}$ is $\overline{\mathbf{e}}L_{\frac{n}{m}}(\frac{m}{d}\mu)$. This proves the first statement of the Corollary. The second statement is obtained from the first one by applying e_j and $\mathbf{e}_j$, respectively. □

Remark 7.16. Note that by virtue of the above results, the algebras $\overline{\mathbf{e}}H_{\frac{m}{n}}(n)\overline{\mathbf{e}}$ and $\overline{\mathbf{e}}H_{\frac{n}{m}}(m)\overline{\mathbf{e}}$ act on the same space $(M\otimes(\oplus_{j\ \mathrm{even}}S^{n-j}V^*\otimes\wedge^jV^*))^{\mathfrak{g}}$ (the first algebra via the CEE construction and the second one via the generalized Gan-Ginzburg construction), and the images of these algebras under these actions coincide.

8. Symmetrized Koszul-BGG complexes

8.1. **Quasiisomorphism of Koszul-BGG complexes.** Consider the BGG resolution $K^\bullet_{m,n}$. As a vector space, it is just the space $\Omega^\bullet\mathfrak{h}_n$ of differential forms on the reflection representation, and the homological degree is given by the degree of a form. The differential is a contraction ι_ξ with an S_n-invariant vector field $\xi = \sum_i f_i\frac{\partial}{\partial x_i}$, where f_i are the singular polynomials for Dunkl operators. In other words, ι_ξ is defined by the identity

$$\iota_\xi(dx_i \wedge \alpha) = f_i\alpha - dx_i \wedge \iota_\xi(\alpha).$$

By definition, $K^\bullet_{m,n}$ is a Koszul complex associated to the polynomials f_i.

Lemma 8.1. The symmetrized complex $(K^\bullet_{m,n})^{S_n}$ coincides with the Koszul complex associated to the polynomials $\sum_i x_i^j f_i$, $1 \le j \le n-1$.

Proof. By a theorem of Solomon [S] we have

$$(\Omega^\bullet\mathfrak{h}_n)^{S_n} = \Omega^\bullet(\mathfrak{h}_n/S_n).$$

The functions on $\mathfrak{h}_n/S_n$ are symmetric functions on $\mathfrak{h}_n$ and form a polynomial ring in power sums $p_2,\dots,p_n$. The statement now follows from the identity

$$\iota_\xi(dp_j) = \iota_\xi(j\sum_i x_i^{j-1}dx_i) = j\sum_i x_i^{j-1}f_i.$$

□

Recall that by (19) the singular polynomials are given by the equation

$$f_i = \frac{\partial}{\partial x_i} \mathrm{Coef}_{m+1} \prod_i (1 - zx_i)^{\frac{m}{n}}.$$

Theorem 8.2. The symmetrized complex $(K^{\bullet}_{m,n})^{S_n}$ is quasi-isomorphic to the Koszul complex associated to the sequence of polynomials

$$\mathrm{Coef}_j \left[\left(1 + \sum_{k=2}^{n} u_k z^k\right)^m - \left(1 + \sum_{k=2}^{m} v_k z^k\right)^n \right], 2 \le j \le m+n-1 \tag{22}$$

in variables $u_2, \ldots, u_n, v_2, \ldots, v_m$.

Proof. By Lemma 8.1 $(K^{\bullet}_{m,n})^{S_n}$ is isomorphic to the Koszul complex associated to the polynomials $g_j = \sum_i x_i^j f_i$, $1 \le j \le n-1$. Similarly to the proof of [Gor2, Theorem 4.3], one can deduce from (19) that g_j are related to $\hat{g}_j = \mathrm{Coef}_{m+j} \prod_i (1 - zx_i)^{\frac{m}{n}}$, $1 \le j \le n-1$ by a triangular change. Therefore the Koszul complexes associated to g_j and $\hat{g}_j$ are quasi-isomorphic. If we denote $\prod_i (1 - zx_i) = 1 + \sum_{k=2}^{n} u_k z^k$, we conclude that $(K^{\bullet}_{m,n})^{S_n}$ is quasi-isomorphic to the Koszul complex associated to the polynomials

$$\mathrm{Coef}_{m+j} (1 + \sum_{k=2}^{n} u_k z^k)^{\frac{m}{n}}, \ 1 \le j \le n-1$$

in variables u_k. The latter complex is quasi-isomorphic to the Koszul complex associated to the polynomials

$$\mathrm{Coef}_j \left[(1 + \sum_{k=2}^{n} u_k z^k)^{\frac{m}{n}} - (1 + \sum_{k=2}^{m} v_k z^k) \right], \ 2 \le j \le m+n-1$$

in variables u_k, v_k, while this set of polynomials is related to (22) by a triangular change which does not affect its quasi-isomorphism class. □

