A dynamic model of barter exchange*

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Abstract

We consider the problem of efficient operation of a barter exchange platform for indivisible goods. We introduce a dynamic model of barter exchange where in each period one agent arrives with a single item she wants to exchange for a different item. We study a homogeneous and stochastic environment: an agent is interested in the item possessed by another agent with probability p, independently for all pairs of agents. We consider two settings with respect to the types of allowed exchanges: a) Only two-way cycles, in which two agents swap their items, b) Two or three-way cycles. The goal of the platform is to minimize the average waiting time of an agent.

Somewhat surprisingly, we find that in each of these settings, a policy that conducts exchanges in a greedy fashion is near optimal, among a large class of policies that includes batching policies. Further, we find that for small p, allowing three-cycles can greatly improve the waiting time over the two-cycles only setting. Specifically, we find that a greedy policy achieves an average waiting time of $\Theta(1/p^2)$ in setting a), and $\Theta(1/p^{3/2})$ in setting b). Thus, a platform can achieve the smallest waiting times by using a greedy policy, and by facilitating three cycles, if possible.

Our findings are consistent with empirical and computational observations which compare batching policies in the context of kidney exchange programs.

Keywords: barter, random graphs, dynamics, kidney exchange

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1 Introduction

A marketplace for barter exchange provides opportunities for agents to exchange items directly, without monetary payments. There is a growing number of such marketplaces facilitating the exchange of a variety of items. We consider the problem faced by an exchange platform for indivisible items, that seeks to enable users to complete a desirable trade as early as possible. A major lever a platform operator has at her disposal is the *policy* employed in conducting exchanges, for example, greedy, batching, etc. We study which policy should be used in order to minimize the waiting time of users in the face of stochastic individual demands for items. We investigate this question under different settings determined by the feasible *types* of exchanges, which are often driven by the technology adopted by the marketplace.

Background and motivation. A number of barter exchange platforms exist for swapping a variety of items. For instance, www.homeexchange.com and www.ReadItSwapIt.com are decentralized marketplaces that enable pairwise swapping by mutual agreement of homes for vacation and books respectively. Finding a pairwise exchange is challenging because it requires two users to each possess an item that the other desires [9]. www.Swap.com, which allows for swapping (and selling) of preowned childrens items, books and DVDs, is a centralized platform Centralization allows it to execute multi-way exchanges, where each agent in the cycle has an item that the next agent in the cycle desires.

Kidney exchange clearinghouses have also noticed the limitations of pairwise exchanges [11]. Kidney exchange can be a suitable solution when a healthy person (with two kidneys) wishes to donate one to a friend or family member but is biologically incompatible with the intended recipient.[1] An incompatible donor-patient pair can exchange their donor's kidney with one or more other such pairs in a cycle so that every patient receives a compatible kidney. Kidney exchange is a growing market and there are mul-

[1] In kidney exchange, an incompatible donor-patient pair can be thought of as a single agent.

tiple clearinghouses such as the Alliance for Paired Donation (APD) and the National Kidney Registy (NKR) in the United States. Since it is desired that a donor gives her kidney no later than her associated patient receives a kidney, cyclic exchanges are conducted simultaneously and are therefore rarely longer than three.

Each of these markets evolve dynamically with agents arriving over time. At any point in time, there is a pool of agents waiting for an exchange, and possibly some feasible exchanges supported by the platform technology. In centralized platforms, the *policy* employed by the platform determines which feasible exchanges are executed, and when. We ask which policy centralized platforms should use, given the types of supported exchanges. One natural option is the *Greedy* policy, where the platform executes a feasible exchange as soon as the possibility emerges. Alternatively, the platform can adopt a *Batching* policy where it waits for a 'batch' of agents to accumulate, and then identifies a set of exchanges that maximizes the number of matched agents, with the remaining agents carried forward into the next batch. More complex policies are possible as well. Kidney exchange clearinghouses in the US have experimented with a variety of batch lengths in the past, and have subsequently shifted to very short batches resembling a greedy approach. When does the greedy policy perform well? What is the optimal policy, and how does it depend on the feasible types of exchanges? We address these questions in this paper.

We simultaneously also quantify the benefits resulting from facilitating multi-way exchanges, relative to allowing only pairwise exchanges.

Approach and contributions. We consider a stylized dynamic model of agent arrival and departure that allows us to study how agent outcomes depend on the policy and types of allowed exchanges.

