Equilibrium correlations in charged fluids coupled to the radiation field

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Abstract

We provide an exact microscopic statistical treatment of particle and field correlations in a system of quantum charges in equilibrium with a classical radiation field. Using the Feynman-Kac-Itô representation of the Gibbs weight, the system of particles is mapped onto a collection of random charged wires. The field degrees of freedom can be integrated out, providing an effective pairwise magnetic potential. We then calculate the contribution of the transverse field coupling to the large-distance particle correlations. The asymptotics of the field correlations in the plasma are also exactly determined.

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1 Introduction

Thermal states of non relativistic particles interacting by the sole Coulomb potential are known to provide an adequate description of many states of matter. The introduction of magnetic interactions between the particles poses a novel problem since they are mediated by the coupling to the transverse part of the electromagnetic field. This immediately leads to consider the full system of matter in equilibrium with radiation : the relevant theory becomes then the thermal quantum electrodynamics (thermal QED).

In order to go beyond pure electrostatics without facing the full QED, a number of studies rely on the Darwin approximation. Darwin has shown [1], [2] that one can eliminate the transverse degrees of freedom of the field within the Lagrangian formalism up to order

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c^{-2} (c is the speed of light). A nice review of the derivation of the Darwin Lagrangian and a lucid discussion of its consequences can be found in [3]. The resulting Darwin Hamiltonian can be used to investigate the equilibrium properties of the so called weakly relativistic plasmas; see the recent works of Appel and Alastuey [4], [5], [6] and earlier references therein. These authors have done a careful analysis of the domain of validity of the Darwin approximation and shown in particular that the predictions of the Darwin Hamiltonian on the tail of particle correlations in thermal states cannot be correct. Indeed the well-known Bohr–van Leeuwen theorem [7] asserts that classical (non-quantum) matter completely decouples from the radiation field. Thus the Darwin Hamiltonian, which treats the particles classically, should not predict any effect of the transverse field when used for thermal equilibrium computations. The Darwin approximation is, however, not deprived of any meaning in statistical physics. Indeed, the authors show in [6] that Darwin predictions about current correlations coincide with those of thermal QED in the restricted window of distances $\lambda_{\text{part}} \ll r \ll \lambda_{\text{ph}}$, where $\lambda_{\text{part}} = \hbar\sqrt{\beta/m}$ is the de Broglie thermal wavelength of the particles and $\lambda_{\text{ph}} = \beta\hbar c$ the thermal wavelength of the photons. But to determine the tail $r \gg \lambda_{\text{ph}}$ of the correlations in the presence of the radiation field, matter has to be treated quantum mechanically to avoid the conclusion of the Bohr–van Leeuwen theorem. The situation is similar to orbital diamagnetism in equilibrium, which is of quantum-mechanical origin.

In this work, we consider equilibrium states of non-relativistic spinless quantum charges coupled with the radiation field in the standard way (section 2). We shall, however, treat the field classically on the ground that the large distances $r \gg \lambda_{\text{ph}}$ are controlled by the small wave numbers $k \sim \frac{1}{r} \ll \frac{1}{\lambda_{\text{ph}}}$, implying $\beta\hbar\omega_{\mathbf{k}} \sim \frac{\lambda_{\text{ph}}}{r} \ll 1$. Hence only long-wavelength photons will contribute to the asymptotics which is expected to be adequately described by classical fields. The full QED model with quantized electromagnetic field will be studied in a subsequent work (see also comments in the concluding remarks, section 8).

Our main tool will be the Feynman-Kac-Itô path integral representation of the degrees of freedom of the charges. The Feynman-Kac integral representation has been widely used to derive various properties of quantum Coulomb systems, in particular to determine the exact large-distance behaviour of the correlations; see [8], [9], and [10], [11] for reviews. In this representation quantum charges become fluctuating charged loops (closed Brownian paths), formally analogous to classical fluctuating wires carrying multipoles of all orders. These fluctuations are responsible for the lack of exponential screening in the quantum plasma and for an algebraic tail $\sim r^{-6}$ of the particle correlations [12].

Adding an external magnetic field produces a phase factor in the Feynman-Kac-Itô formula, whose argument is the flux of the magnetic field across the random loop. Correlations in the case of an homogeneous external magnetic field have been studied in [13]. When the particles are thermalized with the field, the latter becomes itself random and distributed according to the thermal weight of the free radiation. The system can be viewed as a classical-like system of random loops immersed in a random electromagnetic field. At this point, the field degrees of freedom can be exactly integrated out by means of a simple Gaussian integral since the Hamiltonian of free radiation is quadratic in the field amplitudes. One is then left with an effective pairwise current-current interaction between the loops which has a form similar to the magnetostatic energy between a pair of classical currents. For the sake of illustrating the basic mechanisms in a simple setting, this program is carried out in section 3 with particles obeying Maxwell-Boltzmann statistics. Appropriate modifications needed to take into account the particle statistics (Bose or Fermi) are given in section 7.

In section 4 we apply the formalism to the determination of the asymptotic form of the correlation between two quantum particles embedded in a classical plasma. This simple model already illustrates the main features occurring in the general system. The effective magnetic interaction contributes to the r^{-6} tail, but its ratio to the Coulombic contribution is of the order of the square of the relativistic parameter $(\beta mc^2)^{-1} = (\lambda_{\text{part}}/\lambda_{\text{ph}})^2$.

In section 5 we consider the generalization of the results obtained for two particles to the full system of quantum charges. The analysis relies on the technique of quantum Mayer graphs previously developed for Coulomb systems, and we merely point out the few changes that are needed to include the effective magnetic interactions.

Field fluctuations in plasmas have been studied for a long time at macroscopic scales, much larger than interparticle distances; see [2], [14] and references cited therein. In section 6, we reexamine this question from a microscopic viewpoint and show that electromagnetic field correlations are always long ranged due to the quantum nature of the particles. This is in disagreement with the prediction of macroscopic theories. We come back to this point in the concluding remarks (section 8). However, in the classical limit, we recover the fact already observed in [14] that the long-range behaviour of the longitudinal and transverse parts of the electric field correlations compensate exactly.

In section 7, we generalise the formalism developed in section 3 to include Bose and Fermi particle statistics. This is done as usual by decomposing the permutation group into cycles and grouping particles belonging to a cycle into an extended Brownian loop. When this is combined with the Feynman-Kac-Itô path integral representation of the particles, the system takes again a classical-like form: a collection of Brownian loops immersed in a classical random electromagnetic field. At this point the physical quantities can again be analyzed in terms of Mayer graphs comprising pairwise Coulomb and effective magnetic interactions, as in section 5.

The methods presented in this paper have been applied to the study of the semi-classical Casimir effect [15], [16].

2 The model

We first consider the QED model for non-relativistic quantum charges (electrons, nuclei, ions) with masses m_γ and charges e_γ contained in a box $\Lambda \in \mathbb{R}^3$ of linear size L and appropriate statistics. The index γ labels the $\mathcal{S}$ different species and runs from 1 to $\mathcal{S}$. The particles are in equilibrium with the radiation field at temperature T. The field is itself enclosed into a large box K with sides of length R, $R \gg L$. The Hamiltonian of the total finite volume system reads, in Gaussian units,

$$H_{L,R} = \sum_{i=1}^{n} \frac{\left(\mathbf{p}_i - \frac{e_{\gamma_i}}{c}\mathbf{A}(\mathbf{r}_i)\right)^2}{2m_{\gamma_i}} + \sum_{i<j}^{n} \frac{e_{\gamma_i} e_{\gamma_j}}{|\mathbf{r}_i - \mathbf{r}_j|} + \sum_{i=1}^{n} V_{\text{walls}}(\gamma_i, \mathbf{r}_i) + H_0^{\text{rad}}. \tag{1}$$

The sums run on all particles with position $\mathbf{r}_i$, momentum $\mathbf{p}_i$, and species index γ_i; $V_{\text{walls}}(\gamma_i, \mathbf{r}_i)$ is a steep external potential that confines a particle in Λ. It can eventually be taken infinitely steep at the wall's position, implying Dirichlet boundary conditions—*i.e.*, vanishing of the particle wave functions at the boundaries of Λ. The electromagnetic field is written in the Coulomb (or transverse) gauge so that the vector potential $\mathbf{A}(\mathbf{r})$ is divergence free and H_0^{rad} is the

Hamiltonian of the free radiation field. The Coulomb gauge is usually preferred for simplicity in situations where the particles are non-relativistic and high-energy processes are neglected [17]. It has the advantage to clearly disentangle electrostatic and magnetic couplings in the Hamiltonian.

We impose periodic boundary conditions on the faces of the large box K [2]. Hence expanding $\mathbf{A}(\mathbf{r})$ and the free photon energy H_0^{rad} in the plane-wave modes $\mathbf{k} = (\frac{2\pi n_x}{R}, \frac{2\pi n_y}{R}, \frac{2\pi n_z}{R})$ gives

$$\mathbf{A}(\mathbf{r}) = \left(\frac{4\pi\hbar c^2}{R^3}\right)^{1/2} \sum_{\mathbf{k}\lambda} g(\mathbf{k}) \frac{\mathbf{e}_{\mathbf{k}\lambda}}{\sqrt{2\omega_{\mathbf{k}}}} (a^*_{\mathbf{k}\lambda} \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{r}} + a_{\mathbf{k}\lambda} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}}) \tag{2}$$

$$H_0^{\mathrm{rad}} = \sum_{\mathbf{k}\lambda} \hbar\omega_{\mathbf{k}}\, a^*_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda} \tag{3}$$

where $a^*_{\mathbf{k}\lambda}$ and $a_{\mathbf{k}\lambda}$ are the creation and annihilation operators for photons of modes $(\mathbf{k}\lambda)$, $\mathbf{e}_{\mathbf{k}\lambda}$ $(\lambda = 1, 2)$ are two unit polarization vectors orthogonal to $\mathbf{k}$, and $\omega_{\mathbf{k}} = ck$, $k = |\mathbf{k}|$. In (2), $g(\mathbf{k})$, $g(0) = 1$, is a real spherically symmetric smooth form factor needed to take care of the ultraviolet divergencies. It is supposed to decay rapidly beyond the characteristic wave number $k_c = mc/\hbar$ (see [17], chap. 3). Since we are interested in the large-distance $r \to \infty$ asymptotics, related to the small-$\mathbf{k}$ behaviour $k \to 0$, the final result will be independent of this cut-off function.

The total partition function

$$Z_{L,R} = \mathrm{Tr}\ \mathrm{e}^{-\beta H_{L,R}} \tag{4}$$

is obtained by carrying the trace $\mathrm{Tr} = \mathrm{Tr}_{\mathrm{mat}}\mathrm{Tr}_{\mathrm{rad}}$ of the total Gibbs weight over particles' and the field's degrees of freedom : namely, on the particle wave functions with appropriate quantum statistics and on the Fock states of the photons. The average values of observables $\langle O_{\mathrm{mat}} \rangle = Z_{L,R}^{-1} \mathrm{Tr} \left(\mathrm{e}^{-\beta H_{L,R}} O_{\mathrm{mat}} \right)$ concerning only the particle degrees of freedom can be computed from the reduced thermal weight

$$\rho_{L,R} = \frac{\mathrm{Tr}_{\mathrm{rad}}\ \mathrm{e}^{-\beta H_{L,R}}}{Z_{0,R}^{\mathrm{rad}}}, \tag{5}$$

[2]Periodic conditions are convenient here. We could as well choose metallic boundary conditions. Since the field region K will be extended over all space right away, the choice of conditions on the boundaries of K are expected to make no differences for the particles confined in Λ.

where $Z_{0,R}^{\mathrm{rad}} = \mathrm{Tr}_{\mathrm{rad}} \exp(-\beta H_0^{\mathrm{rad}})$ is the partition function of the free radiation field, as follows from the obvious identity

$$\langle O_{\mathrm{mat}} \rangle = \frac{\mathrm{Tr}_{\mathrm{mat}}\,(O_{\mathrm{mat}}\,\rho_{L,R})}{\mathrm{Tr}_{\mathrm{mat}}\,\rho_{L,R}}. \tag{6}$$

We shall perform the thermodynamic limit in two stages by first letting $R \to \infty$. Then $\rho_L = \lim_{R\to\infty} \rho_{L,R}$ defines the effective statistical weight of the particles in Λ immersed in an infinitely extended thermalized radiation field.

As discussed in the Introduction, in this paper we treat the electromagnetic field classically. This amounts to replacing the photon creation and annihilation operators in (2) and (3) by complex amplitudes $\alpha^*_{\mathbf{k}\lambda}$ and $\alpha_{\mathbf{k}\lambda}$. In this case, the free field distribution factorizes out as $\exp(-\beta H_{R,L}) = \exp\left(-\beta H_0^{\mathrm{rad}}\right) \exp(-\beta H_{\mathbf{A}})$, where

$$H_{R,L} = H_{\mathbf{A}} + H_0^{\mathrm{rad}}, \quad H_{\mathbf{A}} = \sum_{i=1}^{n} \frac{\left(\mathbf{p}_i - \frac{e_{\gamma_i}}{c}\mathbf{A}(\mathbf{r}_i)\right)^2}{2m_{\gamma_i}} + U_{\mathrm{pot}}(\mathbf{r}_1, \gamma_1, \ldots, \mathbf{r}_n, \gamma_n), \tag{7}$$

and U_{pot} is the total potential energy. Since the free radiation weight $\exp\left(-\beta H_0^{\mathrm{rad}}\right)$ is Gaussian, $\mathbf{A}(\mathbf{r}) = \mathbf{A}(\mathbf{r}, \{\alpha_{\mathbf{k}\lambda}\})$ can be viewed as a realization of a Gaussian random field, and the term $H_{\mathbf{A}} = H_{\mathbf{A}}(\{\alpha_{\mathbf{k}\lambda}\})$ becomes the energy of the particles in a given realization of the vector potential having Fourier amplitudes $\{\alpha_{\mathbf{k}\lambda}\}$.

