Resource Destroying Maps

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(Received 13 June 2016; published 8 February 2017)

Resource theory is a widely applicable framework for analyzing the physical resources required for given tasks, such as computation, communication, and energy extraction. In this Letter, we propose a general scheme for analyzing resource theories based on resource destroying maps, which leave resource-free states unchanged but erase the resource stored in all other states. We introduce a group of general conditions that determine whether a quantum operation exhibits typical resource-free properties in relation to a given resource destroying map. Our theory reveals fundamental connections among basic elements of resource theories, in particular, free states, free operations, and resource measures. In particular, we define a class of simple resource measures that can be calculated without optimization, and that are monotone nonincreasing under operations that commute with the resource destroying map. We apply our theory to the resources of coherence and quantum correlations (e.g., discord), two prominent features of nonclassicality.

DOI: 10.1103/PhysRevLett.118.060502

Introduction.—Resource theory originates from the observation that certain properties of physical systems become valuable resources when the operations that can be performed are restricted so that such properties are hard to create. A prototypical example of such a property is quantum entanglement [1,2], which becomes a key resource for many quantum information processing tasks, when one is restricted to local operations and classical communication (LOCC). The framework of resource theory has been applied to various other concepts in quantum information, such as purity [3], magic states [4], and coherence [5,6], and to broader areas, such as asymmetry [7] and thermodynamics [8].

Theories of different resources share a similar structure. In general, quantum resource theories contain three basic elements: free states, free (allowed) operations, and resource measures (monotones). These elements are closely related to one another. For example, free operations should not be able to create resource from free states, and resource measures are expected to be monotone nonincreasing under free operations. In recent years, considerable effort has been devoted to developing a unified framework of resource theories [9–11]. In particular, Ref. [9] studies the general case where the set of free operations is maximal, i.e., all (asymptotically) resource nongenerating operations are allowed, and when the resource satisfies several postulates (e.g., the set of free states is convex).

Some key aspects of resource theories are not addressed by existing frameworks, however. For example, characterizing a proper set of free operations is frequently a major difficulty in establishing a resource theory, and we do not yet have general principles and understandings for nonmaximal theories. Indeed, a successful resource theory is usually specified by physical restrictions on the set of allowed operations: LOCC and thermal operations [8,12,13] are prominent examples. But such restrictions are often stronger than merely nongenerating, and may lead to mathematical difficulties in characterizing and calculating monotones. Moreover, existing results do not apply to some resources, such as discord, where the set of free states is nonconvex.

In this Letter, we introduce a simple but universally applicable theory of resource-free properties of quantum operations that addresses these issues. Our theory is based on the notion of resource destroying maps: for a given resource, a resource destroying map leaves free states unchanged, but destroys the resource otherwise. Key features of resource destroying maps are discussed. For example, an immediate observation is that a resource destroying map is not linear (thus cannot be represented by a quantum channel) if the set of free states is nonconvex. As will be seen, many important properties of our framework sharply contrast linear resource destroying maps with nonlinear ones. We demonstrate that the concept of resource destroying maps helps unify and simplify the analysis of resource theories, allowing us to determine whether a quantum operation exhibits a group of fundamental resource-free properties, in addition to nongenerating. A basic result of our theory is that any contractive distance between a state and its resource-free version is monotone nonincreasing under all such operations. Finally, we apply the framework of resource destroying maps to coherence and discord. In particular, we find that the theory of discord, which is poorly understood in terms of resource theory (largely due to its nonconvexity), can exhibit a simple structure in this framework. Moreover, the analysis of discord helps illustrate several peculiar properties of nonlinear resource destroying maps.