In each period, an agent arrives with a single indivisible item that she wishes to exchange. Our model has a homogeneous and independent stochastic demand structure, in which every agent A is willing to exchange his item for any other agent B's item with probability p. Figure 1 shows a "compatibility graph" representation of a snapshot of a barter exchange marketplace, which captures the interest of agents in the items of other agents. Exchanges are conducted via (directed) cycles, and a policy determines which potential exchanges to conduct. The compatibility graph evolves in time as new agents arrive, and existing agents depart after completing exchanges.

Agents who arrive to a barter marketplace want to quickly complete desirable exchanges. Motivated by this, the performance metric we consider is the average waiting time of agents in steady state. (In various contexts, it may be appropriate to consider some other cost function that is non-linear in the waiting time. However, for this foundational work we choose the simple linear cost function.) Thus, the optimal policy is one that minimizes the average waiting time.

We discuss two settings distinguished by the types of possible/allowed exchanges: (i) only 2-way cycles, (ii) 2 and 3-way cycles. In each setting, we seek to identify an approximately optimal policy. Further, we compare the best achievable expected waiting time across settings, with a view to quantifying the benefits of facilitating 3-way cycles. Our key findings, informally stated, are as follows.

- In each of the settings, the Greedy policy is approximately optimal in a large class of policies that includes all batching policies. Thus, in particular, batching does not provide any significant advantage.

- Under the Greedy policy, allowing three-way cycles leads to substantially smaller average waiting time than only allowing two-way cycles.

More precisely, we show that as $p \to 0$, the average waiting time under the Greedy policy scales as $\Theta(1/p^2)$ for the setting based on two-way cycles, as $\Theta(1/p^{3/2})$ for the setting based on two and three-way cycles. Furthermore, for the first setting we show that a Greedy policy achieves the optimal scaling among essentially all possible policies[2]. For the second setting with two and three-way cycles, we prove that greedy is scaling optimal in a broad class of policies called *monotone* policies (see the next section), which includes batching policies. We remark here that a small value of p is reasonable in many practical contexts, since agents are often interested in only a small fraction of the items offered by other agents.[3] Our results imply that, in each setting, for all $p \in (0, 1)$, the waiting time under greedy is within a constant factor of the waiting time using any other batch size. (Simulation experiments suggest that a batch of size of 1 is, in fact, truly optimal in each setting for any p [2].)

[2] In fact, for the two-way cycles setting we show that even the constant factor is tight.

[3] Even kidney exchange clearinghouses observe a substantial fraction of highly sensitized patients that have probability $1-5\%$ of matching.

Interestingly, our results are consistent with computational experiments in kidney exchange using clinical data [4, 1], despite significant heterogeneity and other deviations from our model. These studies find that the benefit of batching relative to greedy, if any, is marginal, in line with our "greedy is approximately optimal among batching policies" finding. (This also matches practice.) Computational experiments have further demonstrated significant benefits from using chains[4] and 3-way cycles over 2-way cycles in dynamic settings [4, 7]. This is consistent with our theoretical findings regarding the benefits of 3-way cycles in our model. We note that our results provide the first theoretical explanation of near optimality of greedy, whereas benefits from 3-cycles have been previously found in related models (see below and [2]).

Our model, while simplistic in its compatibility structure (which is described by a single parameter p), has several advantages. It notably avoids a "market size" parameter altogether (faster arrival of agents simply leads to an inverse rescaling of time), and further avoids a key drawback of previous models involving stochastic compatibilities [3, 4] that artificially require p to scale in a particular way with "market size".[5] Further, studying steady state behavior allows us to quantify performance exclusively in terms of waiting times. The alternative approach of studying a finite time horizon [4], involves end-of-period effects that make it necessary to simultaneously consider both the waiting times and the number of matches, hindering performance comparisons.

These advantages of our model come at the cost of substantial new technical challenges. The most technical part of the paper involves obtaining the bounds of the form $\Theta(1/p^2)$, $\Theta(1/p^{3/2})$. A key challenge we overcome is that the compatibility graph between currently waiting agents, conditional on running greedy so far, is not a directed Erdos-Renyi graph and has a complex distribution. It is sparser in terms of compatibilities in a very specific way: there are no possible exchanges, since the greedy policy would already have executed them. We develop methods to analytically control the graph with this distribution and the associated dynamical system. Another contribution is the technique we develop to prove lower bounds on average waiting times: this technique involves proof by contradiction, and is used in the case of two and three cycles.

