Microscopic theory of the Casimir force at thermal equilibrium: large-separation asymptotics

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Abstract

We present an entirely microscopic calculation of the Casimir force $f(d)$ between two metallic plates in the limit of large separation d. The models of metals consist of mobile quantum charges in thermal equilibrium with the photon field at positive temperature T. Fluctuations of all degrees of freedom, matter and field, are treated according to the principles of quantum electrodynamics and statistical physics without recourse to approximations or intermediate assumptions. Our main result is the correctness of the asymptotic universal formula $f(d) \sim -\frac{\zeta(3) k_{\mathrm{B}} T}{8\pi d^3}$, $d \to \infty$. This supports the fact that, in the framework of Lifshitz' theory of electromagnetic fluctuations, transverse electric modes do not contribute in this regime. Moreover the microscopic origin of universality is seen to rely on perfect screening sum rules that hold in great generality for conducting media.

1 Introduction

Motivations

In 1948, Casimir [1] predicted that two neutral metallic plates placed in vacuum at distance d attract one another due to the electromagnetic field's zero-point fluctuations. In his calculation the microscopic structure of the conductors is not taken into account. The latter are treated as macroscopic bodies imposing metallic boundary conditions on the Maxwell fields.

Early experiments aimed at probing this theoretical prediction remained inconclusive [2] until the late 1990s where first experimental demonstrations were performed [3, 4, 5], opening the way to many others (see, *e.g.*, [6, Sec. 3.5] and [7] for short reviews). A quantitative comparison with experiments requires to include a number of effects not accounted for in Casimir's simple treatment, such as the finite conductivity of the plates, the roughness of the surfaces and the dependence on the temperature T.

Lifshitz [8, 9] provided a first major generalisation by considering plates whose electric properties are described by a frequency-dependent dielectric function $\epsilon(\omega)$. The plates are in thermal equilibrium with a stochastic electromagnetic field, whose random nature is generated by the quantum and thermal fluctuations of photons and matter. The general force formula obtained in this way covers in principle a broad diversity of media, and should be valid for all regimes of temperature T and plate separation d. These regimes are characterized by the single dimensionless parameter

$$\alpha = \frac{\hbar c}{k_{\rm B} T d} = \frac{\lambda_{\rm ph}}{d} \tag{1}$$

which measures the ratio of the photon thermal wavelength $\lambda_{\rm ph} \equiv \beta\hbar c$ to the separation distance d ($\hbar$ is the Planck constant, c the speed of light, $k_{\rm B}$ the Boltzmann constant, $\beta = 1/k_{\rm B}T$).

Although the Lifshitz theory (along with its various reformulations [10, 11, 12, 6, 7]) is commonly used to interpret the experimental data, its predictions are uncertain when applied to *conducting media* at nonzero temperature. Indeed, the value of the force then depends crucially on the behaviour of the dielectric function at vanishing frequencies — a behaviour not directly accessible to experiments. This has led to several theoretical and numerical studies, resulting in a debate that has not yet evolved to a concensual end (see [7, 13, 14, 15, 16] and references cited therein). In short, the controversy amounts to knowing whether the reflection coefficient of the transverse electric (TE) mode of the field, $r^{\rm TE}(\omega, \boldsymbol{k})$ (depending on $\epsilon(\omega)$), vanishes or not in the limit of zero frequency. In the low-temperature, small-separation regime $\alpha \gg 1$ one finds the force by unit surface[1]

$$f(d) \sim -\frac{\pi^2 \hbar c}{240 d^4} + {\rm O}(T^4), \qquad \text{if } r^{\rm TE}(0, \boldsymbol{k}) = 1, \tag{2}$$

$$f(d) \sim -\frac{\pi^2 \hbar c}{240 d^4} + \frac{\zeta(3) k_{\rm B} T}{8\pi d^3} + {\rm O}(T^4), \qquad \text{if } r^{\rm TE}(0, \boldsymbol{k}) = 0, \tag{3}$$

[1] In these formulas, negative/positive terms stand for attractive/repulsive contributions.

while in the high-temperature, large-separation regime $\alpha \ll 1$, one has:

$$f(d) \sim -\frac{\zeta(3) k_{\mathrm{B}} T}{4\pi d^3}, \qquad \text{if } r^{\mathrm{TE}}(0, \boldsymbol{k}) = 1 \tag{4}$$

$$f(d) \sim -\frac{\zeta(3) k_{\mathrm{B}} T}{8\pi d^3}, \qquad \text{if } r^{\mathrm{TE}}(0, \boldsymbol{k}) = 0. \tag{5}$$