The partial trace (5) becomes, explicitly,

$$\rho_{L,R} = \left\langle \mathrm{e}^{-\beta H_{\mathbf{A}}} \right\rangle_{\mathrm{rad}}, \tag{8}$$

where for a general function $F(\{\alpha_{\mathbf{k}\lambda}\})$ of the mode amplitudes $\langle F \rangle_{\mathrm{rad}}$ denotes the normalized Gaussian average over all modes [3]

$$\langle F \rangle_{\mathrm{rad}} = \prod_{\mathbf{k}\lambda} \int \frac{\mathrm{d}^2\alpha_{\mathbf{k}\lambda}}{\pi} \left[\beta\hbar\omega_{\mathbf{k}} \mathrm{e}^{-\beta\hbar\omega_{\mathbf{k}}|\alpha_{\mathbf{k}\lambda}|^2}\right] F(\{\alpha_{\mathbf{k}\lambda}\}). \tag{9}$$

Note that the stability of Coulombic matter and the existence of thermodynamics for extended systems are assured if at least one of the species obeys Fermi statistics [18]. In the next section, merely

[3] The classical field is expanded as in (2) and (3) with dimensionless amplitudes $\alpha_{\mathbf{k}\lambda}$. In fact there will be no $\hbar$ dependence arising from the field, as seen by changing everywhere $\alpha_{\mathbf{k}\lambda} \mapsto \alpha_{\mathbf{k}\lambda}/\sqrt{\hbar}$.

as a matter of simplifying the presentation, we compute the effective particle interactions defined by ρ_L ignoring quantum statistics. In this case, Maxwell-Boltzmann statistics requires the presence of an additional short-range repulsive potential $V_{\rm sr}(\gamma_i, \gamma_j, |\mathbf{r}_i - \mathbf{r}_j|)$ in the Hamiltonian (1) to prevent the collapse of opposite charges and guarantee thermodynamical stability. The generalization to Fermi and Bose statistics will be given in section 7.

3 The gas of charged loops and the effective magnetic interaction

We now introduce the Feynman-Kac-Itô path integral representation of the configurational matrix element $\langle \mathbf{r}_1, ..., \mathbf{r}_n | \mathrm{e}^{-\beta H_{\mathbf{A}}} | \mathbf{r}_1, ..., \mathbf{r}_n \rangle$ for the particles interacting with a fixed realization of the field. For a single particle of mass m and charge e in a scalar potential $V^{\rm ext}(\mathbf{r})$ and vector potential $\mathbf{A}(\mathbf{r})$, we first recall that this matrix element reads [19], [20], [21]

$$\langle \mathbf{r} | \exp\left(-\beta \left[\frac{\left(\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{r})\right)^2}{2m} + V^{\rm ext}(\mathbf{r}) \right] \right) | \mathbf{r} \rangle = \left(\frac{1}{2\pi\lambda^2} \right)^{3/2} \int \mathrm{D}(\boldsymbol{\xi})$$
$$\times \exp\left(-\beta \left[\int_0^1 \mathrm{d}s \; V^{\rm ext}\big(\mathbf{r} + \lambda\boldsymbol{\xi}(s)\big) - i \frac{e}{\sqrt{\beta m c^2}} \int_0^1 \mathrm{d}\boldsymbol{\xi}(s) \cdot \mathbf{A}\big(\mathbf{r} + \lambda\boldsymbol{\xi}(s)\big) \right] \right). \tag{10}$$

Here $\boldsymbol{\xi}(s),\ 0 \le s \le 1,\ \boldsymbol{\xi}(0) = \boldsymbol{\xi}(1) = \mathbf{0}$, is a closed dimensionless Brownian path and $\mathrm{D}(\boldsymbol{\xi})$ is the corresponding conditional Wiener measure normalized to 1. It is Gaussian, formally written as
$\exp\Big(-\frac{1}{2}\int_0^1 \mathrm{d}s \left| \frac{\mathrm{d}\boldsymbol{\xi}(s)}{\mathrm{d}s} \right|^2 \Big) \mathrm{d}[\boldsymbol{\xi}(\cdot)]$, with zero mean and covariance

$$\int \mathrm{D}(\boldsymbol{\xi})\, \xi^\mu(s_1)\xi^\nu(s_2) = \delta^{\mu\nu}(\min(s_1,\ s_2) - s_1 s_2) \tag{11}$$

where $\xi^\mu(s)$ are the Cartesian coordinates of $\boldsymbol{\xi}(s)$. In this representation a quantum point charge looks like a classical charged closed loop denoted by $\mathcal{F} = (\mathbf{r}, \boldsymbol{\xi})$, located at $\mathbf{r}$ and with a random shape $\boldsymbol{\xi}(s)$ having an extension given by the de Broglie length $\lambda = \hbar\sqrt{\beta/m}$ (the quantum fluctuation). The magnetic phase in (10) is a stochastic line integral: it is the flux of the magnetic field across the closed loop. The correct interpretation of this stochastic integral is given by the rule of the middle point; namely, the integral on a small element of line $\mathbf{x}-\mathbf{x}'$ is defined by

$$\int_{\mathbf{x}}^{\mathbf{x}'} \mathrm{d}\boldsymbol{\xi} \cdot \mathbf{f}(\boldsymbol{\xi}) = (\mathbf{x}-\mathbf{x}') \cdot \mathbf{f}\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right), \quad \mathbf{x}-\mathbf{x}' \to 0 \tag{12}$$

We shall stick to this rule when performing explicit calculations.[4] Note the dimensionless relativistic factor $(\beta mc^2)^{-1/2}$ in front of the vector potential term.

This is readily generalized to a system of n interacting particles : The weight in the space of n loops $\mathcal{F}_1 = (\mathbf{r}_1, \gamma_1, \boldsymbol{\xi}_1), \ldots, \mathcal{F}_n = (\mathbf{r}_n, \gamma_n, \boldsymbol{\xi}_n)$ coming from the path integral representation of $\langle \mathbf{r}_1, ..., \mathbf{r}_n | \mathrm{e}^{-\beta H_{\mathbf{A}}} | \mathbf{r}_1, ..., \mathbf{r}_n \rangle$ is $\exp(-\beta U(\mathcal{F}_1, ..., \mathcal{F}_n, \mathbf{A}))$ where

$$\begin{aligned} U(\mathcal{F}_1, ..., \mathcal{F}_n, \mathbf{A}) = & \sum_{i<j}^{n} e_{\gamma_i} e_{\gamma_j} V_{\mathrm{c}}(\mathcal{F}_i, \mathcal{F}_j) \\ & - i \sum_{j=1}^{n} \frac{e_{\gamma_j}}{\sqrt{\beta m_{\gamma_j} c^2}} \int_0^1 \mathrm{d}\boldsymbol{\xi}_j(s) \cdot \mathbf{A}(\mathbf{r}_j + \lambda_{\gamma_j} \boldsymbol{\xi}_j(s)) \end{aligned} \tag{13}$$

The matrix element $\langle \mathbf{r}_1, ..., \mathbf{r}_n | \mathrm{e}^{-\beta H_{\mathbf{A}}} | \mathbf{r}_1, ..., \mathbf{r}_n \rangle$ is obtained by integrating $\exp(-\beta U(\mathcal{F}_1, ..., \mathcal{F}_n, \mathbf{A}))$ over the random shapes $\boldsymbol{\xi}_1, ..., \boldsymbol{\xi}_n$ of the loops, as in (10). In (13),

$$V_{\mathrm{c}}(\mathcal{F}_i, \mathcal{F}_j) = \int_0^1 \mathrm{d}s \; \frac{1}{|\mathbf{r}_i + \lambda_{\gamma_i} \boldsymbol{\xi}_i(s) - \mathbf{r}_j - \lambda_{\gamma_j} \boldsymbol{\xi}_j(s)|} \tag{14}$$

is the Coulomb potential between two loops, and for the sake of brevity, we have omitted the non electromagnetic terms

$$\sum_{i<j}^{n} V_{\mathrm{sr}}(\mathcal{F}_i, \mathcal{F}_j) + \sum_{i=1}^{n} V_{\mathrm{walls}}(\mathcal{F}_i) \tag{15}$$

corresponding to the short-range regularization and to the confinement potential. The vector potential term can be written as

[4] Other prescriptions are possible for the path integral to correctly represent the quantum mechanical Gibbs weight in presence of a magnetic field. The Itô rule may be used when $\mathbf{f}$ is divergence free [20].

$-i \int \mathrm{d}\mathbf{x}\ \mathbf{A}(\mathbf{x}) \cdot \boldsymbol{\mathcal{J}}(\mathbf{x})$ in terms of current densities associated with the Brownian loops :

$$\boldsymbol{\mathcal{J}}(\mathbf{x}) = \sum_{i=1}^{n} \mathbf{j}(\mathcal{F}_i, \mathbf{x}), \quad \mathbf{j}(\mathcal{F}_i, \mathbf{x}) = \frac{e_{\gamma_i}}{\sqrt{\beta m_{\gamma_i} c^2}} \int_0^1 \mathrm{d}\boldsymbol{\xi}_i(s)\ \delta(\mathbf{x} - \mathbf{r}_i - \lambda_{\gamma_i} \boldsymbol{\xi}_i(s)). \tag{16}$$

If one interprets the (ill-defined) derivative $\lambda_{\gamma_i} \mathrm{d}\boldsymbol{\xi}_i(s)/\mathrm{d}s = \mathbf{v}_i(s)$ as the "velocity" of a particle of charge e_{γ_i} moving along the loop $\boldsymbol{\xi}_i(s)$, the quantity $e_{\gamma_i} \mathbf{v}_i(s) \delta(\mathbf{x} - \mathbf{r}_i - \lambda_{\gamma_i} \boldsymbol{\xi}_i(s))$ corresponds to a classical current density. This is just a formal analogy. In subsequent calculations of stochastic integrals arising from (16), we will always use the mathematically well-defined rule of the middle point (12). Moreover, such "imaginary time" currents appearing in the Feynman-Kac-Itô representation are not the physical "real-time" current observables. Our definition (16) also includes the relativistic factor $(\beta m_{\gamma_i} c^2)^{-1/2}$.

A remarkable fact is that the transverse part of the field enters in $\exp(-\beta U(\mathcal{F}_1, ..., \mathcal{F}_n, \mathbf{A}))$ as a phase factor linear in $\mathbf{A}$ and its Fourier amplitudes (contrary to the Hamiltonian (1) written in operatorial form). Since the statistical weight $\mathrm{e}^{-\beta H_0^{\mathrm{rad}}}$ (3) is a Gaussian function of these Fourier amplitudes, it makes it possible to perform explicitly the partial trace over the field degrees of freedom in (8) according to the following steps :

$$\left\langle \exp\left[i\beta \int \mathrm{d}\mathbf{x}\ \mathbf{A}(\mathbf{x}) \cdot \boldsymbol{\mathcal{J}}(\mathbf{x}) \right] \right\rangle_{\mathrm{rad}} = \left\langle \prod_{\mathbf{k}\lambda} \exp\left[i(u^*_{\mathbf{k}\lambda} \alpha_{\mathbf{k}\lambda} + u_{\mathbf{k}\lambda} \alpha^*_{\mathbf{k}\lambda}) \right] \right\rangle_{\mathrm{rad}} =$$
$$\exp\left[-\frac{\beta}{2R^3} \sum_{\mathbf{k}\lambda} \frac{4\pi g^2(\mathbf{k})}{k^2} |\boldsymbol{\mathcal{J}}(\mathbf{k}) \cdot \mathbf{e}_{\mathbf{k}\lambda}|^2 \right] = \exp\left[-\frac{\beta}{2} \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3} (\mathcal{J}^\mu(\mathbf{k}))^* G^{\mu\nu}(\mathbf{k}) \mathcal{J}^\nu(\mathbf{k}) \right]. \tag{17}$$

The first equality is obtained by introducing the mode expansion (2), yielding

$$u_{\mathbf{k}\lambda} = \beta \left(\frac{4\pi\hbar c^2}{R^3} \right)^{1/2} \frac{g(\mathbf{k})}{\sqrt{2\omega_{\mathbf{k}}}} \boldsymbol{\mathcal{J}}(\mathbf{k}) \cdot \mathbf{e}_{\mathbf{k}\lambda}, \quad \boldsymbol{\mathcal{J}}(\mathbf{k}) = \int \mathrm{d}\mathbf{x}\ \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}} \boldsymbol{\mathcal{J}}(\mathbf{x}) \tag{18}$$

The second equality results from (8), (9) and the Gaussian integral $\int \frac{d^2\alpha}{\pi}\, \mathrm{e}^{-b|\alpha|^2 + i(u^*\alpha + u\alpha^*)} = b^{-1} \mathrm{e}^{-b^{-1}|u|^2}$, $b > 0$, whereas the infinite volume limit $R \to \infty$ and the polarization sum have been performed in the last equality. We have denoted by $G^{\mu\nu}(\mathbf{k})$ the covariance of the free transverse field :

$$G^{\mu\nu}(\mathbf{k}) = \frac{4\pi g^2(\mathbf{k})}{k^2}\delta^{\mu\nu}_{\mathrm{tr}}(\mathbf{k}), \quad \delta^{\mu\nu}_{\mathrm{tr}}(\mathbf{k}) = \delta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}, \quad k^\mu G^{\mu\nu}(\mathbf{k}) \equiv 0 \tag{19}$$

($\delta^{\mu\nu}_{\mathrm{tr}}(\mathbf{k})$ is the transverse Kronecker symbol). In (17) and throughout the paper, summation on repeated vector components $\mu, \nu = 1, 2, 3$ is understood. In the configuration space, the asymptotic behaviour of $G^{\mu\nu}(\mathbf{x})$ is obtained by approximating $g^2(\mathbf{k}) \sim 1$ in the inverse Fourier transform of $G^{\mu\nu}(\mathbf{k})$:

$$G^{\mu\nu}(\mathbf{x}) \sim \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\,\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\,\frac{4\pi}{k^2}\left(\delta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) = \frac{1}{2r}\left(\delta_{\mu\nu} + \frac{x^\mu x^\nu}{r^2}\right), \; r = |\mathbf{x}| \to \infty. \tag{20}$$