Our model and analysis bring together the rich literature on (static) random graph models, e.g., see [6, 8] with the rich literature on queuing systems [10, 5]. In our model, the queue of waiting agents has a graph structure (i.e., the compatibility graph). Our stochastic model of compatibilities mirrors the canonical Erdos-Renyi model of a directed (static) random graph (but the dynamics make it much more complex). Comparing with common models of queueing systems, our system is peculiar in that the queueing system does not contain "servers" per se. Instead, the queue, in some sense, serves itself by executing exchanges that the compatibility graph allows. Nodes form cycles with other nodes. Each time an exchange is executed, the corresponding agents/nodes leave the system. As a result, it turns out that for any reasonable policy the system is stable, irrespective of the rate of arrival of agents. If we speed up the arrival rate of agents, the entire system speeds up by the same factor, and waiting time reduces by the same factor. Thus, without loss of generality we consider an arrival rate of 1, with one agent arriving in each time slot.

Notation. We write that $f(p) = O(g(p))$ where $p \in (0, 1]$, if there exists $C < \infty$ such that $|f(p)| \leq Cg(p)$ for all $p \in (0, 1]$, and similarly for $\Theta(\cdot)$ and $\Omega(\cdot)$. We write that $f(p) = o(g(p))$ where $p \in (0, 1]$, if for any $C > 0$, there exists $p_0 > 0$ such that we have $|f(p)| \leq Cg(p)$ for all $p \leq p_0$, and similarly for $\omega(\cdot)$. Given a Markov chain $\{X_t\}$ defined on a state space $\mathcal{X}$ and given a function $f : \mathcal{X} \to \mathbb{R}$, for $x \in \mathcal{X}$, we use the shorthand $\mathbb{E}_x[f(X_t)] \triangleq \mathbb{E}[f(X_t) \mid X_0 = x]$..

Organization of the paper. We describe our model formally in Section 2 and state the main results of the paper in Section 3. In Section 4 we prove our main results for cycles of length two only. Section 5 describes the proof ideas for the case of two and three-cycles (technically the most challenging). We refer to the full version of the paper [2] for complete proofs and for further discussion of related work.

2 Model

Consider the following model of a barter exchange where each agent arrives with an item that she wants

[4] Unbounded chains of exchanges are initiated by 'altruistic donors' who donate a kidney without expecting anything in return. Chains have been increasing more important in practice. We analyze the case of chains as well in the full version of this paper [2], but choose to focus here on the technically more challenging case (especially for three-way cycles) of cycle removal. For chain removal we again find that greedy is near optimal.

[5] These works have a market size parameter n, corresponding to the number of agents in a static or finite horizon setting, and require that $p = \Theta(1/n)$. Such a requirement is problematic, since p is typically exogenous.

to exchange for another item. In our simple binary model, each agent is (equally) interested in the items possessed by some of the other agents, and not interested in the items possessed by the rest.

Compatibility graph representation. The state of the system at any time can be represented by a directed graph where each agent is represented by a node, and a directed edge (i, j) exists if agent j wants the item of agent i. Let $\mathcal{G}(t) = (\mathcal{V}(t), \mathcal{E}(t))$ denote the directed graph of compatibilities observed before time t.

Dynamics. Initially the system may start in any state with a finite number of waiting agents. We consider discrete times $t = 0, 1, 2, \ldots$ At each time, one new agent arrives[6]. The new node representing this agent v has an incoming edge from each waiting agent who wants the item of v, and an outgoing edge to each waiting agent whose item v wants.

Stochastic compatibility model. The item of the new agent v is of interest to each of the waiting agents independently with probability p, and independently, the agent v is interested in the item of each waiting agent independently with probability p. Mathematically, there is a directed edge (in each direction) with probability p between the arriving node v and each other node that currently exists in the system, independently for all nodes and directions.

Allocation and policies. An *allocation* in a compatibility graph is a set of disjoint exchanges, namely a set of disjoint cycles. We say that a node that is part of an allocation is *matched*. When an allocation consisting of cycles is executed, the compatibility graph is updated by eliminating the matched nodes and all their incident edges. Immediately after the arrival of a new node, the platform can choose to perform one or more exchanges, based on its chosen *policy*. Here, a policy is a mapping from the history of the system so far to an allocation. An exchange can happen via a cycle, where a *k-way cycle* is a directed cycle in the graph involving k nodes.