Resource destroying maps.—Here we formally define the notion of resource destroying maps, the key concept of our theory. Let F be the set of free states for a certain theory. For all input states ρ, a resource destroying map λ satisfies the following requirements: (i) resource destroying: if $\rho \notin F$, $\lambda(\rho) \in F$; (ii) nonresource fixing: if $\rho \in F$, $\lambda(\rho) = \rho$. In other words, a resource destroying map outputs a free state if the input is not free, and leaves the input unchanged otherwise. The resource destroying map characterizes the resource-free space: F consists precisely of the fixed points of λ. Resource destroying maps are idempotent due to (ii). They are also surjections onto codomain F since every free state is a preimage of itself. It is helpful to draw an analogy between the structure of resource destruction and a fiber bundle: λ defines a bundle projection onto F. Call a nonfree state a *parent* state of its image free state. Then each free state defines a *family* consisting of corresponding parent states (the fiber) and the free state itself.

Note that a resource destroying map does not have to be completely positive or linear, and can be highly nonuniform. However, we are mostly interested in the physically motivated maps, usually with simple descriptions that work universally for all inputs. For example, the simplest case is when the resource destroying map can be represented by a quantum channel. However, it can be shown that λ cannot be linear (thus not a channel) when F is nonconvex. (See Supplemental Material [14] for details.) In addition, for theories of correlations among multiple parties, local resource destroying maps cannot be a channel either. Notably, entanglement breaking channels [24] do not necessarily leave separable (unentangled) states unchanged, and so are not entanglement destroying maps. Consider uncorrelated states: the channel that stabilizes all local states can only be the identity, which does not destroy resource. Necessary and sufficient conditions for the existence of resource destroying channels are recently investigated in Ref. [25].

For many theories, a simple resource destroying map is easy to identify. For example, complete dephasing in the preferred basis is an obvious coherence destroying map; Haar (uniform) twirling over the group G is a G-asymmetry destroying map [26]. For discord-type quantum correlations, the resource destroying map cannot be a channel (whether local or not) since discord-free (classically correlated) states form a nonconvex set [27], but it can simply be a local measurement in an eigenbasis of the reduced density operator. In the following, we use upper and lower case Greek letters to denote channels and general maps, respectively.

Resource-free conditions.—Now we are ready to introduce a group of general conditions with simple mathematical forms, based on resource destroying maps, which correspond to various typical resource-free properties of quantum operations.

Consider a theory with resource destroying map λ. Let $\mathcal{E}$ be some quantum operation. We start from

$$\mathcal{E}\circ\lambda = \lambda\circ\mathcal{E}\circ\lambda, \tag{1}$$

where $\circ$ is the composition of maps. An equivalent form of this condition is the following: $\mathcal{E}(\lambda(\rho)) = \lambda(\mathcal{E}[\lambda(\rho)])$ for all ρ. Recall that only free states are fixed points of λ. This condition indicates that the output of $\mathcal{E}\circ\lambda$ is always a fixed point of λ, thus free. In other words, the set of free states is closed under $\mathcal{E}$. So we call this condition the *nongenerating* condition, and, correspondingly, the operations satisfying this condition resource nongenerating operations. This is a necessary constraint on free operations, since any other operation can create resource, thus trivializing the theory. Theories that allow all such operations (under some assumptions including convexity) possess a common structure: they are reversible and have regularized relative entropy as the unique monotone asymptotically [9,10].

Next, we consider the following dual form of the nongenerating condition:

$$\lambda\circ\mathcal{E} = \lambda\circ\mathcal{E}\circ\lambda. \tag{2}$$

Think of the output of λ as the free part of an input state. This condition means that $\mathcal{E}$ cannot make use of the resource stored in any input to affect the free part. We call this condition the *nonactivating* condition. An alternative interpretation is that such operations never break up a family: members of the same family must be mapped to the same target family (not necessarily the original one though). An illustration of the nongenerating and nonactivating conditions is given in Fig. 1.

In general, the nongenerating and nonactivating conditions can hold independently. Because of the idempotence of λ, the sufficient and necessary condition for an operation to be resource nongenerating and nonactivating simultaneously is that it commutes with λ:

$$\lambda\circ\mathcal{E} = \mathcal{E}\circ\lambda. \tag{3}$$

We call this condition the *commuting* condition.