Corollary 8.3. The complexes $(K^{\bullet}_{m,n})^{S_n}$ and $(K^{\bullet}_{n,m})^{S_m}$ are quasi-isomorphic as complexes of modules over the ring of symmetric functions. In particular, $(L_{\frac{m}{n}}((n)))^{S_n} \cong (L_{\frac{n}{m}}((m)))^{S_m}$.

Proof. The first statement follows from Theorem 8.2. The second statement follows from the first one, since $H_0(K^{\bullet}_{m,n}) = L_{\frac{m}{n}}((n))$. □

8.2. **Action of the Hamiltonian.** Recall that the quantum Calogero-Moser Hamiltonian is defined by the formula $H_2 = \sum_{i=1}^{n} D_i^2$. Let us compute the action of H_2 on

$$\mathrm{Hom}_{S_n}(\wedge^{\bullet}\mathfrak{h}_n, \mathbb{C}[V]) = (\Omega^{\bullet}\mathfrak{h}_n)^{S_n} = \Omega^{\bullet}(\mathfrak{h}_n/S_n).$$

This action is defined because the space $\mathrm{Hom}_{S_n}(\wedge^{\bullet}\mathfrak{h}_n, \mathbb{C}[V])$ is the S_n-invariants in the Verma module $M_c(\wedge^{\bullet}\mathfrak{h}_n^*)$ over the rational Cherednik algebra. We have to compute $H_2(f(x_1, \ldots, x_n) dp_{\alpha_1} \wedge \ldots \wedge dp_{\alpha_k})$, where p_i are the power sum symmetric functions (providing a coordinate system on $\mathfrak{h}_n/S_n$), f is a symmetric polynomial in x_i (and thus a polynomial in p_i), and $dp_{\alpha_1} \wedge \ldots \wedge dp_{\alpha_k}$ denotes a copy of $\wedge^k \mathfrak{h}_n$ in $\mathbb{C}[\mathfrak{h}_n]$ spanned by the coefficients of $dp_{\alpha_1} \wedge \ldots dp_{\alpha_k}$ in its expansion in dx_i. Since H_2 commutes with the action of S_n, its action on S_n-equivariant differential forms is well-defined and preserves the exterior degree.

Recall that H_2 is a second order differential operator with $\sum \left(\frac{\partial}{\partial x_i}\right)^2$ as second order part, so one has the identity $H_2(fg) = H_2(f)g + fH_2(g) + 2(\nabla f, \nabla g)$, where $(\nabla f, \nabla g) = \sum_i \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_i}$.

Lemma 8.4. The following equation holds:

$$H_2(dp_i \wedge dp_j) = H_2(dp_i) \wedge dp_j + dp_i \wedge H_2(dp_j).$$

Proof. By definition, $dp_i \wedge dp_j$ is a copy of $\wedge^2 V$ spanned by $\frac{\partial p_i}{\partial x_\mu}\frac{\partial p_j}{\partial x_\nu} - \frac{\partial p_i}{\partial x_\nu}\frac{\partial p_j}{\partial x_\mu}$. Therefore

$$H_2(dp_i \wedge dp_j) - H_2(dp_i) \wedge dp_j - dp_i \wedge H_2(dp_j) = \sum_l \left(\frac{\partial^2 p_i}{\partial x_\mu \partial x_l}\frac{\partial^2 p_j}{\partial x_\nu \partial x_l} - \frac{\partial^2 p_i}{\partial x_\nu \partial x_l}\frac{\partial^2 p_j}{\partial x_\mu \partial x_l}\right)$$

Note that $\frac{\partial^2 p_i}{\partial x_\mu \partial x_l}$ vanish for $\mu \neq l$, so the right hand side can be rewritten as

$$\sum_l \left(\frac{\partial^2 p_i}{\partial x_l^2}\frac{\partial^2 p_j}{\partial x_l^2} - \frac{\partial^2 p_i}{\partial x_l^2}\frac{\partial^2 p_j}{\partial x_l^2}\right) = 0.$$