Two settings (or technologies) are considered, differing by the exchanges permitted in an allocation: allocations can output only cycles of length at most k, for $k = 2, 3$.

One natural policy that will play a key role in our results is the *greedy* policy. The greedy policy attempts to match the maximum number of nodes upon each arrival.

Definition 2.1. *The greedy policy for each of the settings is defined as follows: At the beginning of each time period the compatibility graph does not contain cycles with length at most k. Upon arrival of a new node, if a cycle with length at most k can be formed with the newly arrived node, it is removed, with a uniformly random cycle being chosen if multiple cycles are formed. Clearly, at the beginning of the next time period the compatibility graph again does not contain any cycles with length at most k. The procedure is described in figure Figure 1.*

The performance measure we focus on in this paper is the average (steady state) waiting time, which we define to be the average steady state time interval between the arrival of a node and the time when this node is removed. This is a natural metric in our setting, since for any reasonable policy, each node eventually gets matched and is removed, so policies differ only in how long it takes for nodes to match.

The system described above operated under the greedy policy, cf. Definition 2.1 is a Markov chain with a countably infinite number of states, each state corresponding to a compatibility graph. Further, this Markov chain is irreducible since an empty graph is reachable from any other state. This raises the question of whether this Markov chain is positive recurrent. If the answer is positive one can further study various performance measures.[7]

We also consider policies other than the greedy policy, in general the class of policies under which the system is stationary/periodic and ergodic in the $t \to \infty$ limit. This includes[8] the following class of policies that generalize Markov policies:

Definition 2.2. *We call a policy a* periodic Markov *policy if it employs τ homogenous first order Markov policies in round robin for some $\tau \in \mathbb{N}$.*

In other words, a periodic Markov policy implements a heterogeneous first order Markov chain, where the transition matrices repeat cyclically every τ rounds. Now suppose the resulting Markov chain is irreducible and periodic with period τ'. Without loss

[6] One can instead consider a stochastic model of arrivals, e.g., Poisson arrivals in continuous time. In our setting, such stochasticity would leave the behavior of the model essentially unchanged, and indeed, each of our main results extend easily to the case of Poisson arrivals at rate 1.

[7] The Markov chain turns out to be aperiodic for $k = 3$, but for $k = 2$ it is periodic with period 2. In any case, average (steady state) waiting time as defined below is a natural metric for any periodicity.

[8] More precisely, positive recurrent periodic Markov policies (that stabilize the system) lead to a periodic and ergodic system. In any case we are not interested in policies that do not stabilize the system.

of generality, assume that τ is a multiple of τ' (if not, redefine τ as per $\tau \leftarrow \tau\tau'$). Now, clearly the subsequence of states starting with the state at time ℓ and then including states at time intervals of τ, i.e., times $t = \ell, \ell+\tau, \ell+2\tau, \ldots$ forms an irreducible aperiodic first order Markov chain. If this ℓ-th 'outer' Markov chain is positive recurrent, we conclude that it converges to its unique steady state, leading to a periodic steady state for the original system with period τ. Define

$$W_\ell \equiv \text{Expected number of nodes in the system in the steady state of the } \ell\text{-th outer Markov chain.}$$

Thus, W_ℓ is the expected number of nodes in the system at times that are ℓ mod τ in steady state. Then we define the average waiting time for a periodic Markov policy as $W = (1/\tau)\sum_{\ell=0}^{\tau-1} W_\ell$. Note that this is the average number of nodes in the original system over a long horizon in steady state. Recalling Little's law, this is hence identical to the average waiting time for agents who arrive to the system in steady state.

Remark 2.1. *We state our results formally for this broad class of periodic Markov policies, though our bounds extend also to other general policies that lead to a stationary/periodic and ergodic system in the $t \to \infty$ limit.*

3 Main results

We consider two different settings: a) two-way cycles only, b) two-way cycles and three-way cycles.

