Supercuspidal part of the mod l cohomology of GU$(1, n − 1)$-Shimura varieties

By *Sug Woo Shin* at Cambridge, MA

Abstract. Let l be a prime. In this paper we are concerned with GU$(1, n − 1)$-type Shimura varieties with arbitrary level structure at l and investigate the part of the cohomology on which $G(\mathbb{Q}_p)$ acts through mod l supercuspidal representations, where $p \neq l$ is any prime such that $G(\mathbb{Q}_p)$ is a general linear group. The main theorem establishes the mod l analogue of the local-global compatibility. Our theorem also encodes a global mod l Jacquet–Langlands correspondence in that the cohomology is described in terms of mod l automorphic forms on some compact inner form of G.

1. Introduction

It has been observed by numerous mathematicians that torsion in the cohomology of locally symmetric spaces has a deep relationship with arithmetic such as congruences of automorphic forms, Galois representations with torsion coefficients, and the mod l or l-adic versions of the Langlands program. Ultimately one would like to understand the torsion for all locally symmetric spaces, but considering the difficulty it is reasonable to start investigation in the case of Shimura varieties. In fact the latter have some advantage in that they carry interesting Galois actions. The goal of this paper is a representation-theoretic description of the l-torsion cohomology of certain GU$(1, n − 1)$-type Shimura varieties when the level at l is arbitrary.

Let Sh_U be a Shimura variety associated with a reductive group G over $\mathbb{Q}$ with level U, an open compact subgroup of $G(\mathbb{A}^\infty)$. Write E for the reflex field and $\mathcal{H}_{\overline{\mathbb{F}}_l}(G(\mathbb{A}^\infty), U)$ for the Hecke algebra of U-bi-invariant functions with values in $\overline{\mathbb{F}}_l$. The étale cohomology $H^i(\mathrm{Sh}_U, \overline{\mathbb{F}}_l)$ has rich structure as an $\mathcal{H}_{\overline{\mathbb{F}}_l}(G(\mathbb{A}^\infty), U) \times \mathrm{Gal}(\overline{E}/E)$-module and should realize the mod l version of the global Langlands correspondence, which is expected to incorporate a suitable generalization of Serre's conjecture but has yet to be made precise.

A nearly complete result has been given by Emerton ([7]) in the case of elliptic modular curves, and it is reasonable to believe that the same can be done for Shimura curves. For arbitrary PEL-type Shimura varieties, recent work of Lan and Suh ([17, 18]) proves the vanishing of the mod l interior cohomology outside the middle degree assuming that the weight for the coefficient satisfies an effective condition and that the PEL data is unramified at l so that U is hyperspecial at l ("l does not divide the level") and Sh_U has good reduction modulo l. Despite

the beauty and uniformity of their argument, they excluded the case of constant $\overline{\mathbb{F}}_l$-coefficient, which would be essential in that any reasonable mod l automorphic form should be represented by a class in $H^i(\mathrm{Sh}_U, \overline{\mathbb{F}}_l)$. (In the case of modular forms, the corresponding fact is that a form of higher weight is congruent modulo l with a weight 2 form at the cost of introducing l in the level.) To our knowledge little is known about $H^i(\mathrm{Sh}_U, \overline{\mathbb{F}}_l)$ in relation to the mod l Langlands program especially in the higher dimensional case when l divides the level.

As a first step toward the general case, we analyze the mod l étale cohomology in the part on which $G(\mathbb{Q}_p)$ acts through supercuspidal representations (this is philosophically resemblant to working with the simple trace formula rather than the full trace formula) under the two assumptions that

- $G(\mathbb{R})$ is isomorphic (ignoring similitude) to a product $U(1, n-1) \times U(0, n) \times \cdots \times U(0, n)$,

- p is a prime different from l such that $G(\mathbb{Q}_p)$ is isomorphic to a product of general linear groups.

We stress that U is allowed to be arbitrary at l here. This case was generally believed to be difficult but can be treated in our work, as far as the supercuspidal part is concerned, for the simple reason that l plays no special role in the proof, as we consider geometry over a mod p or p-adic base for a prime $p \neq l$ (rather than $p = l$). Our approach is based on an adaptation of Harris–Taylor's method ([13]) to the mod l setting and very different from Lan–Suh's approach and Emerton's. We also include non-compact Shimura varieties in our result although this only occurs when $G(\mathbb{R})$ has no $U(0, n)$-factors.

To describe our results in detail let us set up some notation. Let $U^p \subset G(\mathbb{A}^{\infty,p})$ be an open compact subgroup and define $H^i_c(\mathrm{Sh}_{U^p}, \overline{\mathbb{F}}_l)$ to be the direct limit of $H^i_c(\mathrm{Sh}_{U^p U_p}, \overline{\mathbb{F}}_l)$ as U_p runs over the open compact subgroups of $G(\mathbb{Q}_p)$. The reflex field E may be identified with the CM field F which is used to define G. By a standard procedure $H^i_c(\mathrm{Sh}_{U^p}, \overline{\mathbb{F}}_l)$ is equipped with a commuting action of $\mathcal{H}_{\overline{\mathbb{F}}_l}(G(\mathbb{A}^{\infty,p}), U^p)$ and $\mathrm{Gal}(\overline{F}/F)$. By assumption $G(\mathbb{Q}_p)$ is isomorphic to $\mathrm{GL}_1(\mathbb{Q}_p)$ times a product of groups $\mathrm{GL}_n(F_v)$ and there is a distinguished place w among the indices v which corresponds to the infinite place for the signature $(1, n-1)$. Let us write $H^i_c(\mathrm{Sh}_{U^p}, \overline{\mathbb{F}}_l)_{w\text{-sc}}$ for the maximal subspace which has only supercuspidal subquotients as an $\overline{\mathbb{F}}_l[\mathrm{GL}_n(F_w)]$-module. Let $D_{n,w}$ denote the central division algebra over F_w with invariant $1/n$. Let q_w denote the cardinality of the residue field of F_w.

Now let G' be an inner form of G which is isomorphic to G over $\mathbb{A}^{\infty,p}$, compact mod center at ∞ and a certain group at p with a direct factor $D^\times_{n,w}$ (see Section 4.2 for details). The following is our main result, whose precise version is stated in Theorem 4.3.

Theorem 1.1. *If $i \neq n-1$, then $H^i_c(\mathrm{Sh}_{U^p}, \overline{\mathbb{F}}_l)_{w\text{-sc}} = 0$ and (up to an explicit constant and a twist, and ignoring the $\mathrm{GL}_1(\mathbb{Q}_p)$-factor in $G'(\mathbb{Q}_p)$)*

$$\left[H^{n-1}_c(\mathrm{Sh}_{U^p}, \overline{\mathbb{F}}_l)_{w\text{-sc}} \right]$$
$$= \sum_{\rho_w} \left[\mathrm{Hom}_{D^\times_{n,w}}(\rho_w, C^\infty_c(G'(\mathbb{Q}) \backslash G'(\mathbb{A}^\infty)/U^p, \overline{\mathbb{F}}_l)^{\mathrm{ss}}) \right] [\mathrm{JL}(\rho_w)] [\mathrm{rec}(\mathrm{JL}(\rho_w))]$$

in the Grothendieck group of $\mathcal{H}_{\overline{\mathbb{F}}_l}(G(\mathbb{A}^{\infty,p}), U^p) \times G(\mathbb{Q}_p) \times W_{F_w}$-modules where ρ_w runs over the set of irreducible smooth $D^\times_{n,w}$-representations (up to isomorphism) such that $\mathrm{JL}(\rho_w)$ is supercuspidal. If l does not divide $q_w(q^n_w - 1)$, then the above isomorphism holds without passing to the Grothendieck group.

The theorem identifies $H_c^{n-1}(\mathrm{Sh}_{U^p}, \overline{\mathbb{F}}_l)_{w\text{-sc}}$ as an $\mathcal{H}_{\overline{\mathbb{F}}_l}(G(\mathbb{A}^{\infty,p}), U^p) \times G(\mathbb{Q}_p) \times W_{F_w}$-module in terms of $\overline{\mathbb{F}}_l$-valued automorphic forms on $G'(\mathbb{A}^\infty)$ via the mod l local Jacquet–Langlands and local Langlands correspondences for $D_{n,w}^\times$ and $\mathrm{GL}_n(F_w)$, respectively denoted by JL and rec. In particular the theorem may be thought of as providing a mod l global Jacquet–Langlands type correspondence between part of H_c^0 for G' and part of H_c^{n-1} for G.

Even in the non-banal case we can be precise (though not as explicit) about the subspace $H_c^{n-1}(\mathrm{Sh}_{U^p}, \overline{\mathbb{F}}_l)_{w\text{-sc}}$ without passing to the Grothendieck group. As the answer is somewhat complicated, we do not state it explicitly in this paper but see Remark 4.13 below.

If Sh_{U^p} is compact, the $G(\mathbb{A}^\infty)$-action on $\varinjlim_{U^p} H_c^{n-1}(\mathrm{Sh}_{U^p}, \overline{\mathbb{Q}}_l)$ is semisimple. This can be seen from the corresponding fact on the discrete automorphic spectrum via Matsushima's formula. However the analogous assertion has no reason to be true for $H_c^{n-1}(\mathrm{Sh}_{U^p}, \overline{\mathbb{F}}_l)$ (or its direct limit) and may well be false. A nice aspect of our result is that it identifies the mod l cohomology of Sh_{U^p} rather precisely, at least in the supercuspidal part, without taking Euler-Poincaré characteristic or working in the Grothendieck group. Another advantage is that the assumptions (in the bulleted list above) are easy to state and check.

Here is the sketch of proof. First, the theorem is reduced to the same assertion about $H_c^i(\overline{\mathrm{Sh}}_{U^p}, R\Psi\overline{\mathbb{F}}_l)_{w\text{-sc}}$, namely the supercuspidal part of the compact support cohomology of the special fiber with nearby cycle coefficient. This is standard if the integral models for Sh_U are proper over the base (which is the case if $[F : \mathbb{Q}] > 2$) but still true without the condition thanks to [14]. Second, we show that the cohomology of the Newton strata has no supercuspidal part except the basic stratum $\overline{\mathrm{Sh}}_{U^p}^{(0)}$, which has dimension 0. This is deduced from the fact that the corresponding non-basic Lubin–Tate spaces have only induced representations in their cohomology. The third step is to write $H_c^i(\overline{\mathrm{Sh}}_{U^p}^{(0)}, R\Psi\overline{\mathbb{F}}_l)_{w\text{-sc}}$ in terms of the cohomology of the basic Lubin–Tate space $\mathrm{LT}^{(0)}$ and Igusa variety $\mathrm{Ig}_{U^p}^{(0)}$ by using a formula due to Harris–Taylor and Mantovan (Proposition 4.9). Finally the conclusion follows from the description of the mod l cohomology of $\mathrm{LT}^{(0)}$ due to Dat ([5]) and that of $\mathrm{Ig}_{U^p}^{(0)}$ as in [13] or [24].

Our method should apply well to non-constant coefficients as long as the weight parameter is small relative to p. It would be certainly interesting to extend our results for other Shimura varieties and beyond the supercuspidal part. We comment on such possibilities at the end (Section 5).

