Discrete Comput Geom (2012) 48:847–878
DOI 10.1007/s00454-012-9452-2

Ehrhart h^*-Vectors of Hypersimplices

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Received: 4 May 2011 / Revised: 4 August 2012 / Accepted: 10 August 2012 /
Published online: 29 August 2012

Abstract We consider the Ehrhart h^*-vector for the hypersimplex. It is well-known that the sum of the h_i^* is the normalized volume which equals the Eulerian numbers. The main result is a proof of a conjecture by R. Stanley which gives an interpretation of the h_i^* coefficients in terms of descents and exceedances. Our proof is geometric using a careful book-keeping of a shelling of a unimodular triangulation. We generalize this result to other closely related polytopes.

Keywords hypersimplex · h^*-Vector · Shellable triangulation

1 Introduction

Hypersimplices appear naturally in algebraic and geometric contexts. For example, they can be considered as moment polytopes for torus actions on Grassmannians or weight polytopes of the fundamental representations of the general linear groups GL_n. Fix two integers $0 < k \leq n$. The (k,n)-th *hypersimplex* is defined as follows

$$\overline{\Delta}_{k,n} = \left\{(x_1, \ldots, x_n) \mid 0 \leq x_1, \ldots, x_n \leq 1;\ x_1 + \cdots + x_n = k\right\},$$

or equivalently,

$$\Delta_{k,n} = \left\{(x_1, \ldots, x_{n-1}) \mid 0 \leq x_1, \ldots, x_{n-1} \leq 1;\ k-1 \leq x_1 + \cdots + x_{n-1} \leq k\right\}.$$

They can be considered as the slice of the hypercube $[0,1]^{n-1}$ located between the two hyperplanes $\sum x_i = k-1$ and $\sum x_i = k$.

For a permutation $w \in \mathfrak{S}_n$, we call $i \in [n-1]$ a *descent* of w, if $w(i) > w(i+1)$. We define $\mathrm{des}(w)$ to be the number of descents of w. We call $A_{k,n-1}$ the Eulerian

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number, which equals the number of permutations in $\mathfrak{S}_{n-1}$ with $\mathrm{des}(w) = k - 1$. The following result is well-known (see for example, [8, Exercise 4.59(b)]).

Theorem 1.1 (Laplace) *The normalized volume of $\Delta_{k,n}$ is the Eulerian number $A_{k,n-1}$.*

Let $S_{k,n}$ be the set of all points $(x_1, \dots, x_{n-1}) \in [0, 1]^{n-1}$ for which $x_i < x_{i+1}$ for exactly $k - 1$ values of i (including by convention $i = 0$). Foata asked whether there is some explicit measure-preserving map that sends $S_{k,n}$ to $\Delta_{k,n}$. Stanley [5] gave such a map, which gave a triangulation of the hypersimplex into $A_{k,n-1}$ unit simplices and provided a geometric proof of Theorem 1.1. Sturmfels [9] gave another triangulation of $\Delta_{k,n}$, which naturally appears in the context of Gröbner bases. Lam and Postnikov [4] compared these two triangulations together with the alcove triangulation and the circuit triangulation. They showed that these four triangulations are identical. We call a triangulation of a convex polytope *unimodular* if every simplex in the triangulation has normalized volume one. It is clear that the above triangulations of the hypersimplex are unimodular.

Let $\mathcal{P} \in \mathbb{Z}^N$ be any n-dimensional integral polytope (its vertices are given by integers). Then Ehrhart's theorem tells us that the function

$$i(\mathcal{P}, r) := \#\big(r\mathcal{P} \cap \mathbb{Z}^N\big)$$

is a polynomial in r, and

$$\sum_{r\geq 0} i(\mathcal{P}, r)t^r = \frac{h^*(t)}{(1-t)^{n+1}},$$

where $h^*(t)$ is a polynomial in t with degree $\leq n$. We call $h^*(t)$ the *h^*-polynomial* of $\mathcal{P}$, and the vector $(h_0^*, \dots, h_n^*)$, where h_i^* is the coefficient of t^i in $h^*(t)$, is called the *h^*-vector* of $\mathcal{P}$. We know that the sum $\sum_{i=0}^{i=n} h_i^*(\mathcal{P})$ equals the normalized volume of $\mathcal{P}$.

Katzman [3] proved the following formula for the h^*-vector of the hypersimplex $\Delta_{k,n}$. In particular, we see that $\sum_{i=0}^{i=n} h_i^*(\Delta_{k,n}) = A_{k,n-1}$. Write $\binom{n}{r}_\ell$ to denote the coefficient of t^r in $(1 + t + t^2 + \cdots + t^{\ell-1})^n$. Then the h^*-vector of $\Delta_{k,n}$ is $(h_0^*(\Delta_{k,n}), \dots, h_{n-1}^*(\Delta_{k,n}))$, where for $d = 0, \dots, n-1$

$$h_d^*(\Delta_{k,n}) = \sum_{i=0}^{k-1} (-1)^i \binom{n}{i} \binom{n}{(k-i)d - i}_{k-i}. \tag{1.1}$$

Moreover, since all the $h_i^*(\Delta_{k,n})$ are nonnegative integers [6] (this is not clear from (1.1)), it will be interesting to give a combinatorial interpretation of the $h_i^*(\Delta_{k,n})$.

The *half-open hypersimplex* $\Delta'_{k,n}$ is defined as follows. If $k > 1$,

$$\Delta'_{k,n} = \big\{(x_1, \dots, x_{n-1}) \mid 0 \leq x_1, \dots, x_{n-1} \leq 1;\ k-1 < x_1 + \cdots + x_{n-1} \leq k\big\},$$

and

$$\Delta'_{1,n} = \Delta_{1,n}.$$

We call $\Delta'_{k,n}$ "half-open" because it is basically the normal hypersimplex with the "lower" facet removed. From the definitions, it is clear that the volume formula and triangulations of the usual hypersimplex $\Delta_{k,n}$ also work for the half-open hypersimplex $\Delta'_{k,n}$, and it is nice that for fixed n, the half-open hypersimplices $\Delta'_{k,n}$, for $k = 1, \dots, n-1$, form a disjoint union of the hypercube $[0, 1]^{n-1}$. From the following formula for the h^*-polynomial of the half-open hypersimplices, we can compute the h^*-polynomial of the usual hypersimplices inductively. Also, we can compute its Ehrhart polynomial.

For a permutation w, we call i an *exceedance* of w if $w(i) > i$ (a *reversed exceedance* if $w(i) < i$). We denote by $\mathrm{exc}(w)$ the number of exceedances of w. The main theorems of the paper are the following.

Theorem 1.2 *The h^*-polynomial of the half-open hypersimplex $\Delta'_{k,n}$ is given by,*

$$\sum_{\substack{w \in \mathfrak{S}_{n-1} \\ \mathrm{exc}(w)=k-1}} t^{\mathrm{des}(w)}.$$

We prove this theorem first by a generating function method (in Sect. 2) and second by a geometric method, i.e., giving a shellable triangulation of the hypersimplex (in Sects. 3, 4 and 5).

We can define a different shelling order on the triangulation of $\Delta'_{k,n}$, and get another expression of its h^*-polynomial using descents and a new permutation statistic called *cover* (see its definition in Lemma 6.5).

Theorem 1.3 *The h^*-polynomial of $\Delta'_{k,n}$ is*

$$\sum_{\substack{w \in \mathfrak{S}_{n-1} \\ \mathrm{des}(w)=k-1}} t^{\mathrm{cover}(w)}.$$

Combine Theorem 1.3 with Theorem 1.2, we have the equal distribution of $(\mathrm{exc}, \mathrm{des})$ and $(\mathrm{des}, \mathrm{cover})$:

Corollary 1.4

$$\sum_{w \in \mathfrak{S}_n} t^{\mathrm{des}(w)} x^{\mathrm{cover}(w)} = \sum_{w \in \mathfrak{S}_n} t^{\mathrm{exc}(w)} x^{\mathrm{des}(w)}.$$

Finally, we study the generalized hypersimplex $\Delta_{k,\alpha}$ (Sect. 7). This polytope is related to algebras of Veronese type. For example, it is known [1] that every algebra of Veronese type coincides with the Ehrhart ring of a polytope $\Delta_{k,\alpha}$. We can extend this second shelling to the generalized hypersimplex $\Delta'_{k,\alpha}$ (defined in (7.1)), and express its h^*-polynomial in terms of a colored version of descents and covers (see Theorem 7.3).

2 Proof of Theorem 1.2 by Generating Functions

Here is a proof of this theorem using generating functions.

Proof Suppose we can show that

$$\sum_{r\geq 0}\sum_{k\geq 0}\sum_{n\geq 0} i\big(\Delta'_{k+1,n+1},r\big)u^n s^k t^r = \sum_{n\geq 0}\sum_{\sigma\in\mathfrak{S}_n} t^{\operatorname{des}(\sigma)} s^{\operatorname{exc}(\sigma)} \frac{u^n}{(1-t)^{n+1}}. \tag{2.1}$$

By considering the coefficient of $u^n s^k$ in (2.1), we have

$$\sum_{r\geq 0} i\big(\Delta'_{k+1,n+1},r\big)t^r = (1-t)^{-(n+1)}\Bigg(\sum_{\substack{w\in\mathfrak{S}_n\\ \operatorname{exc}(w)=k}} t^{\operatorname{des}(w)}\Bigg),$$

which implies Theorem 1.2. By the following equation due to Foata and Han [2, Eq. (1.15)]:

$$\sum_{n\geq 0}\sum_{\sigma\in\mathfrak{S}_n} t^{\operatorname{des}(\sigma)} s^{\operatorname{exc}(\sigma)} \frac{u^n}{(1-t)^{n+1}} = \sum_{r\geq 0} t^r \frac{1-s}{(1-u)^{r+1}(1-us)^{-r}-s(1-u)},$$

we only need to show that

$$\sum_{k\geq 0}\sum_{n\geq 0} i\big(\Delta'_{k+1,n+1},r\big)u^n s^k = \frac{1-s}{(1-u)^{r+1}(1-us)^{-r}-s(1-u)}.$$

By the definition of the half-open hypersimplex, we have, for any $r\in\mathbb{Z}_{\geq 0}$,

$$\begin{aligned} r\Delta'_{k+1,n+1} &\\ &= \big\{(x_1,\dots,x_n) \mid 0\leq x_1,\dots,x_n\leq r, rk+1\leq x_1+\cdots+x_n\leq (k+1)r\big\}, \end{aligned}$$

if $k>0$, and for $k=0$,

$$r\Delta'_{1,n+1} = \big\{(x_1,\dots,x_n) \mid 0\leq x_1,\dots,x_n\leq r, 0\leq x_1+\cdots+x_n\leq r\big\}.$$

So

$$i\big(\Delta'_{k+1,n+1},r\big) = \big(\big[x^{kr+1}\big]+\cdots+\big[x^{(k+1)r}\big]\big)\bigg(\frac{1-x^{r+1}}{1-x}\bigg)^n, \tag{2.2}$$

if $k>0$, and when $k=0$, we have

$$i\big(\Delta'_{1,n+1},r\big) = \big(\big[x^0\big]+[x]+\cdots+\big[x^r\big]\big)\bigg(\frac{1-x^{r+1}}{1-x}\bigg)^n. \tag{2.3}$$

Notice that the case of $k=0$ is different from $k>0$ and $i(\Delta'_{1,n+1},r)$ is obtained by evaluating $k=0$ in (2.2) plus an extra term $[x^0](\frac{1-x^{r+1}}{1-x})^n$. Since the coefficient of x^k

of a function $f(x)$ equals the constant term of $\frac{f(x)}{x^k}$, we have

$$\begin{aligned}
&\big(\big[x^{kr+1}\big]+\cdots+\big[x^{(k+1)r}\big]\big)\left(\frac{1-x^{r+1}}{1-x}\right)^n\\
&\quad=\big[x^0\big]\left(\frac{1-x^{r+1}}{1-x}\right)^n\big(x^{-kr-1}+\cdots+x^{-(k+1)r}\big)\\
&\quad=\big[x^{kr}\big]\left(\frac{1-x^{r+1}}{1-x}\right)^n\big(x^{-kr-1}+\cdots+x^{-(k+1)r}\big)x^{kr}\\
&\quad=\big[x^{kr}\big]\frac{(1-x^r)(1-x^{r+1})^n}{(1-x)^{n+1}x^r}.
\end{aligned}$$

So we have, for $k>0$,

$$\begin{aligned}
\sum_{n\ge 0} i\big(\Delta'_{k+1,n+1},r\big)u^n &= \sum_{n\ge 0}\big[x^{kr}\big]\frac{(1-x^r)(1-x^{r+1})^n}{(1-x)^{n+1}x^r}u^n\\
&=\big[x^{kr}\big]\frac{(1-x^r)}{(1-x)x^r}\sum_{n\ge 0}\left(\frac{(1-x^{r+1})u}{1-x}\right)^n\\
&=\big[x^{kr}\big]\frac{x^r-1}{x^r(u-ux^{r+1}-1+x)}.
\end{aligned}$$

For $k=0$, based on the difference between (2.2) and (2.3) observed above, we have

$$\begin{aligned}
\sum_{n\ge 0} i\big(\Delta'_{1,n+1},r\big)u^n &= \sum_{n\ge 0}\big[x^0\big]\frac{(1-x^r)(1-x^{r+1})^n}{(1-x)^{n+1}x^r}u^n\\
&\quad+\sum_{n\ge 0}\big[x^0\big]\left(\frac{1-x^{r+1}}{1-x}\right)^n u^n\\
&=\left(\big[x^0\big]\frac{x^r-1}{x^r(u-ux^{r+1}-1+x)}\right)+\frac{1}{1-u}.
\end{aligned}$$