Decomposing the total current (16) into the individual loop currents we see that the effective weight (17) takes the form

$$\left\langle \exp\left[i\beta\int \mathrm{d}\mathbf{x}\;\mathbf{A}(\mathbf{x})\cdot\boldsymbol{\mathcal{J}}(\mathbf{x})\right]\right\rangle_{\mathrm{rad}} = \prod_{i=1}^{n}\exp\left(-\frac{\beta e^2_{\gamma_i}}{2}W_{\mathrm{m}}(i,i)\right) \times \exp\left(-\beta\sum_{i<j}^{n} e_{\gamma_i}e_{\gamma_j}W_{\mathrm{m}}(i,j)\right), \tag{21}$$

where for two loops $i = \mathcal{F}_i$ and $j = \mathcal{F}_j$ we have introduced the loop-loop effective magnetic potential

$$e_{\gamma_i}e_{\gamma_j}W_{\mathrm{m}}(i,j) = \int\mathrm{d}\mathbf{x}\int\mathrm{d}\mathbf{y}\;(j^\mu(\mathcal{F}_i,\mathbf{x}))^* G^{\mu\nu}(\mathbf{x}-\mathbf{y})j^\nu(\mathcal{F}_j,\mathbf{y}) = \tag{22}$$

$$= \frac{e_{\gamma_i}e_{\gamma_j}}{\beta\sqrt{m_{\gamma_i}m_{\gamma_j}}c^2}\int\frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\,\mathrm{e}^{i\mathbf{k}\cdot(\mathbf{r}_i-\mathbf{r}_j)}\int_0^1 \mathrm{d}\xi^\mu_i(s_1)\,\mathrm{e}^{i\mathbf{k}\cdot\lambda_{\gamma_i}\boldsymbol{\xi}_i(s_1)}\int_0^1\mathrm{d}\xi^\nu_j(s_2)\,\mathrm{e}^{-i\mathbf{k}\cdot\lambda_{\gamma_j}\boldsymbol{\xi}_j(s_2)}G^{\mu\nu}(\mathbf{k}).$$

As a consequence of Gaussian integration, one recovers pairwise interactions (22) between loops. The product in (21) contains the magnetic self-energies of the loops.

It is pleasing and convenient that after averaging over the field modes, the energy of the system of loops becomes an exact and explicit sum of pair potentials (and self-energies)[5] :

$$\left\langle \mathrm{e}^{-\beta U(\mathcal{F}_1,\ldots,\mathcal{F}_n,\mathbf{A})} \right\rangle_{\mathrm{rad}} = \Big[\prod_{i=1}^{n} \mathrm{e}^{-\frac{\beta e_{\gamma_i}^2}{2} W_{\mathrm{m}}(i,i)} \Big] \; \mathrm{e}^{-\beta \sum_{i<j} e_{\gamma_i} e_{\gamma_j} \left(V_{\mathrm{c}}(i,j) + W_{\mathrm{m}}(i,j) \right)}. \tag{23}$$

It is interesting to ask for the status of the partial density matrix (5) compared to that generated by the Darwin Hamiltonian $\rho_{\mathrm{Darwin}} \propto \mathrm{e}^{-\beta H_{\mathrm{Darwin}}}$ or, more generally, if $\rho_{L,R}$ can be cast in the form $\rho_{L,R} \propto \mathrm{e}^{-\beta H_{\mathrm{eff}}}$ for some tractable Hamiltonian $H_{\mathrm{eff}}(\{\mathbf{p}_i, \mathbf{r}_i\})$ depending on the canonical variables of the particles. The answer to this last question is very presumably negative. Indeed the magnetic interaction (22) is a two times functional of the Brownian loops; namely, it lacks the equal-time constraint occurring in the Coulomb potential (14) (see the discussion before (26) below) necessary to come back to a simple operator form by using the Feynman-Kac-Itô formula backwards. This is a well-known common feature of interactions resulting from integrating out external degrees of freedom [19].

The long-distance asymptotics of $W_{\mathrm{m}}(i,j)$ as $|\mathbf{r}_i - \mathbf{r}_j| \to \infty$ is determined by the small $\mathbf{k}$ behaviour in the integrand of (22). Noting that $\int_0^1 \mathrm{d}\boldsymbol{\xi}(s) = 0$ for a closed loop (Itô's lemma), one has

$$\int_0^1 \mathrm{d}\xi_i^\mu(s)\, \mathrm{e}^{i\mathbf{k}\cdot\lambda_{\gamma_i}\boldsymbol{\xi}_i(s)} \sim i\lambda_{\gamma_i} \int_0^1 \mathrm{d}\xi_i^\mu(s)\; \mathbf{k}\cdot\boldsymbol{\xi}_i(s), \quad \mathbf{k} \to 0, \tag{24}$$

and thus

$$\begin{aligned} W_{\mathrm{m}}(i,j) \sim & \qquad (25) \\ \sim \frac{\lambda_{\gamma_i}\lambda_{\gamma_j}}{\beta\sqrt{m_{\gamma_i} m_{\gamma_j}} c^2} & \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3} \mathrm{e}^{i\mathbf{k}\cdot(\mathbf{r}_i - \mathbf{r}_j)} \int_0^1 \mathrm{d}\xi_i^\mu(s_1)(\mathbf{k}\cdot\boldsymbol{\xi}_i(s_1)) \int_0^1 \mathrm{d}\xi_j^\nu(s_2)(\mathbf{k}\cdot\boldsymbol{\xi}_j(s_2)) G^{\mu\nu}(\mathbf{k}) \\ = \frac{\lambda_{\gamma_i}\lambda_{\gamma_j}}{\beta\sqrt{m_{\gamma_i} m_{\gamma_j}} c^2} & \int_0^1 \mathrm{d}\xi_i^\mu(s_1)(\boldsymbol{\xi}_i(s_1)\cdot\nabla_{\mathbf{r}_i}) \int_0^1 \mathrm{d}\xi_j^\nu(s_2)(\boldsymbol{\xi}_j(s_2)\cdot\nabla_{\mathbf{r}_j})\, G^{\mu\nu}(\mathbf{r}_i - \mathbf{r}_j), \end{aligned}$$

as $|\mathbf{r}_i - \mathbf{r}_j| \to \infty$. Upon using the asymptotic form (20) of $G^{\mu\nu}(\mathbf{r}_i - \mathbf{r}_j)$, it is clear that for fixed loop shapes $\boldsymbol{\xi}_i$ and $\boldsymbol{\xi}_j$ the decay of $W_{\mathrm{m}}(i,j)$ is $\sim |\mathbf{r}_i - \mathbf{r}_j|^{-3}$. It is of dipolar type modified by the constraint imposed by the transversality.

The Coulombic part (14) of the loop-loop interaction still decays as r^{-1} and deserves the following remark. From the Feynman-Kac

[5] We omit again in (23) the non-electromagnetic terms (15).

formula the potential (14) inherits the quantum-mechanical equal-time constraint; *i.e.*, every element of charge $e_{\gamma_i}\lambda_{\gamma_i}\mathrm{d}\boldsymbol{\xi}_i(s_1)$ of the first loop does not interact with every other element $e_{\gamma_j}\lambda_{\gamma_j}\mathrm{d}\boldsymbol{\xi}_j(s_2)$ as would be the case in classical physics, but the interaction takes place only if $s_1 = s_2$. It is therefore of interest to split

$$V_{\mathrm{c}}(i,j) = V_{\mathrm{elec}}(i,j) + W_{\mathrm{c}}(i,j), \tag{26}$$

where

$$V_{\mathrm{elec}}(i,j) = \int_0^1 \mathrm{d}s_1 \int_0^1 \mathrm{d}s_2 \, \frac{1}{|\mathbf{r}_i + \lambda_{\gamma_i}\boldsymbol{\xi}_i(s_1) - \mathbf{r}_j - \lambda_{\gamma_j}\boldsymbol{\xi}_j(s_2)|} \tag{27}$$

is a genuine classical electrostatic potential between two charged loops and

$$W_{\mathrm{c}}(i,j) = \int_0^1 \mathrm{d}s_1 \int_0^1 \mathrm{d}s_2 \, (\delta(s_1 - s_2) - 1) \frac{1}{|\mathbf{r}_i + \lambda_{\gamma_i}\boldsymbol{\xi}_i(s_1) - \mathbf{r}_j - \lambda_{\gamma_j}\boldsymbol{\xi}_j(s_2)|} \tag{28}$$

is the part of $V_{\mathrm{c}}(i,j)$ due to intrinsic quantum fluctuations ($W_{\mathrm{c}}(i,j)$ vanishes if $\hbar$ is set equal to zero). Because of the identities

$$\int_0^1 \mathrm{d}s_1 \, (\delta(s_1 - s_2) - 1) = \int_0^1 \mathrm{d}s_2 \, (\delta(s_1 - s_2) - 1) = 0, \tag{29}$$

the large-distance behaviour of W_{c} originates again from the term bilinear in $\boldsymbol{\xi}_i$ and $\boldsymbol{\xi}_j$ in the multipolar expansion of the Coulomb potential in (28)

$$W_{\mathrm{c}}(i,j) \sim \int_0^1 \mathrm{d}s_1 \int_0^1 \mathrm{d}s_2 \, (\delta(s_1 - s_2) - 1) \, (\lambda_{\gamma_i}\boldsymbol{\xi}_i(s_1) \cdot \nabla_{\mathbf{r}_i}) \left(\lambda_{\gamma_j}\boldsymbol{\xi}_j(s_2) \cdot \nabla_{\mathbf{r}_j}\right) \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|}. \tag{30}$$

It is dipolar and formally similar to that of two electrical dipoles of sizes $e_{\gamma_i}\lambda_{\gamma_i}\boldsymbol{\xi}_i$ and $e_{\gamma_j}\lambda_{\gamma_j}\boldsymbol{\xi}_j$.

4 Two quantum charges in a classical plasma

In order to exhibit the effect of the magnetic potential on the particle correlations, we consider the simple model of two quantum charges

e_a and e_b with corresponding loops $\mathcal{F}_a = (\mathbf{r}_a, \boldsymbol{\xi}_a)$ and $\mathcal{F}_b = (\mathbf{r}_b, \boldsymbol{\xi}_b)$ immersed in a configuration ω of classical charges, following section VII of [12] or section IV.C of [11]. According to (26) one can decompose the total energy as $\mathcal{U}(\mathcal{F}_a, \mathcal{F}_b, \omega) = e_a e_b W(\mathcal{F}_a, \mathcal{F}_b) + U_{\rm cl}(\mathcal{F}_a, \mathcal{F}_b, \omega)$ where $W(\mathcal{F}_a, \mathcal{F}_b) = W_{\rm c}(\mathcal{F}_a, \mathcal{F}_b) + W_{\rm m}(\mathcal{F}_a, \mathcal{F}_b)$ is the sum of the electric and magnetic quantum dipolar interactions and $U_{\rm cl}(\mathcal{F}_a, \mathcal{F}_b, \omega)$ is the purely classical Coulomb energy (27) of the two loops $\mathcal{F}_a$ and $\mathcal{F}_b$ together with that of the particles in the configuration ω. The correlation $\rho(\mathcal{F}_a, \mathcal{F}_b)$ between the loops is obtained by integrating out the coordinates ω of the classical charges :

$$\rho(\mathcal{F}_a, \mathcal{F}_b) = \frac{1}{\Xi_{\rm cl}} \int_\Lambda {\rm d}\omega \ {\rm e}^{-\beta \mathcal{U}(\mathcal{F}_a, \mathcal{F}_b, \omega)} = {\rm e}^{-\beta e_a e_b W(\mathcal{F}_a, \mathcal{F}_b)} \rho_{\rm cl}(\mathcal{F}_a, \mathcal{F}_b), \tag{31}$$

where $\Xi_{\rm cl}$ is the partition function of the classical plasma and $\rho_{\rm cl}(\mathcal{F}_a, \mathcal{F}_b)$ is the correlation of the two loops embedded in the plasma interacting with genuine classical Coulomb forces. In the latter quantity, the classical theory of screening applies so that effective interaction between the loops decay exponentially fast [6]. Thus one can approximate $\rho_{\rm cl}(\mathcal{F}_a, \mathcal{F}_b)$ in (31) by $\rho(\mathcal{F}_a)\rho(\mathcal{F}_b)$ up to a term exponentially decaying as $|\mathbf{r}_a - \mathbf{r}_b| \to \infty$. Furthermore, integrating $\rho(\mathcal{F}_a, \mathcal{F}_b)$ on the loop shapes leads to the following expression for the positional correlation of the quantum charges

$$\begin{aligned}
\rho(\mathbf{r}_a, \mathbf{r}_b) &= \int {\rm D}(\boldsymbol{\xi}_a) \int {\rm D}(\boldsymbol{\xi}_b) \ {\rm e}^{-\beta e_a e_b W(\mathcal{F}_a, \mathcal{F}_b)} \rho(\mathcal{F}_a)\rho(\mathcal{F}_b) + \mathcal{O}({\rm e}^{-C|\mathbf{r}_a - \mathbf{r}_b|}) = \\
&= \rho_a \rho_b - \beta e_a e_b \int {\rm D}(\boldsymbol{\xi}_a) \int {\rm D}(\boldsymbol{\xi}_b) \ W(\mathcal{F}_a, \mathcal{F}_b) \rho(\boldsymbol{\xi}_a)\rho(\boldsymbol{\xi}_b) + \\
&\quad + \frac{1}{2}\beta^2 e_a^2 e_b^2 \int {\rm D}(\boldsymbol{\xi}_a) \int {\rm D}(\boldsymbol{\xi}_b) \ W^2(\mathcal{F}_a, \mathcal{F}_b) \rho(\boldsymbol{\xi}_a)\rho(\boldsymbol{\xi}_b) + ... + \mathcal{O}({\rm e}^{-C|\mathbf{r}_a - \mathbf{r}_b|})
\end{aligned} \tag{32}$$