As a somewhat different approach, an adaptation of the argument of [9] to the mod l setting might lead to our main theorem. However for future generalization, it is much less clear to us how to go beyond the supercuspidal part in such an approach than in the approach of this paper.

The organization of the paper is as follows. In Section 2 we review some preliminaries in mod l representation theory of p-adic groups when $p \neq l$. Section 3 is devoted to key results on the mod l cohomology of basic and non-basic Lubin–Tate spaces which are mostly due to Dat. Section 4 is the heart of the paper where the main theorem is stated and proved. The concluding remarks are in Section 5.

Remark on related work. In a lecture in early 2011, Emerton and Gee reported on a vanishing theorem in the case of compact $GU(1, n-1)$-type Shimura varieties.[1] Their methods and announced results overlap little with mine. When I spoke about the results of this paper at a workshop in Kyoto in July 2011, when the paper was being written, Imai and Mieda told

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[1] Now their work is available online, cf. `http://arxiv.org/abs/1203.4963`.

me about their forthcoming results, which later appeared in a preprint. Their result ([14, Theorem 5.5]) implies that $H^i_c(\mathrm{Sh}_{U^p}, \overline{\mathbb{F}}_l)_{\text{w-sc}}$ vanishes unless $i = n - 1$ (by a somewhat different argument) in the same setting as in this paper but does not make connection with the mod *l* Langlands correspondence. More importantly, the main points of their paper and mine are quite different. On the other hand I benefited from their preprint to include the case of non-compact GU(1, *n* − 1)-type Shimura varieties, which was missing in the version presented in Kyoto.

2. Preliminaries

In this section we set up notation and recall various facts in the mod *l* representations theory of *p*-adic reductive groups when $p \neq l$. For a category $\mathscr{C}$, we write $A \in \mathscr{C}$ to mean that A is an object of $\mathscr{C}$.

2.1. Notation. The following notation will be used in this section and afterward.

- *l* and *p* are distinct primes and $n \in \mathbb{Z}_{\geq 1}$,

- $\mathbb{A}$ (resp. $\mathbb{A}^\infty$, $\mathbb{A}^{\infty, p}$) is the ring of adeles (resp. finite adeles, finite adeles away from *p*),

- K is a finite extension of $\mathbb{Q}_p$ with integer ring $\mathcal{O}_K$, residue field cardinality q_K and valuation $v_K : K^\times \twoheadrightarrow \mathbb{Z}$,

- $D_n = D_{n,K}$ is a division algebra with $[D_n : K] = n^2$, Hasse invariant $1/n$ and center K,

- $D^0_n := \ker(D^\times_n \xrightarrow{v_K \cdot \det} \mathbb{Z})$, $\mathrm{GL}_n(K)^0 := \ker(\mathrm{GL}_n(K) \xrightarrow{v_K \cdot \det} \mathbb{Z})$,

- $U_m := \{g \in \mathrm{GL}_n(\mathcal{O}_K) : g \equiv 1 \pmod{p^m}\}$ for $m \geq 1$,

- W_K is the local Weil group for K,

- $|\cdot|_K : W_K \to \overline{\mathbb{F}}_l^\times$ is the character trivial on the inertia group and sends a geometric Frobenius to q_K^{-1},

- every representation of W_K will be *continuous* (which is the case for the W_K-representations realized on cohomology in this paper) or assumed continuous.

When R is a commutative $\mathbb{Z}_l$-algebra and G is the group of K-points of a reductive group,

- $\mathrm{Rep}_R(G)$ is the category of *smooth* R-representations (R-modules V with G-action such that each $v \in V$ has an open stabilizer in G),

- $\mathrm{Rep}^{\mathrm{adm}}_R(G) \subset \mathrm{Rep}_R(G)$ is the subcategory of admissible R-representations (i.e. V such that V^H is a finitely generated R-module for all open compact open subgroup H of G),

- $\mathrm{Groth}_R(G)$ is the Grothendieck group of admissible R-representations when R is a field,

- $\mathrm{Irr}_R(G)$ is the set of isomorphism classes of irreducible representations in $\mathrm{Rep}_R(G)$,

- Rep$_R^{sc}(G)$ and Irr$_R^{sc}(G)$ are defined by restricting to supercuspidal representations when G is a general linear group (but the definition is different for inner forms like $D_n^\times$; see Section 2.3),

- $D_R(G)$ (resp. $D_R^b(G)$) is the derived category (resp. of bounded complexes) associated with Rep$_R(G)$,

- $\mathcal{H}_R(G, H)$ is the Hecke algebra of locally constant compactly supported H-bi-invariant functions on G, where H is an open compact subgroup of G,

- Rep$(\mathcal{H}_R(G, H))$ is the category of $\mathcal{H}_R(G, H)$-modules,

- Hom$_{RG}$ and RHom$_{RG}$ denote Hom and RHom taken in the category Rep$_R(G)$,

- $\Pi \in$ Rep$_R(G)$ of finite length is said to be ϖ-*isotypic* for some $\varpi \in$ Irr$_R(G)$ if every irreducible subquotient of Π is isomorphic to ϖ.

2.2. Mod l supercuspidal representations. As in [5, Definition 3.0.1], a representation $\varpi \in$ Rep$_{\mathbb{Z}_l}($GL$_n(K))$ is called *supercuspidal* if it does not occur as a $\mathbb{Z}_l[$GL$_n(K)^0]$-module subquotient of any representation parabolically induced from a smooth representation on a proper Levi subgroup of GL$_n(K)$. This recovers the notion of supercuspidality for the set Irr$_{\overline{\mathbb{F}}_l}(GL_n(K))$, cf. [5, Corollary B.1.3]. Write Rep$_{\mathbb{Z}_l}^{sc}($GL$_n(K))$ for the full subcategory of Rep$_{\mathbb{Z}_l}($GL$_n(K))$ whose objects are supercuspidal representations. It is known ([5, Proposition 3.0.2]) that the former is a direct factor of Rep$_{\mathbb{Z}_l}($GL$_n(K))$, so any $\varpi \in$ Rep$_{\mathbb{Z}_l}($GL$_n(K))$ decomposes canonically as

$$(2.1) \qquad\qquad \varpi = \varpi_{sc} \oplus \varpi_{\text{non-sc}}$$

such that ϖ_{sc} belongs to Rep$_{\mathbb{Z}_l}^{sc}($GL$_n(K))$ and $\varpi_{\text{non-sc}}$ has no $\mathbb{Z}_l[$GL$_n(K)^0]$-subquotient which is supercuspidal, and the decomposition (2.1) is functorial in ϖ. In particular any representation $\varpi \in$ Rep$_{\overline{\mathbb{F}}_l}(GL_n(K))$ admits a canonical decomposition (2.1).

Let H be a pro-p principal congruence subgroup of GL$_n(K)$, and Rep$_{\mathbb{Z}_l}($GL$_n(K))_{H\text{-gen}}$ denote the subcategory of Rep$_{\mathbb{Z}_l}($GL$_n(K))$ consisting of the representations generated by H-fixed vectors. According to [3, Section 3.5.8] (cf. [4, Lemma A.3]), $\varpi \mapsto \varpi^H$ induces a categorical equivalence

$$(2.2) \qquad\qquad \text{Rep}_{\mathbb{Z}_l}(\text{GL}_n(K))_{H\text{-gen}} \xrightarrow{\sim} \text{Rep}(\mathcal{H}_{\mathbb{Z}_l}(\text{GL}_n(K), H)).$$

Since any object of Rep$_{\mathbb{Z}_l}($GL$_n(K))_{H\text{-gen}}$ is decomposed as in (2.1), it follows that any object M of Rep$(\mathcal{H}_{\mathbb{Z}_l}(GL_n(K), H))$ admits a functorial decomposition $M = M_{sc} \oplus M_{\text{non-sc}}$ via (2.2). By construction $(\varpi_{sc})^H = (\varpi^H)_{sc}$.

2.3. Mod l local Langlands and Jacquet–Langlands. We are concerned with the mod l Langlands bijection of Vigneras ([27]) only in the supercuspidal case. Denote by rec such a bijection from Irr$_{\overline{\mathbb{F}}_l}^{sc}(GL_n(K))$ onto the n-dimensional representations in the set Irr$_{\overline{\mathbb{F}}_l}(W_K)$. Denote the mod l Jacquet–Langlands map ([6], [5, Section 2.2.2]) by

$$\text{LJ} : \text{Irr}_{\overline{\mathbb{F}}_l}(\text{GL}_n(K)) \to \text{Groth}_{\overline{\mathbb{F}}_l}(D_n^\times).$$

Both rec and LJ are compatible with the classical local Langlands and Jacquet–Langlands correspondence via the reduction mod l map $r_l : \text{Groth}_{\overline{\mathbb{Q}}_l}(GL_n(K)) \to \text{Groth}_{\overline{\mathbb{F}}_l}(GL_n(K))$.

There is also a Jacquet–Langlands map due to Dat ([6, Section 1.2.4]), for which he writes ^{z}JL but we write JL for brevity:

$$JL : \operatorname{Irr}_{\overline{\mathbb{F}}_l}(D_n^\times) \hookrightarrow \operatorname{Irr}_{\overline{\mathbb{F}}_l}(GL_n(K))$$

uniquely characterized by the conditions that

(2.3) $\forall \rho \in \operatorname{Irr}_{\overline{\mathbb{F}}_l}(D_n^\times), \quad LJ(JL(\rho)) = \pm\rho$

in the Grothendieck group and that the image of JL is the set of "super-Speh" representations (which include supercuspidal representations). More detail can be found in the articles by Dat and Vigneras cited above.

Lemma 2.1. *If* $\varpi \in \operatorname{Irr}_{\overline{\mathbb{F}}_l}^{sc}(GL_n(K))$, *then* $JL(LJ(\varpi)) = \varpi$. *If* $\rho \in \operatorname{Irr}_{\overline{\mathbb{F}}_l}(D_n^\times)$ *is such that* $JL(\rho)$ *is supercuspidal, then* $LJ(JL(\rho)) = \rho$.

Proof. When ϖ is supercuspidal, $LJ(\varpi)$ is (not a virtual but) a genuine representation in view of the definition of LJ in [5, (2.2.2)] (which is equivalent to that of [6] as explained there) and Step 1 in the proof of [5, Proposition 3.1.1]. This together with (2.3) implies the second assertion. The first assertion follows as any $\varpi \in \operatorname{Irr}_{\overline{\mathbb{F}}_l}^{sc}(GL_n(K))$ is in the image of JL. $\square$

From now on $\operatorname{Irr}_{\overline{\mathbb{F}}_l}^{sc}(D_n^\times)$ will denote the collection of ρ as in Lemma 2.1 above. Define $\operatorname{Rep}_{\overline{\mathbb{F}}_l}^{sc}(D_n^\times)$ to be the full subcategory of $\operatorname{Rep}_{\overline{\mathbb{F}}_l}(D_n^\times)$ consisting of representations whose irreducible subquotients are in $\operatorname{Irr}_{\overline{\mathbb{F}}_l}^{sc}(D_n^\times)$. For $\rho \in \operatorname{Irr}_{\overline{\mathbb{F}}_l}^{sc}(D_n^\times)$, define

$$\mathscr{L}(\rho) := \operatorname{rec}(JL_{\overline{\mathbb{F}}_l}(\rho)^\vee) \otimes |\cdot|_{W_K}^{-(n-1)/2}.$$

Let $\operatorname{Irr}_{\overline{\mathbb{F}}_l}^{sc}(\mathbb{Q}_p^\times \times D_n^\times)$ denote the collection of $\widetilde{\rho} = \rho_0 \otimes \rho$ such that $\rho \in \operatorname{Irr}_{\overline{\mathbb{F}}_l}^{sc}(D_n^\times)$. Define

$$JL(\widetilde{\rho}) := \rho_0 \otimes JL(\rho) \in \operatorname{Irr}_{\overline{\mathbb{F}}_l}(\mathbb{Q}_p^\times \times GL_n(K)), \quad \mathscr{L}(\widetilde{\rho}) := \mathscr{L}(\rho_0) \otimes \mathscr{L}(\rho) \in \operatorname{Irr}_{\overline{\mathbb{F}}_l}(W_K)$$

for $\widetilde{\rho} \in \operatorname{Irr}_{\overline{\mathbb{F}}_l}^{sc}(\mathbb{Q}_p^\times \times D_n^\times)$.