So

$$\sum_{k\ge 0}\sum_{n\ge 0} i\big(\Delta'_{k+1,n+1},r\big)u^n s^k=\left(\sum_{k\ge 0}\big[x^{kr}\big]\frac{x^r-1}{x^r(u-ux^{r+1}-1+x)}s^k\right)+\frac{1}{1-u}.$$

Let $y=x^r$. We have

$$\sum_{k\ge 0}\sum_{n\ge 0} i\big(\Delta'_{k+1,n+1},r\big)u^n s^k=\sum_{k\ge 0}\big[x^{kr}\big]\frac{y-1}{y(u-uxy-1+x)}s^k+\frac{1}{1-u}.$$

Expand $\frac{y-1}{y(u-uxy-1+x)}$ in powers of x, we have

$$\begin{aligned}\frac{y-1}{y(u-uxy-1+x)}&=\frac{y-1}{y}\cdot\frac{1}{u-1-(uxy-x)}\\&=\frac{y-1}{y(u-1)}\cdot\frac{1}{1-\frac{x(uy-1)}{u-1}}\\&=\frac{1-y}{y(1-u)}\sum_{i\geq 0}\left(\frac{(1-uy)x}{1-u}\right)^i.\end{aligned}$$

Since we only want the coefficient of x^i such that r divides i, we get

$$\begin{aligned}\frac{1-y}{y(1-u)}\sum_{j\geq 0}\left(\frac{(1-uy)x}{1-u}\right)^{rj}&=\frac{1-y}{y(1-u)}\cdot\frac{1}{1-\frac{(1-uy)^r x^r}{(1-u)^r}}\\&=\frac{1-y}{y(1-u)}\cdot\frac{(1-u)^r}{(1-u)^r-(1-uy)^r x^r}\\&=\frac{(1-u)^{r-1}(1-y)}{y(1-u)^r-y^2(1-yu)^r}.\end{aligned}$$

So

$$\sum_{k\geq 0}\sum_{n\geq 0} i\big(\Delta'_{k+1,n+1},r\big)u^n s^k=\left(\sum_{k\geq 0}s^k\big[y^k\big]\frac{(1-u)^{r-1}(1-y)}{y(1-u)^r-y^2(1-yu)^r}\right)+\frac{1}{1-u}.$$

To remove all negative powers of y, we do the following expansion:

$$\begin{aligned}\frac{(1-u)^{r-1}(1-y)}{y(1-u)^r-y^2(1-yu)^r}&=\frac{1-y}{(1-u)y}\cdot\frac{1}{1-\frac{y(1-yu)^r}{(1-u)^r}}\\&=\sum_{i\geq 0}\left(\frac{y^{i-1}(1-uy)^{ri}}{(1-u)^{ri+1}}-\frac{y^i(1-uy)^{ri}}{(1-u)^{ri+1}}\right)\\&=\frac{1}{1-u}y^{-1}+\text{nonnegative powers of } y.\end{aligned}$$

Notice that $\sum_{k\geq 0}s^k[y^k]\frac{(1-u)^{r-1}(1-y)}{y(1-u)^r-y^2(1-yu)^r}$ is obtained by taking the sum of nonnegative powers of y in $\frac{(1-u)^{r-1}(1-y)}{y(1-u)^r-y^2(1-yu)^r}$ and replacing y by s. So

$$\sum_{k\geq 0}s^k\big[y^k\big]\frac{(1-u)^{r-1}(1-y)}{y(1-u)^r-y^2(1-yu)^r}=\frac{(1-u)^{r-1}(1-s)}{s(1-u)^r-s^2(1-su)^r}-\frac{1}{s(1-u)}.$$

Therefore,

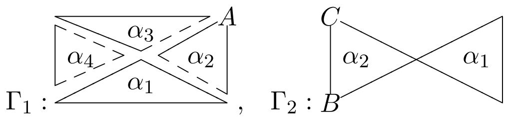
$$\sum_{k\geq 0}\sum_{n\geq 0} i\big(\Delta'_{k+1,n+1}, r\big)u^n s^k = \frac{(1-u)^{r-1}(1-s)}{s(1-u)^r - s^2(1-su)^r} - \frac{1}{s(1-u)} + \frac{1}{1-u}$$

$$= \frac{1-s}{(1-u)^{r+1}(1-us)^{-r} - s(1-u)}. \qquad \square$$

3 Background

3.1 Shellable Triangulation and the h^*-Polynomial

Let Γ be a triangulation of an n-dimensional polytope $\mathcal{P}$, and let $\alpha_1, \ldots, \alpha_s$ be an ordering of the simplices (maximal faces) of Γ. We call $(\alpha_1, \ldots, \alpha_s)$ a *shelling* of Γ [6], if for each $2 \leq i \leq s$, $\alpha_i \cap (\alpha_1 \cup \cdots \cup \alpha_{i-1})$ is a union of facets ($(n-1)$-dimensional faces) of α_i. For example (ignore the letters A, B, and C for now) Γ_1 is a shelling, while any order starting with Γ_2 cannot be a shelling:



An equivalent condition (see e.g., [7]) for a shelling is that every simplex has a *unique minimal non-face*, where by a "non-face", we mean a face that has not appeared in previous simplices. For example, for $\alpha_2 \in \Gamma_1$, the vertex A is its unique minimal non-face, while for $\alpha_2 \in \Gamma_2$, both B and C are minimal and have not appeared before α_2. We call a triangulation with a shelling a *shellable triangulation*. Given a shellable triangulation Γ and a simplex $\alpha \in \Gamma$, define the *shelling number* of α (denoted by $\#(\alpha)$) to be the number of facets shared by α and some simplex preceding α in the shelling order. For example, in Γ_1, we have

$$\#(\alpha_1) = 0, \quad \#(\alpha_2) = 1, \quad \#(\alpha_3) = 1, \quad \#(\alpha_4) = 2.$$

The benefit of having a shelling order for Theorem 1.2 comes from the following result.

Theorem 3.1 ([6] Shelling and Ehrhart Polynomial) *Let Γ be a unimodular shellable triangulation of an n-dimensional polytope $\mathcal{P}$. Then*

$$\sum_{r\geq 0} i(\mathcal{P}, r)t^r = \left(\sum_{\alpha\in\Gamma} t^{\#(\alpha)}\right)(1-t)^{-(n+1)}.$$

To be self-contained, we include a short proof here.

Proof Given a shellable triangulation, we get a partition of $\mathcal{P}$: for any simplex α, let $\alpha' \subset \alpha$ be obtained from α by removing the facets that α shares with the simplices preceding it in the shelling order. The fact that Γ is shellable will guarantee that this is a well-defined partition, i.e., there is no overlap and no missing area. So we can sum over all the parts to compute $i(\mathcal{P}, r)$ (the number of integer points of $r\mathcal{P}$). If $\mathcal{F}$ is a d-dimensional simplex, then

$$\sum_{r\geq 0} i(\mathcal{F}, r)t^r = \frac{1}{(1-t)^{d+1}}.$$

Since the triangulation is unimodular, α is an n-dimensional simplex. Let $k := \#(\alpha)$. Since α' is obtained from α by removing k simplices of dimension $n-1$ from α, the inclusion-exclusion formula implies that

$$\sum_{r\geq 0} i\big(\alpha', r\big)t^r = (1-t)^{-(n+1)}\left(\sum_{i=0}^{k}(-1)^i\binom{k}{i}(1-t)^i\right) = \frac{t^{\#(\alpha)}}{(1-t)^{n+1}}. \qquad \square$$

For example, Γ_1 in the previous example gives us a partition as shown above, and we have

$$\sum_{r\geq 0} i\big(\alpha_1', r\big)t^r = \frac{1}{(1-t)^3},$$

$$\sum_{r\geq 0} i\big(\alpha_2', r\big)t^r = \frac{1}{(1-t)^3} - \frac{1}{(1-t)^2} = \frac{t}{(1-t)^3},$$

and

$$\sum_{r\geq 0} i\big(\alpha_4', r\big)t^r = \frac{1}{(1-t)^3} - 2\frac{1}{(1-t)^2} + \frac{1}{(1-t)} = \frac{t^2}{(1-t)^3}.$$

3.2 Exceedances and Descents

Let $w \in \mathfrak{S}_n$. Define its *standard representation of cycle notation* to be a cycle notation of w such that the first element in each cycle is its largest element and the cycles are ordered with their largest elements increasing. We define the *cycle type* of w to be the composition of n: $\mathrm{C}(w) = (c_1, \dots, c_k)$ where c_i is the length of the ith cycle in its standard representation. The *Foata map* $F\colon w \to \hat{w}$ maps w to $\hat{w}$ obtained from w by removing parentheses from the standard representation of w. For example, consider a permutation $w\colon [5] \to [5]$ given by $w(1) = 5$, $w(2) = 1$, $w(3) = 4$, $w(4) = 3$ and $w(5) = 2$ or in one line notation $w = 51432$. Its standard representation of cycle notation is $(43)(521)$, so $\hat{w} = 43521$. The inverse Foata map $F^{-1}\colon \hat{w} \to w$ allows us to go back from $\hat{w}$ to w as follows: first insert a left parenthesis before every left-to-right maximum and then close each cycle by inserting a right parenthesis accordingly. In the example, the left-to-right maximums of $\hat{w} = 43521$ are 4 and 5, so we get back $(43)(521)$. Based on the Foata map, we have the following result for the equal distribution of exceedances and descents.

Theorem 3.2 (Exceedances and Descents) *The number of permutations in $\mathfrak{S}_n$ with k exceedances equals the number of permutations in $\mathfrak{S}_n$ with k descents.*

Proof First notice that we can change a permutation with k exceedances u to a permutation w with k reverse exceedances and vice versa by applying a *reverse map*: first reverse the letters by changing $u(i)$ to $n+1-u(i)$, then reverse the positions by defining $n+1-u(i)$ to be $w(n+1-i)$. This way, i is an exceedance of u if and only if $n+1-i$ is a reverse exceedance of w. Then the hard part is the connection between descents and reverse exceedances, which will involve the Foata map.

Let $\hat{w}$ be a permutation with k descents $\{(\hat{w}(i_1), \hat{w}(i_1+1)), \dots, (\hat{w}(i_k), \hat{w}(i_k+1))\}$ with $\hat{w}(i_s) > \hat{w}(i_s+1)$ for $s = 1, \dots, k$. We want to find its preimage w in the above map. After inserting parentheses in $\hat{w}$, each pair $(\hat{w}(i_s), \hat{w}(i_s+1))$ lies in the same cycle. So in w, we have $w(\hat{w}(i_s)) = \hat{w}(i_s+1) < \hat{w}(i_s)$, therefore, $\hat{w}(i_s)$ is a reverse exceedance of w. We also find that each reverse exceedance of w corresponds to a descent in $\hat{w}$ by the definition of the Foata map. This finishes the proof. □

For example, to change a permutation with three exceedances to a permutation with three descents, first

$$\dot{4}\dot{3}2\dot{5}1 \xrightarrow{(6-)} \dot{2}\dot{3}4\dot{1}5 \xrightarrow[\text{position}]{\text{reverse}} 5\dot{1}4\dot{3}\dot{2},$$

changes an exceedance in position i to a reverse exceedance in position $6-i$, and then

$$5\dot{1}4\dot{3}\dot{2} \xrightarrow[\text{of cycle structure}]{\text{standard representation}} (43)(521) \xrightarrow{\text{remove parentheses}} \underline{43}\,\underline{521},$$

changes a reverse exceedance in position i to a descent with the first letter i. The above two maps are both reversible.

3.3 Triangulation of the Hypersimplex

We start form a unimodular triangulation $\{t_w \mid w \in \mathfrak{S}_n\}$ of the hypercube, where

$$t_w = \left\{(y_1, \dots, y_n) \in [0,1]^n \mid 0 \le y_{w_1} \le y_{w_n} \le \cdots \le y_{w_n}\right\}.$$

It is easy to see that t_w has the following $n+1$ vertices: $v_0 = (0, \dots, 0)$, and $v_i = (y_1, \dots, y_n)$ given by $y_{w_1} = \cdots = y_{w_{n-i}} = 0$ and $y_{w_{n-i+1}} = \cdots = y_{w_n} = 1$. It is clear that $v_{i+1} = v_i + e_{w_{n-i}}$. Now define the map φ [4, 5] that maps t_w to s_w, sending $(y_1, \dots, y_n)$ to $(x_1, \dots, x_n)$, where

$$x_i = \begin{cases} y_i - y_{i-1}, & \text{if } (w^{-1})_i > (w^{-1})_{i-1}, \\ 1 + y_i - y_{i-1}, & \text{if } (w^{-1})_i < (w^{-1})_{i-1}, \end{cases} \tag{3.1}$$

where we set $y_0 = 0$. For each point $(x_1, \dots, x_n) \in s_w$, set $x_{n+1} = k+1-(x_1 + \cdots + x_n)$. Since v_{i+1} and v_i only differ in $y_{w_{n-i}}$, by (3.1), $\varphi(v_i)$ and $\varphi(v_{i+1})$ only differ in $x_{w_{n-i}}$ and $x_{w_{n-i}+1}$. More explicitly, we have

Lemma 3.3 *Denote w_{n-i} by r. For $\varphi(v_i)$, we have $x_r x_{r+1} = 01$ and for $\varphi(v_{i+1})$, we have $x_r x_{r+1} = 10$. In other words, from $\varphi(v_i)$ to $\varphi(v_{i+1})$, we move a 1 from the $(r+1)$th coordinate forward by one coordinate.*

Proof First, we want to show that for $\varphi(v_i)$, we have $x_r = 0$ and $x_{r+1} = 1$. We need to look at the segment $y_{r-1} y_r y_{r+1}$, of v_i. We know that $y_r = 0$, so there are four cases for $y_{r-1} y_r y_{r+1}$: 000, 001, 100, 101. If $y_{r-1} y_r y_{r+1} = 000$ for v_i, then $y_{r-1} y_r y_{r+1} = 010$ for v_{i+1}. Therefore, $w^{-1}_{r-1} < w^{-1}_r > w^{-1}_{r+1}$. Then by (3.1), we have $x_r x_{r+1} = 01$. Similarly, we can check in the other three cases that $x_r x_{r+1} = 01$ for $\varphi(v_i)$.