Two-way cycles only. Our first result considers only 2-way cycles:

Theorem 3.1. *Under the setting $k = 2$, i.e., removal of two-way cycles only, the greedy policy (cf. Definition 2.1) achieves an average waiting time of $\ln 2/p^2 + o(1/p^2)$. This is optimal, in the sense that for every periodic Markov policy, cf. Definition 2.2, the average waiting time is at least $\ln 2/(-\ln(1-p^2)) = \ln 2/p^2 + o(1/p^2)$.*

The key fact leading to this theorem is that the prior probability of having a two-cycle between a given pair of nodes is p^2, so an agent needs $\Theta(1/p^2)$ options in order to find another agent with whom a mutual swap is desirable. This result is technically by far the simpler one, but of equal interest in its implications. We prove Theorem 3.1 in Section 4.

Two-way cycles and three-way cycles. Our second result considers the case of cycle removals with $k = 3$. Our lower bound in this case applies to a specific class of policies which we now define.

Let $\mathcal{G}$ denote the *global compatibility graph* that includes all nodes that ever arrive to the system, and directed edges representing compatibilities between them.

Definition 3.1. *A deterministic policy is said to be* monotone *if it satisfies the following property: Consider any pair of nodes (i, j) and an arbitrary global compatibility graph $\mathcal{G}$ such that the edge (i, j) is present. Let $\bar{\mathcal{G}}$ be the graph obtained from $\mathcal{G}$ when edge (i, j) is removed. Let T_i and T_j be the times of removal of nodes i and j respectively when the compatibility graph is $\mathcal{G}$ and let $T_{ij} = \min(T_i, T_j)$. Then the policy must act in an identical fashion on $\bar{\mathcal{G}}$ and $\mathcal{G}$ for all $t < T_{ij}$, i.e., the same cycles are removed at the same times in each case, up to time T_{ij}. This property must hold for every pair of nodes (i, j) and every possible $\mathcal{G}$ containing the edge (i, j).*

A randomized policy is said to be monotone if it randomizes between deterministic monotone policies.

Remark 3.1. *Consider the greedy policy for cycle removal defined above. It is easy to see that we can suitably couple the execution of greedy on different global compatibility graphs such that the resulting policy is monotone. The same applies to a batching policy which matches periodically (after arrival of x nodes), by finding a maximum packing of node disjoint cycles and removing them*[9].

Note that the class of monotone policies includes a variety of policies in addition to simple batching policies. For instance, a policy that assigns weights to nodes and finds an allocation with maximum weight (instead of simply maximizing the number of nodes matched) is also monotone.

Theorem 3.2. *Under the setting $k = 3$, i.e., removal of two and three-way cycles, the average waiting time under the greedy policy (cf. Definition 2.1) is $O(1/p^{3/2})$. Furthermore, there exists a constant $C < \infty$ such that, for any monotone policy that is periodic Markov (see Definitions 3.1 and 2.2), the average waiting time is at least $1/(Cp^{3/2})$.*

Theorem 3.2 says that we can achieve a much smaller waiting time with $k = 3$, i.e., two and three-cycle removal, than the removal of two-cycles only (for small p). Further, for $k = 3$ greedy is again near

[9] Note that such a policy is periodic Markov with a period equal to the batch size.

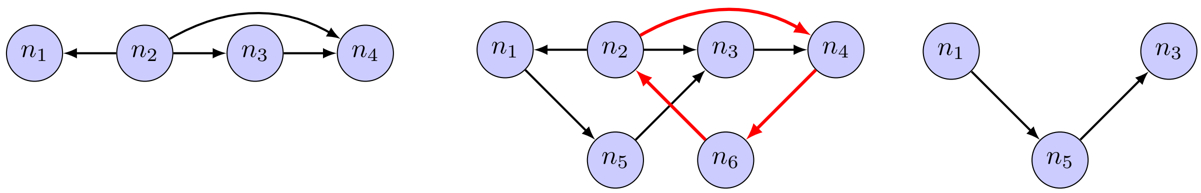

Figure 1: An illustration of cycle matching under the greedy policy, with a maximum cycle length of 3. Initially, nodes n_1, n_2, n_3, and n_4 are all waiting, as shown on the left. Node n_5 arrives, but no directed cycles can be formed. Then n_6 arrives, forming the three cycle $n_6 \to n_2 \to n_4 \to n_6$. On the right, the three cycle is removed, along with the edges incident to any node in the three cycle. Note that when n_6 arrives, a six cycle is also formed, but under our assumptions, the maximum length cycle that can be removed is a three cycle.

optimal in the sense that no monotone policy can beat greedy by more than a constant factor. Theorem 3.2 is proved in Section 5. The proof overcomes a multitude of technical challenges arising from the complex distribution of the compatibility graph at a given time, and introduces several new ideas.

