Saturation of the Tsirelson bound for the Clauser-Horne-Shimony-Holt inequality with random and free observables

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Maximal violation of the CHSH-Bell inequality is usually said to be a feature of anticommuting observables. In this work we show that even random observables exhibit near-maximal violations of the CHSH-Bell inequality. To do this, we use the tools of free probability theory to analyze the commutators of large random matrices. Along the way, we introduce the notion of "free observables," which can be thought of as infinite-dimensional operators that reproduce the statistics of random matrices as their dimension tends towards infinity. We also study the fine-grained uncertainty of a sequence of free or random observables and use this to construct a steering inequality with a large violation.

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I. INTRODUCTION

The notion of quantum mechanics violating local realism was first raised by the work of Einstein, Podolsky and Rosen [1]. This was put on a rigorous and general footing by the revolutionary 1964 paper of Bell [2], which derived an inequality (now known as the Bell inequality) involving correlations of two observables. Bell showed that there is a constraint on any possible correlations obtained from local hidden variable models which can be violated by quantum measurements of entangled states. Later, another Bell-type inequality which is more experimentally feasible was derived by Clauser *et al.* [3]. Since then, Bell inequalities have played a fundamental role in quantum theory and have had applications in quantum information science including cryptography, distributed computing, randomness generation, and many others (see [4] for a review).

In this paper, we mainly focus on the maximal violation of the CHSH-Bell inequality [3]. It is well known that the *Tsirelson bound* $2\sqrt{2}$ for the CHSH-Bell inequality was first obtained by Tsirelson [5]. And he also proved that the bound can be realized by using proper Pauli observables. Apart from the above qubit case, it is possible to find dichotomic observables in high dimensions [6,7], as well as in the continuous-variable (infinite-dimension) case [8], to obtain the Tsirelson bound. Recently, Liang *et al.* [9] have studied the possibility of violation of CHSH-Bell inequality by random observables. For the bipartite qubit case, if two observers share a Bell state, random measure settings lead to a violation with probability ≈ 0.283. However, for two qubits, the probability of the maximal violation is 0, and the probability of near-maximal violation is negligible.

Contrary to the case of qubits, our results show that the probability of near-maximal violation is large in high dimensions. Here near-maximal violations are approximately achieved with a high probability by random high-dimensional observables. Previous methods of showing maximal violation were based on specific algebraic relations, namely, anti-commuting, and indeed there is a sense in which maximal violations imply anticommutation in some subspace [10]. However, this random approach reveals that there is another type of algebraic relation between observables which might lead to the Tsirelson bound of the CHSH-Bell inequality. We call the observables which satisfy those relations *free observables.* This terminology is from a mathematical theory called free probability [11,12]. As we explain below, these free observables are freely independent in some quantum probability space, which is a quantum analog of the classical probability space (see Sec. IV for the definition). A crucial point is that free observables can only exist in infinite dimension and, thus, are experimentally infeasible. We also discuss finite-dimensional approximations (Sec. IV B), which are more experimentally plausible and for which the Tsirelson bound can be approximately obtained.

In another part of this work we study the fine-grained uncertainty relation of free or random observables, which was introduced by Oppenheim and Wehner [13]. It is more fundamental than the usual entropic uncertainty relations and it relates to the degree of violation of Bell inequalities (nonlocal games) [13,14]. For a pair of free (random) observables, we can show that the degree of their uncertainty is 0. On the other hand, it is interesting that for a sequence of free (random) observables $A_1,\dots,A_n$ with $n>4$, the fine-grained uncertainty is upper bounded by $\frac{1}{2}+\frac{1}{\sqrt{n}}$, which is the same as the one given by the anticommuting observables. Therefore as a by-product of the above results, by using free (random) observables we can obtain one type of steering inequality with a large violation that recently was studied in [15].

II. PRELIMINARIES

First, we introduce terminology. For a bipartite dichotomic Bell scenario, there are two spacelike separated observers, say, Alice and Bob. Each of them is described by an N-dimensional Hilbert space H_N, and Alice (Bob) chooses one of n dichotomic (i.e., two-outcome) observables A_i (B_j) that will take results α_i (β_j) from set $\{1,-1\}$. Thus the observables are self-adjoint unitaries.

Next, recall the famous CHSH-Bell inequality [3]. If α_1, α_2, β_1, and β_2 are classically correlated random variables, then

$$|\langle\alpha_1\beta_1\rangle + \langle\alpha_1\beta_2\rangle + \langle\alpha_2\beta_1\rangle - \langle\alpha_2\beta_2\rangle| \leqslant 2, \tag{1}$$

so we say that 2 is the largest *classical value* obtained by any local hidden variable model. In [5], Tsirelson first proved that if the correlations are obtained by quantum theory, then the *quantum value* of the CHSH-Bell inequality is $2\sqrt{2}$ (i.e., the Tsirelson bound). To see this, consider the CHSH-Bell operator

$$B_{\text{CHSH}} = A_1 \otimes B_1 + A_1 \otimes B_2 + A_2 \otimes B_1 - A_2 \otimes B_2, \tag{2}$$

where $A_i, B_j, i, j = 1,2$ are dichotomic observables. By choosing proper observables, e.g., $A_1 = \sigma_x, A_2 = \sigma_z$, $B_1 = (\sigma_x + \sigma_z)/\sqrt{2}$, $B_2 = (\sigma_x - \sigma_z)/\sqrt{2}$, the norm (largest singular value) of the CHSH-Bell operator is $2\sqrt{2}$. If $\mathcal{B} = B_{\text{CHSH}}^2$, then

$$\mathcal{B} = 4\mathbb{1} - [A_1, A_2] \otimes [B_1, B_2]. \tag{3}$$

If both parties choose compatible (commutative) observables, then $\mathcal{B} = 4\mathbb{1}$. Hence incompatible (noncommutative) observables are necessary for the violation of CHSH-Bell inequality [7]. The Tsirelson bound is also determined by the eigenvalues of the commutators $[A_1, A_2]$ and $[B_1, B_2]$. More precisely, suppose the local dimension for each party is N, and the eigenvalues of $[A_1, A_2]$ ($[B_1, B_2]$) are $s_1, \ldots, s_N$ ($t_1, \ldots, t_N$). Then we have [7]

$$\|\mathcal{B}\| = \max_{i,j}\{4 - s_i t_j\}. \tag{4}$$

It is clear that if there exist eigenstates such that the eigenvalues of $[A_1, A_2]$ ($[B_1, B_2]$) are ± 2, then $\|B_{\text{CHSH}}\| = 2\sqrt{2}$. In particular, anticommuting dichotomic local observables, such as σ_x and σ_z, will saturate the Tsirelson bound.