Similarly, we can check the four cases for $y_{r-1} y_r y_{r+1}$: 010, 011, 110, 111 in $\varphi(v_{i+1})$ and get $x_r x_{r+1} = 10$ in all cases. □

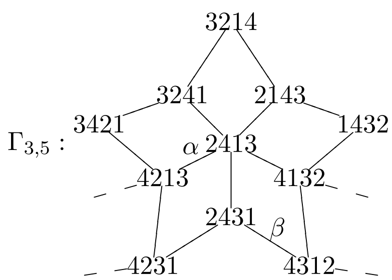
Let $\mathrm{des}(w^{-1}) = k$. It follows from Lemma 3.3 that the sum of the coordinates $\sum_{i=1}^n x_i$ for each vertex $\varphi(v_i)$ of s_w is either k or $k+1$. So we have the triangulation [5] of the hypersimplex $\Delta_{k+1,n+1}$: $\Gamma_{k+1,n+1} = \{s_w \mid w \in \mathfrak{S}_n, \mathrm{des}(w^{-1}) = k\}$.

Now we consider a graph $G_{k+1,n+1}$ on the set of simplices in the triangulation of $\Delta_{k+1,n+1}$. There is an edge between two simplices s and t if and only if they are adjacent (they share a common facet). We can represent each vertex of $G_{k+1,n+1}$ by a permutation and describe each edge of $G_{k+1,n+1}$ in terms of permutations [4]. We call this new graph $\Gamma_{k+1,n+1}$. It is clear that $\Gamma_{k+1,n+1}$ is isomorphic to $G_{k+1,n+1}$.

Proposition 3.4 [4, Lemma 6.1 and Theorem 7.1] *The graph $\Gamma_{k+1,n+1}$ can be described as follows: its vertices are permutations $u = u_1 \cdots u_n \in \mathfrak{S}_n$ with* $\mathrm{des}(u^{-1}) = k$. *There is an edge between u and v, if and only if one of the following two holds:*

1. (*type one edge*) $u_i - u_{i+1} \neq \pm 1$ *for some* $i \in \{1, \dots, n-1\}$, *and v is obtained from u by exchanging u_i, u_{i+1}.*
2. (*type two edge*) $u_n \neq 1, n$, *and v is obtained from u by moving u_n to the front of u_1, i.e., $v = u_n u_1 \cdots u_{n-1}$; or this holds with u and v switched.*

Example 3.5 Here is the graph $\Gamma_{3,5}$ for $\Delta'_{3,5}$.



In the above graph, the edge α between $u = 2413$ and $v = 4213$ is a type one edge with $i = 1$, since $4 - 2 \neq \pm 1$ and one is obtained from the other by switching 2 and 4; the edge β between $u = 4312$ and $v = 2431$ is a type two edge, since $u_4 = 2 \neq 1, 4$ and $v = u_4 u_1 u_2 u_3$. The dotted line attached to a simplex s indicates that s is adjacent to some simplex t in $\Delta_{2,5}$. Since we are considering the half-open hypersimplices, the common facet $s \cap t$ is removed from s.

4 Proof of Theorem 1.2 by a Shellable Triangulation

We want to show that the h^*-polynomial of $\Delta'_{k+1,n+1}$ is

$$\sum_{\substack{w \in \mathfrak{S}_n \\ \mathrm{exc}(w)=k}} t^{\mathrm{des}(w)}.$$

Compare this to Theorem 3.1: if $\Delta'_{k+1,n+1}$ has a shellable unimodular triangulation $\Gamma_{k+1,n+1}$, then its h^*-polynomial is

$$\sum_{\alpha \in \Gamma_{k+1,n+1}} t^{\#(\alpha)}.$$

We will define a shellable unimodular triangulation $\Gamma_{k+1,n+1}$ for $\Delta'_{k+1,n+1}$, label each simplex $\alpha \in \Gamma_{k+1,n+1}$ by a permutation $w_\alpha \in \mathfrak{S}_n$ with $\mathrm{exc}(w_\alpha) = k$. Then show that $\#(\alpha) = \mathrm{des}(w_\alpha)$.

We start from the triangulation $\Gamma_{k+1,n+1}$ studied in Sect. 3.3. By Theorem 3.4, each simplex is labeled by a permutation $u \in \mathfrak{S}_n$ with $\mathrm{des}(u^{-1}) = k$. Based on the Foata map defined in Sect. 3.2, after the following maps, the vertices of $S_{k+1,n+1}$ are permutations in $\mathfrak{S}_n$ with k exceedances:

$$\Gamma_{k+1,n+1} \xrightarrow{\mathrm{rev}} R_{k+1,n+1} \xrightarrow{-1} P_{k+1,n+1} \xrightarrow{F^{-1}} Q_{k+1,n+1} \xrightarrow{\mathrm{rev}} S_{k+1,n+1}, \tag{4.1}$$

where the map $F^{-1} : P_{k+1,n+1} \to Q_{k+1,n+1}$ sending $\hat{w}$ to w is the inverse of the Foata map and the map "rev" is the reverse map we defined in the proof of Theorem 3.2, reversing both the letters and positions of a permutation.

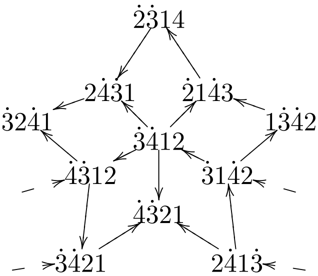
Example 4.1 For an example of the above map from $\Gamma_{3,5}$ to $S_{3,5}$, consider $u = 3241$. It is in $\Gamma_{3,5}$ since $u^{-1} = 4213$ has exactly two descents. Applying the above map to u, we have

$$3241 \xrightarrow{\mathrm{rev}} 4132 \xrightarrow{-1} 2431 \xrightarrow{F^{-1}} 4213 \xrightarrow{\mathrm{rev}} 2431,$$

where 2431 has 2 exceedances.

Apply the above maps to vertices of $\Gamma_{k+1,n+1}$, we call the new graph $S_{k+1,n+1}$. We will define the shelling order on the simplices in the triangulation by orienting each edge in the graph $S_{k+1,n+1}$. If we orient an edge (u, v) such that the arrow points to u, then in the shelling, let the simplex labeled by u be after the simplex labeled by v. We can orient each edge of $S_{k+1,n+1}$ (see Definition 4.7) such that the directed graph is acyclic (Corollary 4.18). This digraph therefore defines a partial order on the simplices of the triangulation. We will prove that any linear extension of this partial order gives a shelling order (Theorem 4.21). Given any linear extension obtained from the digraph, the shelling number of each simplex is the number of incoming edges. Let w_α be the permutation in $S_{k+1,n+1}$ corresponding to the simplex α. Then we can show that for each simplex, its number of incoming edges equals $\mathrm{des}(w_\alpha)$ (Theorem 4.12).

Example 4.2 Here is the graph $S_{3,5}$ for $\Delta'_{3,5}$ with each edge oriented according to Definition 4.7.



For example, the vertex labeled by 3412 with $\mathrm{des}(3412) = 1$ has one incoming edge. Another example, consider the vertex labeled by 3142. It has two incoming edges (including the dotted edge), which is the same as its number of descents. So we can see that it is crucial here that we are looking at the half-open hypersimplex instead of the usual hypersimplex.

In the following three subsections, we will first define how we orient each edge in $S_{k+1,n+1}$ and each vertex has the correct number of incoming edges, then we will show that the digraph is acyclic, and finally, any linear extension gives a shelling.

4.1 Correct Shelling Number

We need a closer look of each graph $R_{k+1,n+1}$, $P_{k+1,n+1}$, $Q_{k+1,n+1}$ obtained in the process of getting $S_{k+1,n+1}$ from $\Gamma_{k+1,n+1}$. First, from the description of $\Gamma_{k+1,n+1}$ (Proposition 3.4) and the maps in (4.1):

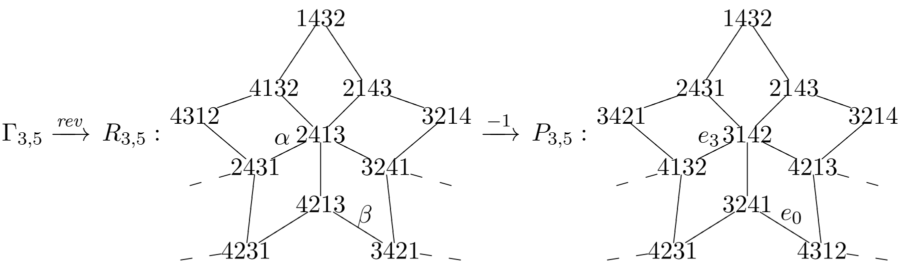
$R_{k+1,n+1}$: its vertices are $u \in \mathfrak{S}_n$ with $\mathrm{des}(u^{-1}) = k$. There are two types of edge:

1. type one edge is the same as in Γ;
2. u and v has a type two edge if and only if $u_1 \neq 1, n$, and v is obtained from u by moving u_1 to the end of u_n, i.e., $v = u_2 \cdots u_n u_1$; or switch the role of u and v.

$P_{k+1,n+1}$: its vertices are $u \in \mathfrak{S}_n$ with $\mathrm{des}(u) = k$. There are two types of edge:

1. (u, v) is a type one edge if and only if the numbers i and $i+1$ are not next to each other in u, and v is obtained from u by exchanging the numbers i and $i+1$. We label this edge e_i.
2. (u, v) is a type two edge if and only if $u_1 \neq 1$ and $u_n \neq 1$, and $v_i = u_i - 1 \pmod{n}$ for $i = 1, \dots, n$ (we denote this by $v = u - 1 \pmod{n}$), or switch the role of u and v. We label this edge e_0.

Example 4.3 Here are the graphs $R_{3,5}$ and $P_{3,5}$ for $\Delta'_{3,5}$.



In the graph $R_{3,5}$ above, the edge labeled α is of type one switching 1 and 3; and β is of type two, with $u = 3421$ and $v = u_2u_3u_4u_1 = 4213$. In the above graph $P_{3,5}$, the edge e_3 is an edge of type one between $u = 4132$ and 3142 switching 3 and 4 since they are not next to each other; and the edge e_1 between $u = 4312$ and $v = 3241 = u - 1 \pmod 4$ is of type two.

Definition 4.4 Let $w \in \mathfrak{S}_n$. Define its *descent set* to be $\mathrm{Des}(w) = \{i \in [n-1] \mid w_i > w_{i+1}\}$ its *leading descent set* to be the actual numbers on these positions, $\mathrm{LdDes}(w) = \{w_i \mid i \in \mathrm{Des}(w)\}$.

For $w \in P_{k+1,n+1}$, since $\mathrm{des}(w) = k$, we have $\mathrm{Des}(w) = \{i_1, \ldots, i_k\}$. By the description of edges in $P_{k+1,n+1}$, we have the following relation of Des and LdDes for an edge in $P_{k+1,n+1}$:

Lemma 4.5 *Let v be a vertex in $P_{k+1,n+1}$.*

1. *Define u by $v = u - 1 \pmod n$. There are three cases depending on the position of the letter n in v:*
   (a) *if $v_1 = n$, then $\mathrm{Des}(u) = \mathrm{Des}(v)\backslash\{1\}$, thus $u \in P_{k,n+1}$;*
   (b) *if $v_n = n$, then $\mathrm{Des}(u) = \mathrm{Des}(v) \cup \{n-1\}$ and $u \in P_{k+2,n+1}$;*
   (c) *if $v_i = n$ with $i \neq 1, n$, then $\mathrm{Des}(u) = \mathrm{Des}(v) \cup \{i-1\}\backslash\{i\}$ and $u \in P_{k+1,n+1}$.*
2. *Let $e_i = (u, v)$ be a type one edge in $P_{k+1,n+1}$. Then we have $\mathrm{Des}(u) = \mathrm{Des}(v)$. In this case, we also compare $\mathrm{LdDes}(u)$ and $\mathrm{LdDes}(v)$:*
   (a) *if $i, i+1 \in \mathrm{LdDes}(u)$ or $i, i+1 \notin \mathrm{LdDes}(u)$, we have $\mathrm{LdDes}(u) = \mathrm{LdDes}(v)$;*
   (b) *otherwise, if $i \in \mathrm{LdDes}(u)$ and $i+1 \notin \mathrm{LdDes}(u)$, we have $\mathrm{LdDes}(v) = \mathrm{LdDes}(u) \cup \{i+1\}\backslash\{i\}$.*

Now consider the map from $P_{k+1,n+1}$ to $S_{k+1,n+1}$. Notice that this map is the same as defined in Theorem 3.2. Therefore, we have

Corollary 4.6 *Vertices in $Q_{k+1,n+1}$ are permutations $w \in \mathfrak{S}_n$ with k reverse exceedances (i such that $w_i < i$), and vertices in $S_{k+1,n+1}$ are permutation $v \in \mathfrak{S}_n$ with $\mathrm{exc}(v) = k$. Moreover, the* reverse exceedances set *in $w \in Q_{k+1,n+1}$, denoted by $\mathrm{Rexc}(w) = \{i \mid w_i < i\}$ is the same as $\mathrm{LdDes}(\hat{w})$, where $\hat{w} = F(w) \in P_{k+1,n+1}$. So part 2 of Lemma 4.5 for $\mathrm{LdDes}(\hat{w})$ also apply for $\mathrm{Rexc}(w)$.*

For $w \in Q$, decompose $[n-1]$ by $A_w \cup B_w \cup C_w$ (disjoint union), where

$$A_w = \big\{ i \in [n-1] \mid i \notin \operatorname{Rexc}(w), i+1 \in \operatorname{Rexc}(w) \big\}, \tag{4.2}$$

$$B_w = \big\{ i \in [n-1] \mid i+1 \notin \operatorname{Rexc}(w), i \in \operatorname{Rexc}(w) \big\}, \quad \text{and} \tag{4.3}$$

$$C_w = \big\{ i \in [n-1] \mid i, i+1 \notin \operatorname{Rexc}(w) \text{ or } i, i+1 \in \operatorname{Rexc}(w) \big\}. \tag{4.4}$$

For example, consider $v = 54\dot{1}\dot{2}6\dot{3}8\dot{7}9$, where the dotted positions are in $\operatorname{Rexc}(v)$. Then $A_v = \{2, 5, 7\}$, $B_v = \{4, 8, 6\}$ and $C_v = \{1, 3\}$.