□

Lemma 8.5. The following identity holds:

$$H_2(dp_k) = (1+c)k(k-1)dp_{k-2} - 2kc\sum_{s=0}^{k-2} p_s dp_{k-2-s}.$$

Proof. By [GORS, Lemma 2.6] $H_2(p_k) = (1+c)k(k-1)p_{k-2} - kc\sum_{s=0}^{k-2} p_s p_{k-2-s}$, and dp_k denotes a copy of $\mathfrak{h}$ spanned by $\frac{\partial p_k}{\partial x_\mu} = D_\mu(p_k)$. Therefore $H_2(dp_k)$ is spanned by

$$H_2(dp_k) = \langle H_2(D_\mu(p_k))\rangle = \langle D_\mu(H_2(p_k))\rangle = dH_2(p_k) =$$

$$(1+c)k(k-1)dp_{k-2} - 2kc\sum_{s=0}^{k-2} p_s dp_{k-2-s}.$$

□

Lemma 8.6. The following equation holds:

$$(\nabla f, \nabla dp_k) = \sum_s \frac{sk(k-1)}{k+s-2}\frac{\partial f}{\partial p_s} dp_{k+s-2}.$$

Proof. By definition, $(\nabla f, \nabla (dp_k)_\mu) = \sum_l \frac{\partial f}{\partial x_l}\frac{\partial^2 p_k}{\partial x_\mu \partial x_l} = k(k-1)x_\mu^{k-2}\frac{\partial f}{\partial x_\mu}$. This is a first order differential operator in f, so it is sufficient to compute it for $f = p_s$:

$$(\nabla p_s, \nabla dp_k)_\mu = k(k-1)x_\mu^{k-2}\frac{\partial p_s}{\partial x_\mu} = sk(k-1)x_\mu^{k+s-3} = \frac{sk(k-1)}{k+s-2}(dp_{k+s-2})_\mu.$$

□

Theorem 8.7. The action of H_2 on the S_n-invariant differential forms is given by the equation

$$H_2(f dp_{\alpha_1}\wedge\ldots dp_{\alpha_k}) = H_2(f)dp_{\alpha_1}\wedge\ldots dp_{\alpha_k}+ \tag{23}$$
$$2\sum_{j=1}^{k}\sum_{s}\frac{s\alpha_j(\alpha_j-1)}{\alpha_j+s-2}\frac{\partial f}{\partial p_s}dp_{\alpha_1}\wedge\ldots\wedge dp_{\alpha_j+s-2}\wedge\ldots dp_{\alpha_k}-$$
$$2cf\sum_{j=1}^{k}\sum_{s}p_s\alpha_j dp_{\alpha_1}\wedge\ldots\wedge dp_{\alpha_j-2-s}\wedge\ldots dp_{\alpha_k}+$$
$$(1+c)f\sum_{j=1}^{k}\alpha_j(\alpha_j-1)dp_{\alpha_1}\wedge\ldots\wedge dp_{\alpha_j-2}\wedge\ldots dp_{\alpha_k}.$$

Proof. By Lemma 8.4 one has

$$H_2(f\cdot dp_{\alpha_1}\wedge\ldots dp_{\alpha_k}) = H_2(f)dp_{\alpha_1}\wedge\ldots dp_{\alpha_k}+\sum_{j=1}^{k} f dp_{\alpha_1}\wedge\ldots\wedge H_2(dp_{\alpha_j})\wedge\ldots dp_{\alpha_k}$$

$$+2\sum_{j=1}^{k}(-1)^{j-1}(\nabla f,\nabla dp_{\alpha_j})dp_{\alpha_1}\wedge\ldots\wedge\widehat{dp_{\alpha_j}}\wedge\ldots dp_{\alpha_k}.$$

Now the theorem follows from Lemma 8.6 and Lemma 8.5. □

Corollary 8.8. Let us consider two sets of coordinates $\{x_i\}$, $\{\widetilde{x_i}\}$ such that $\widetilde{p_i}=cp_i$. Then $H_2^{1/c}(\widetilde{p_i})=\frac{1}{c}H_2^c(p_i)$.

Proof. The statement was proved in [GORS, Theorem 2.9] for symmetric functions. Let us extend it to the differential forms. Indeed, $d\widetilde{p_k}=c\cdot dp_k$, and

$$H_2^{1/c}(f)=\frac{1}{c}H_2^c(f),\ \frac{\partial f}{\partial\widetilde{p_s}}=\frac{1}{c}\frac{\partial f}{\partial p_s},$$

$$\frac{1}{c}\widetilde{p_k}=\frac{1}{c}(cp_k),\ 1+\frac{1}{c}=\frac{1}{c}(1+c).$$

Therefore every term in (23) is multiplied by $\frac{1}{c}$. □

It follows from Proposition 6.1 that the actions of H_2 and ι_ξ commute.