III. A RANDOM APPROACH TO THE TSIRELSON BOUND

Suppose D is an $N \times N$ deterministic diagonal matrix, where the diagonal terms of D are either 1 or -1 and $\text{Tr}(D) = 0$ where Tr is the usual trace for matrices. It is easy to see that $D^2 = \mathbb{1}$. Suppose unitaries $U_i, i = 1, \ldots, n$ are independent Haar-random matrices in the group of unitary matrices $U(N)$. Define the following random dichotomic observables:

$$A_i = U_i D U_i^\dagger, \quad i = 1, \ldots, n. \tag{5}$$

We would like to establish results that hold with a "high probability" over some natural distribution. Recall that we call a sequence of random variables $\{X_N\}_N$ convergent to X almost surely in probability space (Ω, P), if $P(\lim_{N\to\infty} X_N = X) = 1$. With these notions, we claim that the Tsirelson bound of the CHSH-Bell inequality can be obtained with a high probability by using random dichotomic observables in sufficiently large dimensions. More precisely, we have the following theorem:

Theorem 1. Let $A_i = U_i D U_i^\dagger$ and $B_i = V_i D V_i^\dagger$, $i = 1,2$, where U_i and V_i are independent Haar-random unitaries in $U(N)$. Then we have

$$\lim_{N\to\infty} \|B_{\text{CHSH}}\| = 2\sqrt{2}, \quad \text{almost surely}. \tag{6}$$

The above theorem could be understand as the following: with a sufficiently large dimension, the random dichotomic observables may saturate the Tsirelson bound of the CHSH-Bell inequality. We note here that in this approximate scenario, the shared state for Alice and Bob should not be fixed; otherwise it may not obtain any violation at all. To prove this theorem, we first need the following lemma from [12]:

Lemma 1 [12]. Let $\mathcal{M}_N$ be the set of $N \times N$ matrices. Then for every $A \in \mathcal{M}_N$,

$$\|A\| = \lim_{k\to\infty} \{\text{tr}_N[(A^\dagger A)^k]\}^{\frac{1}{2k}}, \tag{7}$$

where $\text{tr}_N = \text{Tr}/N$.

Now denote $A = [A_1, A_2]$ and $B = [B_1, B_2]$. For any $k \in \mathbb{N}_0$, we can use the binomial formula and Eq. (3) to obtain

$$\begin{aligned} \text{tr}_{N^2}(\mathcal{B}^k) &= \text{tr}_{N^2}(4\mathbb{1} - A \otimes B)^k \\ &= \sum_{j=0}^{k} \binom{k}{j} 4^{k-j} (-1)^j \, \text{tr}_N(A^j) \cdot \text{tr}_N(B^j). \end{aligned} \tag{8}$$

Let us consider the term $\text{tr}(A^j)$. Since A_1A_2 and A_2A_1 commute, again by the binomial formula, we have

$$\text{tr}_N(A^j) = \sum_{l=0}^{j} \binom{j}{l} (-1)^{j-l} \, \text{tr}_N((A_1A_2)^{|2l-j|}). \tag{9}$$

Now we need the second key lemma (see Appendix B for details of the proof).

Lemma 2. Let $A_i = U_i D U_i^\dagger$, where $U_i, i = 1, \ldots, n \in U(N)$ are independent Haar-random unitaries. Consider a sequence $i(1), \ldots, i(k) \in [n]$ satisfying $i(1) \neq i(2) \neq i(3) \neq \ldots i(k-1) \neq i(k)$. Then

$$\lim_{N\to\infty} \text{tr}_N(A_{i(1)}A_{i(2)} \cdots A_{i(k)}) = 0, \quad \text{almost surely}. \tag{10}$$

This lemma is mostly due to the work of Collins [16,17], where he and other coauthors developed a method to calculate the moments of polynomial random variables in unitary groups. This method is called the Weingarten calculus and is in turn based on [18]. As we see in the next section, this lemma can be thought of as establishing the "asymptotic freeness" of these random matrices. Thus by Lemma 2, we have (almost surely)

$$\begin{aligned} \lim_{N\to\infty} \text{tr}_N(A^j) &= \sum_{l=0}^{j} \binom{j}{l} (-1)^{j-l} \lim_{N\to\infty} \text{tr}_N\left((A_1A_2)^{|2l-j|}\right) \\ &= \begin{cases} (-1)^{j/2}\binom{j}{j/2} & \text{if} \quad j \quad \text{is even}, \\ 0 & \text{otherwise}. \end{cases} \end{aligned} \tag{11}$$

A similar estimate is also valid for the term $\text{tr}_N(B^j)$. Therefore

$$\begin{aligned} \lim_{N\to\infty} \text{tr}_{N^2}(\mathcal{B}^k) \; &= \sum_{j=0,\ j\text{ is even}}^{k} \binom{k}{j} 4^{k-j} \binom{j}{j/2}^2 \\ &:= Q_k, \quad \text{almost surely}. \end{aligned} \tag{12}$$

By Stirling's formula, we have $\lim_{k\to\infty}(Q_{2k})^{1/2k} = 8$. In other words, for any $\epsilon > 0$, we can choose $k \in \mathbb{N}$, such that $(Q_{2k})^{1/2k} > 8 - \epsilon$. Since $(\text{tr}_{N^2}\,\mathcal{B}^k)^{1/k} \leqslant \|\mathcal{B}\|$ for all $k \geqslant 1$, then we have

$$\liminf_{N\to\infty} \|\mathcal{B}\| \geqslant (Q_{2k})^{1/2k} > 8-\epsilon, \quad \text{almost surely.} \tag{13}$$

On the other hand, due to Tsirelson's inequality [5] we have $\|\mathcal{B}\| \leqslant 8$. Thus we complete our proof of Theorem 1.