In (2)–(3), the dominant term is in both cases the standard Casimir result, whereas in the high-temperature, large-separation regime (4)–(5), one sees a striking reduction of the force amplitude by a factor 1/2 when the TE field modes are assumed not to contribute. Let us add that Formulae (2), (4) can be obtained under the assumption of the plasma relation for the low-frequency dielectric function ($\epsilon(\omega) \sim 1 - \omega_{\mathrm{p}}^2/\omega^2$, $\omega \to 0$, ω_{p} the plasmon frequency), whereas the Drude expression ($\epsilon(\omega) \sim 1 - 4\pi i\sigma/\omega$, σ the conductivity) leads to the results (3), (5). The force (4) is also retrieved by extending Casimir's original calculation to finite temperature between macroscopic plates that are not subject to charge fluctuation [17, 18].

Most of the actual debate focused on the finite-temperature corrections in the low-temperature, small-separation regime. In particular, the fact that the corrective term linear in T in (3) reflects a nonzero entropy at $T = 0$ consists in an unacceptable violation of the Nernst postulate for some authors [7, 14], who thereby favor a nonvanishing reflection coefficient $r^{\mathrm{TE}}(0, \boldsymbol{k})$. Other authors [15, 19], however, argue that this linear correction no longer holds at very low temperature, and favor (3), (5).

In this paper, we pronounce on the controversy in the large-separation regime (with fixed positive temperature). In order to decide which of the two alternatives (4) and (5) is correct, we present a fully microscopic treatment of the Casimir effect based on the principles of quantum electrodynamics and statistical mechanics which does not suffer of intermediate models, assumptions or approximations.[2] By fully microscopic treatment we mean that all degrees of freedom, matter and field, are taken into account, contrary to Casimir's original calculation that ignores particle fluctuations inside the plates. Microscopic models have been produced to retrieve and justify Lifshitz' formula in the case of dielectric matter [23, 24, 25], but conducting media offer more difficulties as far as one has to deal with screening phenomena due to free charges and magnetic forces between free currents.

[2]Experimental setups in cylinder-plane and parallel plate geometries are currently being developed with the purposes of discriminating between the different proposed values for the force [20, 21, 22].

In [26], we computed the average force by unit surface between slabs containing purely classical charges and interacting via the static Coulomb potential, finding

$$f(d) \sim -\frac{\zeta(3) k_{\mathrm{B}} T}{8\pi d^3}, \qquad d \to \infty. \tag{6}$$

In the letter [27] we improved the calculation by considering slabs made of quantum charges and interacting, in addition to the Coulomb force, with the transverse part of a classical electromagnetic field. These features do not alter the form (6) of the large-separation asymptotic force. Reference [27] together with the companion letter [28] stress the importance of including in the calculation the effects of the charge fluctuations in the metals, which are responsible for reducing the asymptotic force amplitude (4) by the factor 1/2. It is also understood why the entirely classical model treated in [26] correctly predicts the high-temperature, large-separation result (6): this is a consequence of the Bohr–van Leeuwen theorem. The theorem states that in classical systems at thermal equilibrium, matter decouples from the transverse electromagnetic field. Since high-temperature conducting phases tend to behave classically, the corresponding Casimir force will be determined at leading order by purely electrostatic interactions.

Statement of results

The present paper addresses the question of the Casimir force in the general framework of nonrelativistic thermal quantum electrodynamics (TQED), namely nonrelativistic quantum charges in interaction with the quantized electromagnetic field. The model (described in more detail in Section 2) consists of mobile quantum charges confined in two slabs A and B of thickness a and b with lateral faces of surface L^2, set at distance d from each other. The charges interact with a quantum electromagnetic field enclosed in a large box Λ. The Hamiltonian $H_{\Lambda,L,d}$ of the system is specified in Formula (10) of Section 2. The photons and the particles are supposed to be in thermal equilibrium at temperature T, so that all the relevant information is contained in the grand-canonical potential $\Phi_{\Lambda,L,d}$ associated with the Hamiltonian $H_{\Lambda,L,d}$. The average force by unit surface exerted between the plates is defined by the rate of change occasioned in $\Phi_{\Lambda,L,d}$ when varying the separating distance d:

$$f_{\Lambda,L}(d) = -\frac{1}{L^2}\frac{\partial}{\partial d}\Phi_{\Lambda,L,d}. \tag{7}$$