For an edge $(u, v) \in Q_{k+1,n+1}$, we label it e_i according to the labeling of the corresponding edge $e_i = (\hat{u}, \hat{v}) \in P_{k+1,n+1}$. Then we orient each edge in $Q_{k+1,n+1}$ in the following way:

Definition 4.7 Let $e_i = (u, v)$ be an edge in $Q_{k+1,n+1}$.

1. For type one edge ($i \neq 0$),
   (a) if $\operatorname{Rexc}(u) \neq \operatorname{Rexc}(v)$, then define $u \to v$ if and only if $i \in \operatorname{Rexc}(v)$ (this implies $i \notin \operatorname{Rexc}(u)$ by Lemma 4.5, part 2(b));
   (b) if $\operatorname{Rexc}(u) = \operatorname{Rexc}(v)$, then define $u \to v$ if and only if $v_i > v_{i+1}$ (this implies $u_i < u_{i+1}$ by Corollary 4.15).
2. For type two edge ($i = 0$), define $u \to v$ if and only if $\hat{v} = \hat{u} - 1 \pmod n$, where $(\hat{u}, \hat{v})$ is the corresponding edge in $P_{k+1,n+1}$.

Based on the above definition and the Foata map, we have the following description of incoming edges for $v \in Q_{k+1,n+1}$.

Lemma 4.8 *Let v be a vertex in $Q_{k+1,n+1}$. Then*

1. *v has an incoming type one edge (e_i with $i \neq 0$) if and only if one of the following two holds*:
   (a) $i \in B_v$;
   (b) $i \in C_v \cap \operatorname{Des}(v)$.
2. *v has an incoming type two edge (e_0) if and only if $v_n \neq n$.*

Proof

1. First, by Definition 4.7, and Lemma 4.5, it is clear that if there exists an edge e_i with $i \neq 0$, $u \to v$ for some $u \in Q_{k+1,n+1}$, then v satisfies one of conditions (a) and (b). On the other hand, we need to show that, if (a) or (b) is true for v, then there exists an edge $e_i = (u, v) \in Q_{k+1,n+1}$. Then, by Definition 4.7, the edge will be oriented as $u \to v$. In fact, consider the corresponding permutation $\hat{v} \in P_{k+1,n+1}$. From the description of $P_{k+1,n+1}$, $\hat{v}$ has a type one edge e_i if and only if i and $i+1$ are not next to each other in $\hat{v}$. But with a careful look at the inverse Foata map, we can see that if (a) or (b) is true for v, then neither the case $\hat{v} = \cdots i(i+1) \cdots$ nor $\hat{v} = \cdots (i+1)i \cdots$ can be true.
2. Let $\hat{u} = \hat{v} + 1 \pmod n$ in $P_{k+1,n+1}$. If $\operatorname{des}(\hat{u}) = \operatorname{des}(\hat{v})$, then we have $\hat{u} \in P_{k+1,n+1}$, so $\hat{v}$ has a type two edge, and this edge points to v by Definition 4.7.

If $\mathrm{des}(\hat{u}) = \mathrm{des}(\hat{v}) - 1$, then $\hat{v}$ still has an incoming edge e_1, since we are considering the half-open hypersimplex and this edge indicates that the common facet $u \cap v$ is removed from v. Then by Lemma 4.5, part 1, $\mathrm{des}(\hat{u}) \le \mathrm{des}(\hat{v})$ if and only if case (b) does not happen, i.e., $\hat{v}_n \neq n$ in P. This is equivalent to $v_n \neq n$ in $Q_{k+1,n+1}$ by the inverse Foata map. □

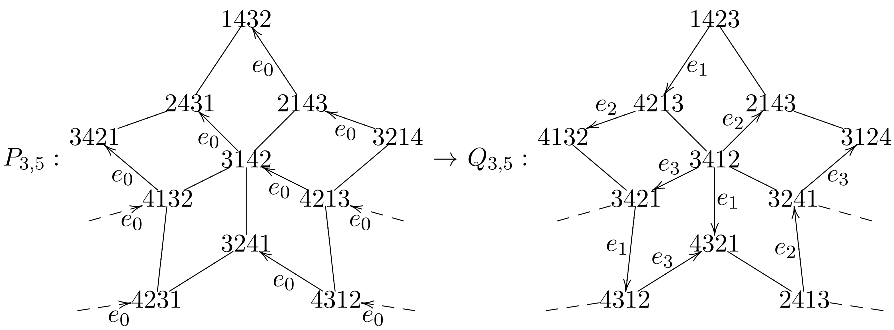
Definition 4.9 Let $I, J \subset [n-1]$. Define a *big block* of $Q_{k+1,n+1}$ to be $b_I = \{w \in Q_{k+1,n+1} \mid \mathrm{Des}(\hat{w}) = I\}$, where $\hat{w} = F(w) \in P_{k+1,n+1}$. Define a *small block* $s_{I,J} = \{w \in b_I \mid \mathrm{Rexc}(w) = J\}$. We say the small block $s_{I,J}$ is smaller than $s_{I',J'}$ if (1) $I < I'$ or (2) $I = I'$ and $J > J'$.

For two different sets $I, I' \subset [n-1]$ with $I = \{i_1 \le \cdots \le i_k\}$ and $I' = \{i'_1 \le \cdots \le i'_\ell\}$, we define $I < I'$ if 1) $k < \ell$ or 2) $k = \ell$ and $i_j \le i'_j$ for all $j = 1, \ldots, k$. Then by Lemmas 4.5 and 4.8, we have

Corollary 4.10 *For an edge $e_i = u \to v \in Q_{k+1,n+1}$ with $u \in s_{I,J}$ and $v \in s_{I',J'}$,*

1. *if $i = 0$, then $I' > I$;*
2. *if $i \neq 0$ and $i \in B_v$, then $I = I'$ and $J' < J$;*
3. *if $i \neq 0$ and $i \in C_v$, then $I = I'$ and $J = J'$.*

Example 4.11 Here is an example of Definition 4.7, Lemma 4.8 and Corollary 4.10, with a type one edge drawn in $Q_{3,5}$ and a type two (e_0) in $P_{3,5}$ for $\Delta'_{3,5}$.



It is clear from the graph $P_{3,5}$ that $\hat{v} \in P_{3,5}$ has an incoming e_0 if and only if $\hat{v}_4 \neq 4$, which is equivalent to $v_4 \neq 4$ in $Q_{3,5}$. Consider $v = 4321 \in Q_{3,5}$. It has $\mathrm{Rexc}(v) = \{3, 4\}$. Since $v_4 \neq 4$, it has an incoming e_0 edge (shown in $P_{3,5}$); since v with $i = 1, 3$ satisfies condition (b) in Lemma 4.8, there are two incoming edges e_1 and e_3 of type two, and these are all the incoming edges of v.

Consider the edge $e_0 = u \to v \in Q_{3,5}$ whose corresponding edge in $P_{3,5}$ is between $\hat{u} = 4312$ and $\hat{v} = 3241$. We have $I = \mathrm{Des}(\hat{u}) = \{1, 2\}$ and $I' = \mathrm{Des}(\hat{v}) = \{1, 3\}$, with $I' > I$. Consider the edge $e_2 = u \to v \in Q_{3,5}$ with $u = 34\dot{1}\dot{2}$ and $v = 2\dot{1}4\dot{3}$, where the dotted positions are in Rexc. Since $2 \in B_v$, we have $J = \mathrm{Rexc}(u) = \{3, 4\}$ and $J' = \mathrm{Rexc}(v) = \{2, 4\}$ with $J > J'$. Finally, consider $e_1 = u \to v \in Q_{3,5}$ with $u = 34\dot{2}\dot{1}$ and $v = 43\dot{1}\dot{2}$. Since $1 \in C_v$, we have $\mathrm{Rexc}(u) = \{3, 4\} = \mathrm{Rexc}(v)$.

With the orientation of $Q_{k+1,n+1}$ by Definition 4.7, we have

Theorem 4.12 *For each vertex $v \in Q_{k+1,n+1}$, the number of its incoming edges equals* $\operatorname{des}(v)$.

Proof First, notice that if $i \in B_v$, then $i \notin \operatorname{Des}(v)$; and if $j \in A_v$, then $j \in \operatorname{Des}(v)$. So $\operatorname{Des}(v) = A_v \cup (C_v \cap \operatorname{Des}(v))$. Now we will define a bijection between the set $\operatorname{Des}(v)$ and the set of incoming edges of v as listed in Lemma 4.8. First notice that $i \in \operatorname{Des}(v) \cap C_v$ corresponds to an incoming edge e_i described in case (b) of Lemma 4.8. Then we need to match A_v with the set of incoming edges in Lemma 4.8, parts 1(a) and 2. There are two cases:

1. If $v_n = n$, by Lemma 4.8, v does not have a type two incoming edge. Then we have a bijection between the sets A_v and B_v by matching $i \in A$ to $\min\{j \in B \mid j > i\}$. For example, $v = 54\dot{1}\dot{2}63\dot{8}\dot{7}9$ where the dotted positions are in $\operatorname{Rexc}(v)$. Then $A_v = \{2, 5, 7\}$ is in bijection with $B_v = \{4, 8, 6\}$. This gives us the desired bijection since the set of i's such that e_i is a type one incoming edge of case 1(a) is exactly B_v.
2. If $v_n \neq n$, A_v has one element more than B_v, since the largest number in A_v does not have image in B_v. But since in this case, v has a type two incoming edge by Lemma 4.8, the extra descent can be taken care by this incoming edge. □

4.2 Acyclicity

We want to show that the digraph defined in the previous subsection gives a shelling order. First, we need to show that any linear extension of the above ordering is well defined, i.e., there is no cycle in the directed graph $S_{k+1,n+1}$ (equivalently, $Q_{k+1,n+1}$ is acyclic). In this section, we restrict to the connected component of a small block of $Q_{k+1,n+1}$, i.e., the subgraph of $Q_{k+1,n+1}$ consisting of permutations with the same Rexc, or equivalently, the subgraph of $P_{k+1,n+1}$ consisting of permutations with the same LdDes. By Lemmas 4.5 and 4.8, $e_i = (u, v) \in Q_{k+1,n+1}$ with u and v in the same small block if and only $i \in C_u$, where C_u is defined in (4.4). We want to show that there is no directed cycle in each small block of $Q_{k+1,n+1}$.

For a permutation w, let $t_i(w)$ be the permutation obtained by switching letters i and $i + 1$ in w, and $s_i(w)$ be the permutation obtained by switching letters in positions i and $i + 1$. Now consider $e_i = (u, v) \in Q_{k+1,n+1}$ and the corresponding edge $(\hat{u}, \hat{v}) \in P$. By definition of $P_{k+1,n+1}$, we have $\hat{u} = t_i \hat{v}$. Then in $Q_{k+1,n+1}$, we have

Lemma 4.13 *Let $e_i = (u, v) \in Q_{k+1,n+1}$ and $i \in C_u$. Then*

$$u = \begin{cases} s_i(v), & C(u) \neq C(v), \\ t_i s_i(v), & C(u) = C(v), \end{cases}$$

where $C(w)$ stands for the cycle type of w defined in Sect. 3.2.

Proof In $P_{k+1,n+1}$, we have $\hat{u} = \cdots i \cdots (i + 1) \cdots$ and $\hat{v} = \cdots (i + 1) \cdots i \cdots$. By the inverse Foata map and the condition that u and v are in the same small block, i.e., $\mathrm{LdDes}(u) = \mathrm{LdDes}(v)$, we can see that the only case when $\mathrm{C}(u) \neq \mathrm{C}(v)$ is $u = \cdots(i\cdots)((i+1)\cdots)\cdots$ and $v = \cdots((i+1)\cdots i\cdots)\cdots$ (in standard cycle notation). Then the conclusion follows from the inverse Foata map. □

Example 4.14 Consider $u = 4321 \in Q_{3,5}$ with standard cycle notation $u = (32)(41)$ in Example 4.11. For $e_3 = (u, v)$ with $v = 4312 = (4231)$, since $\mathrm{C}(u) \neq \mathrm{C}(v)$, we have $u = s_3(v)$. For $e_1 = (u, v')$ with $v' = 3412 = (31)(42)$, since $\mathrm{C}(u) = \mathrm{C}(v)$, we have $u = t_3 s_3(v)$, i.e., u is obtained from v' by switching 3 and 4, which is 4312, and then switching v'_3 and v'_4, which is $4321 = u$.