IV. A FREE APPROACH TO THE TSIRELSON BOUND

The random dichotomic observables do not satisfy the anticommuting relations. In fact, random dichotomic observables are "asymptotically" freely independent, which was first established by Voiculescu [19] in the case of the Gaussian unitary ensemble. That result builds a gorgeous bridge across two distinct mathematical branches: random matrix theory and free probability. In free probability theory, we treat observables A_i and B_j as elements of a C^*-algebra $\mathcal{A}$, equipped with an unital (faithful) state ϕ, where "state" means a linear map from $\mathcal{A}$ to $\mathbb{R}$, unital means $\phi(\mathbb{1})=1$, and faithful means $\phi(AA^*)=0 \Rightarrow A=0$. The pair $(\mathcal{A},\phi)$ is called a C^*-probability space, which is a quantum analog of a classical probability space and we can call it a "quantum" probability space. For example, $(\mathcal{M}_N,\mathrm{tr}_N)$ is a C^*-probability space, where $\mathcal{M}_N$ is the set of $N\times N$ matrices. We refer to [12] for more details of quantum probability.

Lemma 2 inspires us to consider the following adaptation of the definition of freeness to the case of dichotomic observables.

Definition 1. For a given C^*-probability space $(\mathcal{A},\phi)$, dichotomic observables $A_i, i\in I$ are called freely independent if

$$\phi(A_{i(1)}A_{i(2)}\cdots A_{i(k)})=0 \tag{14}$$

whenever we have the following:

(i) k is positive, $i(1),i(2),\dots,i(k)\in I$;

(ii) $\phi(A_{i(k)})=0$ for all k;

(iii) $i(1)\neq i(2), i(2)\neq i(3),\dots,i(k-1)\neq i(k)$.

For the special case $I=\{1,2\}$, the above conditions are equivalent to

$$\begin{aligned}\phi(A_1)&=\phi(A_2)=\phi(A_1A_2)=\phi(A_2A_1)\\ &=\phi(A_1A_2A_1)=\phi(A_2A_1A_2)=\cdots=0.\end{aligned} \tag{15}$$

However, finite-dimensional observables cannot be freely independent. In other words, for fixed N, the C^*-probability space (M_N,tr_N) is too small to talk about freeness, and Definition 1 refers to an empty set. Fortunately, if we consider the observables in infinite-dimensional Hilbert space, it is possible for them to be freely independent in some C^*-probability space $(\mathcal{A},\phi)$. Furthermore, the derivations in Sec. III do not depend on the dimension. In order to use an infinite-dimensional C^*-probability space $(\mathcal{A},\phi)$ instead of (M_N,tr), we need only update Lemma 1 with an appropriate formula, which is achieved by (A2) below. We conclude as follows.

Theorem 2. For the CHSH-Bell inequality, the Tsirelson bound can be obtained by using observables which are freely independent in their respective local system. More precisely, if A_1,A_2 and B_1,B_2 are freely independent in some C^*-probability space $(\mathcal{A},\phi)$, then we have $\|B_{\mathrm{CHSH}}\|=2\sqrt{2}$.

This result is rather abstract, but in the next subsection, we provide a concrete example which satisfies the conditions in this theorem.

A. A concrete example in infinite dimension

For infinite-dimensional C^*-probability space, Definition 1 is meaningful. Now consider a group $G=*_n\mathbb{Z}_2$ and its associated Hilbert space $\ell_2(G)$. This notation refers to the n-fold free product of $\mathbb{Z}_2$ with itself; i.e., the infinite group G with the the following elements: $g_{i_1}, g_{i_1}g_{i_2},\dots,g_{i_1}g_{i_2}\cdots g_{i_n}$, $i_1,\dots,i_n=1,\dots,n$, where $g_1,\dots,g_n$ are the generators of group G whose only relations are $g_i^2=1$. The set $\{|g\rangle : g\in G\}$ forms an orthonormal basis of $\ell_2(G)$, thus the dimension of $\ell_2(G)$ is infinite. Let $\lambda: G\to B(\ell^2(G))$ be the left regular group representation, which is defined as

$$\lambda(g)|h\rangle = |gh\rangle, \quad \forall h\in G. \tag{16}$$

The reduced C^* algebra $C^*_{\mathrm{red}}(G)$ is defined as the norm closure of the linear span $\{\lambda(g), g\in G\}$, where the norm is the operator norm of $B(\ell_2(G))$. There is a faithful trace state ϕ in $C^*_{\mathrm{red}}(G)$ defined as

$$\phi\left(\sum_g \alpha_g\lambda(g)\right) := \alpha_e. \tag{17}$$

Obviously, $\phi(\mathbb{1})=1$. Hence $(C^*_{\mathrm{red}}(G),\phi)$ is a C^*-probability space. If g_i is the generator of the ith copy of $*_n\mathbb{Z}_2, i=1,2,\dots,n$, then

$$A_i=\lambda(g_i), \quad i=1,\dots,n. \tag{18}$$

It is easy to check that $A_i, i=1,\dots,n$ are self-adjoint unitaries and freely independent in $(C^*_{\mathrm{red}}(G),\phi)$. We choose the local Hilbert spaces of Alice and Bob to be $\ell_2(G)$, where $n=2$. By using these free observables, we can obtain the quantum value $2\sqrt{2}$ for the CHSH-Bell inequality. Note that conjugating by a unitary preserves the freeness of observables, i.e., if A_1 and A_2 are freely independent, then $UA_1U^\dagger$ and $UA_2U^\dagger$ are still freely independent for any unitary U. Since the norm of the Bell operator does not change under the local unitary operation, we can simply assume $A_1=B_1=\lambda(g_1)$, $A_2=B_2=\lambda(g_2)$.

B. Truncated free observables in finite dimension

In order to see how the freeness behaves in a simple and direct way, we truncate the free observables given in the last subsection to finite dimension. Denote the elements in $\ell_2(*_2\mathbb{Z}_2)$ as follows:

$$\begin{array}{ccccccccc}\cdots & |g_2g_1g_2\rangle & |g_2g_1\rangle & |g_2\rangle & |e\rangle & |g_1\rangle & |g_1g_2\rangle & |g_1g_2g_1\rangle & \cdots\\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow\\ \cdots & |-3\rangle & |-2\rangle & |-1\rangle & |0\rangle & |1\rangle & |2\rangle & |3\rangle & \cdots\end{array}. \tag{19}$$

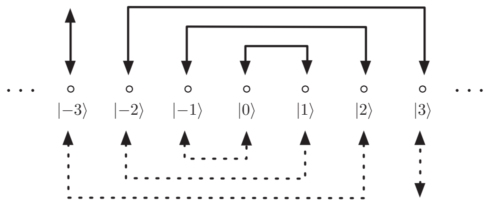

FIG. 1. Solid lines represent $A_1^{(N)}$ and dashed lines represent $A_2^{(N)}$, where $N = 7$. For example, the operator $A_1^{(N)}$ maps the vector $|0\rangle$ to $|1\rangle$, and vice versa. We only need to be careful regarding the vector $|-3\rangle$, where A_1^N maps it to itself.