Theorem 8.9. The quasi-isomorphism of Corollary 8.3 between the complexes $(K^\bullet_{m,n})^{S_n}$ and $(K^\bullet_{n,m})^{S_m}$ commutes with the action of H_2. In other words, if Y is the algebra freely generated by the symbol H_2 and symmetric functions in infinitely many variables, then $(K^\bullet_{m,n})^{S_n}$ and $(K^\bullet_{n,m})^{S_m}$ are complexes of Y-modules, and the quasi-isomorphism of Corollary 8.3 is a quasi-isomorphism of complexes of Y-modules.

Proof. Let us extend the action of H_2 to the constructions of Theorem 8.2. Consider the polynomial ring in variables $u_2,\ldots,u_n,v_2,\ldots,v_m$. We identify u_i with elementary symmetric polynomials in variables $x_1,\ldots,x_n$ and v_i with the elementary symmetric polynomials in variables $\widetilde{x_1},\ldots,\widetilde{x_m}$. Consider the operator $H := nH_2 + m\widetilde{H_2}$. It is sufficient to check that H preserves the Koszul complex associated with the equations

$$\mathrm{Coef}_i\left[(1+\sum(-1)^i u_i z^i)^m - (1+\sum(-1)^i v_i z^i)^n\right],\ i = 2\ldots m+n-1.$$

We can change variables and consider instead power sums in x_i and $\widetilde{x_i}$: the generators will be $p_2,\ldots,p_n,\widetilde{p_2},\ldots,\widetilde{p_m}$, and the equations $E_i := mp_i - n\widetilde{p_i} = 0,\ i = 2\ldots m+n-1$. The corresponding Koszul complexes will be quasi-isomorphic, and it follows from [GORS, Lemma 2.6] that

$$\frac{1}{mn}H(E_i) = H_2(p_i) - \widetilde{H_2}(\widetilde{p_i}) = \left(\frac{m+n}{n}i(i-1)p_{i-2} - \frac{m+n}{m}i(i-1)\widetilde{p}_{i-2}\right) - \left(i\frac{m}{n}\sum_{s=0}^{i-2}p_s p_{i-2-s} - i\frac{n}{m}\sum_{s=0}^{i-2}\widetilde{p_s}\widetilde{p}_{i-2-s}\right) = \frac{m+n}{mn}i(i-1)E_{i-2} - \frac{i}{mn}\sum_{s=0}^{i-2}(mp_{i-2-s}E_s + n\widetilde{p_s}E_{i-2-s}).$$

Since $H(E_i)$ belongs to the ideal generated by E_j with $j < i$, the Koszul complex associated with E_i is invariant under H. □

8.3. **Yet another proof of Theorem 6.3(ii).** Here is a third proof of Theorem 6.3(ii), based on Theorem 8.9. First, note that the statement holds if m is divisible by n. In this case, $d = n$, the differential is zero, so the statement is trivial. Next, by the results of [BEG], the spherical subalgebra $\mathbf{e}H_c(n)\mathbf{e}$ is generated by symmetric functions of the x_i and $H_2 := \sum_i y_i^2$. Therefore, Theorem 8.9 and Proposition 7.8 imply that if the statement of Theorem 6.3(ii) holds for (m,n) then it holds for (n,m). Finally, by Corollary 6.12, using the fact that the shift functor is an equivalence ([BE]), we see that if the statement holds for (m,n) with $m \geq n$, then it holds for $(m-n,n)$. This implies the result by using the Euclidean algorithm (more precisely, any pair (m,n) can be reduced to one with m divisible by n by transformations $(m,n)\mapsto(n,m)$ for $m<n$ and $(m,n)\mapsto(m-n,n)$ for $m\geq n$).

Remark 8.10. Instead of the shift functors (i.e., Corollary 6.12), we could have used Rouquier equivalences of highest weight categories $\mathcal{O}_{\frac{m}{n}} \cong \mathcal{O}_{\frac{m'}{n}}$, where $m,m' > 0$ and $GCD(m,n) = GCD(m',n)$ ([R]). Note that for $GCD(m,n) = 2$, these equivalences were constructed later in [L4].

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