Recall that a quantum operation $\mathcal{E}$ can be specified by Kraus decomposition $\mathcal{E}(\cdot) = \sum_\mu K_\mu \cdot K_\mu^\dagger$, where $\{K_\mu\}$ are Kraus operators satisfying $\sum_\mu K_\mu^\dagger K_\mu \le I$. Each Kraus arm $\mathcal{E}_\mu(\cdot) \equiv K_\mu \cdot K_\mu^\dagger$ corresponds to a (unnormalized) generalized measurement outcome with probability $\mathrm{tr}(K_\mu \cdot K_\mu^\dagger)$. In practice, one may want to require that the nongenerating, nonactivating, or commuting conditions be satisfied even when considering selective measurements; i.e., the outcome of the measurement is accessible. This leads to the following modification of each condition: there is some Kraus decomposition of $\mathcal{E}$ such that all $\mathcal{E}_\mu$ satisfies

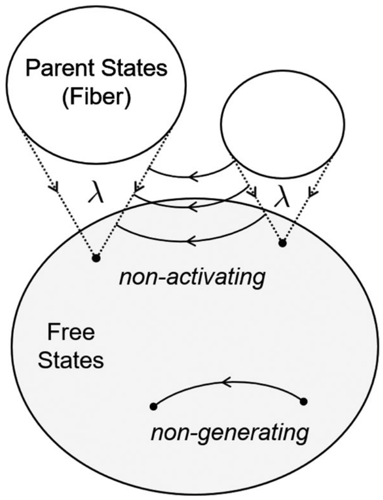

FIG. 1. An illustration of the resource-free conditions. The set of free states is closed under resource nongenerating operations. States belonging to the same family are mapped to the same target family by resource nonactivating operations.

the condition. We call such counterparts *selective* conditions. In other words, selective operations can be implemented by some POVM that exhibits corresponding resource-free properties, even if measurement outcomes are retained. Here we do not impose these conditions on every Kraus decomposition: typically, the relevant decomposition is specified by how we implement the operation, and this can be an overly strong requirement that places extra constraints irrelevant to the resource under study [28]. We shall compare the strength of the original conditions and their selective counterparts in the next section.

For a given resource-free set F, the definition of λ is in general nonunique. Since λ is surjective, the set of resource nongenerating operations is not affected by different choices of λ. In contrast, resource nonactivating operations and thus commuting operations can depend on the bundle structure specified by λ. These observations also hold for the selective version of each condition. Explicit examples are given in the Supplemental Material [14].

General properties.—Here we introduce some typical features of our framework that hold generally in different theories. We shall see that some of these features manifestly contrast linear resource destroying maps with nonlinear ones. Denote the sets of resource nongenerating, nonactivating, and commuting operations as $\bar{\mathbb{X}}$, $\bar{\mathbb{X}}^*$, and $\mathbb{X}$, respectively, and their selective versions by subscript s. By definition, they satisfy $\mathbb{X} = \bar{\mathbb{X}} \cap \bar{\mathbb{X}}^*$ and $\mathbb{X}_s = \bar{\mathbb{X}}_s \cap \bar{\mathbb{X}}_s^*$.

For a theory with resource destroying channel Λ, one can easily construct these operations. Notice that $\Lambda \circ \Omega \in \bar{\mathbb{X}}$, where Ω is an arbitrary operation, by the idempotence of Λ. Meanwhile, $\Lambda \circ \Omega$ belongs to $\bar{\mathbb{X}}^*$ only if Ω itself does. Similarly, $\Omega \circ \Lambda \in \bar{\mathbb{X}}^*$. Destroying the resource in both the input and output allows both conditions to be satisfied: $\Lambda \circ \Omega \circ \Lambda \in \mathbb{X}$. Selective operations can be constructed by similar procedures on each Kraus arm. Let $\{M_\mu\}$ be a Kraus decomposition of Ω, and $\Omega_\mu(\cdot) \equiv M_\mu \cdot M_\mu^\dagger$ denote the action of each Kraus arm. It can be directly verified that each $\Lambda \circ \Omega_\mu$ specifies a resource nongenerating Kraus arm, i.e., $\sum_\mu \Lambda \circ \Omega_\mu \in \bar{\mathbb{X}}_s$. Similarly, $\sum_\mu \Omega_\mu \circ \Lambda \in \bar{\mathbb{X}}_s^*$ and $\sum_\mu \Lambda \circ \Omega_\mu \circ \Lambda \in \mathbb{X}_s$.