With the above notation, we have

$$\begin{aligned} \lambda(g_1)|i\rangle &= |j\rangle, \quad i+j=1, \\ \lambda(g_2)|i\rangle &= |j\rangle, \quad i+j=-1, \end{aligned} \tag{20}$$

where $i,j = \dots,-1,0,1,\dots$.

Now define $A_1^{(N)}$ and $A_2^{(N)}$ to be the truncation of the free observables to dimension $N = 2l+1$ [i.e., we truncated the operators $\lambda(g_1)$ and $\lambda(g_2)$ into the operators acting on N-dimension Hilbert space.). Then we have (see Fig. 1)

$$A_1^{(N)}|i\rangle = |1-i\rangle, \quad i = -l+1,\dots,l, \tag{21a}$$

$$A_1^{(N)}|-l\rangle = |-l\rangle, \tag{21b}$$

and

$$A_2^{(N)}|i\rangle = |-1-i\rangle, \quad i = -l,\dots,l-1, \tag{22a}$$

$$A_2^{(N)}|l\rangle = |l\rangle, \tag{22b}$$

where $|i\rangle, i = -l,\dots,l$ denotes the basis of the N-dimensional Hilbert space.

It is clear that $A_1^{(N)}$ and $A_2^{(N)}$ are self-adjoint unitaries. Thus they can be treated as a pair of dichotomic observables in an N-dimensional Hilbert space. Denote $S = A_2^{(N)} \circ A_1^{(N)}$, so that

$$S|j\rangle = |j-2\rangle, \quad j = -l+2,\dots,l, \tag{23a}$$

$$S|-l+1\rangle = |l\rangle, \tag{23b}$$

$$S|-l\rangle = |l-1\rangle. \tag{23c}$$

In the following diagram it is easy to see that S is a cycle in the permutation group S_N:

$$\begin{array}{ccccccccccccc} |l\rangle & \longrightarrow & |l-2\rangle & \longrightarrow & |l-4\rangle & \longrightarrow & \cdots & \longrightarrow & |-l+2\rangle & \longrightarrow & |-l\rangle \\ \nwarrow & & & & & & & & & & \swarrow \\ & |-l+1\rangle & \longleftarrow & |-l+3\rangle & \longleftarrow & \cdots & \longleftarrow & |l-3\rangle & \longleftarrow & |l-1\rangle. & \end{array} \tag{24}$$

Now for the CHSH-Bell operator B_{CHSH}, by using those truncated free observables, we can show that the quantum value tends to $2\sqrt{2}$ as $N \to \infty$. Then due to the fact that the eigenvalues of S are $\lambda_j = \exp^{2\pi i j/N}$, $j = 0,\dots,N-1$, we have

$$\begin{aligned} \|B_{\text{CHSH}}^2\| &= \|4\mathbb{1} - [A_1^{(N)},A_2^{(N)}] \otimes [A_1^{(N)},A_2^{(N)}]\| \\ &= \|4\mathbb{1} - (S^\dagger - S) \otimes (S^\dagger - S)\| \\ &= \max_j \{4 + 4(\mathrm{Im}(\lambda_j))^2\} \\ &= \max_j \left\{4 + 4\sin^2\frac{2\pi j}{N}\right\} \\ &\approx 4 + 4\left(1 - O\left(\frac{1}{N^2}\right)\right) = 8 - O\left(\frac{1}{N^2}\right). \end{aligned} \tag{25}$$

Here, for simplicity, we have assumed that Alice and Bob take same measurements. Therefore, we have the following proposition:

Proposition 1. By using truncated free observables $A_1^{(N)}$, $A_2^{(N)}$, $N = 2l+1$, we can asymptotically obtain the Tsirelson bound for the CHSH-Bell inequality, i.e., $\|B_{\text{CHSH}}\| = 2\sqrt{2} - O(1/N^2)$.

This result suggests the speed of the convergence mentioned in Theorem 1, namely, the Tsirelson bound will be saturated with the speed of $O(1/N)$ by using the random observables. However, the rigorous proof would need very careful and subtle analysis by Weingarten calculus.

V. FINE-GRAINED UNCERTAINTY RELATIONS FOR RANDOM (FREE) OBSERVABLES

The uncertainty principle and nonlocality are two fundamental and intrinsic concepts of quantum theory which were quantitatively linked by Oppenheim and Wehner's work [13]. There they introduced a notion called "fine-grained uncertainty relations" to quantify the "amount of uncertainty" in a particular physical theory. Suppose we have n dichotomic observables A_i, $i = 1,\dots,n$, corresponding to measurement settings $P_i^a = \frac{\mathbb{1}+(-1)^a A_i}{2}$, $i = 1,\dots,n$, $a = 0,1$. The uncertainty of measurement settings P_i^0, $i = 1,\dots,n$, is defined as

$$\xi_{\vec{0}} = \sup_\rho \left\{\frac{1}{n}\sum_{i=1}^n \mathrm{Tr}\left(P_i^0\rho\right)\right\} = \frac{1}{2} + \frac{1}{2n}\sup_\rho \mathrm{Tr}\left(\sum_{i=1}^n A_i\rho\right). \tag{26}$$

Similarly, the uncertainty of P_i^1, $i = 1,\dots,n$, is

$$\xi_{\vec{1}} = \sup_\rho \left\{\frac{1}{n}\sum_{i=1}^n \mathrm{Tr}\left(P_i^1\rho\right)\right\} = \frac{1}{2} - \frac{1}{2n}\sup_\rho \mathrm{Tr}\left(\sum_{i=1}^n A_i\rho\right). \tag{27}$$

Note that

$$\sup_\rho \left|\mathrm{Tr}\left(\sum_{i=1}^n A_i\rho\right)\right| = \left\|\sum_{i=1}^n A_i\right\|. \tag{28}$$

Hence $\xi_{\vec{0}} = \xi_{\vec{1}} = \frac{1}{2} + \frac{1}{2n}\|\sum_{i=1}^n A_i\|$. The state ρ which can obtain $\xi_{\vec{x}}$ is called the maximally certain state for these measurement settings. If we assume that A_i are freely independent observables, then we have the following proposition (see Appendixes C and D for the proof):

Proposition 2. The fine-grained uncertainty for free observables A_i, $i = 1,\dots,n,n > 4$, is

$$\xi_{\vec{0}} = \xi_{\vec{1}} \leqslant \frac{1}{2} + \frac{1}{\sqrt{n}} < 1. \tag{29}$$

The same results approximately hold for random observables $A_i = U_i D U_i^\dagger$, $i = 1,\dots,n,n > 4$, with a high probability.