2.4. Decomposing admissible representations.   For two representations

$$\varpi_1, \varpi_2 \in \operatorname{Irr}_{\overline{\mathbb{F}}_l}(GL_n(K))$$

it is known (see [8, Theorem 3.2.13] for instance) that $\operatorname{Ext}^i(\varpi_1, \varpi_2) = 0$ for all i unless ϖ_1 and ϖ_2 have the same supercuspidal support. In particular this implies that any finite length

$$\Pi \in \operatorname{Rep}_{\overline{\mathbb{F}}_l}^{sc}(GL_n(K))$$

decomposes as a direct sum of ϖ-isotypic (Section 2.1) subrepresentations $\Pi[\varpi]$ as ϖ runs over $\operatorname{Irr}_{\overline{\mathbb{F}}_l}^{sc}(GL_n(K))$. In view of the equivalence (implied by [5, Theorem 3.2.5]) between the categories $\operatorname{Rep}_{\overline{\mathbb{F}}_l}^{sc}(GL_n(K))$ and $\operatorname{Rep}_{\overline{\mathbb{F}}_l}^{sc}(D_n^\times)$, any finite length

$$\Pi \in \operatorname{Rep}_{\overline{\mathbb{F}}_l}^{sc}(D_n^\times)$$

similarly decomposes as

$$\Pi \simeq \bigoplus_{\rho \in \operatorname{Irr}_{\overline{\mathbb{F}}_l}^{sc}(D_n^\times)} \Pi[\rho].$$

Lemma 2.2. *Any admissible $\Pi \in \mathrm{Rep}^{\mathrm{sc}}_{\overline{\mathbb{F}}_l}(D_n^\times)$ decomposes as*

$$\Pi \simeq \bigoplus_{\rho \in \mathrm{Irr}^{\mathrm{sc}}_{\overline{\mathbb{F}}_l}(D_n^\times)} \Pi[\rho]. \tag{2.4}$$

The same holds if $D_n^\times$ is replaced with $\mathbb{Q}_p^\times \times D_n^\times$.

Proof. By [5, Proposition B.2.1], $\Pi \simeq \bigoplus_\rho \Pi_\rho$ where ρ runs over the set of inertia equivalence classes (i.e. up to a twist by an unramified character of $K^\times$) such that the irreducible subquotients of each Π_ρ belong to the class of ρ. Since Π is admissible, Π_ρ has finite length. Applying the discussion above the lemma, we obtain (2.4). The last assertion of the lemma is proved in the same way. □

Remark 2.3. The lemma is not immediately obvious since a mod l supercuspidal representation may not be itself projective in the category with fixed central character, unlike in the case of coefficients in characteristic 0. Also note that without the admissibility condition, (2.4) may not hold even in characteristic 0 unless ρ is taken up to inertia equivalence. (Consider a compact induction of a supercuspidal representation from D_n^0 to $D_n^\times$.)

2.5. Banal primes. Vigneras ([26]) introduced the notion of banal primes l for a p-adic reductive group G where $l \neq p$ so that when l is banal, mod l smooth representations of the given group behave as if they were complex representations. One key property for a banal prime l is that mod l (super)cuspidal representations of G are injective and projective in the category of smooth G-representations with fixed central character. For $D_n^\times$, non-banal primes are those dividing the pro-order of the maximal order of $D_n^\times$, namely those l which divide $q_K(q_K^n - 1)$.

3. Lubin–Tate spaces

In our convention a Lubin–Tate space refers to the deformation space of a one-dimensional (not necessarily formal) Barsotti–Tate group given by quasi-isogenies, as a special case of Rapoport–Zink spaces ([22]). The usual Lubin–Tate space for *formal* Basotti–Tate groups is called the basic Lubin–Tate space in this paper. The case of zero-dimensional (i.e. étale) Barsotti–Tate groups is also considered although the relevant geometric spaces are not Lubin–Tate spaces. The heart of this section is Dat's results on the compact support cohomology complex of Lubin–Tate spaces, built upon work of Harris–Taylor and Faltings–Fargues among others.

3.1. Definitions. Let $n, h \in \mathbb{Z}$ be such that $n \geq 1$ and $0 \leq h \leq n$. Fix a nontrivial ring homomorphism $\mathcal{O}_K \to \overline{\mathbb{F}}_p$. (In the global setting of the next section, we will use the reduction map $\mathcal{O}_{F_w} \to \overline{k(w)}$ for this.) Let Σ^0_{n-h} be a Barsotti–Tate group of height $[K : \mathbb{Q}_p](n - h)$ and dimension 1 over $\overline{\mathbb{F}}_p$ with $\mathcal{O}_K$-action, and $\Sigma^{\text{ét}}_h$ an étale Barsotti–Tate group of height $[K : \mathbb{Q}_p]h$ with $\mathcal{O}_K$-action. As usual, $\mathcal{O}_K$ is assumed to act on the one-dimensional $\overline{\mathbb{F}}_p$-vector space Lie Σ^0_{n-h} via the fixed morphism $\mathcal{O}_K \to \overline{\mathbb{F}}_p$. Then Σ^0_{n-h} and $\Sigma^{\text{ét}}_h$ are unique up to isomorphisms as Barsotti–Tate groups with $\mathcal{O}_K$-actions. Set

$$\Sigma_{n-h,h} := \Sigma^0_{n-h} \times \Sigma^{\text{ét}}_h \quad \text{and} \quad D_{n-h,h} := \mathrm{End}_{\overline{\mathbb{F}}_p}(\Sigma_{n-h,h}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

It is a standard fact that

$$D_{n-h,h} \simeq D_{n-h} \times M_h(K)$$

where D_{n-h} is a central division algebra over K with invariant $\frac{1}{n-h}$.

Write U_m for the kernel of the natural projection

$$GL_n(\mathcal{O}_K) \to GL_n(\mathcal{O}_K/\varpi_K^m).$$

Let $\{LT_{n-h,h,U_m}\}_{m\geq 1}$ denote the tower of Lubin–Tate spaces of level U_m, each of which is a $\widehat{K^{ur}}$-analytic space. The reader is referred to [9, Section 2.3.9] for a precise definition. (Also see [3, Section 3.2] for the crucial $h=0$ case.) We follow Dat ([3, Section 3.3]) to define a cohomology complex

$$R\Gamma_c(LT_{n-h,h}, \Lambda) \in D^b_\Lambda(GL_n(K) \times D^\times_{n-h,h} \times W_K)$$

where $\Lambda \in \{\mathbb{F}_l, \mathbb{Z}_l, \mathbb{Q}_l\}$. Even though Dat did this only for $h=0$, the same construction works for any h by using [2, Appendix B] if D in his paper is replaced with our $D_{n-h,h}$. When $h=0$, we will simply write LT_n for $LT_{n,0}$. As usual, define

$$H^i_c(LT_{n-h,h}, \Lambda) := \varinjlim_m H^i_c(LT_{n-h,h,U_m}, \Lambda), \quad i \geq 0,$$

which is equipped with a Λ-linear action by $GL_n(K) \times D^\times_{n-h,h} \times W_K$. It is explained in [3, Section 3.3] that $H^i_c(LT_n, \Lambda)$ computes the cohomology of the complex $R\Gamma_c(LT_n, \Lambda)$.

3.2. Dat's results in the basic case ($h = 0$).   In this subsection we write $R\Gamma_{c,\Lambda}$ (resp. $R\Gamma_{c,U_m,\Lambda}$) for $R\Gamma_c(LT_n, \Lambda)$ (resp. $R\Gamma_c(LT_{n,U_m}, \Lambda)$) and G for $GL_n(K)$ to simplify notation. Write $R\Gamma_{c,\Lambda,sc} := (R\Gamma_{c,\Lambda})_{sc}$ for the G-supercuspidal part (Section 2.2). Recall from [3, Lemma 3.5.9] that $R\Gamma_{c,U_m,\Lambda} \simeq (R\Gamma_{c,\Lambda})^{U_m}$ in $D^b_\Lambda(D^\times_n \times \mathcal{H}_\Lambda(G, U_m) \times W_K)$.

Proposition 3.1.   *The complex $R\Gamma_{c,\mathbb{Z}_l}$ as an object of the derived category $D^b_{\mathbb{Z}_l}(D^\times_n)$ is isomorphic to a bounded complex of projective $D^\times_n$-representations of locally finite type contained in degrees $[n-1, 2n-2]$.*

Remark 3.2.   The analogous fact holds if the roles of $D^\times_n$ and $GL_n(K)$ are interchanged, cf. [5, Propositions 2.1.3, 3.1.1]. As our proposition is not fully stated in Dat's article (although it is implicit there), we derive it from his results.

Proof.   This is [5, Proposition A.2.1].   □

Our major concern is the supercuspidal part of the cohomology. Projectivity is an essential ingredient in the proof of the main theorem.

Proposition 3.3.   *The following statements hold:*

(i) $R\Gamma_{c,\mathbb{Z}_l,sc} \simeq H^{n-1}_c(LT_n, \mathbb{Z}_l)_{sc}[1-n]$.

(ii) $H^{n-1}_c(LT_{n,U_m}, \mathbb{Z}_l)_{sc}$ *is projective in* $\text{Rep}_{\mathbb{Z}_l}(D^\times_n)$ *for each* $m \geq 1$.

(iii) $H^{n-1}_c(LT_n, \mathbb{Z}_l)_{sc}$ *is projective and of locally finite type in* $\text{Rep}_{\mathbb{Z}_l}(D^\times_n)$.