Since $W(\mathcal{F}_a, \mathcal{F}_b) \sim |\mathbf{r}_a - \mathbf{r}_b|^{-3}$ (see (25), (30)), the above expansion in powers of W generates algebraically decaying terms at large separation. It is known that in a homogeneous and isotropic phase, the electric dipole part $W_{\rm c}$ does not contribute at linear order [12], [11]. The same is true for the magnetic part. To see this, it is convenient

[6] The usual Debye theory of screening has been rigorously shown to be valid at least at sufficiently high temperature [22].

to write the linear $W_{\rm m}$ term of (32) as

$$
\begin{aligned}
&-\beta e_a e_b \int \mathrm{D}(\boldsymbol{\xi}_a)\int \mathrm{D}(\boldsymbol{\xi}_b)\ W_{\rm m}(\mathcal{F}_a,\mathcal{F}_b)\rho(\boldsymbol{\xi}_a)\rho(\boldsymbol{\xi}_b) \\
&= -\frac{\beta e_a e_b}{\sqrt{\beta m_a c^2}\sqrt{\beta m_b c^2}} \int \frac{d^3\mathbf{k}}{(2\pi)^3}\ \mathrm{e}^{i\mathbf{k}\cdot(\mathbf{r}_a-\mathbf{r}_b)}\ t_a^{\mu *}(\mathbf{k}) t_b^{\nu}(\mathbf{k}) G^{\mu\nu}(\mathbf{k}). \qquad (33)
\end{aligned}
$$

The stochastic $\boldsymbol{\xi}_a$-line-integral is now included in the definition of the tensor

$$
t_a^{\mu}(\mathbf{k}) = \int \mathrm{D}(\boldsymbol{\xi}_a)\rho(\boldsymbol{\xi}_a)\int_0^1 \mathrm{d}\xi_a^{\mu}(s)\ \mathrm{e}^{-i\lambda_a \mathbf{k}\cdot\boldsymbol{\xi}_a(s)} \qquad (34)
$$

and likewise for $t_b^{\nu}(\mathbf{k})$. Since both the measure $\mathrm{D}(\boldsymbol{\xi}_a)$ and $\rho(\boldsymbol{\xi}_a)$ are invariant under a rotation of $\boldsymbol{\xi}_a$ in an isotropic system, $t_a^{\mu}(\mathbf{k})$ transforms in a covariant manner under rotations of $\mathbf{k}$. Thus it is necessarily of the form $t_a^{\mu}(\mathbf{k}) = k^{\mu} f_a(|\mathbf{k}|)$, implying the vanishing of (33) because of the transversality of $G^{\mu\nu}(\mathbf{k})$. One concludes that the slowest non-vanishing contribution comes from the W^2 term in (32)

$$
\rho(\mathbf{r}_a,\mathbf{r}_b) - \rho_a\rho_b = \frac{A(\beta)}{|\mathbf{r}_a-\mathbf{r}_b|^6} + \mathrm{O}\left(\frac{1}{|\mathbf{r}_a-\mathbf{r}_b|^8}\right). \qquad (35)
$$

The temperature-dependent amplitude $A(\beta) = A_{\rm cc}(\beta) + A_{\rm mm}(\beta) + A_{\rm cm}(\beta)$ involves in principle electric and magnetic contributions from ${W_{\rm c}}^2$ and ${W_{\rm m}}^2$, as well as a cross contribution from $2W_{\rm c}W_{\rm m}$. These contributions can be calculated explicitly at lowest order in $\hbar$ (or equivalently in the high-temperature limit $\beta \to 0$). The electric contribution in this limit is known to be [12], [11]

$$
A_{\rm cc}(\beta) \sim \hbar^4 \frac{\beta^4}{240}\frac{e_a^2 e_b^2}{m_a m_b}\rho_a\rho_b. \qquad (36)
$$

To compute the magnetic contribution in the same limit, we write the quadratic term

$$
\begin{aligned}
&\frac{\beta^2 e_a^2 e_b^2}{2}\int \mathrm{D}(\boldsymbol{\xi}_a)\rho(\boldsymbol{\xi}_a)\int \mathrm{D}(\boldsymbol{\xi}_b)\rho(\boldsymbol{\xi}_b)\ {W_{\rm m}}^2(\mathcal{F}_a,\mathcal{F}_b) = \frac{e_a^2 e_b^2}{2m_a c^2 m_b c^2} \\
&\times \int \frac{d^3\mathbf{k}_1}{(2\pi)^3}\int \frac{d^3\mathbf{k}_2}{(2\pi)^3}\ \mathrm{e}^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot(\mathbf{r}_a-\mathbf{r}_b)}\ (T_a^{\mu\nu}(\mathbf{k}_1,\mathbf{k}_2))^* T_b^{\sigma\tau}(\mathbf{k}_1,\mathbf{k}_2) G^{\mu\sigma}(\mathbf{k}_1) G^{\nu\tau}(\mathbf{k}_2) \\
&\qquad (37)
\end{aligned}
$$

in terms of the tensors

$$T_a^{\mu\nu}(\mathbf{k}_1,\mathbf{k}_2) = \int \mathrm{D}(\boldsymbol{\xi}_a)\rho(\boldsymbol{\xi}_a)\int_0^1 \mathrm{d}\xi_a^\mu(s_1)\int_0^1 \mathrm{d}\xi_a^\nu(s_2)\ \mathrm{e}^{-i\lambda_a\mathbf{k}_1\cdot\boldsymbol{\xi}_a(s_1)}\mathrm{e}^{-i\lambda_a\mathbf{k}_2\cdot\boldsymbol{\xi}_a(s_2)} \tag{38}$$

and $T_b^{\sigma\tau}(\mathbf{k}_1,\mathbf{k}_2)$, defined likewise. As usual the behaviour at large distances is controlled by that of the integrand of (37) at small wave numbers. Expanding (38) at lowest order in $\mathbf{k}_1$ and $\mathbf{k}_2$ gives

$$\begin{aligned} T_a^{\mu\nu}(\mathbf{k}_1,\mathbf{k}_2) &\sim \int \mathrm{D}(\boldsymbol{\xi}_a)\rho(\boldsymbol{\xi}_a)\int_0^1 \mathrm{d}\xi_a^\mu(s_1)\int_0^1 \mathrm{d}\xi_a^\nu(s_2)\,(-i\lambda_a\mathbf{k}_1\cdot\boldsymbol{\xi}_a(s_1))\,(-i\lambda_a\mathbf{k}_2\cdot\boldsymbol{\xi}_a(s_2)) \\ &= -\lambda_a^2 k_1^\epsilon k_2^\eta \int \mathrm{D}(\boldsymbol{\xi}_a)\rho(\boldsymbol{\xi}_a)\int_0^1 \mathrm{d}\xi_a^\mu(s_1)\int_0^1 \mathrm{d}\xi_a^\nu(s_2)\ \xi_a^\epsilon(s_1)\xi_a^\eta(s_2) \end{aligned} \tag{39}$$

and likewise for $T_b^{\sigma\tau}(\mathbf{k}_1,\mathbf{k}_2)$. One sees that because of the factor $\lambda_a^2\lambda_b^2$, the overall contribution in (37) will have a $\hbar^4$ factor so that at this order we can neglect the quantum fluctuation in the density setting $\rho(\boldsymbol{\xi}_a) \sim \rho_a$ independent of $\boldsymbol{\xi}_a$. Thus the stochastic integral to be calculated becomes (appendix A)

$$\int \mathrm{D}(\boldsymbol{\xi})\int_0^1 \mathrm{d}\xi^\mu(s)\int_0^1 \mathrm{d}\xi^\nu(t)\ \xi^\epsilon(s)\xi^\eta(t) = \frac{1}{12}(\delta^{\mu\nu}\delta^{\eta\epsilon} - \delta^{\mu\eta}\delta^{\nu\epsilon}), \tag{40}$$

leading to

$$\begin{aligned} T_a^{\mu\nu}(\mathbf{k}_1,\mathbf{k}_2) &\sim -\frac{\lambda_a^2\rho_a}{12}(\delta^{\mu\nu}\mathbf{k}_1\cdot\mathbf{k}_2 - k_2^\mu k_1^\nu), \\ T_b^{\sigma\tau}(\mathbf{k}_1,\mathbf{k}_2) &\sim -\frac{\lambda_b^2\rho_b}{12}(\delta^{\sigma\tau}\mathbf{k}_1\cdot\mathbf{k}_2 - k_2^\sigma k_1^\tau). \end{aligned} \tag{41}$$

When this is inserted into (37) and summation on vectorial indices are performed, one finds the expression

$$A\int\frac{\mathrm{d}\mathbf{k}_1}{(2\pi)^3}\int\frac{\mathrm{d}\mathbf{k}_2}{(2\pi)^3}\,\mathrm{e}^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot(\mathbf{r}_a-\mathbf{r}_b)}\,(4\pi)^2|g(\mathbf{k}_1)|^2|g(\mathbf{k}_2)|^2\left[1+\frac{(\mathbf{k}_1\cdot\mathbf{k}_2)^2}{k_1^2k_2^2}\right], \tag{42}$$

with $A = \frac{\lambda_a^2\lambda_b^2e_a^2e_b^2\rho_a\rho_b}{288m_am_bc^4}$. The first term in the large brackets gives a rapidly decaying contribution since it involves the Fourier transform of the form factor $g^2(\mathbf{k})$. The algebraic large-distance contribution comes from the second term which reads, after Fourier transformation (approximating $g(\mathbf{k}) \sim 1$, $\mathbf{k} \to 0$),

$$A \left(\partial_\mu \partial_\nu \frac{1}{|\mathbf{r}_a - \mathbf{r}_b|} \right) \left(\partial_\mu \partial_\nu \frac{1}{|\mathbf{r}_a - \mathbf{r}_b|} \right) = A \, \frac{6}{|\mathbf{r}_a - \mathbf{r}_b|^6}. \tag{43}$$

Finally one checks that there is no cross Coulomb-magnetic contribution $A_{\mathrm{cm}}(\beta)$ at the dominant order $\mathbf{r}^{-6}$ as a consequence of transversality (appendix B). So adding (36) and (43) gives the final result

$$\rho(\mathbf{r}_a, \mathbf{r}_b) - \rho_a \rho_b \sim \hbar^4 \beta^4 \frac{\rho_a \rho_b e_a^2 e_b^2}{240 \, m_a m_b} \left[1 + \frac{5}{(\beta m_a c^2)(\beta m_b c^2)} \right] \frac{1}{|\mathbf{r}_a - \mathbf{r}_b|^6} \tag{44}$$

as $|\mathbf{r}_a - \mathbf{r}_b| \to \infty$ and at lowest order in $\hbar$. One sees from (14) and (22) that the order of magnitude of the ratio $W_{\mathrm{m}}/V_{\mathrm{c}}$ is $(\beta mc^2)^{-1}$. In an electrolyte at room temperature $T = 300K$, this ratio is found to be $\approx 10^{-11}$. The magnetic correction to the correlation decay (44) is negligible in this case.

5 Particle correlations in the many-body system

We apply the formalism developed in section 3 to the determination of the large-distance decay of the particle density correlations in the more general case where all particles are quantum-mechanical, but still obeying Maxwell-Boltzmann statistics.

We show hereafter that the algebraic r^{-6} decay of the (truncated) particle density correlations

$$\rho_{\mathrm{T}}(\gamma_a, \mathbf{r}_a, \gamma_b, \mathbf{r}_b) \sim \frac{A_{ab}(\beta, \{\rho_\gamma\})}{|\mathbf{r}_a - \mathbf{r}_b|^6}, \qquad |\mathbf{r}_a - \mathbf{r}_b| \to \infty \tag{45}$$

found in the absence of the radiation field [9], [11] is not altered, but that the coefficient $A_{ab}(\beta, \{\rho_\gamma\})$ contains in addition small magnetic terms of the order $(\beta mc^2)^{-2}$, as in (44). As an illustration, we give the lowest order contribution of this coefficient with respect to Planck's constant $\hbar$.

By the Feynman-Kac-Itô representation, the full system composed of quantum point charges coupled to the radiation field has reduced to a classical-like system of extended charged loops $\mathcal{F} = (\mathbf{r}, \gamma, \boldsymbol{\xi})$ for which all the methods of classical statistical mechanics apply. The only novelty comes from the additional magnetic potential $W_{\rm m}$. In the following, we merely summarize the arguments since they are essentially the same as those found in [9], [11] when no radiation field is present.

As usual, we express the truncated two-loop correlation $\rho_{\rm T}(\mathcal{F}_a,\mathcal{F}_b) = \rho(\mathcal{F}_a)\rho(\mathcal{F}_b)h(\mathcal{F}_a,\mathcal{F}_b)$ in terms of the loop Ursell function $h(\mathcal{F}_a,\mathcal{F}_b)$. The latter function can be expanded in a formal diagrammatic Mayer series of powers of the loop densities $\rho(\mathcal{F})$. One needs to resum the long-range part of the Coulomb potential $V_{\rm c}$, which is responsible for the non-integrability of the Mayer bonds $f(\mathcal{F}_i,\mathcal{F}_j) = \exp(-\beta e_{\gamma_i}e_{\gamma_j}[V_{\rm c}(\mathcal{F}_i,\mathcal{F}_j) + W_{\rm m}(\mathcal{F}_i,\mathcal{F}_j)]) - 1$ at infinity. Using the decomposition (26) we resum the convolution chains built with the purely electrostatic long-range part $V_{\rm elec}(\mathcal{F},\mathcal{F}')$ into a Debye-Hückel-type screened potential $\Phi_{\rm elec}(\mathcal{F},\mathcal{F}')$. Then reorganizing the diagrams leads to a representation of the loop Ursell function by terms of so-called prototype diagrams, built with the two kinds of bonds

$$F(\mathcal{F},\mathcal{F}') = -\beta e_\gamma e_{\gamma'}\Phi_{\rm elec}(\mathcal{F},\mathcal{F}'), \tag{46}$$

$$F^{\rm R}(\mathcal{F},\mathcal{F}') = {\rm e}^{-\beta e_\gamma e_{\gamma'}[\Phi_{\rm elec}(\mathcal{F},\mathcal{F}')+W(\mathcal{F},\mathcal{F}')]} - 1 + \beta e_\gamma e_{\gamma'}\Phi_{\rm elec}(\mathcal{F},\mathcal{F}'), \tag{47}$$

where we have defined $W = W_{\rm c} + W_{\rm m}$ as in section 4. [7]

The potential $\Phi_{\rm elec}(\mathcal{F},\mathcal{F}')$ has been studied in [23]. It corresponds to the term $n = 0$ of the full quantum analog of the Debye-Hückel potential given by formula (89) of [23]. This contribution $n = 0$ is shown to be decaying at infinity faster than any inverse power of $|\mathbf{r}-\mathbf{r}'|$ (see formula (58) of [23], and the comment following it).