For the special case $n = 2$, we have $\|A_1 + A_2\| = 2$. Thus for $n = 2$, $\xi_{\vec{0}} = \xi_{\vec{1}} = 1$ (see Appendix D for random observables and Appendix E for free observables). Interestingly, for truncated free observables we have

$$\begin{aligned}\left\|A_1^{(N)} + A_2^{(N)}\right\|^2 &= 2\mathbb{1} + S + S^\dagger = \max_j\{2 + 2\mathrm{Re}(\lambda_j)\}\\ &= \max_j\left\{2 + 2\cos\frac{2\pi j}{N}\right\} = 4.\end{aligned} \tag{30}$$

Thus for the truncated free observables, we always have $\xi_{\vec{0}} = \xi_{\vec{1}} = 1$, regardless of what dimension we truncate to.

In a recent work, some of us show that there is a tight relationship between fine-grained uncertainty and violation of one specific steering inequality, called the linear steering inequality, which was first used in [20] to verify steering by experiment. It has the form

$$S_n = \sum_{i=1}^n \langle \alpha_i A_i \rangle \leqslant C_n, \tag{31}$$

where C_n is called the *local hidden state* bound of S_n. This bound can be calculated easily as follows [20]:

$$C_n = \sup_{\alpha_i=\pm 1}\left\|\sum_{i=1}^n \alpha_i A_i\right\|. \tag{32}$$

If the observables A_i are chosen to be operators of a Clifford algebra, which are anticommutative, a large (unbounded) violation can be obtained [15]. Because the degree of the fine-grained uncertainty of free or random observables is of the same order as that of anticommuting observables, we find:

Corollary 1. If $A_i, i = 1,\dots,n$ are chosen to be free observables, then the local hidden state bound of the steering inequality, $S_n = \sum_{i=1}^n \langle \alpha_i A_i \rangle \leqslant C_n$, is upper bounded by $2\sqrt{n}$. The similar result holds for random observables with a high probability.

Here we note that for the free case, we should also care about the quantum values of steering inequalities. Due to the work of Navascués and Pérez-García's, there are two ways to define them [21]. One is a commuting way that means the system is described by a total Hilbert space, and the other is the total system described in a tensor form. As a matter of fact, they also used the free observables $\lambda(g_i)$, $i = 1,\dots,n$, to define the linear steering inequality. They showed that the quantum value defined in the commuting sense is n, while in the tensor scenario it is upper bounded by $2\sqrt{n-1}$. So from their work, we can easily see that the local hidden state bound is upper bounded by $2\sqrt{n-1}$ for free observables. Their bound is even sharper than ours. However, we have provided another proof which is more focused on the freeness property and is applicable to random observables.

VI. CONCLUSIONS

In this paper, we show that random dichotomic observables generically achieve near-maximal violation of the CHSH-Bell inequality, approaching the Tsirelson bound in the limit of large dimension. This is despite the fact that these observables are not anticommuting. Instead, due to Voiculescu's theory, they are asymptotically freely independent. This means that when the dimension increases, their behaviors tend to those of free observables in some quantum probability space. However, the quantum state that is optimal for the random observables is random as well, as, in general, it will depend on the observables. For a fixed state, random observables might not lead to any violation. Another main result of this paper is that we have considered the fine-grained uncertainty of a sequence of free or random observables. The degree of their uncertainty is of the same order as the one which is given by the anticommuting observables. As a by-product of this result, we can construct a linear steering inequality with a large violation by using free or random observables. For further applications, free observables may be used for studying the quantum value of other types of Bell inequalities. Thus a natural question arises: *Do free observables always maximally violate any Bell inequalities?* Unfortunately, a quick answer is that we can consider the linear Bell operator $\sum_{i=1}^n A_i \otimes A_i$. It is trivial since its quantum and classical values are both n, while the quantum value given by free observables is upper bounded by $2\sqrt{n}$. However, it seems promising when considering other specific Bell inequalities. Since the free observables and their truncated ones are deterministic (constructive), another possible application is that this may be a new source of constructive examples of Bell inequality violations where previously only random ones were known.

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APPENDIX A: C^*-PROBABILITY SPACE AND FREELY INDEPENDENT

Definition 2. A $*$-probability space $(\mathcal{A},\phi)$ consists of a unital $*$-algebra $\mathcal{A}$ over $\mathbb{C}$ and a unital linear positive functional,

$$\phi : \mathcal{A} \to \mathbb{C}; \quad \phi(1_{\mathcal{A}}) = 1. \tag{A1}$$

The elements $a \in \mathcal{A}$ are called noncommutative random variables in $(\mathcal{A},\phi)$. A C^*-probability space is a $*$-probability space $(\mathcal{A},\phi)$ where $\mathcal{A}$ is an unital C^* algebra.

If, additionally, we assume THAT ϕ is faithful, we have, for any $a \in \mathcal{A}$,

$$\|a\| = \lim_{k\to\infty} [\phi((a^*a)^k)]^{\frac{1}{2k}}. \tag{A2}$$

Definition 3 [12,22]. For a given C^*-probability space $(\mathcal{A},\varphi)$, let $\mathcal{A}_1,\dots,\mathcal{A}_n$ be $*$ subalgebras of $\mathcal{A}$. They are said to be free if, for all $a_i \in \mathcal{A}_{j(i)}$, $i = 1,\dots n$, $j(i) \in \{1,\dots,n\}$, such that $\phi(a_i) = 0$, one has

$$\phi(a_1a_2\dots a_n) = 0 \tag{A3}$$

whenever $j(1) \neq j(2)$, $j(2) \neq j(3),\dots, j(n-1) \neq j(n)$. A sequence of random variables is said to be free if the unital subalgebras they generate are free.