We remark that we could not think of any good candidate policy in our homogeneous model of compatibility that violates monotonicity but should do well on average waiting time. As such, we conjecture (but were unable to prove) that our lower bound on average waiting time applies to arbitrary and not just monotone policies.

The following fact may provide some intuition for the $\Theta(1/p^{3/2})$ scaling of average waiting time[10]: In a static directed Erdős-Rényi graph with (small) edge probability p, one needs the number of nodes n to grow as $\Omega(1/p^{3/2})$ in order to, with high probability, cover a fixed fraction (e.g., 50%) of the nodes with node disjoint two and three cycles[11]. Our rigorous analysis leading to Theorem 3.2 shows that this coarse calculation in fact leads to the correct scaling for average number of nodes in the dynamic system under the greedy policy, and that no monotone policy can do better.

Our result leaves open the case of larger cycles, i.e. $k > 3$, under the greedy, arbitrary monotone and arbitrary general policies. Based on intuition similar to the above, we conjecture that for the setting with removal of cycles of length up to k, the greedy policy achieves the average waiting time of $\Theta(p^{-\frac{k}{k-1}})$, and furthermore for every policy the average waiting time is lower bounded by $\Omega(p^{-\frac{k}{k-1}})$.

[10] Recall that the average number of nodes is the same as the average waiting time, using Little's law.

[11] The expected total number of three cycles is n^3p^3 and the expected number of node disjoint three cycles is of the same order for $n^3p^3 \lesssim n$. We need $n^3p^3 \sim n$ in order to cover a given fraction of nodes with node disjoint three cycles, leading to $n \gtrsim 1/p^{3/2}$. For $n \sim 1/p^{3/2}$, the number of two-cycles is $n^2p^2 \sim 1/p = o(n)$, i.e., very few nodes are part of two-cycles.

4 Two-way cycle removal

In this section we consider removal of two-cycles only, i.e., $k = 2$. The greedy policy corresponding $k = 2$ is simple to characterize, since, as we show below, the underlying process behaves as a simple random walk. We observe that the random walk has a negative drift when $|\mathcal{V}(t)| \geq \log(2)/p^2$, and obtain a tight characterization of waiting time under greedy using a simple coupling argument. The key idea for the lower bound is that regardless of the implemented policy, the rate at which 2-cycles which will be eventually removed are formed must equal to the half of the rate at which new nodes arrive, which is equal to unity. Further, the probability that we do not form any cycles which will be eventually removed is lower bounded by the probability that we do not form any cycles at all. This probability depends only on the number of nodes in the system, the desired quantity.

Proof. [Theorem 3.1] We first compute the expected steady state waiting time under the greedy policy. Observe that for all $t \geq 0$,

$$|\mathcal{V}(t+1)| = \begin{cases} |\mathcal{V}(t)| + 1 & \text{w.p. } (1-p^2)^{|\mathcal{V}(t)|}, \\ |\mathcal{V}(t)| - 1 & \text{w.p. } 1 - (1-p^2)^{|\mathcal{V}(t)|}. \end{cases}$$

Let $\varepsilon > 0$ be arbitrary. If $|\mathcal{V}(t)| > (1+\varepsilon)\ln(2)/p^2$, then there exists a sufficiently small $p = p(\varepsilon)$ such that for all $p > p(\varepsilon)$

$$\mathbb{P}(|\mathcal{V}(t+1)| = |\mathcal{V}(t)| + 1) = (1-p^2)^{|\mathcal{V}(t)|} \leq \frac{1}{2^{1+\varepsilon}}.$$

Let $q = 1/2^{1+\varepsilon} < 1/2$, and let X_t be a sequence of i.i.d. random variables with distribution

$$X_t = \begin{cases} 1 & \text{w.p. } q\,, \\ -1 & \text{w.p. } 1-q\,. \end{cases}$$