For a permutation w, define its *inversion set* to be $\mathrm{inv}(w) = \{(w_i, w_j) \mid i\langle j, w_i\rangle w_j\}$ and denote $\#\mathrm{inv}(w)$ by $i(w)$. By Lemma 4.13, we have

Corollary 4.15 *For $e_i = (u, v) \in Q_{k+1,n+1}$ with $i \in \mathrm{Des}(u)$ and $i \in C_u$, we have $i \notin \mathrm{Des}(v)$ and $i(v) \le i(u)$.*

Now consider a sequence of edges E in a small block of $Q_{k+1,n+1}$: $u \leftarrow \cdots \leftarrow v$. By Corollary 4.15, we have $i(v) \le i(u)$. In order to show that there is no cycle in each small block, we find an invariant that strictly decreases along any directed path. We define the *E-inversion set* to be

$$\mathrm{inv}_E(w) = \bigl\{(w_i, w_j) \in \mathrm{inv}(w) \mid \{e_i, \dots, e_{j-1}\} \subset E\bigr\} \tag{4.5}$$

and claim that $i_E(w) = \#\mathrm{inv}_E(w)$ is such an invariance (Lemma 4.17).

Example 4.16 For $w = 361452798$ and $E = \{e_i \mid i \in \{2, 3, 5\}\}$, we first cut w into blocks (indicated by lines):

$$w = \underline{3}\,\underline{614}\,\underline{52}\,\underline{7}\,\underline{9}\,\underline{8},$$

with the property that each block can be permutated arbitrarily by $\{s_i \mid i \in \{2, 3, 5\}\}$. Then $\mathrm{inv}_E(w) = \{(6, 1), (6, 4), (5, 2)\}$, i.e., $(w_i, w_j) \in \mathrm{inv}(w)$ with w_i, w_j in the same block. This is the same as in (4.5).

Here are three extremal examples. If $E = \{e_i \mid i \in [n-1]\}$, then $\mathrm{inv}_E(w) = \mathrm{inv}(w)$ for all $w \in \mathfrak{S}_n$. If $E = \{e_i\}$ and $e_i = u \leftarrow v \in Q$, then $i_E(u) = 1$ and $i_E(v) = 0$. If $E = \{e_i \mid i \in I \subset [n-1]\}$ and $i \notin \mathrm{Des}(w)$ for all $i \in I$, then $i_E(w) = 0$. This is the situation in Lemma 4.17.

Lemma 4.17 *Let $u \leftarrow \cdots \leftarrow v$ be a sequence of edges in a small block of $Q_{k+1,n+1}$ with edge set E. Then $i_E(v) < i_E(u)$.*

Proof By Lemmas 4.13 and 4.8, part 1, we have $i_E(v) \le i_E(u)$. Suppose we have $i_E(v) = i_E(u)$. We will show that no edge can belong to E. First, we show that e_{n-1} cannot be in E. Let w be any permutation in the above path from v to u. If $n-1 \notin C_w$, then certainly $e_{n-1} \notin E$. Suppose $n-1 \in C_w$. Let $\hat{w}' = t_{n-1}(\hat{w})$ in P. Notice that we always have $\mathrm{C}(w') \neq \mathrm{C}(w)$. Then by Lemma 4.13, we have $w' = s_i(w)$, and thus $i_E(w') < i_E(w)$. So $e_{n-1} \notin E$.

Now consider u. Notice that the last cycle of u in its standard cycle notation must start with n. Let the cycle be $(n\, a_1\, a_2 \cdots a_k)$. We claim that $e_{a_1} \notin E$. First, for u, since $u_n = a_1$ and $e_{n-1} \notin E$, all pairs (u_i, u_n) are not in $\mathrm{inv}_E(u)$ by definition of inv_E in (4.5). Let $\hat{u}' = t_{a_1}(\hat{u})$ in P. Independent of the fact that $\mathrm{C}(u) = \mathrm{C}(u')$, we have $i_E(u') < i_E(u)$. Now consider any w appearing in the path from v to u. Suppose all edges before w are not e_{a_1}. Then we still have $w_n = a_1$. Let $\hat{w}' = t_{a_1}(\hat{w})$. Then again consider both cases $\mathrm{C}(w) = \mathrm{C}(w')$ or not, by Lemma 4.13, we have $i_E(w') < i_E(w)$. Therefore, $e_{a_1} \notin E$.

With the same argument, we can show that $e_{a_2} \notin E, \ldots, e_{a_k} \notin E$. Then we can move to the previous cycle, until we have $e_i \notin E$ for all $i \in [n-1]$. □

Corollary 4.18 *$Q_{k+1,n+1}$ is acyclic.*

Proof First it is not hard to see that there is no cycle that involves vertices in different small blocks, since both big blocks and small blocks have the structure of a poset, and edges between two small/big blocks all have the same direction. Therefore, if $Q_{k+1,n+1}$ has a cycle, it has to be within a small block.

Suppose there is a directed cycle within a small block with edge set E. Consider some w in the cycle and let $w_1 = w_2 = w$ in Lemma 4.17, we will have $i_E(w) < i_E(w)$, a contradiction. □

4.3 Shellable Triangulation

In this section, we will show that any linear extension of the ordering of the simplices in $Q_{k+1,n+1}$ is shellable. We will prove this by showing that each simplex has a unique minimal nonface (see Sect. 3.1).

Let us first assign a face F to each simplex. Each incoming edge $\alpha \xleftarrow{e_i} \alpha_i$ defines a unique vertex M_i of α that α has but α_i does not have. Then let $F = \{M_i\}$ be given by all the incoming edges of α. We want to show that F is the unique minimal nonface of α. First, let us assume F is a nonface, i.e., it has not appeared before α in a given order of simplices. We can see that F is the unique minimal nonface, i.e., any proper subface of F has appeared before. In fact, let M_i be a vertex in F but not in $F' \subset F$. Then we have $F' \subset \alpha_i$ since α_i has every vertex of α except for M_i.

In the rest of this section, we will show that F is a nonface. To show this, let Q_F be the (connected) component of $Q_{k+1,n+1}$ consisting of all simplices containing F. Then it suffices to show that α is the only source of Q_F, and any other simplices are reachable by α, i.e., there exists a directed path from α to that simplex. We will first show this within each small block and then connect different small blocks.

Let $Q_{F,s}$ be a (connected) component of $Q_{k+1,n+1}$ consisting of all simplices in a small block s containing F. In Sect. 5, we define a "vertex expression" for each simplex in Δ. Let the vertex expression of two simplices be $\alpha = M_1 \cdots M_{n+1}$ and $\beta = M'_1 \cdots M'_{n+1}$. Assume α and β are connected by an edge e_i. Then by Corollary 5.3, α and β differs only by the $(i+1)$th vertex, i.e., $M_{i+1} \neq M'_{i+1}$ and $M_j = M'_j$ for all $j \neq i+1$. Then it follows that there exists an edge set E for $Q_{F,s}$, such that $Q_{F,s}$ is closed under this edge set: if β is connected to α by an edge $e \in E$ and $\alpha \in Q_{F,s}$, then $\beta \in Q_{F,s}$. In fact, let $\alpha \in Q_{F,s}$ and say the vertices of F are in the positions $J \subset [n]$

of α. Then we have $E = \{e_i \mid i \notin J, i \in C_u, \text{ for any } u \in s\}$. To show the nonface property for each small block (Corollary 4.20), we need the following lemma about the Foata map.

Lemma 4.19 *Let $I \subset \{1, 2, \ldots, n-1\}$. For a permutation $w \in \mathfrak{S}_n$, consider the set $E(w)$ of all the permutations obtained by applying any sequence of t_i, $(i \in I)$ to w, i.e.,*

$$E(w) = \{u = t_{i_1} \cdots t_{i_k}(w) \mid i_j \in I \text{ for some } k\}.$$

Then there exists a unique $u \in E(w)$ such that $F^{-1}(u)$ has ascents in I.

Proof We can describe an algorithm to determine this u uniquely. First, notice that the group generated by t_i, $(i \in I)$ is a subset of the symmetric group S_n, and has the form $S_{a_1} \times S_{a_2} \times \cdots \times S_{a_k}$, where $a = (a_1, a_2, \ldots, a_k)$ is a composition of n. For example, if $n = 9$, and $I = \{2, 3, 5, 7, 8\}$, then $a = (1, 3, 2, 3)$. A composition in k parts divides the numbers $1, 2, \ldots, n$ into k parts, and numbers in each region can be permuted freely by t_i, $(i \in I)$.

Now in the given w, replace numbers in each region by a letter and order the letters by the linear order of the regions. In the previous example, replace $\{1\}$, $\{2, 3, 4\}$, $\{5, 6\}$ and $\{7, 8, 9\}$ by a, b, c, d respectively and we have the order $a < b < c < d$. For example, if $w = 253496187$, then we get a word $bcbbdcadd$.

Next, add parentheses to the word in front of each left-to-right maximum, as in the inverse Foata map. For $bcbbdcadd$, we have $(b)(cbb)(dcadd)$. Notice that we do not have parentheses before the second and third d. No matter how we standardize this word, the cycles we get will be a refinement of the cycles for the word.

Now comes the most important part. We want to standardize the word in a way such that $v = F^{-1}(w)$ is increasing in all positions of I. To do this, we look at a letter in the word and compare it to the next word it goes to in the cycle notation. For example, consider the b's in $(b_1)(cb_2b_3)(dcadd)$. $v_{b_1} = b_1 \in \{2, 3, 4\}$, $v_{b_2} = b_3 \in \{2, 3, 4\}$ and $v_{b_3} \in \{5, 6\}$. Since $v_{b_3} > v_{b_1}$ and v_{b_2}, to keep v increasing in positions $\{2, 3, 4\}$, we have $b_3 > b_1$ and b_2, so $b_3 = 4$. Now continue to compare b_1 and b_2. Since $v_{b_2} = b_3 > v_{b_1} = b_1$, we have $b_1 < b_2$, and thus $b_1 = 2$, $b_2 = 3$. Notice that if there are no periodic cycles, then we can always choose a unique way to standardize the letters to a permutation with the required property. For a periodic cycle, there is still a unique way to standardize them, which is to standardize each letter in the cycle increasingly. For example, for $(baba)$, (3142) is the unique way. This completes the algorithm and proof. □

Corollary 4.20 (Small Block Shelling) *For any face $F \subset \Delta'$, if $Q_{F,s} \neq \emptyset$, then $Q_{F,s}$ has only one source and any other simplices are reachable by that source.*

Proof Let E be the edge set corresponding to $Q_{F,s}$. By Lemma 4.8, part 1(b), if α is a source in $Q_{F,s}$, then $i_E(\alpha) = 0$. First, by Lemma 4.17, we know that there exists at least one such source. In fact, let $\alpha \in Q_{F,s}$. If $i_E(\alpha) \neq 0$, then by Lemma 4.8, part 1(b), we can keep going along the incoming edges of $Q_{F,s}$. Since there is no cycle within the small block and there are only finitely many simplices in $Q_{F,s}$, we will reach a source.

Now by Lemma 4.19, there is at most one source for $Q_{F,s}$. Then the proposition is proved since the above "tracing back along arrows" will guarantee that each simplex in $Q_{F,s}$ is reachable by that unique source. □

Theorem 4.21 *Any linear extension of the above defined ordering between adjacent simplices will give a shelling order for the half-open hypersimplex.*

Proof It suffices to show that for each face F in Δ', Q_F has only one source and any other simplices are reachable by that source. First by Proposition 5.5, Q_F starts with a unique minimal connected small block. By Lemma 4.8, parts 1(b) and 2, each simplex in Δ' has an incoming edge from a simplex in a smaller small block. Therefore, the source α_F in the unique minimal small block of Q_F is the unique source of Q_F, and each simplex in Q_F is reachable from α_F via the unique source in each $Q_{F,s}$. □

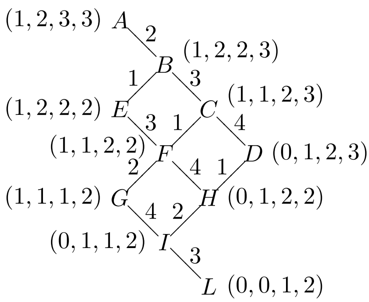
5 Vertex Expression for Simplices in the Triangulation

Let $z_i = x_1 + \cdots + x_i$, we have an equivalent definition for $\Delta_{k+1,n+1}$:

$$\Delta_{k+1,n+1} = \big\{(z_1, \ldots, z_n) \mid 0 \le z_1, z_2 - z_1, \ldots, z_n - z_{n-1} \le 1;\ k \le z_n \le k+1\big\}.$$

In this new coordinate system, the triangulation of $\Delta_{k+1,n+1}$ is called the alcoved triangulation [4].

Now all the integral points will be vertices of some simplex in the triangulation. Denote the set of all the integral points in $\Delta_{k+1,n+1}$ by $V_{k+1,n+1} = \{\mathbb{Z}^n \cap \Delta_{k+1,n+1}\}$. Now we define a partial order on $V_{k+1,n+1}$ (we will drop the indices from now on). For $M = (m_1, \ldots, m_n)$, $N = (m'_1, \ldots, m'_n) \in V$, we define $M > N$ if and only if $m_i \ge m'_i$ for $i = 1, \ldots, n$. If $M = N + e_i$, where e_i is the vector with 1 in the ith position and 0 elsewhere, then label this edge in the Hasse diagram by $n + 1 - i$. We still call the Hasse diagram of this poset on $V_{k+1,n+1}$ by $V_{k+1,n+1}$ itself. Here is an example of $V_{3,5}$.



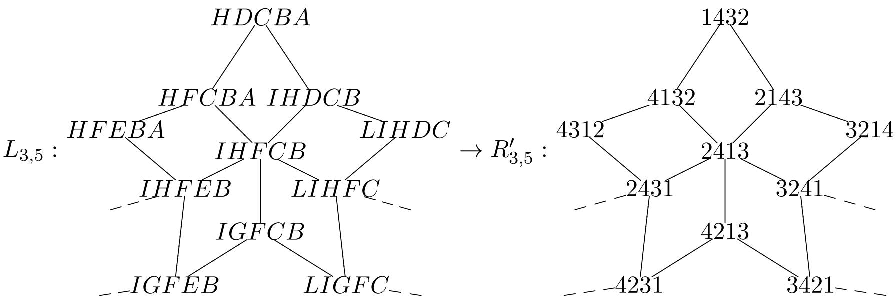
Lemma 5.1 $n+1$ *points of* $V_{k+1,n+1}$ *form a simplex in the triangulation of* $\Delta_{k+1,n+1}$ *if and only if these points form an* n*-chain in the poset* V *and the labels of edges are distinct. Moveover, vertex expressions with the same starting letter will also have the same ending letter.*

For example, $HFCBA$ is a simplex in $\Delta_{3,5}$, since the labels along the path form a permutation 4132.