One may also ask if the resource-free properties hold for compositions and convex combinations. The answer is yes for compositions for any λ. For example, $\mathbb{X}$ is obviously closed under composition: given two operations $\mathcal{E}_1$ and $\mathcal{E}_2$ satisfying $\mathcal{E}_{1,2} \circ \lambda = \lambda \circ \mathcal{E}_{1,2}$ for some resource destroying map λ, it holds that $\mathcal{E}_2 \circ \mathcal{E}_1$ is also a λ-commuting operation: by using the respective commuting conditions, we obtain $(\mathcal{E}_2 \circ \mathcal{E}_1) \circ \lambda = \mathcal{E}_2 \circ \lambda \circ \mathcal{E}_1 = \lambda \circ (\mathcal{E}_2 \circ \mathcal{E}_1)$. This conclusion also holds for $\bar{\mathbb{X}}$, $\bar{\mathbb{X}}^*$, and selective classes, which can be proven by similar arguments. On the other hand, all classes are closed under convex combination when λ is a linear map. Again, take the commuting condition as an example: $[p\mathcal{E}_1 + (1-p)\mathcal{E}_2] \circ \lambda = p\mathcal{E}_1 \circ \lambda + (1-p)\mathcal{E}_2 \circ \lambda = p\lambda \circ \mathcal{E}_1 + (1-p)\lambda \circ \mathcal{E}_2 = \lambda \circ [p\mathcal{E}_1 + (1-p)\mathcal{E}_2]$. Similar arguments work for other conditions. For nonlinear λ, however, the last equality does not necessarily hold. For the same reason, when λ is linear, selective conditions are stronger than their respective original versions (e.g., $\mathbb{X}_s \subset \mathbb{X}$), but otherwise this is not necessarily true.

We now show that the commuting condition plays a special role in the quantification of resources, a central theme of resource theories. The most basic property of a proper resource measure (a non-negative real function of states) is monotonicity under free operations: free operations should not be able to increase the amount of resource. A natural type of measure is the minimal distance to the set of free states, where the distance is given by some function $D(\rho, \sigma)$ defined on two states ρ and σ that is contractive, i.e., obeys the data processing inequality $D[\Gamma(\rho), \Gamma(\sigma)] \leq D(\rho, \sigma)$ for any operation Γ. Note that D is not necessarily a metric. Nonsymmetric distances such as relative Rényi entropies are also valid choices of D. Formally, a distance measure of resource is given by $\mathfrak{D}(\rho) \coloneqq \inf_{\sigma \in F} D(\rho, \sigma)$. Monotonicity holds for such measures due to the minimization. However, such optimizations are often computationally hard. Now consider the following function:

$$\tilde{\mathfrak{D}}(\rho) \coloneqq D(\rho, \lambda(\rho)). \tag{4}$$

Because of the absence of minimization, $\tilde{\mathfrak{D}}(\rho) \geq \mathfrak{D}(\rho)$. However, if we restrict the set of allowed operations to $\mathbb{X}$, this measure also satisfies the monotonicity requirement,

$$\tilde{\mathfrak{D}}(\rho) \geq D(\Gamma(\rho), \Gamma[\lambda(\rho)]) = D(\Gamma(\rho), \lambda[\Gamma(\rho)]) \equiv \tilde{\mathfrak{D}}(\Gamma(\rho)), \tag{5}$$