APPENDIX B: PROOFS OF LEMMA 2

Lemma 2 is a direct corollary of the work of Collins [16]. A random variable $u \in (\mathcal{A},\phi)$ is called a Haar unitary when it is unitary and

$$\phi(u^j) = \begin{cases} 1 & \text{if} \quad j = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{B1}$$

Since we have

$$\lim_{N\to\infty} \mathrm{tr}(D^j) = \begin{cases} 1 & \text{if} \quad j \quad \text{is even,} \\ 0 & \text{if} \quad j \quad \text{is odd.} \end{cases} \tag{B2}$$

Then there will exist a C^*-probability space $(\mathcal{A},\phi)$ and a random variable $d \in \mathcal{A}$, such that

$$\lim_{N\to\infty} \mathrm{tr}(D^j) = \phi(d^j) \quad \text{for all} \quad j \geqslant 0. \tag{B3}$$

Let $u_1,\dots,u_n$ be a sequence of Haar unitaries in $(\mathcal{A},\phi)$ which are freely independent together with d. We give a concrete example of $u_1,\dots,u_n,d$ at the end of this section. Let $E(\cdot) = \int \cdot\, d\mu$, where $d\mu$ is the Haar measure on $U(N)$, then by Theorem 3.1 in [16], we have

$$\begin{aligned}\lim_{N\to\infty} E\ \mathrm{tr}(A_{i(1)}A_{i(2)}\dots A_{i(k)}) &= \phi(u_{i(1)}du^*_{i(1)}\cdots u_{i(k)}du^*_{i(k)}) \\ &= 0,\end{aligned} \tag{B4}$$

where the second equation comes from the freeness of $d,u_1,\dots,u_n$. Moreover, by Theorem 3.5 in [16],

$$P(|\,\mathrm{tr}(A_{i(1)}A_{i(2)}\dots A_{i(k)})| \geqslant \epsilon) = O(N^{-2}). \tag{B5}$$

Then by the Borel-Cantelli lemma, for any $\epsilon > 0$,

$$\limsup_{N\to\infty} \left|\mathrm{tr}(A_{i(1)}A_{i(2)}\dots A_{i(k)})\right| \leqslant \epsilon, \quad \text{almost surely.} \tag{B6}$$

Hence

$$\lim_{N\to\infty} \mathrm{tr}(A_{i(1)}A_{i(2)}\cdots A_{i(k)}) = 0, \quad \text{almost surely.} \tag{B7}$$

A concrete example of $u_1,\dots,u_n$ and d

Let $G = *_{2n+1}\mathbb{Z}_2$ and g_i, $i = 1,\dots,2n+1$ be the generator of the ith copy. Let $u_i = \lambda(g_{2i-1}g_{2i})$, $i = 1,\dots,n$, and $d = \lambda(g_{2n+1})$. Then the C^* probability we consider is $(C^*_{\text{red}}(G),\phi)$, which is defined in Sec. IV A. It is easy to check that Eqs. (B1) and (B3) hold. Thus u_i, $i = 1,\dots,n$, are Haar unitaries in $(C^*_{\text{red}}(G),\phi)$. Moreover, $u_1,\dots,u_n,d$ are freely independent in $(C^*_{\text{red}}(G),\phi)$.

APPENDIX C: PROOF OF PROPOSITION 2

Suppose the dichotomic observables A_i, $i = 1,\dots,n$, are freely independent in some C^*-probability space $(\mathcal{A},\phi)$. Then by Eq. (A2),

$$\begin{aligned}\left\|\sum_{i=1}^n A_i\right\| &= \lim_{k\to\infty}\left(\phi\left(\sum_{i=1}^n A_i\right)^{2k}\right)^{\frac{1}{2k}} \\ &= \lim_{k\to\infty}\left(\phi\left(\sum_{i(1),\dots,i(2k)=1}^{n} A_{i(1)}\dots A_{i(2k)}\right)\right)^{\frac{1}{2k}}.\end{aligned} \tag{C1}$$

To estimate the above equation we need the following definitions and facts from combinatorics [12]. For a given set $\{1,\dots,2k\}$, there is a partition $\pi = \{V_1,\dots,V_s\}$ of this set. π is determined as follows: Two numbers, p and q, belong to the same block V_k of π if and only if $i(p) = i(q)$. There is a particular partition called the pair partition, in which every block contains only two elements. A pair partition of $\{1,\dots,2k\}$ is called noncrossing if there does not exist $1 \leqslant p_1 < q_1 < p_2 < q_2 \leqslant 2k$ such that p_1 is paired with p_2 and q_1 is paired with q_2. The number of noncrossing pair partitions of the set $\{1,\dots,2k\}$ is given by the Catalan number $C_k = \frac{1}{k+1}\binom{2k}{k}$.

Now for the indices $i(1),\dots,i(2k)$, if there exists a pair of adjacent indices which belong to the same block, e.g., $i(s-1) = i(s)$, then we will shrink the indices $i(1),\dots,i(2k)$ to $i(1),\dots,i(s-2)$, $\emptyset, i(s+1),\dots,i(2k)$, since obviously $A_{i(s-1)}A_{i(s)} = \mathbb{1}$. According to this rule, we can shrink π to a new partition $\tilde{\pi}$ on $\{1,\dots,2t\}$, where $t \leqslant k$. Hence we can divide π into two groups:

Case 1. $\tilde{\pi} = \emptyset$.

Case 2. The indices in $\tilde{\pi}$ satisfy condition (iii) in Definition 1; i.e., the adjacent indices are not equal.

We decompose $\phi(\sum_{i=1}^n A_i)^{2k}$ into two terms,

$$\phi\left(\sum_{i=1}^n A_i\right)^{2k} = \phi\sum_{\pi\in\Pi_1} + \phi\sum_{\pi\in\Pi_2} := II_1 + II_2, \tag{C2}$$

where the set of partitions Π_1 and Π_2 is defined as follows: Partition $\pi \in \Pi_1$ if and only if π belongs to case 1. And $\pi \in \Pi_2$ if and only if π belongs to case 2.

By our assumption, i.e., freeness of A_i, $II_2 = 0$. For the term II_1, it is easy to see that II_1 is equal to the cardinality of set Π_1. Due to the shrink process, $\pi \in \Pi_1$ only if there is an even number of elements in every block. These partitions with even elements in every block can be realized in the following process: first choosing an arbitrary noncrossing pair partition, then combining some proper blocks into one block. Hence the number of $\pi \in \Pi_1$ is upper bounded by $C_k n^k$. Thus

$$\phi\left(\sum_{i=1}^{n} A_i\right)^{2k} \leqslant C_k n^k. \tag{C3}$$

Therefore, under our assumption,

$$\left\|\sum_{i=1}^{n} A_i\right\| = \lim_{k\to\infty}\left(\phi\left(\sum_{i=1}^{n} A_i\right)^{2k}\right)^{\frac{1}{2k}} \leqslant 2\sqrt{n}. \tag{C4}$$

Note: For the local hidden state bound C_n of steering inequality S_n in Eq. (32), the variables α_i do not make any contribution to the whole derivation. Thus C_n is also upper bounded by $2\sqrt{n}$.