The Casimir force is defined as

$$f(d) \equiv \lim_{L^2\to\mathbb{R}^2} \lim_{\Lambda\to\mathbb{R}^3} f_{\Lambda,L}(d), \tag{8}$$

where the thermodynamic limit of the system is taken in two stages. We first let the box enclosing the field $\Lambda \to \mathbb{R}^3$, and then extend the plates' surfaces $L^2 \to \mathbb{R}^2$. The plates' thicknesses a and b are kept finite. The main result is that at any fixed temperature $T > 0$ (such that the thermal energy $k_B T$ is much less than the rest mass energies mc^2 of the particles), the large-separation asymptotic force is again given by Formula (6). The amplitude is linear in T, independent of the Planck constant $\hbar$ and of the speed of light c, and universal with respect to the microscopic constitution of the plates. Nonuniversal contributions and contributions depending on $\hbar$ and c, will only occur at the next order $\mathrm{O}(d^{-4})$ in the large-separation expansion of the force. This result not only validates from first principles the second alternative in Eqs. (4), (5) (associated with the vanishing of the reflection coefficient for TE modes), but also establishes universality of the leading term (6) on a microscopic basis. Since our methods might be not quite familiar, we summarize the main steps of our derivation.

Methods

Casting the quantum system in a classical-like form: the space of loops

When we have a conducting medium it is of utmost importance to deal properly with the collective screening effects. The idea is to cast the quantum system in a form which is as close as possible to that of a classical system of charges, where we have for instance the well-developed Debye-Hückel theory of screening. To this effect we introduce a joint functional integral representation of the Gibbs weight associated with the total Hamiltonian of matter and field. In this formalism, developed in [29] and recalled in Section 2, quantum particles appear as extended objects consisting of random closed wires $\mathcal{L}$ (called loops) carrying both a charge and a current. The loop size, which is measured by the thermal de Broglie wavelength, reflects the intrinsic quantum fluctuation of the particle. There are two kinds of pairwise interactions between loops. The first one $V^{\mathrm{c}}(\mathcal{L}_i, \mathcal{L}_j)$ originates from the electrostatic (Coulomb) potential between charges, Formula (20). The second one, $\mathcal{W}^{\mathrm{m}}(\mathcal{L}_i, \mathcal{L}_j)$ (Formula (21)) is called the magnetic potential. It is an effective interaction resulting from integrating out the field degrees of freedom: one can figure it as current interactions between the loops mediated by the transverse part of the electromagnetic field. At this point, although being an exact representation of the quantum TQED system, the statistical mechanics of loops has a classical-like structure which enables a convenient application of the methods of classical statistical mechanics.

Expressing the Casimir force in terms of loops

The force between two loops is given, as in classical physics, by the gradients of the potentials $\partial_x V^{\rm c}(\mathcal{L}_i, \mathcal{L}_j)$ and $\partial_x \mathcal{W}^{\rm m}(\mathcal{L}_i, \mathcal{L}_j)$ (along the x-axis perpendicular to the plates). The average force, as usual, is obtained by averaging these forces with the equilibrium correlation function $\rho^{(2)}(\mathcal{L}_i, \mathcal{L}_j)$ between two loops. The precise expression is found in Formula (38) in Section 3 where we have also singled out the proper Casimir force due to fluctuations. The additional part, called here capacitor force, is the direct Coulomb force that would occur between globally nonneutral plates. One can benefit from the translational invariance along the $\mathbf{y}$ directions parallel to the plates by using the correponding two-dimensional Fourier variable $\mathbf{k}$ and scale it as $\mathbf{k} = \mathbf{q}/d$ where $\mathbf{q}$ is now a dimensionless Fourier variable. The scaling trivially provides a prefactor $1/d^2$ in the force Formula (40). The remaining d-dependence remains embedded in the microscopic expressions of the forces and correlations between loops.