The asymptotic decay of the two-particle correlation $\rho_{\rm T}(\gamma_a,\mathbf{r}_a,\gamma_b,\mathbf{r}_b)$ is inferred from that of the loop correlation $\rho_{\rm T}(\mathcal{F}_a,\mathcal{F}_b)$ by integrating it over the Brownian shapes $\boldsymbol{\xi}_a$ and $\boldsymbol{\xi}_b$. The bond F is rapidly decreasing, and the asymptotic decay of $F^{\rm R}$ is dominated by the dipolar decays of $W_{\rm c}$ and $W_{\rm m}$: $F^{\rm R}(\mathcal{F},\mathcal{F}') \sim -\beta e_\gamma e_{\gamma'}W(\mathcal{F},\mathcal{F}')$ as $|\mathbf{r}-\mathbf{r}'| \to \infty$. We further extract this dipolar part from $F^{\rm R}$ and define the bond

$$\begin{aligned}\widetilde{F^{\rm R}}(\mathcal{F},\mathcal{F}') &= F^{\rm R}(\mathcal{F},\mathcal{F}') + \beta e_\gamma e_{\gamma'}W(\mathcal{F},\mathcal{F}') \\ &\sim \tfrac{1}{2}[\beta e_\gamma e_{\gamma'}W(\mathcal{F},\mathcal{F}')]^2 = {\rm O}(|\mathbf{r}-\mathbf{r}'|^{-6})\end{aligned} \tag{48}$$

[7]Strictly speaking, the short-range repulsive potential needed in the framework of Maxwell-Boltzmann statistics would arise here in the exponent of (47). It has no implication in this discussion about long-range behaviours, and we simply omit it.

and work now with the three bonds F, $\widetilde{F^{\mathrm{R}}}$, and W. [8]

To find out the slowest-decaying diagrams, we write the truncated two-loop correlation $\rho_{\mathrm{T}}(\mathcal{F}_a,\mathcal{F}_b)$ in an exact Dyson series of convolution chains involving W and H :

$$\begin{aligned}\rho_{\mathrm{T}}(\mathcal{F}_a,\mathcal{F}_b) =& \rho(\mathcal{F}_a)\rho(\mathcal{F}_b)H(\mathcal{F}_a,\mathcal{F}_b) - \beta(K\star W\star K)(\mathcal{F}_a,\mathcal{F}_b)\\ &+ \beta^2(K\star W\star K\star W\star K)(\mathcal{F}_a,\mathcal{F}_b) + ...\end{aligned}\tag{49}$$

where H denotes the sum of the diagrams that remain connected under removal of one W-bond and $K(\mathcal{F}_1,\mathcal{F}_2) = \rho(\mathcal{F}_1)\rho(\mathcal{F}_2)H(\mathcal{F}_1,\mathcal{F}_2) + \delta(\mathcal{F}_1,\mathcal{F}_2)\rho(\mathcal{F}_1)$. This topological constraint ensures that H decays at least as r^{-6}. The series (49) is conveniently analysed in Fourier representation with respect to $\mathbf{r}_a - \mathbf{r}_b$. After expanding W into the sum $W_{\mathrm{c}} + W_{\mathrm{m}}$, we have three types of chains : pure W_{c} or W_{m} chains and mixed $W_{\mathrm{c}}, W_{\mathrm{m}}$ chains. It is shown in [9], [11] that the contribution of pure W_{c} chains to the particle correlation $\rho_{\mathrm{T}}(\gamma_a,\mathbf{r}_a,\gamma_b,\mathbf{r}_b) = \int \mathrm{D}(\boldsymbol{\xi}_a)\int \mathrm{D}(\boldsymbol{\xi}_b)\,\rho_{\mathrm{T}}(\mathcal{F}_a,\mathcal{F}_b)$ decays strictly faster than $\mathrm{o}(|\mathbf{r}_a - \mathbf{r}_b|^{-6})$. [9] We show below that all other chains containing W_{m} bonds vanish identically as the consequence of transversality. This implies that the longest-range part of the correlations originates from the function H in the first term of the right-hand side of (49), hence the result (45).

A chain mixing W_{c} and W_{m} bonds must have at least one element $W_{\mathrm{c}}\star K\star W_{\mathrm{m}}$ or $W_{\mathrm{m}}\star K\star W_{\mathrm{c}}$. In Fourier space, one can write, from (28) and (22),

$$\begin{aligned}(W_{\mathrm{c}}\star K\star W_{\mathrm{m}})(\gamma_a,\boldsymbol{\xi}_a,\gamma_b,\boldsymbol{\xi}_b,\mathbf{k}) =& \int_0^1\mathrm{d}s_a\int_0^1\mathrm{d}s_1\,(\delta(s_a-s_1)-1)\frac{4\pi}{k^2}\mathrm{e}^{i\mathbf{k}\cdot\lambda_{\gamma_a}\boldsymbol{\xi}_a(s_a)}\\ &\times\left[T^{\nu_2}(\mathbf{k},s_1)G^{\nu_2,\nu_b}(\mathbf{k})\right]\int_0^1 d\xi_b^{\nu_b}(s_b)\mathrm{e}^{-i\mathbf{k}\cdot\lambda_{\gamma_b}\boldsymbol{\xi}_b(s_b)},\end{aligned}\tag{50}$$

where

$$\begin{aligned}T^{\nu_2}(\mathbf{k},s_1) =& \sum_{\gamma_1}\int \mathrm{D}(\boldsymbol{\xi}_1)\sum_{\gamma_2}\int \mathrm{D}(\boldsymbol{\xi}_2)\,\mathrm{e}^{-i\mathbf{k}\cdot\lambda_{\gamma_1}\boldsymbol{\xi}_1(s_1)}K(\gamma_1,\boldsymbol{\xi}_1,\gamma_2,\boldsymbol{\xi}_2,\mathbf{k})\\ &\times\int_0^1\mathrm{d}\xi_2^{\nu_2}(s_2)\mathrm{e}^{i\mathbf{k}\cdot\lambda_{\gamma_2}\boldsymbol{\xi}_2(s_2)}\end{aligned}\tag{51}$$

[8] In [11], the bond F is further decomposed into a multipole expansion. Our bonds F^{R} and $\widetilde{F^{\mathrm{R}}}$ differ formally from their bonds F_{l} and $\tilde{F}_{\mathrm{l}}$ only by the inclusion of the magnetic contribution W_{m} into W.

[9] In this proof, only the invariance of H under rotations is used, which also holds when the magnetic potential is included.

and $K(\gamma_1, \boldsymbol{\xi}_1, \gamma_2, \boldsymbol{\xi}_2, \mathbf{k})$ is the Fourier transform of $K(\mathcal{F}_1, \mathcal{F}_2)$ with respect to $\mathbf{r}_1 - \mathbf{r}_2$. As the measures $\mathrm{D}(\boldsymbol{\xi}_1)$ and $\mathrm{D}(\boldsymbol{\xi}_2)$ and the function $K(\gamma_1, \boldsymbol{\xi}_1, \gamma_2, \boldsymbol{\xi}_2, \mathbf{k})$ are invariant under spatial rotations, $T^{\nu_2}(\mathbf{k}, s_1)$ transforms as a tensor, implying that it is necessarily of the form $T^{\nu_2}(\mathbf{k}, s_1) = k^{\nu_2}\, a(k, s_1)$ for some rotationally invariant function a of $\mathbf{k}$. Using $k^\mu G^{\mu\nu}(\mathbf{k}) \equiv 0$ one deduces immediately that (50) vanishes. The case of $W_\mathrm{m} \star K \star W_\mathrm{c}$ is similar. To see that there are no chains containing only W_m bonds in $\rho_\mathrm{T}(\gamma_a, \mathbf{r}_a, \gamma_b, \mathbf{r}_b)$, it is sufficient to notice that the integrated root element $\int \mathrm{D}(\boldsymbol{\xi}_a)\, K \star W_\mathrm{m}$ also involves a factor $[T^{\nu_2}(\mathbf{k}) G^{\nu_2,\nu_b}(\mathbf{k})]$ (for another function $T^{\nu_2}(\mathbf{k})$ transforming in a covariant manner), and thereby vanishes for the same reason.

The graphs that do contribute to the coefficient $A_{ab}(\beta, \{\rho_\gamma\})$ of (45) are those of H that contain bonds with algebraic decay : namely, $\widetilde{F^\mathrm{R}}$ and W. To select the lowest contribution in $\hbar$, one notes first that W is at least of order $\hbar^2$, as seen in (25), (30) which correspond to the lowest-order terms in the multipolar expansions of W_c and W_m. (Higher-order multipoles generate higher powers of the de Broglie wavelengths.) Since Φ_elec is rapidly decreasing, the algebraic part of $\widetilde{F^\mathrm{R}}$ is of order $\hbar^4$ and is given by $\frac{1}{2}[\beta e_\gamma e_{\gamma'} W(\mathcal{F}, \mathcal{F}')]^2$, as in (48). Thus, up to order $\hbar^4$, graphs with an algebraic decay can contain only one bond W, two bonds W, or one bond $\widetilde{F^\mathrm{R}}$ belonging to paths connecting the two root points. If there is a single such link W, by the topological structure of H there exists another path connecting the root points made of the more rapidly decreasing bonds F and $\widetilde{F^\mathrm{R}}$. Hence the whole graph has a decay faster than r^{-6}. If there are two W bonds in between the root points, as each of them is of order $\hbar^2$ all the other bonds and vertices can be evaluated in the classical limit $\hbar \to 0$. Consequently, at least one of the extremities of either bond W is attached to a purely classical part of the graph, which is independent of the Brownian shapes. We call such a point a classical end of W. At such points, integration over the Brownian shape of the loop "kills" the r^{-3} decay of W (see Appendix C), leading to an overall decay faster than r^{-6}. Finally, at order $\hbar^4$, the only graphs that contribute to (45) are constituted by a single $\widetilde{F^\mathrm{R}}$ bond linked to the root points by purely classical subgraphs. The sum of such graphs contributes to the particle correlation in the large-distance limit as

$$\rho_{\mathrm{T}}(\gamma_a,\mathbf{r}_a,\gamma_b,\mathbf{r}_b) \sim \sum_{\gamma_1,\gamma_2}\left[\int \mathrm{d}\mathbf{r}\; n_{\mathrm{T}}^{\mathrm{cl}}(\gamma_a,\gamma_1,\mathbf{r})\right]\left[\int \mathrm{d}\mathbf{r}\; n_{\mathrm{T}}^{\mathrm{cl}}(\gamma_2,\gamma_b,\mathbf{r})\right] \tag{52}$$

$$\times \int \mathrm{D}(\boldsymbol{\xi}_1)\int \mathrm{D}(\boldsymbol{\xi}_2)\;\tfrac{1}{2}\left[\beta e_{\gamma_1}e_{\gamma_2}W^{\mathrm{dip}}(\gamma_1,\boldsymbol{\xi}_1,\gamma_2,\boldsymbol{\xi}_2,\mathbf{r}_a-\mathbf{r}_b)\right]^2,$$

where $W^{\mathrm{dip}} = {W_{\mathrm{c}}}^{\mathrm{dip}} + {W_{\mathrm{m}}}^{\mathrm{dip}}$ is the sum of the dipolar parts (30) and (25) of W_{c} and W_{m}, and $n_{\mathrm{T}}^{\mathrm{cl}}(\gamma_a,\gamma_1,\mathbf{r})$ is the classical truncated two-point density correlation (including coincident points). The functional integrals in (52) have been calculated in section 4, see (37)-(44), yielding the final result

$$\rho_{\mathrm{T}}(\gamma_a,\mathbf{r}_a,\gamma_b,\mathbf{r}_b) \sim \frac{\hbar^4\beta^4}{240}\sum_{\gamma_1,\gamma_2}\left[\int \mathrm{d}\mathbf{r}\; n_{\mathrm{T}}^{\mathrm{cl}}(\gamma_a,\gamma_1,\mathbf{r})\right]\left[\int \mathrm{d}\mathbf{r}\; n_{\mathrm{T}}^{\mathrm{cl}}(\gamma_2,\gamma_b,\mathbf{r})\right]$$

$$\times \frac{e_{\gamma_1}^2 e_{\gamma_2}^2}{m_{\gamma_1}m_{\gamma_2}}\left[1+\frac{5}{\beta m_{\gamma_1}c^2\beta m_{\gamma_2}c^2}\right]\frac{1}{|\mathbf{r}_a-\mathbf{r}_b|^6} \tag{53}$$

as $|\mathbf{r}_a-\mathbf{r}_b| \to \infty$ and at lowest order in $\hbar$. To this order, the only difference with (44) is the occurrence of the classical correlation functions $n_{\mathrm{T}}^{\mathrm{cl}}$, a manifestation of the fact that in the quantum many-body problem, every pair of particles contribute to the tail of the correlation function. This generalizes the result of [12], formula (5.12), to the inclusion of the magnetic interactions.