Proof. Part (i) is proved in [5, (3.1.1.1)] based on Mieda's result. Let us prove (ii) and (iii). Proposition 3.1 shows that $R\Gamma_{c,\mathbb{Z}_l,\mathrm{sc}}$ is locally of finite type as the latter is a direct factor of $R\Gamma_{c,\mathbb{Z}_l}$. For projectivity, it suffices to check the projectivity of statement (ii) as $H_c^{n-1}(\mathrm{LT}_{n,U_m}, \mathbb{Z}_l)_{\mathrm{sc}}$ is a direct factor of $H_c^{n-1}(\mathrm{LT}_n, \mathbb{Z}_l)$ as a $G \times D_n^\times$-module. (See [3, paragraph above Lemma 3.5.9].) Since the cohomology of LT_{n,U_m} is compactly induced from that of $\mathrm{LT}_{n,U_m}^{(0)}$, we may as well show that $H_c^{n-1}(\mathrm{LT}_{n,U_m}^{(0)}, \mathbb{Z}_l)_{\mathrm{sc}}$ is projective in $\mathrm{Rep}_{\mathbb{Z}_l}(D_n^0)$. The latter space is admissible (even finite dimensional), supercuspidal (automatic for D_n^0) and has bounded cohomological dimension thanks to Proposition 3.1 and part (i) of the current proposition. Now the projectivity can be proved exactly as in the step 3 of the proof of [5, Proposition 3.1.1]. □

The following deep result relates the cohomology of LT_n with the mod *l* local Langlands and Jacquet–Langlands correspondences.

Proposition 3.4. *We have*

$$H_c^{n-1}(\mathrm{LT}_n, \mathbb{F}_l)_{\mathrm{sc}} \simeq H_c^{n-1}(\mathrm{LT}_n, \mathbb{Z}_l)_{\mathrm{sc}} \otimes_{\mathbb{Z}_l} \mathbb{F}_l$$

equivariantly with $D_n^\times \times \mathrm{GL}_n(K) \times W_K$-*actions. For every* $m \geq 1$ *and* $\rho \in \mathrm{Irr}_{\overline{\mathbb{F}}_l}(D_n^\times)$, *there is an isomorphism of* $\overline{\mathbb{F}}_l[\mathrm{GL}_n(K) \times W_K]$-*module*

$$\mathrm{Hom}_{\mathbb{F}_l D_n^\times}(H_c^{n-1}(\mathrm{LT}_n, \mathbb{F}_l)_{\mathrm{sc}}, \rho)(1-n) \simeq \mathrm{JL}(\rho) \otimes \mathscr{L}(\mathrm{JL}(\rho))$$

if $\rho \in \mathrm{Irr}_{\overline{\mathbb{F}}_l}^{\mathrm{sc}}(D_n^\times)$ *and* $\mathrm{Hom}_{\mathbb{F}_l D_n^\times}(H_c^{n-1}(\mathrm{LT}_n, \mathbb{F}_l)_{\mathrm{sc}}, \rho) = 0$ *otherwise. Here* $(1-n)$ *denotes the Tate twist.*

Proof. The first isomorphism is justified by $H_c^n(\mathrm{LT}_n, \mathbb{Z}_l) = 0$ via a long exact sequence. The second is derived from a deep result of Dat ([5, Theorem 3.2.4]) for the cohomology with $\mathbb{Z}_l$-coefficient. Compare with Théorème 2 of that paper. □

3.3. The non-basic case ($1 \leq h \leq n - 1$). Let $\Lambda \in \{\mathbb{F}_l, \mathbb{Z}_l, \mathbb{Q}_l\}$. We show that supercuspidal representations do not occur in the cohomology of non-basic Lubin–Tate spaces with Λ-coefficients.

Proposition 3.5. *We have* $R\Gamma_c(\mathrm{LT}_{n-h,h}, \Lambda)_{sc} = 0$ *if* $h > 0$.

Proof. It suffices to show that $H_c^i(\mathrm{LT}_{n-h,h}, \Lambda)_{sc} = 0$ for all $h > 0$. Equivalently we may show that H^0 of the formal nearby cycle for the formal model of $\mathrm{LT}_{n-h,h}$ has no supercuspidal part in each degree $i \geq 0$. For this we freely borrow notation and facts from [12, Section 4.3]. What has to be checked is that $(\Psi_{n-h,h}^i)_{\mathrm{sc}} = 0$. This follows from [12, Proposition 4.3.14], which tells us that $\Psi_{n-h,h}^i$ is induced from the parabolic subgroup with Levi factor $\mathrm{GL}_{n-h}(K) \times \mathrm{GL}_h(K)$. (We use Λ-coefficient unlike Harris who uses $\mathbb{Q}_l$, so one could be inverting *l* in the isomorphism (4.3.12) of [12] if not careful. Note that the kernel of the mod ϖ map $\mathrm{GL}(n, \mathcal{O}/\varpi^m) \to \mathrm{GL}(n, \mathcal{O}/\varpi)$, to be denoted K_m, is a *p*-group. In constructing [12, (4.3.12)], we ignore $m = 1$ and for $m \geq 2$, replace $[\mathrm{GL}(n, \mathcal{O}_m) : P_{h,0}(\mathcal{O}_m)]$ with $[\mathrm{GL}(n, \mathcal{O}_m) \cap K_m : P_{h,0}(\mathcal{O}_m) \cap K_m]$. The latter is invertible in Λ and the argument of [12] goes through.) □

3.4. Etale case ($n = h$). In this case $\Sigma_{0,n} = \Sigma_n^{\text{ét}}$ is étale. Let us write $\text{LT}_n^{\text{ét}}$ for $\text{LT}_{0,n}$ and recall the following facts from [9, Examples 2.3.22, 4.4.8]: For each $m \geq 1$, the $\widehat{K^{\text{ur}}}$-analytic space $\text{LT}_{n,U_m}^{\text{ét}}$ is zero-dimensional and has points (all defined over $\widehat{K^{\text{ur}}}$) described as

$$\text{LT}_{n,U_m}^{\text{ét}} = \text{GL}_n(K)/U_m,$$

thus

$$(3.1) \qquad R\Gamma_c(\text{LT}_{n,U_m}^{\text{ét}}, \Lambda) = H_c^0(\text{LT}_{n,U_m}^{\text{ét}}, \Lambda) = C_c^\infty(\text{GL}_n(K)/U_m, \Lambda)$$

(concentrated in degree 0). The $\mathcal{H}_\Lambda(\text{GL}_n(K), U_m)$-module structure is induced by right translation by a U_m-double coset on $\text{GL}_n(K)/U_m$. The action of $D_{0,n}^\times = \text{GL}_n(K)$ is as follows: $(g \cdot \phi)(h) = \phi(g^{-1}h)$ for $g \in \text{GL}_n(K)$ and $\phi \in C_c^\infty(\text{GL}_n(K)/U_m, \Lambda)$. The Weil group W_K acts trivially.

Lemma 3.6. *The space $H_c^0(\text{LT}_{n,U_m}^{\text{ét}}, \Lambda)$ is projective as a $\Lambda D_{0,n}^\times$-module and for every $\rho \in \text{Rep}_\Lambda^{\text{adm}}(D_n^\times)$, there are isomorphisms as $\mathcal{H}_\Lambda(\text{GL}_n(K), U_m) \otimes \Lambda[W_K]$-modules*

$$(3.2) \qquad R\text{Hom}_{\Lambda D_{0,n}^\times}(R\Gamma_c(\text{LT}_{n,U_m}^{\text{ét}}, \Lambda), \rho) \simeq \text{Hom}_{\Lambda D_{0,n}^\times}(H_c^0(\text{LT}_{n,U_m}^{\text{ét}}, \Lambda), \rho) \simeq \rho^{U_m}$$

where ρ is given its natural $\text{GL}_n(K)$-action and trivial W_K-action on the right hand side.

Proof. Let us justify the second isomorphism of (3.2). Write $\mathbf{1}_{U_m}$ for the trivial representation of U_m. Since $C_c^\infty(\text{GL}_n(K)/U_m, \Lambda) = \text{c-ind}_{U_m}^{\text{GL}_n(K)}(\mathbf{1}_{U_m})$, Frobenius reciprocity tells us that canonically

$$\text{Hom}_{\Lambda D_{0,n}^\times}(C_c^\infty(\text{GL}_n(K)/U_m, \Lambda), \rho) \simeq \text{Hom}_{U_m}(\mathbf{1}_{U_m}, \rho) \simeq \rho^{U_m}$$

and the $\mathcal{H}_\Lambda(\text{GL}_n(K), U_m)$-module structure on the space $C_c^\infty(\text{GL}_n(K)/U_m, \Lambda)$ induces the usual $\mathcal{H}_\Lambda(\text{GL}_n(K), U_m)$-module structure on ρ^{U_m}. The projectivity of $H_c^0(\text{LT}_{n,U_m}^{\text{ét}}, \Lambda)$ is easy to see from Frobenius reciprocity as taking U_m-invariants is an exact functor (even an equivalence of categories, cf. [3, Section 3.5.8]). The first isomorphism of (3.2) follows from (3.1) and the projectivity. $\square$

4. Mod l cohomology of Shimura varieties

In this section we fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ for every prime p and $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$.

4.1. GU$(1, n-1)$-Shimura varieties. In order to introduce Shimura varieties of our interest, we consider the following PEL datum $(B, *, V, \langle \cdot, \cdot \rangle, h)$:

- B is a division algebra with center F, which is finite over $\mathbb{Q}$,
- $*$ is a positive involution on B of second kind,
- V is a finite free left B-module,
- $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{Q}$ is a nondegenerate alternating pairing such that

$$\langle bv_1, v_2 \rangle = \langle v_1, b^* v_2 \rangle$$

  for all $b \in B$ and $v_1, v_2 \in V$,

- $h : \mathbb{C} \to \text{End}_{B_\mathbb{R}}(V_\mathbb{R})$ is an $\mathbb{R}$-algebra morphism such that $(v_1, v_2) \mapsto \langle v_1, h(i)v_2 \rangle$ is symmetric and positive definite.

Then F is a CM field and $F^+ := F^{*=1}$ is the maximal totally real subfield. Set

$$n := [B : F]^{1/2} \cdot \text{rank}_B V.$$

Define a $\mathbb{Q}$-group G by the rule

$$G(R) = \{(g, \lambda) \in \text{End}_{B_R}(V_R) \times R^\times : \forall v_1, v_2 \in V, \langle g v_1, g v_2 \rangle = \lambda \langle v_1, v_2 \rangle\}$$

for any $\mathbb{Q}$-algebra R. Put

$$G^1 := \ker(\lambda : G \to \mathbb{G}_m).$$

Then G^1 is an inner form of $\text{Res}_{F^+/\mathbb{Q}} U^*(n)$ where $U^*(n)$ denotes a quasi-split unitary group in n variables relative to F/F^+. Let Φ_∞ be a CM-type for F such that Φ_∞ and its complex conjugate is a partition of the set of complex embeddings of F. The following will be in effect throughout this article.

Hypothesis 4.1. *The space* $V \otimes_\mathbb{Q} \mathbb{R} \simeq \prod_{v \in \Phi_\infty} V \otimes_{F,v} \mathbb{C}$ *has signature* $(1, n - 1)$ *at one* $v = \tau$ *and* $(0, n)$ *at all the other* v. *(Our notion of signature is as in* [13, Section I.6]*.)*[2]

The PEL data satisfying Hypothesis 4.1 were considered in [1], [13] and [25] for instance. (In [25], $B = F$ and G is quasi-split at all finite places but avoids $F^+ = \mathbb{Q}$. In the first two references $\text{rank}_B V = 1$ so that $\text{End}_B(V)$ is a division algebra over F.) Fix a prime p satisfying

Hypothesis 4.2. *Every place of* F^+ *above* p *splits in* F.