where the inequality follows from the contractivity of D. Therefore, for any resource theory with free operations satisfying the commuting condition, we have a class of computationally easy monotones which avoid optimizations (given that λ is suitably defined). We should note that $\tilde{\mathfrak{D}}$ is not necessarily continuous everywhere when λ is nonlinear, which requires more careful analysis in practice (as will be demonstrated for discord). The possibility of retaining measurement outcomes leads to the *selective* monotonicity condition—monotonicity under selective measurements on average. Following a similar argument as Eq. (5), a general result we can obtain at the moment is that $\tilde{\mathfrak{D}}$ obeys selective monotonicity under selective commuting operations, for a restricted class of D including quantum relative entropy (details in the Supplemental Material [14]). Recall that, when λ is linear, $\mathbb{X}_s \subset \mathbb{X}$: selective monotonicity is stronger than monotonicity; however, this is not necessarily the case when λ is nonlinear.

Examples.—We first focus on the theory of quantum coherence. Here, a basis of interest is specified, and density operators that are diagonal in this basis are incoherent (free). The study of coherence from a resource theory perspective has attracted a considerable amount of attention and effort in recent years. A few definitions of coherence-free operations stemmed from various perspectives are proposed and studied lately [5,6,29–35], most of which can directly emerge from our framework as follows. Complete dephasing in the preferred basis, denoted by Π, is a natural coherence-destroying map. Let $\bar{X}(\Pi)$ and $\bar{X}^*(\Pi)$ and $X(\Pi)$ be the sets of coherence nongenerating, nonactivating, and commuting operations given by Π, respectively (an additional subscript s for selective operations). $\bar{X}(\Pi)$ contains all coherence nongenerating operations, which are recently analyzed in Ref. [35]. Members of $\bar{X}^*(\Pi)$ cannot activate the coherence stored in the input in the sense that $\mathcal{E}(\cdot)$ and $\mathcal{E}\circ\Pi(\cdot)$ are always indistinguishable by measuring incoherent observables. So $X(\Pi)$ contains operations that can neither create nor activate coherence. In the preparation of this Letter, we became aware that these operations were very recently studied as dephasing-covariant operations in Refs. [33,34]. $\bar{X}_s(\Pi)$ and $X_s(\Pi)$ are, respectively, the sets of incoherent operations [5] and strictly incoherent operations [32]. Detailed discussions of these classes and further comparisons to other relevant proposals of coherence-free operations are provided in the Supplemental Material [14]. For any theory where the free operations belong to $X(\Pi)$, we know that $D[\cdot, \Pi(\cdot)]$ for any contractive D represents a coherence monotone. In comparison, monotonicity of some D may fail if more operations are allowed. For example, not all relative Rényi entropies are monotone under $\bar{X}(\Pi)$ [34].

Next, we consider discord [36,37], the most general form of nonclassical correlations; see Ref. [27] for a comprehensive review. Discord places a stronger constraint on free states than entanglement in the sense that it can exist in separable states. Discord has been shown to be the underlying resource for various tasks [38–41]. However, a formal treatment of discord in the resource theory framework (e.g., transformation rules) remains elusive, mostly because our understanding of discord-free operations is limited, and most existing general results for resource theory do not directly apply to discord, due to its nonconvexity. Here, we focus on the one-sided discord as measured on subsystem A of a bipartite state ρ_{AB}, and local operations acting on the same subsystem. The ideas can be generalized to nonlocal operations and multipartite cases. A state is regarded as discord-free if there exist local rank-one projective measurements that do not perturb the joint state. Such states take the form $\rho_{AB} = \sum_i p_i |i\rangle_A \langle i| \otimes \rho_B^i$, where $\{|i\rangle\}$ is a complete orthonormal basis of A. These states are conventionally called classical-quantum (CQ) states. Because of the nonconvexity of CQ, discord can be created just by mixing, and discord destroying maps cannot be linear. Suppose the local density operator $\rho_A = \mathrm{tr}_B \rho_{AB}$ admits a spectral decomposition $\rho_A = \sum_i p_i |i\rangle\langle i|$. Then

$$\pi_A(\rho_{AB}) \coloneqq \sum_i (|i\rangle_A \langle i| \otimes I_B) \rho_{AB} (|i\rangle_A \langle i| \otimes I_B), \qquad (6)$$

i.e., a local measurement in an eigenbasis of A, is the most natural discord destroying map. Obviously, π_A is nonlinear and thus not a channel: the basis in which the projection takes place is dependent on the input state, and not uniquely defined within degenerate subspaces. Also note that π_A never changes the local states.