Let $S_0 = 0$ and for $t \geq 1$, $S_{t+1} = (S_t + X_t)^+$, so S_t is a Birth-Death process. Letting $r = q/(1-q) < 1$, in steady state $\mathbb{P}(S_\infty = i) = r^i(1-r)$ for $i = 0, 1, \ldots$, so

$$\mathbb{E}[S_\infty] = r/(1-r) = q/(1-2q) = \frac{1}{2^{1+\varepsilon} - 2}.$$

We can couple the random walk $|\mathcal{V}(t)|$ with S_t such that $|\mathcal{V}(t)| < (1+\varepsilon)\ln(2)/p^2 + S_t$ for all t. This yields

$$\begin{aligned} \mathbb{E}[|\mathcal{V}(\infty)|] &\leq (1+\varepsilon)\frac{\ln(2)}{p^2} + \mathbb{E}[S_\infty] \\ &\leq (1+\varepsilon)\frac{\ln(2)}{p^2} + \frac{1}{2^{1+\varepsilon} - 2}\,. \end{aligned}$$

Thus for every $\varepsilon > 0$, we have

$$\lim_{p \to 0} \frac{\mathbb{E}[|\mathcal{V}(\infty)|] - \ln(2)/p^2}{1/p^2} \leq \varepsilon \ln(2).$$

As ε was arbitrary, the result follows.

Now we establish the lower bound on $|\mathcal{V}(\infty)|$. Let v be a newly arriving node at time t, and $\mathcal{W}$ be the nodes currently in system that are waiting to be matched. Let I be the indicator that at the arrival time of v (just before cycles are potentially deleted), no 2-cycles between v and any node in $\mathcal{W}$ exist. Let $\tilde{I}$ be the indicator that at the arrival time of v, no two cycles *that will be eventually removed* that are between v and any node in $\mathcal{W}$ exist (in particular, $\tilde{I}$ depends on the future). Thus $\tilde{I} \geq I$ a.s. Let $\tilde{V}_t$ be the number of vertices in the system before time t such that the cycle which eventually removes them has not yet arrived. We let $\tilde{V}_\infty$ be the distribution of $\tilde{V}_t$ when the system begins in steady state. By stationarity

$$0 = \mathbb{E}[\tilde{V}_{t+1} - \tilde{V}_t] = \mathbb{E}_{\tilde{V}_\infty}[2\tilde{I} - 1],$$

giving $E[\tilde{I}] = 1/2$. Intuitively, in steady state, the expected change in the number of vertices not yet "matched" must be zero. Thus we obtain

$$\begin{aligned} \frac{1}{2} = \mathbb{E}[\tilde{I}] &\geq \mathbb{E}[I] = \mathbb{E}[\mathbb{E}[I \mid |\mathcal{V}(\infty)|]] \\ &= \mathbb{E}\left[(1-p^2)^{|\mathcal{V}(\infty)|}\right] \geq (1-p^2)^{\mathbb{E}[|\mathcal{V}(\infty)|]}, \end{aligned}$$

by Jensen's inequality. Taking logarithms on both sides and rearranging terms, we get

$$\mathbb{E}[|\mathcal{V}(\infty)|] \geq \frac{\log(1/2)}{\log(1-p^2)} = \frac{\log(2)}{-\log(1-p^2)}.$$

5 Three-way cycle removal: main proof ideas

The proof of Theorem 3.2 (see [2]) is far more involved than for the case $k = 2$, especially the upper bound, and relies on delicate combinatorial analysis of 3-cycles in a random graph formed by nodes present in the system in steady state and those arriving over a certain time interval. We consider a time interval of the order $\Theta(1/p^{3/2})$ and assume that the system starts with at least order $\Theta(1/p^{3/2})$ nodes in the underlying graph. We establish a negative drift in the system and then rely on a Lyapunov function technique in order to establish the required upper bound.

We bound the waiting time under greedy using a natural Lyapunov function: the number of nodes/agents in the system. A challenge here is that we need to establish a negative drift for *any* starting residual graph containing $n_t \geq n_0$ nodes, for some $n_0 = O(1/p^{3/2})$. In particular, we need to show that the expected number of nodes decreases even if the residual graph contain *no* edges. Clearly, considering a single arrival will not allow formation of any three cycles in an empty graph, and the likelihood of formation of a two cycle is very small for $n_t = o(1/p^2)$. In fact, one needs to look at $T = \Theta(1/n^{3/2})$ consecutive arrivals in order to ensure negative drift in expectation during that period of length T, irrespective of the residual graph at the start. The analysis is extremely involved since the greedy policy continues to be executed during this interval of length T, so the graph at each time is difficult to control. We show that even if the graph at the start is empty, for appropriate T and n_0, the next T arrivals mostly form three-cycles containing two new arrivals and one previously present node (few arrivals form cycles between themselves, and few are still present at $t + T$), leading to an expected reduction in the number of nodes in the system. For this, we need to define and control half a dozen different appropriate 'failure' events, show that each failure event is unlikely to occur, and show a reduction in the number of nodes in the system if there is no failure. Note that two-cycles play little role in this setting since formation of a two-cycle is very rare when $\mathbb{E}[n_t] = W = \Theta(1/p^{3/2})$.