As a final comment, we observe that the inclusion of the transverse degrees of freedom of the field does not modify the charge sum rule in the system of loops and hence it also holds for the charge correlations in the particle system. This sum rule reads

$$\int \mathrm{d}\mathbf{r}\int \mathrm{D}(\boldsymbol{\xi})\sum_{\gamma}\frac{e_\gamma \rho_{\mathrm{T}}(\mathcal{F},\mathcal{F}_1)}{\rho(\mathcal{F}_1)} = -e_{\gamma_1}. \tag{54}$$

It states that the charge of the cloud of loops induced around a fixed loop $\mathcal{F}_1$ exactly compensates that of $\mathcal{F}_1$. The proof can be carried out word by word as in [23], section 6.1.2. It relies exclusively on the long-range part r^{-1} of the Coulomb potential V_{c} and is not altered by the presence of the magnetic potential W_{m}.

6 Transverse field correlations

A characteristic feature of charged systems is that longitudinal field correlations always remain long ranged in spite of the screening mechanisms that reduce the range of the particle correlations. It has been established on a microscopic basis that the correlations of the longitudinal electric field $\mathbf{E}_\mathrm{l}$ behave as [24], [25]

$$\langle E_\mathrm{l}^\mu(\mathbf{x}) E_\mathrm{l}^\nu(\mathbf{y})\rangle_\mathrm{T} \sim -\partial_\mu \partial_\nu \frac{1}{|\mathbf{x}-\mathbf{y}|}\left[-\tfrac{2\pi}{3}\int \mathrm{d}\mathbf{r}\ |\mathbf{r}|^2 S(\mathbf{r})\right], \quad |\mathbf{x}-\mathbf{y}| \to \infty, \tag{55}$$

where $S(\mathbf{r})$ is the (classical or quantum-mechanical) charge-charge correlation function.

In order to obtain the correlations of the transverse fields we first consider correlations $\langle A^\mu(\mathbf{x}) A^\nu(\mathbf{y})\rangle_\mathrm{T}$ of the vector potential at free points $\mathbf{x}$ and $\mathbf{y}$ in space. These correlations are easily obtained by functional differentiation, adding to the original Hamiltonian (1) a coupling to an external current $\boldsymbol{\mathcal{J}}_\mathrm{ext}(\mathbf{x})$

$$H_{L,R}(\boldsymbol{\mathcal{J}}_\mathrm{ext}) = H_{L,R} - i\int \mathrm{d}\mathbf{x}\ \boldsymbol{\mathcal{J}}_\mathrm{ext}(\mathbf{x}) \cdot \mathbf{A}(\mathbf{x}), \tag{56}$$

so that

$$\langle A^\mu(\mathbf{x}) A^\nu(\mathbf{y})\rangle_\mathrm{T} = -\frac{1}{\beta^2}\frac{\delta^2}{\delta \mathcal{J}^\mu_\mathrm{ext}(\mathbf{x}) \delta \mathcal{J}^\nu_\mathrm{ext}(\mathbf{y})} \ln \mathrm{Tr}\ \mathrm{e}^{-\beta H_{L,R}(\boldsymbol{\mathcal{J}}_\mathrm{ext})}\Big|_{\boldsymbol{\mathcal{J}}_\mathrm{ext}=0}. \tag{57}$$

Decomposing $H_{L,R}$ as in (7) one can write

$$\begin{aligned}&\langle A^\mu(\mathbf{x}) A^\nu(\mathbf{y})\rangle_\mathrm{T} = \\ &= -\frac{1}{\beta^2}\frac{\delta^2}{\delta \mathcal{J}^\mu_\mathrm{ext}(\mathbf{x}) \delta \mathcal{J}^\nu_\mathrm{ext}(\mathbf{y})} \ln \mathrm{Tr}_\mathrm{mat} \left\langle \mathrm{e}^{-\beta H_\mathbf{A}} \mathrm{e}^{i\beta\int \mathrm{d}\mathbf{x}\ \boldsymbol{\mathcal{J}}_\mathrm{ext}(\mathbf{x})\cdot \mathbf{A}(\mathbf{x})}\right\rangle_\mathrm{rad}\Big|_{\boldsymbol{\mathcal{J}}_\mathrm{ext}=0}.\end{aligned} \tag{58}$$

Using the Feynman-Kac formula as in section 3 one sees that the only modification in (17) is the replacement of the loop current $\boldsymbol{\mathcal{J}}(\mathbf{x})$ by the total current[10]

$$\boldsymbol{\mathcal{J}}_\mathrm{tot}(\mathbf{x}) = \boldsymbol{\mathcal{J}}(\mathbf{x}) + \boldsymbol{\mathcal{J}}_\mathrm{ext}(\mathbf{x}). \tag{59}$$

[10] As a consequence of the imaginary coupling constant in the Hamiltonian (56), the total current is real, so that we can still apply the Gaussian integration formula used in (17).

The Gaussian integration on the field variables replaces (17) by

$$\exp\Big\{-\frac{\beta}{2}\int\frac{d\mathbf{k}}{(2\pi)^3}\,(\mathcal{J}^{\mu}_{\rm tot}(\mathbf{k}))^*\,G^{\mu\nu}(\mathbf{k})\mathcal{J}^{\nu}_{\rm tot}(\mathbf{k})\Big\}=\exp\Big\{-\frac{\beta}{2}\int\frac{d\mathbf{k}}{(2\pi)^3}G^{\mu\nu}(\mathbf{k})$$
$$\times\left[(\mathcal{J}^{\mu})^*\,\mathcal{J}^{\nu}+(\mathcal{J}^{\mu}_{\rm ext})^*\,\mathcal{J}^{\nu}+(\mathcal{J}^{\mu})^*\,\mathcal{J}^{\nu}_{\rm ext}+(\mathcal{J}^{\mu}_{\rm ext})^*\,\mathcal{J}^{\nu}_{\rm ext}\right](\mathbf{k})\Big\}. \tag{60}$$

Therefore, from (60), functional differentiation with respect to $\boldsymbol{\mathcal{J}}_{\rm ext}$ according to (58) produces two terms

$$\langle A^{\mu}(\mathbf{x})A^{\nu}(\mathbf{y})\rangle_{\rm T}=\langle A^{\mu}(\mathbf{x})A^{\nu}(\mathbf{y})\rangle_{\rm T}^{0}+\langle A^{\mu}(\mathbf{x})A^{\nu}(\mathbf{y})\rangle_{\rm T}^{\rm mat}. \tag{61}$$

The first contribution (written in Fourier form)

$$\langle A^{\mu}(\mathbf{x})A^{\nu}(\mathbf{y})\rangle_{\rm T}^{0}=\frac{1}{\beta}\int\frac{d\mathbf{k}}{(2\pi)^3}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}G^{\mu\nu}(\mathbf{k})$$
$$\sim\frac{1}{2\beta r}\left(\delta^{\mu\nu}+\frac{r^{\mu}r^{\nu}}{r^2}\right),\quad r\to\infty,\ \ \mathbf{r}=\mathbf{x}-\mathbf{y}, \tag{62}$$

arises from the part quadratic in $\boldsymbol{\mathcal{J}}_{\rm ext}$ in (60). It describes the thermal fluctuations of the free field, and in view of (20), decays as r^{-1}. The second term, coming from the part linear in $\boldsymbol{\mathcal{J}}_{\rm ext}$,

$$\langle A^{\mu}(\mathbf{x})A^{\nu}(\mathbf{y})\rangle_{\rm T}^{\rm mat}=$$
$$=-\int\frac{d\mathbf{k}}{(2\pi)^3}\,e^{i\mathbf{k}\cdot\mathbf{x}}\int\frac{d\mathbf{k}'}{(2\pi)^3}\,e^{i\mathbf{k}'\cdot\mathbf{y}}G^{\mu\sigma}(\mathbf{k})G^{\nu\tau}(\mathbf{k}')\ \langle\mathcal{J}^{\sigma}(\mathbf{k})\mathcal{J}^{\tau}(\mathbf{k}')\rangle_{\rm T}, \tag{63}$$

represents the modification to the free-field fluctuations caused by the presence of matter. It involves the loop current correlation function $\langle\mathcal{J}^{\sigma}(\mathbf{k})\mathcal{J}^{\tau}(\mathbf{k}')\rangle_{\rm T}$ where the average is taken with respect to the thermal weight (23) for the statistical-mechanical system of loops. Expressing the currents $\boldsymbol{\mathcal{J}}(\mathbf{k})=\int d\mathcal{F}\ \mathbf{j}(\mathcal{F},\mathbf{k})\hat{\rho}(\mathcal{F})$ in terms of the density of loops $\hat{\rho}(\mathcal{F})=\sum_i\delta(\mathcal{F},\mathcal{F}_i)$ (see (16)), one can write this current correlation in terms of the loop density correlation function $n_{\rm T}(\gamma_1,\boldsymbol{\xi}_1,\gamma_2,\boldsymbol{\xi}_2,\mathbf{k})$ (including the contribution of coincident points) :

$$\langle\mathcal{J}^{\sigma}(\mathbf{k})\mathcal{J}^{\tau}(\mathbf{k}')\rangle_{\rm T}=(2\pi)^3\delta(\mathbf{k}+\mathbf{k}')$$
$$\times\sum_{\gamma_1,\gamma_2}\int D(\boldsymbol{\xi}_1)\int D(\boldsymbol{\xi}_2)\ \mathcal{T}^{\sigma}(\gamma_1,\boldsymbol{\xi}_1,\mathbf{k})\big(\mathcal{T}^{\tau}(\gamma_2,\boldsymbol{\xi}_2,\mathbf{k})\big)^*n_{\rm T}(\gamma_1,\boldsymbol{\xi}_1,\gamma_2,\boldsymbol{\xi}_2,\mathbf{k}). \tag{64}$$

The $\delta(\mathbf{k}+\mathbf{k}')$ is the manifestation of the translational invariance of the state, and we have set

$$\mathcal{T}^{\sigma}(\gamma_i,\boldsymbol{\xi}_i,\mathbf{k})=\frac{e_{\gamma_i}}{\sqrt{\beta m_{\gamma_i}c^2}}\int_0^1 d\xi_i^{\sigma}(s_i)\ e^{i\lambda_{\gamma_i}\mathbf{k}\cdot\boldsymbol{\xi}_i(s_i)}. \tag{65}$$

When (64) is introduced into (63), one obtains the final form

$$\langle A^\mu(\mathbf{x})A^\nu(\mathbf{y})\rangle_{\mathrm{T}}^{\mathrm{mat}} = -\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\, \mathrm{e}^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} G^{\mu\sigma}(\mathbf{k})G^{\nu\tau}(\mathbf{k})\mathcal{Q}^{\sigma\tau}(\mathbf{k}), \quad (66)$$

where $\mathcal{Q}^{\sigma\tau}(\mathbf{k})$ is the tensor

$$\mathcal{Q}^{\sigma\tau}(\mathbf{k}) = \sum_{\gamma_1,\gamma_2} \int \mathrm{D}(\boldsymbol{\xi}_1) \int \mathrm{D}(\boldsymbol{\xi}_2)\, \mathcal{T}^\sigma(\gamma_1,\boldsymbol{\xi}_1,\mathbf{k}) \big(\mathcal{T}^\tau(\gamma_2,\boldsymbol{\xi}_2,\mathbf{k})\big)^* n_{\mathrm{T}}(\gamma_1,\boldsymbol{\xi}_1,\gamma_2,\boldsymbol{\xi}_2,\mathbf{k}). \quad (67)$$

To obtain the long-distance behaviour of this correlation we examine the integrand in (67) at small $\mathbf{k}$. Because of isotropy, the tensor $\mathcal{Q}^{\sigma\tau}(\mathbf{k})$ transforms covariantly under the rotations, and thus is of the form

$$\mathcal{Q}^{\sigma\tau}(\mathbf{k}) = a(k)\delta^{\sigma\tau} + b(k)k^\sigma k^\tau, \quad k = |\mathbf{k}| \quad (68)$$

The term $k^\sigma k^\tau$ does not contribute to (67) since $G^{\mu\sigma}(\mathbf{k})$ is transversal. Because of Itô's lemma, $\mathcal{T}^\sigma(\gamma_i,\boldsymbol{\xi}_i,\mathbf{k})$ is linear in $\mathbf{k}$ as $\mathbf{k}\to 0$, implying $a(k) = a\, k^2[1+o(k)]$. Hence, using $\delta_{tr}^{\mu\sigma}(\mathbf{k})\delta_{tr}^{\nu\sigma}(\mathbf{k}) = \delta_{tr}^{\mu\nu}(\mathbf{k})$ one finds

$$G^{\mu\sigma}(\mathbf{k})G^{\nu\tau}(\mathbf{k})\mathcal{Q}^{\sigma\tau}(\mathbf{k}) = 4\pi\, a\, \frac{4\pi}{k^2}\, \delta_{tr}^{\mu\nu}(\mathbf{k})[1+o(k)] = 4\pi\, a\, G^{\mu\nu}(\mathbf{k})[1+o(k)] \quad (69)$$

as $k\to 0$. This shows that $\langle A^\mu(\mathbf{x})A^\nu(\mathbf{y})\rangle_{\mathrm{T}}^{\mathrm{mat}}$ has the same type of decay as the free field part (62) with a modified amplitude. Summing up the two contributions (61) gives

$$\langle A^\mu(\mathbf{x})A^\nu(\mathbf{y})\rangle_{\mathrm{T}} \sim \frac{1}{2r}\left(\delta^{\mu\nu} + \frac{r^\mu r^\nu}{r^2}\right)\left(\frac{1}{\beta} - 4\pi a\right), \quad r\to\infty. \quad (70)$$

For $\mathbf{B}(\mathbf{x}) = \nabla\times\mathbf{A}(\mathbf{x})$, one finds, from (70),

$$\langle B^\mu(\mathbf{x})B^\nu(\mathbf{y})\rangle_{\mathrm{T}} \sim \left(\partial_\mu\partial_\nu \frac{1}{r}\right)\left(\frac{1}{\beta} - 4\pi a\right), \quad r\to\infty. \quad (71)$$

The constant $a = a(\hbar,\beta,\rho)$ embodies the effects of matter on the transverse field fluctuations. It has a relativistic factor $(mc^2)^{-1}$ and vanishes in the classical limit $\hbar\to 0$ (in accordance to the Bohr–van Leeuwen decoupling) as $\mathrm{O}(\hbar^4)$ (see Appendix D).