Proof Starting with a point in $V_{3,5}$, for example $H = (0, 1, 2, 2)$, we need to add one to each coordinate, in order to get a simplex. It always end up with $A = (1, 2, 3, 3)$. □

For each simplex, we define its *vertex expression* to be the expression formed by its $n + 1$ vertices (from small to large in the poset $V_{k+1,n+1}$). For example, $HFCBA$ is a vertex expression.

We denote the set of all such simplices in their vertex expressions by $L_{k+1,n+1}$, and denote the corresponding permutations read from the paths of $V_{k+1,n+1}$ by $R'_{k+1,n+1}$. Since two simplices are adjacent if and only if their vertices differ by one vertex, we can add a graph structure on $L_{k+1,n+1}$ (and thus on $R'_{k+1,n+1}$): we connect two simplices if and only if their vertex expressions differ by one vertex. For example, from $L_{3,5}$, we get $R'_{3,5}$ by reading the labels of the corresponding paths in $V_{3,5}$:



Notice that in $V_{3,5}$, since the vertices E, F, H, G, I, L have $z_4 = 2$, they lie on the lower facet of $\Delta_{3,5}$. Therefore, we have a dotted line attached to each of the simplices $IHFEB$, $LIHFC$, $LIGFC$ and $IGFEB$, indicating that these simplices have a lower facet removed.

We have the following connections between the vertex expressions (graph $L_{k+1,n+1}$ and $R'_{k+1,n+1}$) and the graphs $R_{k+1,n+1}$ (and $P_{k+1,n+1}$, $Q_{k+1,n+1}$) we studied in Sect. 3. For example, compare $R'_{3,5}$ above with $R_{3,5}$ in Sect. 3.

Proposition 5.2 $R'_{k+1,n+1} = R_{k+1,n+1}$.

Proof Since the permutations $r \in R_{k+1,n+1}$ are $\{r \in \mathfrak{S}_n \mid \mathrm{des}(r^{-1}) = k\}$, we first need to show that the permutations in $R'_{k+1,n+1}$ have the same property. For a simplex α, let $M_1 \cdots M_{n+1}$ be its vertex expression, with $M_1 = (m_1, \ldots, m_n)$ and $M_{n+1} = (m'_1, \ldots, m'_n) = M_1 + \sum_{i=1}^{n} e_i$. Let $r'_\alpha = a_1 a_2 \cdots a_n$ be the permutation in $R'_{k+1,n+1}$ corresponding to this simplex α. Then we have $M_{i+1} = M_i + e_{n+1-a_i}$.

Because of the restriction that $k \le z_n \le k + 1$ and $0 \le z_1 \le 1$ for both M_1 and M_{n+1}, we have $m_1 = 0$ and $m_n = k$. By the other restrictions that $0 \le z_{i+1} - z_i \le 1$, we need to go up by 1 k times from m_1 to m_n. So there exists a set $I \subset [n]$ with $\#I = k$, such that $m_{i+1} = m_i + 1$, for each $i \in I$, and $m_{j+1} = m_j$ for $j \in [n]\backslash I$.

To keep the above restrictions for each M_i, $i = 1, \dots, n$, we need to add e_i before e_{i+1} for $i \in I$, and add e_j before e_{j+1} for $j \in [n]\backslash I$. Then by the way we defined r'_α, we have $\mathrm{Des}(r'^{-1}_\alpha) = n + 1 - I$ and thus $\mathrm{des}(r'^{-1}_\alpha) = \#I = k$.

Now we want to show that the edges in the graph $R'_{k+1,n+1}$ are the same as in $R_{k+1,n+1}$. Since each edge in $L_{k+1,n+1}$ corresponds to a vertex-exchange, there are two types of edge in $L_{k+1,n+1}$.

First, exchange a vertex in the middle without touching the other vertices. An edge in $L_{k+1,n+1}$ changing the ith vertex with $i \neq 1$ and $i \neq n + 1$ corresponds to an edge in $R'_{k+1,n+1}$ exchanging the $(i - 1)$th and the ith letters of the permutation $r' \in R'_{k+1,n+1}$. By the restrictions $0 \le z_{j+1} - z_j \le 1$, we can make such a change if and only r'_{i-1} and r'_i are not consecutive numbers. Therefore, this edge is the type one edge in $R_{k+1,n+1}$.

Second, remove the first vertex and attach to the end another vertex. This edge in $L_{k+1,n+1}$ corresponds to the edge in $R'_{k+1,n+1}$ changing $r' = a_1 a_2 \cdots a_n$ to $s' = a_2 \cdots a_n a_1$. We claim that we can make such a change if and only if $a_1 \neq 1$ and $a_1 \neq n$. In fact, if $a_1 = n$, then for the second vertex of the simplex corresponding to r', we have $z_1 = 1$. Since the vertex expression of s' is obtained from that of r' by removing the first vertex of r' and attaching to the end another vertex, the first vertex of s' is the same as the second vertex of r'. So for the first vertex of s', we have $z_1 = 1$, but then we cannot add e_1 to s' any more, since we require $0 \le z_1 \le 1$; if $a_1 = 1$, then $z_n = k + 1$ for the first vertex of the simplex corresponding to s', so we cannot add e_n to s' any more, since we require $k \le z_n \le k + 1$. Therefore, this edge is the type two edge in $R_{k+1,n+1}$. □

Corollary 5.3

1. *Two simplices are in the same big block if and only if the first vertices in their vertex expression* $(L_{k+1,n+1})$ *is the same. This implies that their last vertices are also the same.*
2. *Two simplices only differ by the* $(i + 1)$*th vertex in the vertex expression, if and only if they are connected by an edge* e_i.

For $J \subset [n]$, we call e_i a *backward move* if $i \in J$ and $i + 1 \notin J$; and call it a *forward move* if $i \notin J$ and $i + 1 \in J$. Let $t \in s_{I,J}$ for some $I \subset [n]$. When we apply e_i to t, we get a simplex in a smaller small block if e_i is a backward move and in a bigger small block if e_i is a forward move. We call both backward and forward moves *movable edges*.

For any face F in $\Delta_{k+1,n+1}$, consider the subgraph of $Q_{k+1,n+1}$ with all simplices containing F, denoted by Q_F, and its restriction to a small block s, denoted by $Q_{F,s}$.

Lemma 5.4 *For any connected small block* s, $Q_{F,s}$ *is connected. In particular,* $Q_{F,s_{I,J_0}}$ *is connected, where* $J_0 = \{n - k + 1, \dots, n\}$.

Proof For any two simplices $t_1, t_2 \in Q_{F,s}$, let $t_1 = M_1 \cdots M_{n+1}$ and $t_2 = N_1 \cdots N_{n+1}$ be their vertex expressions. Since s is connected, there exists a path from t_1 to t_2 without any movable edges. So $M_i = N_i$ for all movable edges e_i. On the other hand, there exists a path from t_1 to t_2 using only edges e_j where $M_j \neq N_j$, this path is in Q_F. Since j is not those movable edges, this path is also in s, and thus t_1 to t_2 is connected by a path in $Q_{F,s}$.

We only need to show that s_{I,J_0} is connected, then by the first statement, $Q_{F,s_{I,J_0}}$ is connected.

For any fixed big block I, each permutation $w \in P_{k+1,n+1}$ is obtained by a set partition of $[n-k]$ and J_0 according to I, since $I = \mathrm{Des}(w)$ and $J_0 = \mathrm{LdDes}(w)$. For example, for $n = 9$, $k = 4$ and $I = \{1, 2, 5, 6\}$, each $w \in P$ is obtained as follows. We first choose two from $J_0 = \{6, 7, 8, 9\}$ to be $w_1 w_2$ and the other two to be $w_5 w_6$. Within each of the two 2-blocks, numbers need to be decreasing. Then choose two from $\{1, 2, 3, 4, 5\}$ to be $w_3 w_4$ and the other three to be $w_7 w_8 w_9$. Within each block, numbers need to be increasing. Then it is not hard to see that any two such permutations can be obtained from each other without using an e_{n-k}-edge, so s_{I,J_0} is connected. □

Proposition 5.5 *Q_F starts with a unique minimal connected small block.*

Proof Suppose not. Let $t_1 \in s_{I,J}$, $t_2 \in s_{I',J'}$ in two disconnected minimal small blocks in Q_F. Write them in vertex expression, we have $t_1 = M_1 \cdots M_{n+1}$ and $t_2 = N_1 \cdots N_{n+1}$.

If $I \neq I'$ and they are incomparable, then there exists another simplex $t \in b_{I''}$ in Q_F with $I'' < I'$ and $I'' < I$. In fact, looking at the poset $V_{k+1,n+1}$, both t_1, t_2 are some $n+1$-chains in $V_{k+1,n+1}$, their common vertices contain F, and they have different ending points M_{n+1}, N_{n+1}. Let $E \in t_1 \cap t_2$ be the maximal element of $t_1 \cap t_2$ in V, and let t be the chain ending at E and passing through $t_1 \cap t_2$. Then t has the desired property. So t_1, t_2 are not in minimal small blocks.

Now we assume $I = I'$. If $J = J'$, then by Lemma 5.4, $J \neq J_0$, so J has a backward move. We can show that there exists a backward move i of J such that $M_i \neq N_i$. First, it is easy to see that there exists a movable edge e_i such that $M_i \neq N_i$, otherwise $s_{I,J}$ is connected. Then by symmetry, it is impossible that all of these movable edges are forward moves. Then let t be the simplex obtained from t_1 by an e_i move. Since $M_i \neq N_i$, we have $M_i \notin F$. Therefore, $t \in Q_F$ and t is in a smaller small block, which contradicts the assumption that $s_{I,J}$ is a minimal small block in Q_F.

Now assume $J \neq J'$ and they are incomparable. By Lemma 4.5, part 2, we need to apply a sequence of moves to get from $s_{I,J}$ to $s_{I,J'}$. Since J, J' are incomparable, there exists a backward move for J, which is a necessary move from $s_{I,J}$ to $s_{I,J'}$. It follows that there exists such a move e_i with $M_i \neq N_i$. Then we can apply this move to t_1 and get a smaller small block in Q_F than $s_{I,J}$. □

6 Proof of Theorem 1.3: Second Shelling

We want to show that the h^*-polynomial of $\Delta'_{k+1,n+1}$ is also given by

$$\sum_{\substack{w \in \mathfrak{S}_n \\ \mathrm{des}(w)=k}} t^{\mathrm{cover}(w)},$$

we will define cover in a minute. Compare this to Theorem 3.1: if $\Delta'_{k+1,n+1}$ has a shellable unimodular triangulation $\Gamma_{k+1,n+1}$, then its h^*-polynomial is

$$\sum_{\alpha \in \Gamma_{k+1,n+1}} t^{\#(\alpha)}.$$

Similar to Theorem 1.2, we will define shellable unimodular triangulation for $\Delta'_{k+1,n+1}$, but this shelling is different from the one we use for Theorem 1.2. Label each simplex $\alpha \in \Gamma_{k+1,n+1}$ by a permutation $w_\alpha \in \mathfrak{S}_n$ with $\mathrm{des}(w_\alpha) = k$. Then show that $\#(\alpha) = \mathrm{cover}(w_\alpha)$.

We start from the graph $\Gamma_{k+1,n+1}$ studied in Sect. 3.3. Define a graph $M_{k+1,n+1}$ such that $v \in V(M_{k+1,n+1})$ if and only if $v^{-1} \in V(\Gamma_{k+1,n+1})$ and $(u, v) \in E(M_{k+1,n+1})$ if and only if $(u^{-1}, v^{-1}) \in E(\Gamma_{k+1,n+1})$. By Proposition 3.4, we have

$$V(M_{k+1,n+1}) = \{w \in \mathfrak{S}_n \mid \mathrm{des}(w) = k\},$$

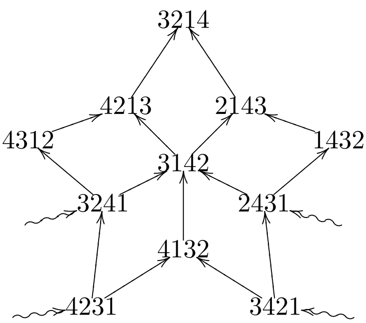
and $(w, u) \in E(M_{k+1,n+1})$ if and only if w and u are related in one of the following ways:

1. type one: exchanging the letters i and $i + 1$ if these two letters are not adjacent in w and u
2. type two: one is obtained by subtracting 1 from each letter of the other (1 becomes $n - 1$).

Now we want to orient the edges of $M_{k+1,n+1}$ to make it a digraph. Consider $e = (w, u) \in E(M_{k+1,n+1})$.

1. if e is of type one, and i is before $i + 1$ in w, i.e., $\mathrm{inv}(w) = \mathrm{inv}(u) - 1$, then orient the edge as $w \leftarrow u$.
2. if edge (w, u) is of type two, and v is obtained by subtracting 1 from each letter of u (1 becomes $n - 1$), then orient the edge as $w \leftarrow u$.