We now plug π_A into the conditions. Let $\mathcal{E}_A$ be a local operation acting on A. Note that we are considering the effect on the joint space: For example, the nongenerating condition reads $(\mathcal{E}_A \otimes I_B)\circ\pi_A = \pi_A\circ(\mathcal{E}_A \otimes I_B)\circ\pi_A$. This condition determines whether an operation always maps a CQ state to another. As opposed to entanglement, discord can be created by certain local operations. Such operations have been studied in Refs. [42,43]. $\bar{X}_A^*(\pi_A)$ and $X_A(\pi_A)$ have not been considered before to our knowledge. We can classify a variety of simple quantum operations according to their behaviors in the theory of π as follows (proofs in the Supplemental Material [14]). Local unitary-isotropic channels (mixture of a unitary channel and depolarization, which are intuitively strongly discord-free) indeed belong to $X_A(\pi_A)$ and $X_{s,A}(\pi_A)$. Rank-one projective measurements, however, are in $\bar{X}_{s,A}(\pi_A) \backslash X_A(\pi_A)$. Furthermore, local mixed-unitary channels belong to all selective classes, but some of them are not in the original classes, supporting our general observation that selective conditions are not necessarily stronger than their original counterparts for nonlinear λ.

As shown earlier, contractive distances between any ρ_{AB} and $\pi_A(\rho_{AB})$, e.g., $S[\rho_{AB} \| \pi_A(\rho_{AB})]$, is monotone under $X_A(\pi_A)$ (including all unitary-isotropic channels), and selectively monotone under $X_{s,A}(\pi_A)$ (including all mixed-unitary channels). This quantity is equivalent to a physically motivated simple measure of discord called diagonal discord [44]. (Similar quantities are independently discussed in Refs. [45–49].) Diagonal discord may suffer from discontinuities (infinitesimal perturbations may lead to a sudden jump in the value of diagonal discord) [50,51]; however, it can be shown that they only occur at degeneracies [52].

Reference [53] adopts an approach similar to the idea of resource destroying maps to study nonclassicality of operations. There, operations that commute with einselection [54] (complete dephasing) in a certain basis are regarded as classical. The key difference between the setup of Ref. [53] and the discord theory discussed here is that the basis for einselection needs to be specified; thus, not all discord-free states are the fixed points of such einselection [55]. Ref. [53] is more about local coherence in some preferred basis rather than discord.

Concluding remarks.—In this Letter, we propose a simple framework for resource theories based on the notion of resource destroying maps. Our theory provides a general scheme for understanding the power of quantum operations in relation to certain resources. The theory shows how to extend results that have been previously derived for specific resources to a more general class of resource theories. In particular, our framework may lead to conceptual advances in understanding nonconvex theories such as discord. It would also be interesting to apply the framework of resource destroying maps to other important resource theories, such as those of entanglement, magic states, asymmetry and thermodynamics.

Z. W. L. and S. L. are supported by AFOSR and ARO. X. H. is supported by NSFC under Grant No. 11504205. We thank Can Gokler, Iman Marvian, Peter Shor, Kevin Thompson, and Yechao Zhu and anonymous referees for helpful discussions.

Note added.—During the final revision of this Letter, we became aware of a recent review on discord [56], which includes a detailed discussion of the importance and difficulties of studying discord under the resource theory framework, and the state of the art of this field (in particular the local commutativity-preserving operations as the maximal set of local free operations).

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