APPENDIX D: FINE-GRAINED UNCERTAINTY FOR RANDOM OBSERVABLES

In fact, the statement is a corollary of the work of Collins and Male [22]. Here we restate their result as follows: Let $A_i = U_i D U_i^\dagger$; then there exist C^*-probability space $(\mathcal{A},\phi)$ and Haar unitaries $u_1,\ldots,u_n$ which are freely independent of element $d \in \mathcal{A}$, such that

$$\lim_{N\to\infty}\left\|\sum_{i=1}^{n} A_i\right\| = \left\|\sum_{i=1}^{n} u_i d u_i^*\right\|, \quad \text{almost surely.} \tag{D1}$$

Denote $a_i = u_i d u_i^*$; it is easy to see that $a_1,\ldots,a_n$ are freely independent in $(\mathcal{A},\phi)$. Hence due to a similar argument in Appendix C, we have $\|\sum_{i=1}^n u_i d u_i^*\| \leqslant 2\sqrt{n}$. Therefore we have the following corollary:

Corollary 2. Let $A_i = U_i D U_i^\dagger$, $i = 1,\ldots,n$, and U_i be independent random matrices in $U(N)$. Then we have

$$\lim_{N\to\infty}\left\|\sum_{i=1}^{n} A_i\right\| \leqslant 2\sqrt{n}, \quad \text{almost surely.} \tag{D2}$$

For the special case $n = 2$, we have the following corollary:

Corollary 3. Let $A_i = U_i D U_i^\dagger, i = 1,2$, and U_i be independent random matrices in $U(N)$. Then we have

$$\lim_{N\to\infty}\left\|\sum_{i=1}^{2} A_i\right\| = 2, \quad \text{almost surely.} \tag{D3}$$

Proof. For all $k \in \mathbb{N}_0$, then almost surely we have

$$\begin{aligned}
&\lim_{N\to\infty} \operatorname{tr}(A_1 + A_2)^{2k} \\
&= \sum_{j=0}^{k}\binom{k}{j}2^{k-j}\sum_{l=0}^{j}\binom{j}{l}\lim_{N\to\infty}\operatorname{tr}(A_1A_2)^{2l-j} \\
&= \sum_{j=0,\text{ even}}^{k}\binom{k}{j}2^{k-j}\binom{j}{j/2}.
\end{aligned} \tag{D4}$$

Since $\lim_{k\to\infty}(\sum_{j=0,\text{ even}}^{k}\binom{k}{j}2^{k-j}\binom{j}{j/2})^{\frac{1}{2k}} = 2$, then by the standard argument in this sequel, we have

$$\liminf_{N\to\infty}\left\|\sum_{i=1}^{2} A_i\right\| \geqslant 2 - \epsilon, \quad \text{almost surely.} \tag{D5}$$

On the other hand, $\|\sum_{i=1}^2 A_i\| \leqslant 2$ is obvious.

APPENDIX E: MAXIMALLY CERTAIN STATES FOR $\xi_{\vec{0}}$ AND $\xi_{\vec{1}}$ IN THE CASE $n = 2$

Let $A_1 = \lambda(g_1)$, $A_2 = \lambda(g_2)$, where g_1, g_2 are generators of group $*_2\mathbb{Z}_2$. We need the following notions.

Definition 4. A group G is amenable if there exists a state μ of $\ell_\infty(G)$ which is invariant under the left translation action: i.e., for all $s \in G$ and $f \in \ell_\infty(G)$, $\mu(s\cdot f) = \mu(f)$.

Definition 5. Let G be a group; a Følner net (sequence) is a net of nonempty finite subsets $F_n \subset G$ such that $|F_n \cap gF_n|/|F_n| \to 1$ for all $g \in G$, where gF_n denotes the subset $\{gh : h \in F_n\}$.

For any $g \in G$, there exists N such that, for all $n \geqslant N$, $g \in F_n$. There are many characterizations of amenable groups.

Proposition 3 [23]. Let G be a discrete group. The following are equivalent:

(i) G is amenable;

(ii) G has a Følner net (sequence);

(iii) For any finite subset $E \subset G$, we have $\frac{1}{|E|}\|\sum_{g\in E}\lambda(g)\| = 1$.

For instance, group $*_2\mathbb{Z}_2$ is amenable. Hence by the above proposition, $\|\lambda(g_1) + \lambda(g_2)\| = 2$. With the above notions, we can formally define a state

$$\rho_n = \frac{1}{|F_n|}\sum_{g,h\in F_n}|g\rangle\langle h|, \tag{E1}$$

where F_n is a Følner sequence of $G = *_2\mathbb{Z}_2$. Now we have

$$\begin{aligned}
&\lim_{n\to\infty}\operatorname{Tr}((\lambda(g_1)+\lambda(g_2))\rho_n) \\
&= \lim_{n\to\infty}\frac{1}{|F_n|}\left(\sum_{g,h\in F_n}\langle h|g_1g\rangle + \sum_{g,h\in F_n}\langle h|g_2g\rangle\right) \\
&= \lim_{n\to\infty}\frac{1}{|F_n|}(|F_n\cap g_1F_n| + |F_n\cap g_2F_n|) = 2,
\end{aligned} \tag{E2}$$

where for the second equation we have used the property of the Følner sequence. Thus in this approximate sense, the fine-grained uncertainty of A_1^0 and A_2^0 is 1. Technically we can construct $\tilde{\rho}_n$ to approximate $\xi_{\vec{1}}$. First, we define two subsets of $G = *_2\mathbb{Z}_2$:

$$\begin{aligned}
&G_1 = \{g_1, g_1g_2, g_1g_2g_1, \ldots\} \\
\text{and}\quad &G_2 = \{g_2, g_2g_1, g_2g_1g_2, \ldots\}.
\end{aligned} \tag{E3}$$

In fact, G_1 (G_2) is the subset of wards which begin with g_1 (g_2). It is easy to see that $G_1 \cup G_2 \cup \{e\} = G$. Now we define a state

$$|\tilde{\phi}_n\rangle = \frac{1}{\sqrt{|F_n|}}\sum_{g\in F_n} e^{i\theta_g}|g\rangle, \tag{E4}$$

where F_n is still a Følner sequence of G and

$$\theta_g = \begin{cases}\pi/2, & g \in G_1,\\ -\pi/2, & g \in G_2,\\ 0, & g = e.\end{cases} \tag{E5}$$