Then Φ_∞ induces the set $\mathcal{V}_p$ of p-adic places of F such that $\mathcal{V}_p$ and its complex conjugate partition the set of all p-adic places of F. In particular there is an isomorphism

$$(4.1) \qquad G_{\mathbb{Q}_p} \simeq \mathbb{G}_m \times \prod_{v \in \mathcal{V}_p} \text{Res}_{F_v/\mathbb{Q}_p} \text{GL}_n.$$

As explained in [15, Section 5] and [13, Section 3.1] (our notation is closest to that of [25, Section 5.2]), the above PEL datum gives rise, via a moduli problem for abelian schemes with additional structure, to the projective system of smooth quasi-projective varieties $\{\text{Sh}_U\}$ over F, where U runs over sufficiently small open compact subgroups of $G(\mathbb{A}^\infty)$. By Hypothesis 4.1 Sh_U has dimension $n - 1$ over F.

Denote by Sh_U^{can} the canonical model over F by Deligne and Shimura. It is known that the variety Sh_U is isomorphic to $\#\ker^1(\mathbb{Q}, G)$-copies of Sh_U^{can} ([15, Section 8]) and that $\#\ker^1(\mathbb{Q}, G) \in \{1, 2\}$ (see the proof of [25, Lemma 3.1]).

4.2. Main theorem. Let G' denote an inner form of G over $\mathbb{Q}$, unique up to isomorphism, such that

- $G' \times_\mathbb{Q} \mathbb{A}^{\infty,p} \simeq G \times_\mathbb{Q} \mathbb{A}^{\infty,p}$,
- $G'(\mathbb{R})$ is compact modulo center,
- $G'(\mathbb{Q}_p) \simeq \mathbb{Q}_p^\times \times D_{n,w}^\times \times \prod_{v \in \mathcal{V}_p \setminus \{w\}} \text{GL}_n(F_v)$ (cf. $J^{(0)}(\mathbb{Q}_p)$ of (4.7).)

The existence of G' is easily checked by the Galois cohomology computation as in [1, Section 2].

---

[2] We may say instead that $G_\mathbb{R}^1$ is isomorphic to $U(1, n - 1) \times U(0, n) \times \cdots \times U(0, n)$. Given such an isomorphism, one can always choose a CM-type Φ_∞ that works.

For any open compact subgroup $U^p \subset G(\mathbb{A}^{\infty,p})$ define for $i \geq 0$

$$H_c^i(\mathrm{Sh}_{U^p}, \overline{\mathbb{F}}_l) := \varinjlim_{U_p} H_c^i(\mathrm{Sh}_{U^p U_p}, \overline{\mathbb{F}}_l)$$

as

$$(4.2) \qquad\qquad U_p = U_{p,0} \times \prod_{v \in \mathcal{V}_p} U_{p,v}$$

runs over the set of pro-p congruence subgroups of $G(\mathbb{Q}_p) \simeq \mathrm{GL}_1(\mathbb{Q}_p) \times \prod_{v \in \mathcal{V}_p} \mathrm{GL}_n(F_v)$, cf. (4.1). Namely $U_{p,0}$ (resp. $U_{p,v}$) is a subgroup of $\mathrm{GL}_1(\mathbb{Z}_p)$ (resp. $\mathrm{GL}_n(\mathcal{O}_{F_v})$) congruent to 1 modulo a power of p (resp. a uniformizer of $\mathcal{O}_{F_v}$). Then $H_c^i(\mathrm{Sh}_{U^p}, \overline{\mathbb{F}}_l)$ is a module over $\mathcal{H}_{\overline{\mathbb{F}}_l}(G(\mathbb{A}^{\infty,p}), U^p) \otimes \overline{\mathbb{F}}_l[G(\mathbb{Q}_p) \times \mathrm{Gal}(\overline{F}/F)]$. We can take the $\mathrm{GL}_n(F_w)$-supercuspidal part of $H_c^{n-1}(\mathrm{Sh}_{U^p}, \overline{\mathbb{F}}_l)$ in the sense of (2.1) and denote it by $H_c^{n-1}(\mathrm{Sh}_{U^p}, \overline{\mathbb{F}}_l)_{w\text{-sc}}$. The latter space is still stable under the action of $\mathcal{H}_{\overline{\mathbb{F}}_l}(G(\mathbb{A}^{\infty,p}), U^p)$, $G(\mathbb{Q}_p)$ and $\mathrm{Gal}(\overline{F}/F)$. Similarly define $H_c^i(\mathrm{Sh}_{U^p}^{\mathrm{can}}, \overline{\mathbb{F}}_l)$ and $H_c^i(\mathrm{Sh}_{U^p}^{\mathrm{can}}, \overline{\mathbb{F}}_l)_{w\text{-sc}}$. Clearly (cf. Section 4.1)

$$(4.3) \qquad\qquad H_c^i(\mathrm{Sh}_{U^p}, \overline{\mathbb{F}}_l) = \#\ker^1(\mathbb{Q}, G) \cdot H_c^i(\mathrm{Sh}_{U^p}^{\mathrm{can}}, \overline{\mathbb{F}}_l)$$

equivariantly with the Hecke and Galois actions.

Put $\widetilde{D}_{n,w}^\times := \mathbb{Q}_p^\times \times D_{n,w}^\times$ and view it as a subgroup of $G'(\mathbb{Q}_p)$ in the obvious manner. Let

$$\widetilde{\rho}_w = \rho_0 \otimes \rho_w \in \mathrm{Irr}_{\overline{\mathbb{F}}_l}^{\mathrm{sc}}(\widetilde{D}_{n,w}^\times).$$

In what follows, denote the semisimplification of the space $C_c^\infty(G'(\mathbb{Q}) \backslash G'(\mathbb{A}^\infty)/U^p, \overline{\mathbb{F}}_l)$ as an $\mathcal{H}_{\overline{\mathbb{F}}_l}(G(\mathbb{A}^{\infty,p}), U^p) \times G'(\mathbb{Q}_p) \times W_{F_w}$-representation by $C_c^\infty(G'(\mathbb{Q}) \backslash G'(\mathbb{A}^\infty)/U^p, \overline{\mathbb{F}}_l)^{\mathrm{ss}}$.

Theorem 4.3. *If l is sufficiently large (in a way depending on U^p), then the subspace $H_c^{n-1}(\mathrm{Sh}_{U^p}^{\mathrm{can}}, \overline{\mathbb{F}}_l)_{w\text{-sc}}$ is isomorphic to*

$$\left(\bigoplus_{\widetilde{\rho}_w \in \mathrm{Irr}_{\overline{\mathbb{F}}_l}^{\mathrm{sc}}(\widetilde{D}_{n,w}^\times)} \mathrm{Hom}_{\widetilde{D}_{n,w}^\times}(\widetilde{\rho}_w, C_c^\infty(G'(\mathbb{Q}) \backslash G'(\mathbb{A}^\infty)/U^p, \overline{\mathbb{F}}_l)) \otimes \mathrm{JL}(\widetilde{\rho}_w) \otimes \mathscr{L}(\mathrm{JL}(\widetilde{\rho}_w)) \right)$$

as an $\mathcal{H}_{\overline{\mathbb{F}}_l}(G(\mathbb{A}^{\infty,p}), U^p) \times G(\mathbb{Q}_p) \times W_{F_w}$-module. In general, the same holds true in the Grothendieck group if Hom *is taken with $C_c^\infty(G'(\mathbb{Q}) \backslash G'(\mathbb{A}^\infty)/U^p, \overline{\mathbb{F}}_l)^{\mathrm{ss}}$ in the second argument. In all cases,*

$$(4.4) \qquad\qquad H_c^i(\mathrm{Sh}_{U^p}^{\mathrm{can}}, \overline{\mathbb{F}}_l)_{w\text{-sc}} = 0 \quad \text{if } i \neq n-1.$$

Remark 4.4. Precisely the condition on l is that $l \notin S(U^p)$ as in Lemma 4.6 below, where $S(U^p)$ is defined via any sequence of U_p as in Lemma 4.5.

Although the mod l global Langlands correspondence in general has not been seamlessly formulated yet to our knowledge, it would be appropriate to view the above result as a local-global compatibility for mod l Langlands correspondence, the analogue of its l-adic counterpart (e.g. [13, Theorem VII.1.9], [25, Theorem 6.4]). Moreover an interesting feature of Theorem 4.3 is that it realizes a mod l Jacquet–Langlands-type correspondence between G and G' in some sense. This is reminiscent of the relationship between elliptic modular forms and quaternionic modular forms as in [23] (there $G = \mathrm{GL}_2$ and G' is its inner form ramified precisely at p and ∞). One difference is that Serre considers coherent cohomology whereas we are concerned with étale cohomology.

4.3. A lemma on semisimple central action. The material of this section will be used in Section 4.6 to take care of central characters at the p-adic place.[3)] Let $Z(\widetilde{D}^\times_{n,w})$ be the center of $\widetilde{D}^\times_{n,w}$, isomorphic to $\mathbb{Q}^\times_p \times F^\times_w$. Let ϖ_w be a uniformizer of F_w so that there is a natural isomorphism

$$Z(\widetilde{D}^\times_{n,w}) \simeq (\mathbb{Z}^\times_p \times p^{\mathbb{Z}}) \times (\mathcal{O}^\times_{F_w} \times \varpi^{\mathbb{Z}}_w).$$

Let $(p, 1), (1, \varpi_w) \in Z(\widetilde{D}^\times_{n,w})$ act on $G'(\mathbb{A}^\infty)$ by (right) multiplication (viewing $\widetilde{D}^\times_{n,w}$ naturally as a subgroup of $G'(\mathbb{Q}_p)$, cf. the product decomposition of $G'(\mathbb{Q}_p)$ at the start of Section 4.2).

Lemma 4.5. *There are only finitely many primes dividing the cardinality of either a $(p, 1)$-orbit or a $(1, \varpi_w)$-orbit on the double quotient $G'(\mathbb{Q})\backslash G'(\mathbb{A}^\infty)/U^p U_p$ as U_p runs over a countable sequence of open compact normal subgroups of a maximal compact subgroup of $G'(\mathbb{Q}_p)$ converging to $\{1\}$ (where U^p is fixed).*

Proof. Since $G'(\mathbb{R})$ is compact modulo center, the double quotient is finite and $G'(\mathbb{Q})$ is discrete in $G'(\mathbb{A}^\infty)$, cf. [11, Propositions 1.4, 4.3]. In particular all orbits are finite.

Here is some reduction step. Fix a maximal compact subgroup U^0_p of $G'(\mathbb{Q}_p)$. Let $\{g_1, \ldots, g_r\}$ be a set of representatives for $G'(\mathbb{Q})\backslash G'(\mathbb{A}^\infty)/U^p U^0_p$. Choose a normal subgroup U_p of U^0_p (in the sequence) which is small enough such that U_p is pro-p and

$$g_i U_p g_i^{-1} \cap G'(\mathbb{Q}) = \{1\}$$

(in $G'(\mathbb{A}^\infty)$) for $1 \leq i \leq r$. The last condition implies that

$$(4.5) \qquad\qquad g U_p g^{-1} \cap G'(\mathbb{Q}) = \{1\}, \quad g \in G(\mathbb{A}^\infty).$$

(To see this, write $g = \gamma g_i u^p u_p$ for $\gamma \in G'(\mathbb{Q})$, $u^p \in U^p$, $u_p \in U^0_p$ and some $1 \leq i \leq r$, and examine the effect of conjugation by each of γ, g_i, u^p and u_p.) Now fix such a U_p. Clearly there are finitely many primes which divide the cardinality of a $(p, 1)$-orbit or a $(1, \varpi_w)$-orbit on $G'(\mathbb{Q})\backslash G'(\mathbb{A}^\infty)/U^p U_p$ (as there are finitely many orbits). For the lemma it suffices to show that there is no new prime divisor other than p as we replace U_p with any finite index normal subgroup $U'_p \subset U_p$.