For the lower bound, we introduce a novel approach that allows us to prove a matching lower bound (up to constants) for monotone policies by contradiction. The rough idea is as follows: if the steady state expected waiting time is small (in this case smaller than $1/(Cp^{3/2})$ for appropriate C), then a typical new arrival sees a small number of nodes currently in the system, and so typically does not form a two or three-cycle with existing nodes or even the next few arrivals. Thus, the typical arrival typically has a long waiting time, which contradicts our initial assumption of a small expected waiting time.

We remark that the following conjecture results if we assume that n_t (the number of agents in the system at time t) concentrates, and that the typical number of edges in a compatibility graph at time t with n_t nodes is close to what it would have been under an Erdos-Renyi graph with n_t nodes and edge probability p.

Conjecture 5.1. *For the removal of two and three-way cycles ($k = 3$), the expected waiting time in steady state under a greedy policy scales as $\sqrt{\ln(3/2)}/p^{3/2} + o(1/p^{3/2})$, and no periodic Markov policy (including non-monotone policies) can achieve an expected waiting time that scales better than this.*

Here the constant $\sqrt{\ln(3/2)}$ results from requiring (under our assumptions) that a newly arrived node forms a triangle with probability $1/3$.

Simulation results were consistent with this conjecture: the predicted expected waiting time for greedy from the leading term $\sqrt{\ln(3/2)}/p^{3/2}$ is $W = 80$ for $p = 0.04$, $W = 43$ for $p = 0.06$, $W = 28$ for $p = 0.08$ and $W = 20.1$ for $p = 0.1$. If proved, this conjecture would be refinement of Theorem 3.2. A proof would require a significantly more refined analysis for both the upper bound and the lower bound.

6 Conclusion

Overcoming the rare coincidence of wants is a major obstacle in facilitating timely exchanges for agents in a barter marketplace. In this paper we studied how the policy adopted by the clearinghouse affect agents' waiting times in a thin marketplace. We investigated this question for a variety of settings determined by the feasible types of exchanges, which are largely driven by the technology adopted by the marketplace. We also studied how the feasible types of exchanges affect the waiting times.

We studied these questions in a dynamic model with a stylized homogenous stochastic demand structure. The market is represented by a compatibility graph: agents are represented by nodes, and each directed edge, which represents that the source agent has an item that is acceptable to the target agent, exists a priori with probability p. Exchanges take place in the form of cycles. The key technical challenge we face is that in our dynamic setting, the compatibility graph between agents present at a particular time has a complicated distribution that depends on the feasible exchanges and the policy employed by the clearinghouse.

We analyzed the long run average time agents spend waiting to exchange their item, in a two settings with respect to feasible exchanges: 2-way cycles only, and 2 and 3-way cycles. Our main finding is that regardless of the setting, the greedy policy which attempts to match upon each arrival, is approximately optimal (minimizes average waiting time) among a large class of policies that includes batching policies. We also find that three-way cycles lead to a large improvement in waiting times relative to two-cycles only. Although we do not model important details of kidney exchange clearinghouses, our findings are consistent with computational experiments and practice in that context.

Our work raises several further questions and we describe here a few of these. Allowing for heterogeneous agents or goods may lead to different qualitative results in some settings. For example, if Bob is a very difficult-to-please agent who is willing to accept only Alice's item but they are not both currently part of any feasible exchange, it may be beneficial to make Alice wait for some time in the hope of finding an exchange that can allow Bob to get Alice's item. Thus, when cycles of more than two agents are permitted, some waiting may improve efficiency in the presence of heterogeneity (some evidence for this has been found [4]).

Finally, unbounded chains of exchanges initiated by 'altruistic donors' have been increasing more important in practice. We analyze the case of chains as well in the full version of this paper [2], and again find that greedy is near optimal in that setting.

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