Let $\tilde{\rho}_n = |\tilde{\phi}_n\rangle\langle\tilde{\phi}_n|$; then we have

$$\lim_{n\to\infty} \mathrm{Tr}((\lambda(g_1)+\lambda(g_2))\tilde{\rho}_n) = \lim_{n\to\infty} \frac{1}{|F_n|} \sum_{g,h\in F_n} e^{i(\theta_g-\theta_h)}(\langle h|g_1 g\rangle + \langle h|g_2 g\rangle)$$
$$= \lim_{n\to\infty} \frac{1}{|F_n|}\left(\sum_{g\in F_n\cap g_1 F_n} e^{i(\theta_g-\theta_{g_1 g})} + \sum_{g\in F_n\cap g_2 F_n} e^{i(\theta_g-\theta_{g_2 g})}\right). \tag{E6}$$

For the first term on the right-hand side, for large enough n, we can say $e,g_1 \in F_n$. Therefore $e,g_1 \in F_n \cap g_1 F_n$ for large enough n. Then we have

$$\frac{1}{|F_n|}\sum_{g\in F_n\cap g_1 F_n} e^{i(\theta_g-\theta_{g_1 g})} = \frac{1}{|F_n|}\sum_{g\in F_n\cap g_1 F_n, g\neq\{e,g_1\}} e^{i(\theta_g-\theta_{g_1 g})} + \frac{1}{|F_n|}e^{i(\theta_e-\theta_{g_1 e})} + \frac{1}{|F_n|}e^{i(\theta_{g_1}-\theta_{g_1 g_1})}$$
$$= \frac{1}{|F_n|}\sum_{g\in F_n\cap g_1 F_n, g\neq\{e,g_1\}} e^{i\pi} = -\frac{|F_n\cap g_1 F_n|-2}{|F_n|}, \tag{E7}$$

where for the second equation we have used (E5). A similar argument is valid for the second term on the right-hand side of (E6). Thus, finally, we have

$$\lim_{n\to\infty} \mathrm{Tr}((\lambda(g_1)+\lambda(g_2))\tilde{\rho}_n) = -2. \tag{E8}$$

APPENDIX F: QUANTUM VALUE OF THE COMPLEX CHSH-BELL INEQUALITY

In this Appendix we consider a Bell inequality which has a form similar to that of the CHSH-Bell inequality. The Bell operator is defined as

$$\mathcal{B} = A_1\otimes B_1 + A_1\otimes B_2 + A_2\otimes B_1 + \omega A_2\otimes B_2, \tag{F1}$$

where $\omega = e^{\frac{2\pi i}{3}}$. Here the observables are not dichotomic. Instead, there are three possible outcomes: 1, ω, and ω^2. Thus A_i and B_j are required to be unitaries and satisfy $A_i^3 = B_j^3 = \mathbb{1}$ for any $i,j = 1,2$. The classical value of this Bell functional is $\sqrt{7}$.

Now for the quantum value, we can assume that $A_1 = B_1$, $A_2 = B_2$, and A_1, A_2 are freely independent in some C^*-probability space. Hence we have

$$\mathcal{B}\mathcal{B}^\dagger = 3\mathbb{1}\otimes\mathbb{1} + (\mathbb{1}-\omega A)\otimes(\mathbb{1}-\omega A), \tag{F2}$$

where $A = A_1 A_2^\dagger + \omega A_2 A_1^\dagger$. By the binomial formula we have

$$\mathrm{tr}(\mathcal{B}\mathcal{B}^\dagger)^k = \sum_{j=0}^{k}\binom{k}{j}3^{k-j}\left(\sum_{l=0,\, l \text{ is even}}^{j}\binom{j}{l}\binom{l}{l/2}\right)^2 := Q_k. \tag{F3}$$

On one hand, by Stirling's formula, for even l, $\binom{l}{l/2} \leqslant 2^l$, thus

$$Q_k \leqslant \sum_{j=0}^{k}\binom{k}{j}3^{k-j}\left(\sum_{l=0}^{j}\binom{j}{l}2^l\right)^2 = 12^k. \tag{F4}$$

By Lemma 1, we have $\|\mathcal{B}\| \leqslant 2\sqrt{3}$. By a slight adaption of the results in [24], where a method to estimate the quantum value for given dichotomic Bell inequalities is provided, we can conclude that $2\sqrt{3}$ is an upper bound for the quantum value of the complex CHSH-Bell inequality. In fact, this upper bound can be obtained by choosing

$$A_1 = B_1 = \begin{pmatrix} 0 & 0 & 1\\ \omega^2 & 0 & 0\\ 0 & \omega & 0\end{pmatrix},$$
$$A_2 = B_2 = \begin{pmatrix} 0 & 0 & -\omega\\ 0 & 1 & 0\\ \omega^2 & 0 & 0\end{pmatrix}. \tag{F5}$$

On the other hand,

$$Q_{2k} = \sum_{j=0}^{k}\binom{2k}{2j}3^{2k-2j}\left(\sum_{l=0}^{j}\binom{2j}{2l}\binom{2l}{l}\right)^2 + \sum_{j=1}^{k}\binom{2k}{2j-1}3^{2k-2j+1}\left(\sum_{l=0}^{j-1}\binom{2j-1}{2l}\binom{2l}{l}\right)^2$$
$$\approx \sum_{j=0}^{k}\binom{2k}{2j}3^{2k-2j}\left(\sum_{l=0}^{j}\binom{2j}{2l}2^{2l}\right)^2 + \sum_{j=1}^{k}\binom{2k}{2j-1}3^{2k-2j+1}\left(\sum_{l=0}^{j-1}\binom{2j-1}{2l}2^{2l}\right)^2$$
$$\gtrsim \sum_{j=0}^{k}\binom{2k}{2j}3^{2k-2j}\left(\sum_{l=0}^{j}\binom{j}{l}2^{2l}\right)^2 + \sum_{j=1}^{k}\binom{2k}{2j-1}3^{2k-2j+1}\left(\sum_{l=0}^{j-1}\binom{j-1}{l}2^{2l}\right)^2$$
$$\approx \sum_{j=0}^{2k}\binom{2k}{j}3^{2k-j}5^j = 8^{2k}. \tag{F6}$$

Therefore $\|\mathcal{B}\| \geqslant 2\sqrt{2} > \sqrt{7}$.

This method is also promising for the famous MABK Bell inequalities [25,26].

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