The proof easily boils down to checking the following abstract statement: Let X be a finite set with a free action by a finite p-group H. Let α be an automorphism of X commuting with H. (Apply this when X is $G'(\mathbb{Q})\backslash G'(\mathbb{A}^\infty)/U^p U_p$, H is U_p/U'_p, and α is either $(p, 1)$ or $(1, \varpi_w)$. Then H acts freely by (4.5).) For each $x \in X$, write $\bar{x} \in X/H$ for x mod H and let $\langle \alpha \rangle \cdot \bar{x} \subset X/H$ (resp. $\langle \alpha \rangle \cdot x \subset X$) denote the orbit of $\bar{x}$ (resp. x) under the α-action. The assertion to show is that for each $x \in X$, $|\langle \alpha \rangle \cdot x|$ is $|\langle \alpha \rangle \cdot \bar{x}|$ times a power of p. This is elementary to check. (If $m, n \in \mathbb{Z}_{\geq 1}$ are minimal such that $\alpha^m \bar{x} = \bar{x}$ and $\alpha^n x = x$, then obviously $|\langle \alpha \rangle \cdot x| = n$, $|\langle \alpha \rangle \cdot \bar{x}| = m$, and $m \mid n$. Moreover $\alpha^m x = xh$ for some $h \in H$, and n/m is nothing but the order of h in H, which should be a power of p since H is a p-group.) $\qquad\square$

Let $S(U^p)$ be the set of primes l which are either non-banal for $\widetilde{D}^\times_{n,w}$ (cf. Section 2.5) or lie in the finite set of primes in the preceding lemma for a fixed choice of a sequence of U_p as in that lemma. Note that $S(U^p)$ is a finite set.

[3)] I thank the referee for pointing out a mistake in an earlier draft, which is fixed in this subsection.

Lemma 4.6. *If $l \notin S(U^p)$, then the center of the group $\widetilde{D}_{n,w}^\times$ acts semisimply on the space $C_c^\infty(G'(\mathbb{Q})\backslash G'(\mathbb{A}^\infty)/U^p, \overline{\mathbb{F}}_l)$.*

Proof. Recall that $Z(\widetilde{D}_{n,w}^\times) \simeq (\mathbb{Z}_p^\times \times p^{\mathbb{Z}}) \times (\mathcal{O}_{F_w}^\times \times \varpi_w^{\mathbb{Z}})$. The groups $\mathbb{Z}_p^\times$ and $\mathcal{O}_{F_w}^\times$ act on each vector through finite quotients, whose orders are prime to l as l is banal. Hence $\mathbb{Z}_p^\times$ and $\mathcal{O}_{F_w}^\times$ act semisimply. It remains to show that $(p, 1), (1, \varpi_w) \in Z(\widetilde{D}_{n,w}^\times)$ act semisimply on $C_c^\infty(G'(\mathbb{Q})\backslash G'(\mathbb{A}^\infty)/U^p U_p, \overline{\mathbb{F}}_l)$ for sufficiently small U_p. Partitioning the finite set $G'(\mathbb{Q})\backslash G'(\mathbb{A}^\infty)/U^p U_p$ into orbits of $(p, 1)$ (resp. $(1, \varpi_w)$), the proof amounts to checking that the action of $1 \in \mathbb{Z}/m\mathbb{Z}$ on $C(\mathbb{Z}/m\mathbb{Z}, \overline{\mathbb{F}}_l)$ by translation is semisimple if $l \nmid m$. This is straightforward. $\square$

4.4. Newton stratification.

One important ingredient of proof is a formula due to Harris and Taylor, which describes the cohomology of GU$(1, n-1)$-type Shimura varieties in terms of the cohomology of Igusa varieties and Lubin–Tate spaces. We will recall the Newton stratification from [13] where more details can be found. Note that we do not need properness for our discussion below although the Shimura varieties of Harris–Taylor are proper over the base.

An integral model $\mathrm{Sh}_{U^p, \vec{m}}$ over $\mathcal{O}_{F_w}$ with Drinfeld level structure at p, prescribed by an r-tuple of nonnegative integers $\vec{m} = (m_1, \ldots, m_r)$, is defined exactly as in [13, Section III.4] (cf. [25, Section 5.2]). Denote by $\overline{\mathrm{Sh}}_{U^p, \vec{m}}$ the special fiber and by $\mathcal{A}$ the universal abelian scheme over $\overline{\mathrm{Sh}}_{U^p, \vec{m}}$. Then the Barsotti–Tate group $\mathcal{A}[w^\infty]$ has dimension 1. In the setting of [25, Section 5.2], following Harris–Taylor, a (set-theoretic) Newton stratification

$$\overline{\mathrm{Sh}}_{U^p, \vec{m}} = \coprod_{h=0}^{n-1} \overline{\mathrm{Sh}}_{U^p, \vec{m}}^{(h)}$$

was defined such that $\overline{\mathrm{Sh}}_{U^p, \vec{m}}^{(h)}$ is the locus where $\mathcal{A}[w^\infty]$ has étale height h. In particular $\overline{\mathrm{Sh}}_{U^p, \vec{m}}^{(0)}$ is the supersingular stratum. The setting of Harris–Taylor is slightly different in that the B in the PEL datum is a central division algebra over F of degree n^2, but the same Newton stratification works as explained in [13, Section III.4]. Harris and Taylor showed the following:

  (i) $\overline{\mathrm{Sh}}_{U^p, \vec{m}}^{(h)}$ has dimension h for $0 \leq h \leq n-1$,

  (ii) $\overline{\mathrm{Sh}}_{U^p, \vec{m}}^{[h]} := \bigcup_{j=0}^h \overline{\mathrm{Sh}}_{U^p, \vec{m}}^{(j)}$ is closed in $\overline{\mathrm{Sh}}_{U^p, \vec{m}}$ for each $0 \leq h \leq n-1$.

In view of (ii) each $\overline{\mathrm{Sh}}_{U^p, \vec{m}}^{(h)}$ is locally closed in $\overline{\mathrm{Sh}}_{U^p, \vec{m}}$ and as such inherits a reduced subscheme structure. Let $\Lambda \in \{\overline{\mathbb{F}}_l, \overline{\mathbb{Z}}_l, \overline{\mathbb{Q}}_l\}$. (We could consider $\mathbb{F}_l$, $\mathbb{Z}_l$ and $\mathbb{Q}_l$ as well.) Define for each $i \geq 0$

$$H_c^i(\overline{\mathrm{Sh}}_{U^p}^{(h)}, R\Psi\Lambda) := \varinjlim_{\vec{m}} H_c^i(\overline{\mathrm{Sh}}_{U^p, \vec{m}}^{(h)} \times_{k(w)} \overline{k(w)}, R\Psi\Lambda),$$

which is equipped with an action of $\mathcal{H}_\Lambda(G(\mathbb{A}^{\infty,p}), U^p) \times G(\mathbb{Q}_p) \times W_{F_w}$, smooth and admissible for $G(\mathbb{Q}_p)$ and continuous for W_{F_w} as explained in [14, Lemma 4.1] and its proof, relying on the ideas of [20].

Lemma 4.7. *For each $1 \leq h \leq n-1$ the following is an exact sequence of Λ-modules equivariant for the action of $\mathcal{H}_\Lambda(G(\mathbb{A}^{\infty,p}), U^p) \times G(\mathbb{Q}_p) \times W_{F_w}$:*

$$\cdots \to H_c^i(\overline{\mathrm{Sh}}_{U^p}^{(h)}, R\Psi\Lambda) \to H_c^i(\overline{\mathrm{Sh}}_{U^p}^{[h]}, R\Psi\Lambda) \to H_c^i(\overline{\mathrm{Sh}}_{U^p}^{[h-1]}, R\Psi\Lambda)$$
$$\to H_c^{i+1}(\overline{\mathrm{Sh}}_{U^p}^{(h)}, R\Psi\Lambda) \to \cdots.$$

Proof.   As Λ-modules the exact sequence follows from taking cohomology of the exact triangle of [10, Section I.8.7.(3)] for

$$(4.6) \qquad \overline{\mathrm{Sh}}_{U^p}^{[h-1]} \hookrightarrow \overline{\mathrm{Sh}}_{U^p}^{[h]} \quad \text{and} \quad \overline{\mathrm{Sh}}_{U^p}^{(h)} \hookrightarrow \overline{\mathrm{Sh}}_{U^p}^{[h]}$$

and taking a direct limit over $\vec{m}$. The assertion on equivariance follows from the equivariance of (4.6) for the above action. The desired equivariance is standard except for the $G(\mathbb{Q}_p)$-action. Recall from the proof of [14, Lemma 4.1] that $G(\mathbb{Q}_p)$ is generated by $G^+(\mathbb{Q}_p)$ and p as a group in their notation. The action of p is trivial on cohomology in all cases, so it suffices to check the $G^+(\mathbb{Q}_p)$-equivariance. (In fact the geometric objects are acted on by not $G(\mathbb{Q}_p)$ but only $G^+(\mathbb{Q}_p)$.) This is a consequence of the fact that the $G^+(\mathbb{Q}_p)$-action disturbs an abelian scheme only within its isogeny class.   □

We are about to recall some basics of Igusa varieties. The reader is invited to refer to [19, 20] for more details. (A short summary for the case at hand is found in [25, Section 5.2].) Igusa varieties $\mathrm{Ig}_{U^p,m}^{(h)}$ are finite étale coverings of $\overline{\mathrm{Sh}}_{U^p,\vec{0}}^{(h)}$ defined by a moduli problem for Igusa level structure, which is roughly an isomorphism of $A[w^\infty]$ with a constant family of a fixed Barsotti–Tate group over $\overline{\mathbb{F}}_p$. The dimension of $\mathrm{Ig}_{U^p,m}^{(h)}$ is h in view of (i) above. The tower $\{\mathrm{Ig}_{U^p,m}^{(h)}\}_{m\geq 1}$ admits a commuting action of $U^p \backslash G(\mathbb{A}^{\infty,p})/U^p$ and furthermore a certain submonoid of $J^{(h)}(\mathbb{Q}_p)$, where

$$(4.7) \qquad J^{(h)}(\mathbb{Q}_p) \simeq \mathbb{Q}_p^\times \times D_{n-h,h,F_w}^\times \times \prod_{v \in \mathcal{V}_p \backslash \{w\}} \mathrm{GL}_n(F_v).$$

The action of the latter submonoid on

$$H_c^i(\mathrm{Ig}_{U^p}^{(h)}, \Lambda) := \varinjlim_m H_c^i(\mathrm{Ig}_{U^p,m}^{(h)}, \Lambda), \quad i \geq 0,$$

uniquely extends to a smooth and admissible action of the whole $J^{(h)}(\mathbb{Q}_p)$, cf. [20, Proposition 7]. When $h = 0$, the action of $J^{(0)}(\mathbb{Q}_p)$ is already defined on

$$\mathrm{Ig}_{U^p}^{(0)} := \varprojlim_m \mathrm{Ig}_{U^p,m}^{(0)}$$

(before passing to cohomology) and admits a concrete description. Let us choose an isomorphism (canonical up to $G(\mathbb{A}^{\infty,p}) \times J^{(0)}(\mathbb{Q}_p)$-conjugacy)

$$\iota_b : G'(\mathbb{A}^\infty) \simeq G(\mathbb{A}^{\infty,p}) \times J^{(0)}(\mathbb{Q}_p).$$

Lemma 4.8.   *As a $G(\mathbb{A}^{\infty,p}) \times J^{(0)}(\mathbb{Q}_p)$-set, we have a non-canonical isomorphism*

$$\varprojlim_{U^p} \mathrm{Ig}_{U^p}^{(0)}(\overline{\mathbb{F}}_p) = (G'(\mathbb{Q}) \backslash G'(\mathbb{A}^\infty))^{\# \ker^1(\mathbb{Q}, G)}$$

where the action on the latter is induced by the right multiplication of $G(\mathbb{A}^{\infty,p}) \times J^{(0)}(\mathbb{Q}_p)$ through ι_b^{-1}. For each U^p, $\mathrm{Ig}_{U^p}^{(0)}(\overline{\mathbb{F}}_p)$ is described as the set of U^p-cosets.