In order to find the correlations of the transverse electric field

$$\begin{aligned}\mathbf{E}_{\mathrm{t}}(\mathbf{x}) &= -\frac{1}{c}\left.\frac{\partial \mathbf{A}(\mathbf{x},t)}{\partial t}\right|_{t=0} \\ &= -\left(\frac{4\pi\hbar c^2}{R^3}\right)^{1/2}\sum_{\mathbf{k}\lambda} g(\mathbf{k})\frac{\mathbf{e}_{\mathbf{k}\lambda}}{\sqrt{2\omega_{\mathbf{k}}}}\Big(\frac{i\omega_{\mathbf{k}}}{c}\alpha^*_{\mathbf{k}\lambda}\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}} - \frac{i\omega_{\mathbf{k}}}{c}\alpha_{\mathbf{k}\lambda}\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\Big),\end{aligned} \tag{72}$$

we couple the latter to an external polarisation $\boldsymbol{\mathcal{P}}_{\mathrm{ext}}(\mathbf{x})$,

$$H_{L,R}(\boldsymbol{\mathcal{P}}_{\mathrm{ext}}) = H_{L,R} - i\int \mathrm{d}\mathbf{x}\ \boldsymbol{\mathcal{P}}_{\mathrm{ext}}(\mathbf{x})\cdot\mathbf{E}_{\mathrm{t}}(\mathbf{x}), \tag{73}$$

and proceed as after (56). This amounts to replacing everywhere $\boldsymbol{\mathcal{J}}_{\mathrm{ext}}(\mathbf{k})$ by $ik\boldsymbol{\mathcal{P}}_{\mathrm{ext}}(\mathbf{k})$ so that the right-hand side of equation (60) is changed into

$$\begin{aligned}&\exp\Big\{-\frac{\beta}{2}\int\frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}G^{\mu\nu}(\mathbf{k}) \\ &\times\left[(\mathcal{J}^\mu)^*\mathcal{J}^\nu - ik\,(\mathcal{P}^\mu_{\mathrm{ext}})^*\mathcal{J}^\nu + ik\,(\mathcal{J}^\mu)^*\mathcal{P}^\nu_{\mathrm{ext}} + k^2\,(\mathcal{P}^\mu_{\mathrm{ext}})^*\mathcal{P}^\nu_{\mathrm{ext}}\right](\mathbf{k})\Big\}.\end{aligned} \tag{74}$$

As $\boldsymbol{\mathcal{P}}_{\mathrm{ext}}(\mathbf{r})$ and $\boldsymbol{\mathcal{J}}(\mathbf{r})$ are real, $\boldsymbol{\mathcal{P}}^*_{\mathrm{ext}}(\mathbf{k}) = \boldsymbol{\mathcal{P}}_{\mathrm{ext}}(-\mathbf{k})$ and likewise for $\boldsymbol{\mathcal{J}}$. From the change of variable $\mathbf{k}\mapsto-\mathbf{k}$, one sees that the second term in the integrand in (74) is exactly compensated by the third term. Only the term quadratic in $\boldsymbol{\mathcal{P}}_{\mathrm{ext}}$ remains, which is responsible upon functional differentiation for the thermal fluctuations of the free field, as in (62). Hence, the correlations of the transverse part of the electric field are decoupled from matter and one finds

$$\langle E^\mu_{\mathrm{t}}(\mathbf{x})E^\nu_{\mathrm{t}}(\mathbf{y})\rangle_{\mathrm{T}} = \langle E^\mu_{\mathrm{t}}(\mathbf{x})E^\nu_{\mathrm{t}}(\mathbf{y})\rangle^0_{\mathrm{T}} \sim \left(\partial_\mu\partial_\mu\frac{1}{r}\right)\frac{1}{\beta}, \quad r\to\infty. \tag{75}$$

The asymptotic correlation of the complete electric field $\mathbf{E}(\mathbf{x}) = \mathbf{E}_{\mathrm{l}}(\mathbf{x}) + \mathbf{E}_{\mathrm{t}}(\mathbf{x})$ follows from (55) and (75) (one can check by similar calculations that the cross correlation $\langle E^\mu_{\mathrm{l}}(\mathbf{x})E^\nu_{\mathrm{t}}(\mathbf{y})\rangle_{\mathrm{T}}$ vanishes identically) :

$$\begin{aligned}\langle E^\mu(\mathbf{x})E^\nu(\mathbf{y})\rangle_{\mathrm{T}} &= \langle E^\mu_{\mathrm{l}}(\mathbf{x})E^\nu_{\mathrm{l}}(\mathbf{y})\rangle_{\mathrm{T}} + \langle E^\mu_{\mathrm{t}}(\mathbf{x})E^\nu_{\mathrm{t}}(\mathbf{y})\rangle_{\mathrm{T}} \\ &= \left(\partial_\mu\partial_\mu\frac{1}{r}\right)\left(\tfrac{2\pi}{3}\int\mathrm{d}\mathbf{r}\ |\mathbf{r}|^2 S(\mathbf{r}) + \frac{1}{\beta}\right), \quad r\to\infty.\end{aligned} \tag{76}$$

In the classical limit, $S(\mathbf{r})$ satisfies the second-moment Stillinger–Lovett sum rule [25] $-\frac{2\pi}{3}\int\mathrm{d}\mathbf{r}\ |\mathbf{r}|^2 S(\mathbf{r}) = 1/\beta$. Hence, the asymptotic longitudinal electric field correlations in the matter are exactly compensated by those of the free radiation part, as noted in [14]. However, this no longer holds for quantum plasmas. As an illustration, for the quantum one-component plasma (jellium), one has [26]

$$-\tfrac{2\pi}{3}\int \mathrm{d}\mathbf{r}\ |\mathbf{r}|^2 S(\mathbf{r}) = \tfrac{\hbar\omega_p}{2}\coth\left(\frac{\hbar\omega_p\beta}{2}\right) = \frac{1}{\beta} + \frac{\beta}{3}\left(\frac{\hbar\omega_p}{2}\right)^2 + \mathrm{O}(\hbar^4), \tag{77}$$

where ω_p is the plasmon frequency. The long range of the electric field correlations is thus a purely quantum-mechanical effect. These findings are further discussed in the concluding remarks (section 8).

7 Bose and Fermi statistics

In this final section we introduce the needed modifications when the Fermionic or Bosonic particle statistics are taken into account.

The Bose or Fermi statistics of the particles can be incorporated in the formalism following the same procedure as described in [8], [11] (section V). The matrix elements of (8), which is still an operator depending on the particle variables, are to be evaluated with properly symmetrized (antisymmetrized) states. When combining the decomposition of the permutation into cycles with the Feynman-Kac-Itô path integral representation this leads to generalize the previous loops $\mathcal{F} = (\mathbf{r}, \gamma, \boldsymbol{\xi})$ to Brownian loops $\mathcal{L} = (q, \mathbf{R}, \gamma, \mathbf{X})$ that carry q particles (a set of particles labeled by indices belonging to a permutation cycle of length q). The loop is located at $\mathbf{R}$ and has a random shape which is a Brownian bridge $\mathbf{X}(s)$, $0 \le s \le q$, $\mathbf{X}(0) = \mathbf{X}(q) = \mathbf{0}$ with zero mean and covariance

$$\int \mathrm{D}(\mathbf{X})\, X_\mu(s_1) X_\nu(s_2) = \delta_{\mu\nu}\, q\left[\min\left(\frac{s_1}{q}, \frac{s_2}{q}\right) - \frac{s_1}{q}\frac{s_2}{q}\right]. \tag{78}$$

We merely give the final formulae since all steps are essentially identical as those presented in the above mentioned works.

The grand canonical partition function of the particle system, with the field degrees of freedom integrated out, takes the following classical-like form in the space of loops

$$\Xi_\Lambda = \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} \mathrm{d}\mathcal{L}_i\ z(\mathcal{L}_i) \exp\big(-\beta\mathcal{U}(\mathcal{L}_1, \ldots, \mathcal{L}_n)\big). \tag{79}$$

Integration on phase space means integration over space and summation over all internal degrees of freedom of the loops :

$$\int \mathrm{d}\mathcal{L} \cdots = \int \mathrm{d}\mathbf{R} \sum_{\gamma} \sum_{q=1}^{\infty} \int \mathrm{D}(\mathbf{X}) \cdots . \tag{80}$$

$\mathcal{U}(\mathcal{L}_1, \ldots, \mathcal{L}_n)$ is the sum of the pair interactions between two different loops $e_{\gamma_i} e_{\gamma_j} [\mathcal{V}_{\mathrm{c}}(\mathcal{L}_i, \mathcal{L}_j) + \mathcal{W}_{\mathrm{m}}(\mathcal{L}_i, \mathcal{L}_j)]$ with

$$\mathcal{V}_{\mathrm{c}}(\mathcal{L}_i, \mathcal{L}_j) = \int_0^{q_i} \mathrm{d}s_1 \int_0^{q_j} \mathrm{d}s_2 \; \delta(\tilde{s}_1 - \tilde{s}_2) \frac{1}{\left|\mathbf{R}_i + \lambda_{\gamma_i} \mathbf{X}_i(s_1) - \mathbf{R}_j - \lambda_{\gamma_j} \mathbf{X}_j(s_2)\right|} \tag{81}$$

the Coulomb potential, and

$$\begin{aligned} \mathcal{W}_{\mathrm{m}}(\mathcal{L}_i, \mathcal{L}_j) = {} & \frac{1}{\beta \sqrt{m_{\gamma_i} m_{\gamma_j}} c^2} \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3} \, \mathrm{e}^{i\mathbf{k}\cdot(\mathbf{r}_i - \mathbf{r}_j)} \\ & \times \int_0^{q_i} \mathrm{d}X_i^{\mu}(s_1) \, \mathrm{e}^{i\mathbf{k}\cdot\lambda_{\gamma_i}\mathbf{X}_i(s_1)} \int_0^{q_j} \mathrm{d}X_j^{\nu}(s_2) \, \mathrm{e}^{-i\mathbf{k}\cdot\lambda_{\gamma_j}\mathbf{X}_j(s_2)} \; G^{\mu\nu}(\mathbf{k}) \end{aligned} \tag{82}$$

the effective magnetic potential obtained after integrating out the field variables. Here $\tilde{s} = s \bmod 1$ and $\delta(\tilde{s}) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{2i\pi n s}$ is the periodic Dirac function of period 1 that takes into account the equal time constraint imposed by the Feynman-Kac formula. Finally, the activity $z(\mathcal{L}_i)$ of a loop

$$z(\mathcal{L}_i) = \frac{(\eta_{\gamma_i})^{q_i - 1}}{q_i} \; \frac{z_{\gamma_i}^{q_i}}{(2\pi q_i \lambda_{\gamma_i}^2)^{3/2}} \; \exp(-\beta[\mathcal{U}_{\mathrm{self}}(\mathcal{L}_i) + \mathcal{V}_{\mathrm{walls}}(\mathcal{L}_i)]), \quad z_{\gamma_i} = \mathrm{e}^{\beta\mu_{\gamma_i}} \tag{83}$$

incorporates the chemical potential μ_{γ_i} of the particle, the effects of quantum statistics ($\eta_{\gamma_i} = 1$ for bosons and $\eta_{\gamma_i} = -1$ for fermions), and the internal interaction $\mathcal{U}_{\mathrm{self}}(\mathcal{L}_i) = -\frac{\beta e_i^2}{2}(\mathcal{V}_{\mathrm{c}} + \mathcal{W}_{\mathrm{m}})(\mathcal{L}_i, \mathcal{L}_i)$ of the particles belonging to the same loop (omitting the infinite Coulomb self-energies of the point particles). The addition of the magnetic potential $\mathcal{W}_{\mathrm{m}}$ is the only modification compared to the formalism previously developed for pure Coulombic interactions. Maxwell-Boltzmann statistics and the potentials (22) and (14) of section 3 are recovered when only single-particle loops ($q = 1$) are retained.

At this point, due to the classical-like structure of the partition function (79), methods of classical statistical mechanics can be used in the auxiliary system of loops, in particular the technique of Mayer graphs, as in section 5. The statistics of the particles affects the coefficients of the tails of the density and field correlations, but not their general forms (45), (71) and (76).

8 Concluding remarks

The Feynman–Kac–Itô path integral representation of the Gibbs weight enables one to integrate out the (classical) field variables. This yields an exact pairwise magnetic potential in the space of loops, which is asymptotically dipolar. It generates small ($\mathrm{O}((\beta mc^2)^{-2})$) corrections to the tail of the particle correlation due to the pure Coulombic interaction.

A word is necessary about spin interactions that have not been included in the above discussion. Spin-1/2 electrons give rise to the additional term $-\nu \sum_{i=1}^{n} \boldsymbol{\sigma}_i \cdot \mathbf{B}(\mathbf{r}_i)$ in the Hamiltonian, with $\mathbf{B}(\mathbf{r}) = \nabla \wedge \mathbf{A}(\mathbf{r})$, $\nu = \frac{g_s e\hbar}{4mc}$, g_s the gyromagnetic factor, and $\boldsymbol{\sigma}$ the Pauli matrices. Using spin coherent states [27], a functional integral representation of the Gibbs weight can be constructed including the coupling of the spins to the field. Since this coupling is linear with respect to the vector potential, Gaussian integration on the field variables leads again to an effective spin-spin interaction $W_{\mathrm{s}}(i,j)$ and effective cross interactions $W_{\mathrm{sm}}(i,j)$ and $W_{\mathrm{ms}}(i,j)$ between spin and orbital magnetism; details can be found in [28]. One finds that these interactions are of dipolar type $\sim r^{-3}, r \to \infty$ and they have to be added to the magnetic potential $W_{\mathrm{m}}(i,j)$. In a homogeneous and isotropic phase, the spin interaction terms contribute to the r^{-6} tail of the particle correlations with the same amplitude $\frac{\lambda_a^2 \lambda_b^2 e_a^2 e_b^2 \rho_a \rho_b}{m_a m_b c^4}$, up to numerical factors, as that found in section 4 in the case of the magnetic potential W_{m}.