Example 6.1 Here is the directed graph $M_{3,5}$ for $\Delta'_{3,5}$:



Lemma 6.2 *There is no cycle in the directed graph $M_{k+1,n+1}$.*

Proof Let us call the subgraph of $M_{k+1,n+1}$ connected by only type one edges a component. Then there is no cycle involving type two edges since they all point in the same direction from one component to another. Then there is no cycle involving only type one edges either, since the number of inversions decreases along the directed path of type one edges. □

Therefore, $M_{k+1,n+1}$ defines a poset on $V(M_{k+1,n+1})$ and $M_{k+1,n+1}$ is the Hasse diagraph of the poset, which we still denote as $M_{k+1,n+1}$. This poset can be seen as a variation of the poset of the weak Bruhat order.

For an element in the poset $M_{k+1,n+1}$, the larger its rank is, the further its corresponding simplex is from the origin. More precisely, notice that each $v = (x_1, \dots, x_n) \in V_{k+1,n+1} = \Delta_{k+1,n+1} \cap \mathbb{Z}^n$ has $|v| = \sum_{i=1}^n x_i = k$ or $k+1$. For $u \in M_{k+1,n+1}$, by which we mean $u \in V(M_{k+1,n+1})$, define

$$A_u = \#\big\{v \text{ is a vertex of the simplex } s_{u^{-1}} \mid |v| = k+1\big\}.$$

Proposition 6.3 *Let $w > u$ in the above poset $M_{k+1,n+1}$. Then $A_w \ge A_u$.*

This proposition follows from the following lemma and the definition of the two types of directed edge.

Lemma 6.4 $A_u = u_n$.

Proof Let $w = u^{-1}$ and use the notations in section 2. Vertices of s_w are $\varphi(v_i)$ for $i = 0, \dots, n$. Since $v_0 = (0, \dots, 0)$, by (3.1), $|\varphi(v_0)| = k$, so $x_{n+1} = 1$ for $\varphi(v_0)$. By Lemma 3.3, from $\varphi(v_{n-u_n})$ to $\varphi(v_{n-u_n+1})$, $x_n x_{n+1}$ is changed from 01 to 10. Moreover $x_{n+1} = 1$, thus $|\varphi(v_i)| = \sum_{j=1}^n x_j = k$ for $i = 0, \dots, n-u_n$, and $x_{n+1} = 0$, thus $|\varphi(v_i)| = k+1$ for $i = n-u_n+1, \dots, n$. Therefore, there are u_n vertices with $|\varphi(v_i)| = k+1$, thus $A_u = u_n$. □

We define *cover* of a permutation $w \in M_{k+1,n+1}$ to be the number of permutations $v \in M_{k+1,n+1}$ it covers, i.e., the number of incoming edges of w in the graph $M_{k+1,n+1}$. From the above definition, we have the following, (in the half-open setting):

Lemma 6.5

1. *If* $w_1 = 1$, *then* $\mathrm{cover}(w) = \#\{i \in [n-1] \mid (w^{-1})_i + 1 < (w^{-1})_{i+1}\}$;
2. *if* $w_1 \ne 1$, *then* $\mathrm{cover}(w) = \#\{i \in [n-1] \mid (w^{-1})_i + 1 < (w^{-1})_{i+1}\} + 1$.

Proof The elements in $\{i \in [n-1] \mid (w^{-1})_i + 1 < (w^{-1})_{i+1}\}$ correspond to the type one edges pointing to w. So we need to show that w has an incoming type two edge in the graph for $\Delta'_{k,n}$ if and only if $w_1 \ne 1$. Let u be the permutation obtained by subtracting one from each letter of w (1 becomes $n-1$).

1. If $w_1 \ne 1$ and $w_{n-1} \ne 1$, then $\mathrm{des}(u) = \mathrm{des}(w)$, so $u \in M_{k,n}$.
2. If $w_{n-1} = 1$, then $\mathrm{des}(u) = \mathrm{des}(w) - 1$, so $u \in M_{k-1,n}$. Since we are considering the half-open setting, this incoming edge is still in $\Delta'_{k,n}$. This corresponds to the waved edges in the above example of $\Delta'_{3,5}$.
3. If $w_1 = 1$, then $\mathrm{des}(u) = \mathrm{des}(w) + 1$, so this edge is not in $\Delta'_{k,n}$. □

Recall the graph $R_{k+1,n+1}$ defined in Sect. 4 is obtained by

$$M_{k+1,n+1} \xrightarrow{w^{-1}} \Gamma_{k+1,n+1} \xrightarrow{\text{rev}} R_{k+1,n+1}.$$

By Proposition 5.2, $R_{k+1,n+1}$ is also obtained from the n-chain expression of each simplex in $\Delta_{k+1,n+1}$. We can describe the same orientation of edges (u, w) in $R_{k+1,n+1}$ with n-chain expression $u = L_1 < \cdots < L_{n+1}$ and $w = I_1 < \cdots < I_{n+1}$:

1. type one edge e_i: if $u_i < u_{i+1}$, then $u \leftarrow w$. We have $L_{i+1} \neq I_{i+1}$ with $\text{rank}(L_{i+1}) = \text{rank}(I_{i+1})$ in the poset V and $L_j = I_j$ for all $j \neq i+1$. $u_i < u_{i+1}$ if and only if the vector $L_{i+1} = (z_1, \ldots, z_n) < I_{i+1} = (z'_1, \ldots, z'_n)$ in dominance order, i.e., $z_n + \cdots + z_{n-\ell} \geq z'_n + \cdots + z'_{n-\ell}$ for all ℓ. Note that by definition, we have $z_n \geq z_{n-1} \geq \cdots \geq z_1$ and $z'_n \geq z'_{n-1} \geq \cdots \geq z'_1$.
2. type two edge: if $w = u_2 \cdots u_n u_1$, then $w \leftarrow u$. This corresponds to the case $w = L_2 < \cdots < L_{n+1} < L_1$ in the poset $V_{k+1,n+1}$.

With the above ordering on the n-chain expressions of simplices in $\Delta_{k+1,n+1}$, we can prove the following:

Proposition 6.6 *Any linear extension of the above ordering gives a shelling order on the triangulation of* $\Delta'_{k+1,n+1}$.

Proof We want to show that for any linear extension of the order in $M_{k+1,n+1}$, every simplex has a unique minimal nonface (see definitions in Sect. 3.1).

For each simplex $\alpha \in \Delta_{k+1,n+1}$, assign to it a face $F \subset \alpha$ in the following way. Each incoming edge $\alpha \xleftarrow{e_i} \alpha_i$ defines a unique vertex L_i of α that α has but α_i does not have. Then let $F = \{L_i\}$ be given by all the incoming edges of α. We want to show that F is the unique minimal face of α and it has never appeared before in any linear extension of the ordering given by the directed graph.

First, assume F has never appeared before, then it is clear that F is the unique minimal face, i.e., any proper subface of F has appeared before. In fact, let L_i be a vertex in F but not in $F' \subset F$. Then we have $F' \subset \alpha_i$ since α_i has every vertex of α except for L_i.

Now we will show that F has never appeared before α in any linear extension, i.e., for any other β which also has F, there exists a directed path from α to β. It suffices to show the following: for any face $F \subset \Delta_{k+1,n+1}$, the component M_F of simplices containing F has a unique source, and any other simplex is reachable from that source (there exists a directed path from α to β).

In M_F, let us first consider the subgraph of simplices starting with the same letter, say A, denoted by $M_{F,A}$. We want to prove that $M_{F,A}$ has a unique source, and any other simplex is reachable from that source. By the description of edges in $M_{k+1,n+1}$, simplices in $M_{F,A}$ are connected by type one edges. For any edge $e_i H = H_1 \cdots H_{n+1} \to W = W_1 \cdots W_{n+1}$, we have $i \neq 0, n$, $H_{i+1} \neq W_{i+1}$ and $H_j = W_j$ for all $j \neq i+1$. Now let $F \cup \{A, B\} = \{F_1 < F_2 < \cdots < F_\ell\}$ ordered as in the poset $V_{k+1,n+1}$. It is clear that all simplices $M_{F,A}$ are $(n+1)$-chain in the interval $[A, B]$, where $B = A + \sum_{i=1}^{n} e_i$ passing through $F_1, \ldots, F_\ell$. Now order the letters of the same rank in each of the intervals $[F_i, F_{i+1}]$ by dominance order. We claim that the unique source is the chain obtained by choosing the dominant maximal element in each rank. First, notice that in the interval $[F_i, F_{i+1}]$, if $\text{rank}(A_1) = \text{rank}(A_2) + 1 = k$ and both A_1 and A_2 are maximal in dominance order compared to other element in $[F_i, F_{i+1}]$ with ranks k and $k-1$ respectively, then we have $A_1 > A_2$. So the dominant maximal elements in each rank of $[F_i, F_{i+1}]$ and $F \cup \{A, B\}$ form a chain. Moreover, for any other chain in $M_{F,A}$, we can apply a simple move to change one vertex to a larger element in dominant order until we reach the chain with dominant maximal in each rank. Then the reachability also follows.

Now consider the whole M_F. We claim that the ending point of the source is the maximal element in F, denoted by F_h. Any chain β not ending with F_h ends with some letter larger than F_h in the poset $V_{k+1,n+1}$, then by moving down steps, there exists a simplex $\gamma \in M_{F,F'_h}$, where $F'_h = F_h - \sum_{i=1}^n e_i$ such that there is a directed path from γ to β. We know that M_{F,F'_h} has its unique source α, which connects to γ by a directed path towards γ. Thus we have a directed path from α to β via γ. □

It is clear that the shelling number of the simplex corresponding to w is $\text{cover}(w)$. Then by Theorem 3.1 and Proposition 6.6, we have a proof of Theorem 1.3. Combine the above with Theorem 1.2, we have an indirect proof of Corollary 1.4.

We want a direct combinatorial proof, which will give another proof of Theorem 1.2, and help us find a colored version of exceedance by Theorem 7.3 in the next section.

7 The h^*-Polynomial for Generalized Half-Open Hypersimplex

We want to extend Theorem 1.3 to the hyperbox $B = [0, a_1] \times \cdots \times [0, a_n]$. Write $\alpha = (a_1, \ldots, a_d)$ and define the generalized half-open hypersimplex as

$$\Delta'_{k,\alpha} = \{(x_1, \ldots, x_n) \mid 0 \le x_i \le a_i; k - 1 < x_1 + \cdots + x_n \le k\}. \tag{7.1}$$

Note that the above polytope is a multi-hypersimplex studied in [4]. For a nonnegative integral vector $\beta = (b_1, \ldots, b_n)$, let $C_\beta = \beta + [0, 1]^n$ be the cube translated from the unit cube by the vector β. We call β the *color* of C_β.

We extend the triangulation of the unit cube to B by translation and assign to each simplex in B a *colored permutation*

$$w_\beta \in \mathfrak{S}_\alpha = \{w \in \mathfrak{S}_n \mid b_i < a_i, i = 1, \ldots, n\}.$$

Let $F_i = \{x_i = 0\} \cap [0, 1]^n$ for $i = 1, \ldots, n$. Define the *exposed facets* for the simplex $s_{u^{-1}}$ in $[0, 1]^n$, with $u \in M$, to be $\text{Expose}(u) = \{i \mid s_{u^{-1}} \cap F_i \text{ is a facet of } s_{u^{-1}}\}$.

We can compute $\text{Expose}(u)$ explicitly as follows

Lemma 7.1 *Set* $u_0 = 0$. *Then* $\text{Expose}(u) = \{i \in [n] \mid u_{i-1} + 1 = u_i\}$.

Proof Denote $u^{-1} = w$. Let $\varphi(v_i)$, $i = 0, \ldots, n$ be the vertices of s_w. Then $i \in \text{Expose}(u)$ if and only if $x_i = 0$ for n vertices of s_w. By the description of vertices of s_w in Lemma 3.3, from $\varphi(v_{n-u_i})$ to $\varphi(v_{n-u_i+1})$, we change $x_i x_{i+1}$ from 01 to 10; and from $\varphi(v_{n-u_{i-1}})$ to $\varphi(v_{n-u_{i-1}+1})$, we change $x_{i-1} x_i$ from 01 to 10. If $u_{i-1} + 1 = u_i$, we have $v_{n-u_{i-1}} = v_{n-u_i+1}$. Then 1 will pass through x_i quickly and thus $x_i = 1$ for only one vertex $\varphi(v_{n-u_i+1})$ of s_w. Otherwise, $x_i = 1$ for more than one vertex. □

Now we want to extend the shelling on the unit cube to the larger rectangle. In this extension, F_i will be removed from C_β if $b_i \neq 0$. Therefore, for the simplex s_{w_β}, we will remove the facet $F_i \cap s_{w_\beta}$ for each $i \in \mathrm{Expose}(w) \cap \{i \mid b_i \neq 0\}$ as well as the $\mathrm{cover}(w_\beta)$ facets for neighbors within C_β. We call this set $\mathrm{Expose}(w) \cap \{i \mid b_i \neq 0\}$ the *colored exposed facet (cef)*, denoted by $\mathrm{cef}(w_\beta)$, for each colored permutation $w_\beta = (w, \beta)$.

Based on the above extended shelling, with some modifications of Proposition 6.6, we can show that the above order is a shelling order. We show the idea of the proof by the following example.

Example 7.2 Consider $\Delta'_{k,\alpha}$ for $\alpha = (1,2,2)$ and $k = 3$. In z-coordinates, where $z_i = x_1 + \cdots + x_i$, we have

$$V_{3,\{1,2,2\}} = \big\{A(0,0,2), B(0,1,2), C(1,1,2), F(0,2,2), G(1,2,2), D(0,1,3), E(1,1,3), H(0,2,3), I(1,2,3), L(1,3,3)\big\}.$$

Drawing them in the poset as described in Sect. 5, we have the following poset on the left. The simplices in the triangulation of $\Delta_{k,\alpha}$ are 3-chains of $V_{3,\{1,2,2\}}$ with distinct labels along the chain. We draw these 3-chains on the right with an edge between each pair of adjacent simplices.