Proof.   This is the $h = 0$ case of [13, Lemma V.1.2] (cf. [24, Lemma 7.3]).   □

The following theorem, essentially due to Harris and Taylor, is a version by Mantovan. (When $h = 0$, a slightly weaker statement can be deduced from the Hochschild–Serre spectral sequence of Fargues, cf. [9, Corollary 4.5.21, Theorem 7.2.1].) Strictly speaking, Mantovan works with PEL data unramified at p, but her method also applies to Harris–Taylor's situation and ours where PEL data may be ramified at p, thanks to Hypotheses 4.1 and 4.2 (and the fact that the deformation of one-dimensional formal $\mathcal{O}_{F_w}$-modules is well understood).

Proposition 4.9. *There is a spectral sequence of* Λ-*modules equivariant for the action of* $\mathcal{H}_\Lambda(G(\mathbb{A}^{\infty,p}), U^p) \times G(\mathbb{Q}_p) \times W_{F_w})$:

$$(4.8) \quad E_2^{i,j} = \varinjlim_{U_p} \bigoplus_{\substack{s+t=j \\ s,t \geq 0}} \operatorname{Ext}^i_{\Lambda J^{(h)}(\mathbb{Q}_p)}(H_c^{2(n-1)-s}(\operatorname{LT}_{U_p}^{(h)}, \Lambda), H_c^t(\operatorname{Ig}_{U^p}^{(h)}, \Lambda))(1 - n)$$

$$\Rightarrow H_c^{i+j}(\overline{\operatorname{Sh}}_{U^p}^{(h)}, R\Psi\Lambda)$$

where U_p *runs over the subgroups of* $G(\mathbb{Q}_p)$ *as in* (4.2) *and* (1 − *n*) *denotes the Tate twist, and*

$$(4.9) \quad \operatorname{LT}_{U_p}^{(h)} \simeq \operatorname{LT}_{1,U_{p,0}} \times \operatorname{LT}_{n-h,h,U_{p,w}} \times \prod_{v \in \mathcal{V}_p \setminus \{w\}} \operatorname{LT}_{n,U_{p,v}}^{\text{ét}}.$$

Proof. Theorem 3.2 of [21] gives us a formula analogous to (4.8) but without Tate twist and with Ext^i and $H_c^{2(n-1)-s}(\operatorname{LT}_{U_p}^{(h)}, \Lambda)$ replaced by Tor^i and the degree s compact support cohomology of the special fiber of $\operatorname{LT}_{U_p}^{(h)}$, respectively. From here one obtains (4.8) by appealing to [19, Theorem 8.7]. The switching from H_c^s to $H^{2(n-1)-s}$ and the introduction of Tate twist are explained by the use of Poincaré duality in the proof of that theorem on [19, p. 323]. The isomorphism (4.9) is explained in [9, Sections 2.3.7.1, 2.3.21]. □

4.5. Passing between the generic and special fibers. As in the last subsection let $\Lambda \in \{\overline{\mathbb{F}}_l, \overline{\mathbb{Z}}_l, \overline{\mathbb{Q}}_l\}$. Under Hypothesis 4.1, the integral models for our Shimura varieties (with Drinfeld level structure) are proper over $\mathcal{O}_{F_w}$ provided that $[F^+ : \mathbb{Q}] > 1$, in which case G is anisotropic modulo center over $\mathbb{Q}$. This follows from [16, Theorem 5.3.3.1, Remark 5.3.3.2]. The proper case is nice in that the standard theory of nearby cycles provides an isomorphism of $\mathcal{H}_{\overline{\mathbb{F}}_l}(G(\mathbb{A}^{\infty,p}), U^p) \times G(\mathbb{Q}_p) \times W_{F_w}$-modules

$$(4.10) \quad H_c^i(\overline{\operatorname{Sh}}_{U^p}, R\Psi\Lambda) \simeq H_c^i(\operatorname{Sh}_{U^p}, \Lambda), \quad i \geq 0.$$

When $F^+ = \mathbb{Q}$, our Shimura varieties are not proper, but a recent result of Imai and Mieda ([14]) shows that the supercuspidal parts are still the same:

Proposition 4.10. *As* $\mathcal{H}_\Lambda(G(\mathbb{A}^{\infty,p}), U^p) \times G(\mathbb{Q}_p) \times W_{F_w}$-*modules,*

$$(4.11) \quad H_c^i(\overline{\operatorname{Sh}}_{U^p}, R\Psi\Lambda)_{w\text{-sc}} \simeq H_c^i(\operatorname{Sh}_{U^p}, \Lambda)_{w\text{-sc}}.$$

Proof. When $[F^+ : \mathbb{Q}] > 1$, this is an immediate consequence of (4.10). In general we appeal to the proof of [14, Theorem 4.2]. That theorem states that the kernel and cokernel of the canonical map $H_c^i(\overline{\operatorname{Sh}}_{U^p}, R\Psi\Lambda) \to H_c^i(\operatorname{Sh}_{U^p}, \Lambda)$ have no supercuspidal subquotients as $G(\mathbb{Q}_p)$-representations, which is slightly weaker than what we want. (Note that a non-supercuspidal representation of $G(\mathbb{Q}_p)$ may still have a supercuspidal $\operatorname{GL}_n(F_w)$-representation

as its w-component.) However (ii) and the proof of (iii) in [14, Proposition 4.12] tell us that the kernel and cokernel as $G(\mathbb{Q}_p)$-representations are induced from $P_j(\mathbb{Q}_p)$ for $\mathbb{Q}_p$-rational proper parabolic subgroups P_j. Each P_j arises as the stabilizer of $\mathcal{F}_{m,r,\overline{x}}$ with $x \in \text{Sh}^{\Delta}_{m,]r[}$ and $r \geq 1$ in [14, Lemma 3.12], where r is the rank of the torus part in the fiber of the semi-abelian scheme over the toroidal compactification. The fact that the nontrivial torus part admits an action of $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ implies that P_j is a *proper* parabolic subgroup in each v-factor in (4.1). In particular the above kernel and cokernel have no GL$_n(F_w)$-supercuspidal subquotients.   □

4.6. Proof of the main theorem. In view of (4.3) we may instead prove the analogous assertion for $H^i_c(\text{Sh}_{U^p}, \overline{\mathbb{F}}_l)_{w\text{-sc}}$ with #ker$^1(\mathbb{Q}, G)$-multiplicity. Thanks to Proposition 4.10 it is enough to prove the main theorem for $H^i_c(\overline{\text{Sh}}_{U^p}, R\Psi\overline{\mathbb{F}}_l)_{w\text{-sc}}$ in place of $H^i_c(\text{Sh}_{U^p}, \overline{\mathbb{F}}_l)_{w\text{-sc}}$. We begin by observing that the non-basic strata have no GL$_n(F_w)$-supercuspidal part.

Lemma 4.11. *We have* $H^i_c(\overline{\text{Sh}}^{(h)}_{U^p}, R\Psi\overline{\mathbb{F}}_l)_{w\text{-sc}} = 0$ *for all* $i \geq 0$ *if* $1 \leq h \leq n - 1$.

Proof. This results from Propositions 3.5 and 4.9.   □

By Lemma 4.11 and Lemma 4.7, for every $i \geq 0$

$$(4.12) \qquad H^i_c(\overline{\text{Sh}}^{[0]}_{U^p}, R\Psi\overline{\mathbb{F}}_l)_{w\text{-sc}} = H^i_c(\overline{\text{Sh}}^{[1]}_{U^p}, R\Psi\overline{\mathbb{F}}_l)_{w\text{-sc}}$$

$$= \cdots = H^i_c(\overline{\text{Sh}}^{[n-1]}_{U^p}, R\Psi\overline{\mathbb{F}}_l)_{w\text{-sc}}$$

as $\mathcal{H}_{\overline{\mathbb{F}}_l}(G(\mathbb{A}^{\infty,p}), U^p) \times G(\mathbb{Q}_p) \times W_{F_w}$-modules. Hence it suffices to prove the assertions of Theorem 4.3 above for the subspace $H^i_c(\overline{\text{Sh}}^{(0)}_{U^p}, R\Psi\Lambda)_{w\text{-sc}}$. To this end it is helpful to understand $H^{n-1}_c(\text{LT}^{(0)}_{U_p}, \overline{\mathbb{F}}_l)_{w\text{-sc}}$. We see from (4.9) that

$$(4.13) \quad R\Gamma_c(\text{LT}^{(0)}_{U_p}) \simeq R\Gamma_c(\text{LT}_{1,U_{p,0}}) \otimes R\Gamma_c(\text{LT}_{n,U_{p,w}}) \otimes \bigotimes_{v \in \mathcal{V}_p \setminus \{w\}} R\Gamma_c(\text{LT}^{\text{ét}}_{n,U_{p,v}})$$

where we suppressed $\overline{\mathbb{F}}_l$-coefficients for brevity. (We suppress in (4.14) as well.) Proposition 3.3 and Lemma 3.6 tell us that $R\Gamma_c(\text{LT}^{(0)}_{U_p})_{w\text{-sc}}$ is concentrated on degree $n - 1$ (up to isomorphism), that

$$(4.14) \quad H^{n-1}_c(\text{LT}^{(0)}_{U_p})_{w\text{-sc}} \simeq H^0_c(\text{LT}_{1,U_{p,0}}) \otimes H^{n-1}_c(\text{LT}_{n,U_{p,w}}) \otimes \bigotimes_{v \in \mathcal{V}_p \setminus \{w\}} H^0_c(\text{LT}^{\text{ét}}_{n,U_{p,v}})$$

and that (4.14) is a projective $\overline{\mathbb{F}}_l J^{(0)}(\mathbb{Q}_p)$-module.