Regarding the electric field correlations in the plasma, we also find that they have an algebraic decay of dipolar type. This is in disagreement with the macroscopic calculation presented by Landau and Lifshitz [2], §88, based on linear response theory and the fluctuation-dissipation theorem. Indeed, the electric field fluctuations obtained in this theory are short ranged (exponentially fast decaying) : unlike in our calculation, the algebraic parts of the longitudinal and transverse correlations compensate exactly in the Landau and Lifshitz formulae [29]. Understanding the relation between our strictly microscopic approach and the macroscopic theory of field fluctuations is an open problem.

Let us, however, briefly point out some differences between the two approaches. The microscopic approach involves all length scales, whereas Landau and Lifshitz assume that the correlations of the polarisation are local (δ correlated in space) and thus deal with a wave-number-independent dielectric function $\epsilon(\omega)$. Taking into account the wave-number dependence, it is likely that $\epsilon(\mathbf{k}, \omega)$ has terms non-analytic in $\mathbf{k}$, reflecting the fact that Coulombic matter has algebraically decaying correlations. In fact, for the jellium model, the static dielectric function $\epsilon(\mathbf{k}, \omega = 0)$ has a singular term $\sim |\mathbf{k}|$ in its small-$\mathbf{k}$ expansion [30]. It is possible that in a non-local generalization of the Landau–Lifshitz theory such singular terms also generate power-law decays of the field correlations. Furthermore, the magnetic permeability is usually set equal to that of the vacuum, thus ignoring the magnetization fluctuations, whereas in our calculation the latter are properly included.

We stress again that the results of this paper hold when the electromagnetic field is classical, which is supposed to correctly depict the small-wave-number regime, as said in the Introduction. Hence, the occurrence of the Planck constant originates exclusively from the quantum-mechanical nature of matter. If the field is quantized, we can, however, not exclude an interplay between $\hbar_{\mathrm{matter}}$ and $\hbar_{\mathrm{field}}$, which could lead to a modification of the asymptotic formulae presented in the paper.

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Appendix A

To establish (40) according to the middle point prescription (12) one has to evaluate the rotationally covariant tensor

$$\int \mathrm{D}(\boldsymbol{\xi}) \int_0^1 \mathrm{d}\xi^\alpha(s) \int_0^1 \mathrm{d}\xi^\gamma(t)\ \xi^\omega(s)\xi^\epsilon(t) = \tag{84}$$
$$= \lim_{n,m\to\infty} \sum_{k,l=1}^{n,m} \int \mathrm{D}(\boldsymbol{\xi}) \left[\xi^\alpha(k_n) - \xi^\alpha\left(k_n - \tfrac{1}{n}\right)\right] \left[\xi^\gamma(l_m) - \xi^\gamma\left(l_m - \tfrac{1}{m}\right)\right]$$
$$\times \frac{1}{2}\left[\xi^\omega(k_n) + \xi^\omega\left(k_n - \tfrac{1}{n}\right)\right] \frac{1}{2}\left[\xi^\epsilon(l_m) + \xi^\epsilon\left(l_m - \tfrac{1}{m}\right)\right] = \delta^{\alpha\gamma}\delta^{\omega\epsilon}A_1 + \delta^{\alpha\omega}\delta^{\gamma\epsilon}A_2 + \delta^{\alpha\epsilon}\delta^{\gamma\omega}A_3,$$

where $k_n = \frac{k}{n}$ and $l_m = \frac{l}{m}$. Setting $C(s,t) = \delta^{\mu\nu}(\min(s,\ t) - st)$ (see (11)), one has

$$\begin{aligned}
A_1 = &\lim_{n,m\to\infty} \frac{1}{4}\left[C(k_n,l_m) - C\left(k_n, l_m - \tfrac{1}{m}\right) - C\left(k_n - \tfrac{1}{n}, l_m\right) + C\left(k_n - \tfrac{1}{n}, l_m - \tfrac{1}{m}\right)\right] \\
&\times \left[C(k_n,l_m) + C\left(k_n, l_m - \tfrac{1}{m}\right) + C\left(k_n - \tfrac{1}{n}, l_m\right) + C\left(k_n - \tfrac{1}{n}, l_m - \tfrac{1}{m}\right)\right], \\
A_2 = &\lim_{n,m\to\infty} \frac{1}{4}\left[C(k_n,k_n) + C\left(k_n, k_n - \tfrac{1}{n}\right) - C\left(k_n - \tfrac{1}{n}, k_n\right) - C\left(k_n - \tfrac{1}{n}, k_n - \tfrac{1}{n}\right)\right] \\
&\times \left[C(l_m,l_m) + C\left(l_m, l_m - \tfrac{1}{m}\right) - C\left(l_m - \tfrac{1}{m}, l_m\right) - C\left(l_m - \tfrac{1}{n}, l_n - \tfrac{1}{n}\right)\right], \\
A_3 = &\lim_{n,m\to\infty} \frac{1}{4}\left[C(k_n,l_m) + C\left(k_n, l_m - \tfrac{1}{m}\right) C\left(k_n - \tfrac{1}{n}, l_m\right) - C\left(k_n - \tfrac{1}{n}, l_m - \tfrac{1}{m}\right)\right] \\
&\times \left[C(l_m,k_n) + C\left(l_m, k_n - \tfrac{1}{n}\right) - C\left(l_m - \tfrac{1}{m}, k_n\right) - C\left(l_m - \tfrac{1}{m}, k_n - \tfrac{1}{n}\right)\right].
\end{aligned} \tag{85}$$

This results from the application of Wick's theorem to the Gaussian average (84) with covariance (11). Expanding $C\left(k_n - \frac{1}{n}, l_m\right) = C(k_n,l_m) - \frac{1}{n}(\partial_1 C)(k_n,l_m)$ and $C\left(k_n, l_m - \frac{1}{m}\right) = C(k_n,l_m) - \frac{1}{m}(\partial_2 C)(k_n,l_m)$ and taking the limits $n,m\to\infty$ gives

$$\begin{aligned}
A_1 &= \int_0^1 \mathrm{d}s \int_0^1 \mathrm{d}t\ C(s,t)(\partial_1\partial_2 C)(s,t) = \frac{1}{12}, \\
A_2 &= \frac{1}{4}\left(\int_0^1 \mathrm{d}s\ \frac{\mathrm{d}}{\mathrm{d}s}\, C(s,s)\right)^2 = 0, \\
A_3 &= \int_0^1 \mathrm{d}s \int_0^1 \mathrm{d}t\ (\partial_1 C)(s,t)\,(\partial_2 C)(s,t) = -\frac{1}{12},
\end{aligned} \tag{86}$$

hence the result (40).

Appendix B

From (22) and (28) the cross Coulomb-magnetic term is

$$\beta^2 e_a^2 e_b^2 \int \mathrm{D}(\boldsymbol{\xi}_a)\rho(\boldsymbol{\xi}_a) \int \mathrm{D}(\boldsymbol{\xi}_b)\rho(\boldsymbol{\xi}_b)\ W_{\mathrm{c}}(\mathcal{F}_a, \mathcal{F}_b) W_{\mathrm{m}}(\mathcal{F}_a, \mathcal{F}_b) =$$
$$\frac{\beta e_a^2 e_b^2}{\sqrt{m_a m_b} c^2} \int \frac{\mathrm{d}\mathbf{k}_1}{(2\pi)^3} \int \frac{\mathrm{d}\mathbf{k}_2}{(2\pi)^3}\, \mathrm{e}^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot(\mathbf{r}_a-\mathbf{r}_b)} \int_0^1 \mathrm{d}s_1 \int_0^1 \mathrm{d}s_2 \left(\delta(s_1 - s_2) - 1\right)$$
$$\times \left(H_a^\mu\right)^* (\mathbf{k}_1, \mathbf{k}_2, s_1) H_b^\nu(\mathbf{k}_1, \mathbf{k}_2, s_2)\, \frac{4\pi}{k_1^2}\, G^{\mu\nu}(\mathbf{k}_2), \tag{87}$$

where

$$H_a^\mu(\mathbf{k}_1, \mathbf{k}_2, s_1) = \int \mathrm{D}(\boldsymbol{\xi}_a)\rho(\boldsymbol{\xi}_a)\, \mathrm{e}^{-i\lambda_a \mathbf{k}_1\cdot\boldsymbol{\xi}_a(s_1)} \int_0^1 \mathrm{d}\xi_a^\mu(s)\ \mathrm{e}^{-i\lambda_a \mathbf{k}_2\cdot\boldsymbol{\xi}_a(s)}. \tag{88}$$

Because of the rotational invariance of $\mathrm{D}(\boldsymbol{\xi}_a)\rho(\boldsymbol{\xi}_a)$, averages of odd powers of $\boldsymbol{\xi}_a$ vanish. This implies that in the small-$\mathbf{k}_1, \mathbf{k}_2$ expansion of $H_a^\mu(\mathbf{k}_1, \mathbf{k}_2, s_1)$ only odd monomials in $\mathbf{k}_1, \mathbf{k}_2$ occur:

$$\begin{aligned} H_a^\mu(\mathbf{k}_1, \mathbf{k}_2, s_1) &\sim \int \mathrm{D}(\boldsymbol{\xi}_a)\rho(\boldsymbol{\xi}_a) \int_0^1 \mathrm{d}\xi_a^\mu(s)\ \left(i\lambda_a \mathbf{k}_2 \cdot \boldsymbol{\xi}_a(s)\right) + \mathrm{O}_3(\mathbf{k}_1, \mathbf{k}_2) \\ &= \mathrm{const} \times k_2^\mu + \mathrm{O}_3(\mathbf{k}_1, \mathbf{k}_2), \end{aligned} \tag{89}$$

where $\mathrm{O}_3(\mathbf{k}_1, \mathbf{k}_2)$ represent monomials of order 3 in the components of $\mathbf{k}_1, \mathbf{k}_2$. The same holds for $H_b^\mu(\mathbf{k}_1, \mathbf{k}_2, s_2)$. Since $k_2^\mu G^{\mu\nu}(\mathbf{k}_2) = 0$ by transversality, one concludes that the term (87) decays at least as $|\mathbf{r}_a - \mathbf{r}_b|^{-8}$.

Appendix C

If point i in $W_{\mathrm{c}}(i,j)$ or $W_{\mathrm{m}}(i,j)$ is a classical end, there is no other $\boldsymbol{\xi}_i$ dependence at this point than that arising from these bonds. In the asymptotic formula (30) for W_{c}, this dependence is linear and thereby vanishes upon the space-inversion invariant $\mathrm{D}(\boldsymbol{\xi}_i)$ integration. In the case of W_{m}, from formula (22), the $\mathrm{D}(\boldsymbol{\xi}_i)$ integration yields the factor

$$\int \mathrm{D}(\boldsymbol{\xi}_i) \int_0^1 \mathrm{d}\xi_i^\mu(s_1)\ \mathrm{e}^{i\mathbf{k}\cdot\lambda_i\boldsymbol{\xi}_i(s_1)} \propto k^\mu \tag{90}$$

because of covariance under rotation. Hence, this contribution vanishes as a consequence of transversality $k^\mu G^{\mu\nu}(\mathbf{k}) = 0$.

Appendix D

An explicit expression for the constant $a = a(\hbar, \beta, \rho)$ follows from taking the trace in equation (69) and using (67) expanded for small k. This yields

$$a = \frac{1}{2} \sum_{\gamma,\gamma'} \frac{e_\gamma \lambda_\gamma e_{\gamma'} \lambda_{\gamma'}}{\beta \sqrt{m_\gamma m_{\gamma'}} c^2} \int \mathrm{D}(\boldsymbol{\xi}) \int \mathrm{D}(\boldsymbol{\xi}') \times \int_0^1 \mathrm{d}\xi^\mu(s) \int_0^1 \mathrm{d}\xi^\nu(s') \left(\hat{\mathbf{k}} \cdot \boldsymbol{\xi}(s)\right) \left(\hat{\mathbf{k}} \cdot \boldsymbol{\xi}'(s')\right) \delta_{\mathrm{tr}}^{\mu\nu}(\hat{\mathbf{k}})\, n_{\mathrm{T}}(\gamma, \boldsymbol{\xi}, \gamma', \boldsymbol{\xi}', \mathbf{k} = \mathbf{0}), \tag{91}$$

where $\hat{\mathbf{k}} = \mathbf{k}/k$. As $\lambda_\gamma \lambda_{\gamma'}$ is of order $\hbar^2$, at lowest order in $\hbar$ one can set $\hbar = 0$ in the correlation function. The latter becomes independent of the quantum fluctuations $\boldsymbol{\xi}, \boldsymbol{\xi}'$ and reduces to the density correlation function of the corresponding classical system. The remaining functional integrals, of the type $\int \mathrm{D}(\boldsymbol{\xi}) \int_0^1 \mathrm{d}\xi^\mu(s) \xi^\sigma(s)$, vanish identically. The terms of order $\mathrm{O}(\hbar)$ in n_{T} are necessarily linear in $\boldsymbol{\xi}$ or $\boldsymbol{\xi}'$. They do not contribute to a since averages of odd powers $\boldsymbol{\xi}$ or $\boldsymbol{\xi}'$ are zero, implying that there are no $\hbar^3$-terms in a. We thus conclude that a is $\mathrm{O}(\hbar^4)$.

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