If two simplices are in the same cube, then we orient the edges as in Sect. 3. If not, then the arrow points to the one whose permutation has fewer descents. With this extension, we can still compare two simplices that only differ by the $(i+1)$th vertices L_{i+1} and I_{i+1} by comparing L_{i+1} and I_{i+1} in the dominance order. So the proof of Proposition 6.6 holds for $\Delta_{k,\alpha}$ too.

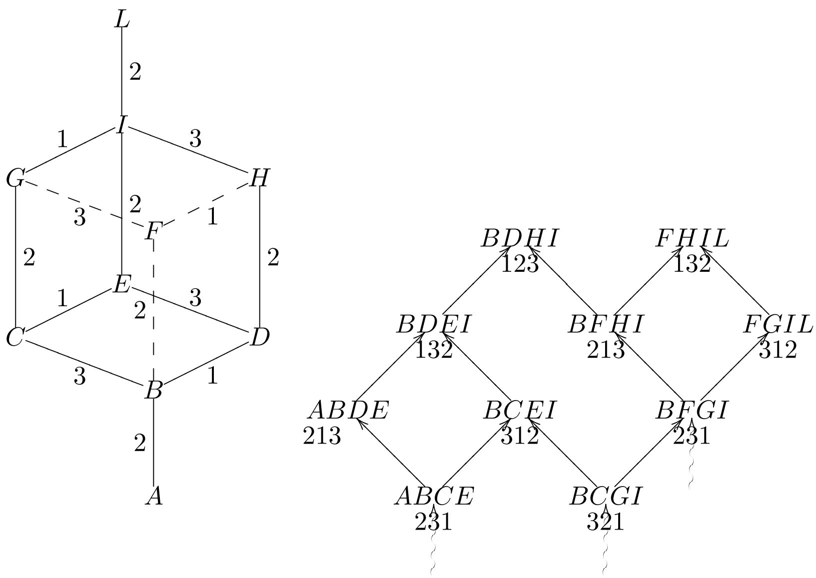


Table 1 $\mathrm{des}(w)=0$

w	$\mathrm{cover}(w)$	$\mathrm{Expose}(w)$	$\mathrm{cef}(w_{(0,0,1,3)})$	$\mathrm{cef}(w_{(0,1,0,3)})$	$\mathrm{cef}(w_{(0,1,1,2)})$
1234	0	$\{1,2,3,4\}$	2	2	3

Table 2 $\mathrm{des}(w)=1$

w	$\mathrm{cover}(w)$	$\mathrm{Expose}(w)$	$\mathrm{cef}(w_{(0,0,0,3)})$	$\mathrm{cef}(w_{(0,0,1,2)})$	$\mathrm{cef}(w_{(0,1,0,2)})$	$\mathrm{cef}(w_{(0,1,1,1)})$
1243	1	$\{1,2\}$	0	0	1	1
1342	1	$\{1,3\}$	0	1	0	1
1423	1	$\{1,4\}$	1	1	1	1
2341	1	$\{2,3\}$	0	1	1	2
3412	1	$\{2,4\}$	1	1	2	2
4123	1	$\{3,4\}$	1	2	1	2
1324	2	$\{1\}$	0	0	0	0
2314	2	$\{2\}$	0	0	1	1
3124	2	$\{3\}$	0	1	0	1
2134	2	$\{4\}$	1	1	1	1
2413	2	$\{\}$	0	0	0	0

Then, by Theorem 3.1 and the fact that the shelling number for w_β is $\mathrm{cover}(w_\beta)+\mathrm{cef}(w_\beta)$, we have the following theorem.

Theorem 7.3 *The h^*-polynomial for $\Delta'_{k,\alpha}$ is*

$$\sum_{\substack{w_\beta\in\mathfrak{S}_\alpha\\ \mathrm{des}(w)+|\beta|=k-1}} t^{\mathrm{cover}(w_\beta)+\mathrm{cef}(w_\beta)}.$$

Example 7.4 Consider $n=5$, $k=5$ and $\alpha=(1,2,2,4)$. We want to compute the h^*-polynomial of $\Delta'_{5,(1,2,2,4)}$ by Theorem 7.3, where the sum is over all (w,β) with $w\in\mathfrak{S}_4$, $\beta=(b_1,\dots,b_4)$ with $b_1=0$, $0\le b_2<2$, $0\le b_3<2$, $0\le b_4<4$ and $\mathrm{des}(w)+|\beta|=4$.

1. If $\mathrm{des}(w)=0$, we have $w=1234$, and the color β with $|\beta|=4$ is one of $(0,0,1,3)$, $(0,1,0,3)$ and $(0,1,1,2)$. From Table 1, we have
$$\sum_{\mathrm{des}(w)=0,|\beta|=4} t^{\mathrm{cover}(w_\beta)+\mathrm{cef}(w_\beta)}=2t^2+t^3.$$
2. If $\mathrm{des}(w)=1$, the color β with $|\beta|=3$ is one of $(0,0,0,3)$, $(0,0,1,2)$, $(0,1,0,2)$ and $(0,1,1,1)$. From Table 2, we have
$$\sum_{\mathrm{des}(w)=1,|\beta|=3} t^{\mathrm{cover}(w_\beta)+\mathrm{cef}(w_\beta)}=5t+26t^2+13t^3.$$

Table 3 $\mathrm{des}(w)=2$

w	$\mathrm{cover}(w)$	$\mathrm{Expose}(w)$	$\mathrm{cef}(w_{(0,0,0,2)})$	$\mathrm{cef}(w_{(0,0,1,1)})$	$\mathrm{cef}(w_{(0,1,0,1)})$	$\mathrm{cef}(w_{(0,1,1,0)})$
1432	1	{1}	0	0	0	0
3421	1	{2}	0	0	1	1
4231	1	{3}	0	1	0	1
4312	1	{4}	1	1	1	0
2143	2	{}	0	0	0	0
2431	2	{}	0	0	0	0
3214	2	{}	0	0	0	0
3241	2	{}	0	0	0	0
4132	2	{}	0	0	0	0
4213	2	{}	0	0	0	0
3142	3	{}	0	0	0	0

Table 4 $\mathrm{des}(w)=3$

w	$\mathrm{cover}(w)$	$\mathrm{Expose}(w)$	$\mathrm{cef}(w_{(0,1,0,0)})$	$\mathrm{cef}(w_{(0,0,1,0)})$	$\mathrm{cef}(w_{(0,0,0,1)})$
4321	1	{}	0	0	0

3. If $\mathrm{des}(w)=2$, the color β with $|\beta|=2$ is one of $(0,0,0,2)$, $(0,0,1,1)$, $(0,1,0,1)$ and $(0,1,1,0)$. From Table 3, we have

$$\sum_{\mathrm{des}(w)=2,|\beta|=2} t^{\mathrm{cover}(w_\beta)+\mathrm{cef}(w_\beta)} = 9t + 31t^2 + 4t^3.$$

4. If $\mathrm{des}(w)=3$, we have $w=4321$, and the color β with $|\beta|=1$ is one of $(0,1,0,0)$, $(0,0,1,0)$ and $(0,0,0,1)$. From Table 4, we have

$$\sum_{\mathrm{des}(w)=3,|\beta|=1} t^{\mathrm{cover}(w_\beta)+\mathrm{cef}(w_\beta)} = 3t.$$

To sum up, the h^*-polynomial of $\Delta'_{5,(1,2,2,4)}$ is $17t + 59t^2 + 18t^3$.

8 Some Identities

Proposition 8.1 *For any* $k \in [n-1]$, *we have*

1. $\#\{w \in \mathfrak{S}_n \mid \mathrm{exc}(w)=k, \mathrm{des}(w)=1\} = \binom{n}{k+1}$.
2. $\{w \in \mathfrak{S}_n \mid \mathrm{des}(w)=k, \mathrm{cover}(w)=1\} = \{w \in \mathfrak{S}_n \mid \#\mathrm{Expose}(w) = n-(k+1)\}$.
3. $\#\{w \in \mathfrak{S}_n \mid \mathrm{des}(w)=k, \mathrm{cover}(w)=1, \mathrm{Expose}(w)=S\} = 1$, *for any* $S \subset [n]$ *with* $|S| = n-(k+1)$.
4. $\#\{w \in \mathfrak{S}_n \mid \mathrm{des}(w)=k, \mathrm{cover}(w)=1\} = \binom{n}{k+1}$.

Proof

1. Notice that if i is an exceedance and $i+1$ is not, then i is a descent. Since $\mathrm{des}(w)=1$, all exceedances are next to each other. Let i be the first exceedance. Then it suffices to choose $i < w_i < w_{i+1} < \cdots < w_{n-k+1}$ to determine w.
2. Let i_0 be the smallest i such that $i \notin \mathrm{Expose}(w)$. Notice that this i_0 will cause one cover. In fact, if $i_0=1$, then $w_1 \neq 1$; if $i_0>1$, then $w_{i_0}-1$ is before w_{i_0} and they are not adjacent. Since $\mathrm{cover}(w)=1$, after the i_0th position of w, there is no $j\cdots(j+1)$. Then it follows that for each $i \notin \mathrm{Expose}(w)$ with $i \neq i_0$, $i-1$ is a descent of w. On the other hand, if $j \in \mathrm{Expose}(w)$, $j-1$ is not a descent. Therefore, to make $\mathrm{des}(w)=k$, we need k elements other than i_0 that are not in $\mathrm{Expose}(w)$.
3. Let $S=\{a_1,\dots,a_{k+1}\}$. It is easy to check that the only w satisfying the condition is the following: $w_1\cdots w_{a_1-1}=1\cdots(a_1-1)$, $w_{a_1}>w_{a_2}>\cdots>w_{a_{k+1}}$ and $w_{j+1}=w_j+1$ for $j=a_i,a_i+1,\dots,a_{i+1}-2$ if $a_{i+1}-a_i>1$ for $i=1,2,\dots,k+1$, where we set $a_{k+1}=n+1$. For example, if $S=\{2,3,5,7\}$ for $n=9$, then $w=197856234$.
4. Follows from (2) and (3). □

Proposition 8.2 *For any* $1<i<n$, *we have*

1. $\#\{w\in\mathfrak{S}_n \mid \mathrm{exc}(w)=1, \mathrm{des}(w)=k\}=\binom{n+1}{2k}$.
2. $\#\{w\in\mathfrak{S}_n \mid \mathrm{des}(w)=1, \#\mathrm{Expose}(w)=n-2k \text{ or } n+1-2k\}=1$
3. $\{w\in\mathfrak{S}_n \mid \mathrm{des}(w)=1, \#\mathrm{Expose}(w)=n-2k \text{ or } n+1-2k\}\subset\{w\in\mathfrak{S}_n \mid \mathrm{cover}(w)=k\}$.
4. $\#\{w\in\mathfrak{S}_n \mid \mathrm{des}(w)=1, \mathrm{cover}(w)=k\}=\binom{n}{2k}+\binom{n}{2k-1}=\binom{n+1}{2k}$.

Proof

1. Let the unique exceedance be i and assume $w_i=j>i$. First, we have $w_\ell=\ell$ for $\ell<i$ and $\ell>j$, also $w_\ell\le\ell$ for $i<\ell<j$. Now notice that if $i<\ell\in\mathrm{Des}(w)$, then we must have $w_\ell=\ell$, otherwise, we cannot have $w_h\le h$ for all $i<h<\ell$. Then, we can show that a $2k$-subset $\{i<i_1<j_1+1<i_2<j_2+1<\cdots<i_{k-1}<j_{k-1}+1<j+1\}\subset[n+1]$ corresponds to a unique such permutation w in the following way: $w_s=s$ for $i_\ell\le s\le j_\ell$, for all $1\le\ell\le k-1$ and then fill the gaps with the left numbers increasingly. We see that $\mathrm{Des}(w)=\{i,j_1,j_2,\dots,j_{k-1}\}$. For example, consider $\{2,3,4,6,8,9\}$ for $n=9$. First we have $w_1=1$, $w_9=9$; then we have $w_2=8$, $w_3=3$, $w_6w_7=67$. Finally we fill the positions w_4,w_5,w_8 with the rest of the numbers $2,4,5$, and get $w=183246759$ with $\mathrm{exc}(w)=1$ and $\mathrm{Des}(w)=\{2,3,7\}$. Conversely, it is easy to define a unique $2k$-subset as above for a given w.
2. Let $[n]-\mathrm{Expose}(w)=\{i_1,\dots,i_\ell\}$, where $\ell=2k-1$ or $2k$. It is not very hard to see that in order to make sure $\mathrm{des}(w)=1$, w has to be the following one. Define $w_i=i$ for $1\le i<i_1$. Then let $r=\lfloor\frac{\ell}{2}\rfloor$, define $B_j=w_{i_j}\cdots w_{i_{j+1}-1}$ for $1\le j\le r$ and $A_j=w_{i_{r+j}}\cdots w_{i_{r+j+1}-1}$ for $1\le j\le\ell-r$, where we set $i_{\ell+1}=n+1$. Then we put numbers $i_1,i_1+1,\dots,n$ into the positions $A_1B_1A_2B_2\cdots A_rB_r(A_{r+1})$ alternatively. For example, Let $[n]-\mathrm{Expose}(w)=\{3,4,6,8,9\}$ with $n=9$. Then $w=125893467$.

3. It is clear from the construction in (2), that w has k covers.
4. Follows from (2) and (3). □

See the relations between cover and Exposed set shown in Tables 2 and 3 for an example of the above two propositions.

Acknowledgements N. Li was partially supported by the US National Science Foundation under Grant DMS-0604423.

I thank my advisor Richard Stanley for introducing me to the problem and giving me help and encouragement, Ira Gessel for helpful communication about the generating function proof, Yan Zhang for a nice idea for the proof of Lemma 4.19, Dorian Croitoru and Steven Sam for helpful discussions and reading the draft carefully. I am also very grateful to the anonymous reviewer for very helpful comments and instructions.

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