Lemma 4.12. *As an* $\overline{\mathbb{F}}_l$*-vector space with* $\mathcal{H}_{\overline{\mathbb{F}}_l}(G(\mathbb{A}^{\infty,p}), U^p) \times G(\mathbb{Q}_p) \times W_{F_w}$*-action,*

$$H^i_c(\overline{\text{Sh}}^{(0)}_{U^p}, R\Psi\overline{\mathbb{F}}_l)_{w\text{-sc}} = 0 \quad \textit{if } i \neq n - 1$$

and

$$(4.15) \quad H^{n-1}_c(\overline{\text{Sh}}^{(0)}_{U^p}, R\Psi\overline{\mathbb{F}}_l)$$

$$= \varinjlim_{U_p} \text{Hom}_{\overline{\mathbb{F}}_l J^{(0)}(\mathbb{Q}_p)}(H^{n-1}_c(\text{LT}^{(0)}_{U_p}, \overline{\mathbb{F}}_l)_{w\text{-sc}}, H^0_c(\text{Ig}^{(0)}_{U^p}, \overline{\mathbb{F}}_l))(1 - n).$$

Proof. Take the supercuspidal part of (4.8) when $h = 0$. The discussion preceding the lemma and the fact that $\mathrm{Ig}_{U^p}^{(0)}$ is zero-dimensional imply that

$$E_2^{i,j} = 0 \quad \text{unless } s = n - 1 \text{ and } t = 0.$$

Moreover

$$E_2^{i,j} = 0 \quad \text{unless } i = 0$$

since $H_c^{n-1}(\mathrm{LT}_{U_p}^{(0)})_{w\text{-sc}}$ is projective (Proposition 3.3). The lemma is proved.   □

The above lemma verifies (4.4). It remains to prove the first assertion of Theorem 4.3. By Lemma 4.8, the right hand side of (4.15) is isomorphic to $\#\ker^1(\mathbb{Q}, G)$-copies of

$$\varinjlim_{U_p} \mathrm{Hom}_{\overline{\mathbb{F}}_l J^{(0)}(\mathbb{Q}_p)}(H_c^{n-1}(\mathrm{LT}_{U_p}^{(0)}, \overline{\mathbb{F}}_l)_{w\text{-sc}}, C_c^\infty(G'(\mathbb{Q})\backslash G'(\mathbb{A}^\infty)/U^p, \overline{\mathbb{F}}_l)).$$

Let us suppress $\overline{\mathbb{F}}_l$ in the notation from here. Set

$$G^w(\mathbb{Q}_p) := \prod_{v \in \mathcal{V}_p \backslash \{w\}} \mathrm{GL}_n(F_v)$$

and

$$U_p^w := \prod_{v \in \mathcal{V}_p \backslash \{w\}} U_v.$$

By Lemma 3.6 and (4.14) the last expression is isomorphic to

$$(4.16) \qquad \varinjlim_{U_p} \mathrm{Hom}_{\widetilde{D}_{n,w}^\times}(H_c^0(\mathrm{LT}_{1,U_{p,0}}) \otimes H_c^{n-1}(\mathrm{LT}_{n,U_{p,w}})_{w\text{-sc}},$$
$$C_c^\infty(G'(\mathbb{Q})\backslash G'(\mathbb{A}^\infty)/U^p U_p^w))(1 - n).$$

Note that $C_c^\infty(G'(\mathbb{Q})\backslash G'(\mathbb{A}^\infty)/U^p U_p^w)$ is admissible and smooth as a $\widetilde{D}_{n,w}^\times$-representation. We can write

$$C_c^\infty(G'(\mathbb{Q})\backslash G'(\mathbb{A}^\infty)/U^p U_p^w)$$
$$= C_c^\infty(G'(\mathbb{Q})\backslash G'(\mathbb{A}^\infty)/U^p U_p^w)_{w\text{-sc}} \oplus C_c^\infty(G'(\mathbb{Q})\backslash G'(\mathbb{A}^\infty)/U^p U_p^w)_{\text{non-}w\text{-sc}}$$

where every irreducible $\widetilde{D}_{n,w}^\times$-subquotient of the first (resp. second) direct summand on the right hand side belongs to (resp. lies outside) $\mathrm{Irr}_{\overline{\mathbb{F}}_l}^{\mathrm{sc}}(\widetilde{D}_{n,w}^\times)$. According to the last vanishing of Proposition 3.4, only the first direct summand contributes nontrivially to the Hom of (4.16). Applying Lemma 2.2, (4.16) may be written as below where $\widetilde{\rho}_w$ runs over $\mathrm{Irr}_{\overline{\mathbb{F}}_l}^{\mathrm{sc}}(\widetilde{D}_{n,w}^\times)$:

$$(4.17) \qquad \varinjlim_{U_p} \bigoplus_{\widetilde{\rho}_w} \mathrm{Hom}_{\widetilde{D}_{n,w}^\times}(H_c^0(\mathrm{LT}_{1,U_{p,0}}) \otimes H_c^{n-1}(\mathrm{LT}_{n,U_{p,w}})_{w\text{-sc}},$$
$$C_c^\infty(G'(\mathbb{Q})\backslash G'(\mathbb{A}^\infty)/U^p U_p^w)[\widetilde{\rho}_w])(1 - n).$$

For the moment let us assume that $l \notin S(U^p)$ as in Lemma 4.6, cf. Remark 4.4. So in particular l is a banal prime for $\widetilde{D}_{n,w}^\times$, and the center of the group $\widetilde{D}_{n,w}^\times$ acts semisimply on $C_c^\infty(G'(\mathbb{Q})\backslash G'(\mathbb{A}^\infty)/U^p U_p^w)$. Hence an extension of $\widetilde{\rho}_w$ by itself in that space has central character, and has to split in light of the projectivity and injectivity from Section 2.5. Since

$C_c^\infty(G'(\mathbb{Q})\backslash G'(\mathbb{A}^\infty)/U^p U_p^w)[\widetilde{\rho}_w]$ has finite length as a $\widetilde{D}_{n,w}^\times$-representation, it is a finite direct sum of $\widetilde{\rho}_w$ as a $\widetilde{D}_{n,w}^\times$-representation. We canonically have

$$(4.18) \qquad C_c^\infty(G'(\mathbb{Q})\backslash G'(\mathbb{A}^\infty)/U^p U_p^w)[\widetilde{\rho}_w]$$
$$= \mathrm{Hom}_{\widetilde{D}_{n,w}^\times}(\widetilde{\rho}_w, C_c^\infty(G'(\mathbb{Q})\backslash G'(\mathbb{A}^\infty)/U^p U_p^w)) \otimes \widetilde{\rho}_w$$

as an $\mathscr{H}(G'(\mathbb{A}^{\infty,p}), U^p) \times \mathscr{H}(G^w(\mathbb{Q}_p), U_p^w) \times \widetilde{D}_{n,w}^\times$-module. Hence in (4.17), the summand is

$$\mathrm{Hom}_{\mathbb{Q}_p^\times}(H_c^0(\mathrm{LT}_{1,U_{p,0}}), \rho_0) \otimes \mathrm{Hom}_{D_{n,w}^\times}(H_c^{n-1}(\mathrm{LT}_{n,U_{p,w}})_{w\text{-sc}}, \rho_w)(1-n)$$
$$\otimes \mathrm{Hom}_{\widetilde{D}_{n,w}^\times}(\widetilde{\rho}_w, C_c^\infty(G'(\mathbb{Q})\backslash G'(\mathbb{A}^\infty)/U^p))^{U_p^w}.$$

Therefore Proposition 3.4 finishes the proof.

In general when *l* is arbitrary (but unequal to *p*), (4.18) holds true in the Grothendieck group if the right hand side is replaced with

$$\left[\mathrm{Hom}_{\widetilde{D}_{n,w}^\times}(\widetilde{\rho}_w, C_c^\infty(G'(\mathbb{Q})\backslash G'(\mathbb{A}^\infty)/U^p U_p^w)^{\mathrm{ss}}\right][\widetilde{\rho}_w].$$

This replacement does not change the summand of (4.17) in the Grothendieck group since $H_c^0(\mathrm{LT}_{1,U_{p,0}}) \otimes H_c^{n-1}(\mathrm{LT}_n)_{w\text{-sc}}$ is a projective $\widetilde{D}_{n,w}^\times$-representation. We conclude again by Proposition 3.4.

Remark 4.13. In the proof we have taken care not to pass to the Grothendieck group of representations until (4.18). This makes it possible to improve Theorem 4.3 by identifying $H_c^{n-1}(\mathrm{Sh}_{U^p}^{\mathrm{can}}, \overline{\mathbb{F}}_l)_{w\text{-sc}}$ as an $\mathscr{H}_{\overline{\mathbb{F}}_l}(G(\mathbb{A}^{\infty,p}), U^p) \times G(\mathbb{Q}_p) \times W_{F_w}$-module precisely. One can do this by substituting the description of $H_c^0(\mathrm{LT}_{1,U_{p,0}})$ and $H_c^{n-1}(\mathrm{LT}_{n,U_w})$ in [5, Théorème 3.2.4 (iii)] into formula (4.17). The result involves projective generators of suitable mod *l* representation categories.

5. Scope of generalization

We conclude with comments on the possibility of generalizing our work.

Non-supercuspidal part. We had some advantage by restricting our attention to the $\mathrm{GL}_n(F_w)$-supercuspidal part. Locally the mod *l* cohomology of Lubin–Tate spaces is completely described in that case. Globally there was a significant simplification that the supercuspidal part only appears in the basic stratum (where $h = 0$). In the basic case Igusa varieties are zero dimensional and their mod *l* cohomology is easy to describe. In the non-supercuspidal case the problem is more difficult in both local and global aspects. In particular the author does not know how to compute the mod *l* cohomology of Igusa varieties without already knowing the mod *l* cohomology of Shimura varieties. A counting point formula of [24] was useful for studying the $\overline{\mathbb{Q}}_l$-cohomology but does not seem to help in the mod *l* context. (The case of $h = 1$ may be doable as Igusa varieties are curves, but this is not satisfactory.)

Non-constant coefficients. The method of this paper would apply equally well to non-constant coefficients which come from "*p*-small weights" in the sense of [17, Definition 2.29].

Basically one has to run the argument of this paper after replacing Proposition 4.9 with the nonconstant coefficient version of [21, Theorem 3.2] (which can be stated in the derived category as in [19]).

     Other Shimura varieties.   One could try to prove the analogue of Theorem 4.3 for other PEL-type Shimura varieties. To do this on the $G(\mathbb{Q}_p)$-supercuspidal part of the mod l cohomology, two ingredients would be needed in our approach. First it ought to be shown that the mod l cohomology of non-basic Rapoport–Zink spaces associated with the Newton strata in the special fibers have no supercuspidal representations in their mod l cohomology. Next one has to establish results which are similar to those of Section 3.2 on the mod l cohomology of the relevant basic Rapoport–Zink spaces.

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Sug Woo Shin, Department of Mathematics, Massachusetts Institute of Technology,
77 Massachusetts Avenue, Cambridge, MA 02139, USA;
and Korea Institute for Advanced Study, 85 Hoegiro, Dongdaemun-gu, Seoul 130-722, Republic of Korea
e-mail: swshin@math.mit.edu