Screening of the electrostatic interactions

This is the subject of Section 4. The main observation is that the Coulomb interaction between loops can be decomposed into $V^{\rm c} = V^{\rm el} + \mathcal{W}^{\rm c}$ (see Formulas (42) and (43)). Here $V^{\rm el}$ is the genuine classical electrostatic interaction between charged wires whereas $\mathcal{W}^{\rm c}$ incorporates the proper effect of the quantum nature of the particles manifested by the fluctuations of the loops. One can easily extend to $V^{\rm el}$ the standard ideas of the classical Debye-Hückel theory, providing an effective resummed potential Φ that becomes integrable at large distances (in the planar geometry one is concerned by the integrability in the $\mathbf{y}$ directions along the plates, see Appendix C). One is therefore left with the screened potential Φ together with the additional interactions $\mathcal{W}^{\rm c}$ and $\mathcal{W}^{\rm m}$. The latter would not be present for classical charges: they result from the intrinsic quantum fluctuations of the particles and behave as electric and magnetic dipole interaction at large distance. At this point on can use in the space of loops the methods of Mayer expansion and integral equations well developed in the context of classical Coulomb fluids. Of particular importance is the perfect screening sum rule stating that any specified loop is surrounded by a screening cloud of loops whose total charge compensates that of the specified loop. This imposes an exact integral constraint on the two-loop correlation function, Formula (52), that turns out to be at the origin of the universality of the Casimir force.

The large-separation asymptotics

Apart from the obvious $1/d^2$ scaling factor, the d-dependence of the force has to be extracted from the forces and correlations between loops. The Coulomb part of the force (expressed in terms of the $\mathbf{q}$ transverse Fourier variable) has a non vanishing limit as $d \to \infty$ whereas the magnetic force vanishes as $1/d^2$ and the correlation between the plates as $1/d$ (Section 6). This implies that the dominant term decays as $1/d^3$ and does not involve a direct contribution of the magnetic force. To obtain the amplitude of this $1/d^3$ term it is necessary to determine the exact asymptotic form of the correlation. The latter is seen to be made of two terms (Formulae (59) and (62)) decaying as $1/d$ times a product of certain arrangements of correlations pertaining to the individual plates. When this is introduced in the force Formula (40) one discovers that the perfect screening sum rules in each of the plates wash out all details of the microscopic structure of the conductors, thereby leading to the wonderfully-simple result (6) and providing a physical explanation of universality (Section 5). The analysis of the correlation in Subsections 6.1 and 6.3, if somewhat lengthy, uses common reasoning in terms of Mayer graph diagramatics. It basically reveals that the electric and magnetic dipolar potentials between the loops $\mathcal{W}^{\rm c}$ and $\mathcal{W}^{\rm m}$ do not eventually contribute to the dominant $1/d^3$ term of the force. This term is entirely due to electrostatics and screening, so explaining why the purely classical model of [26] gives the correct result. The complexity of the full quantum mechanical treatment presented in this paper contrasts with the simplicity of the result, a fact that we could not foresee right away. More comments and perspectives are offered in the Concluding remarks in Section 7.

2 Description of the system

We consider two parallel slabs $A = [-a, 0] \times L^2$ and $B_d = [d, b+d] \times L^2$ with thickness a and b and lateral surface L^2. The x-axis is perpendicular to the plates, the inner face of slab A being fixed at $x = 0$ while the slab B_d is set at a distance d from it. The slabs contain nonrelativistic point particles of several species γ (electrons, ions, nuclei) with charges e_γ, masses m_γ, spins s_γ and appropriate statistics. These particles are confined by walls without electrical properties in the two separate regions and no exchange is possible from one slab to the other. Particles in one plate are always distinguishable from alike particles in the other plate. To ensure the global neutrality of each

plate, we impose

$$\sum_a e_{\gamma_a} = \sum_b e_{\gamma_b} = 0, \tag{9}$$

where the sums are carried over particles in A and B_d, respectively.

This system of interacting charges is coupled to a quantum electromagnetic field which is itself enclosed into a larger box Λ englobing both plates. The N-particle Hamiltonian reads

$$\begin{aligned} H_{\Lambda,L,d} = &\sum_{i=1}^{N} \frac{1}{2m_{\gamma_i}} \Big(\boldsymbol{P}_i - \frac{e_{\gamma_i}}{c} \boldsymbol{A}(\boldsymbol{r}_i) \Big)^2 + \sum_{i<j} e_{\gamma_i} e_{\gamma_j} v(\boldsymbol{r}_i - \boldsymbol{r}_j) \\ &+ \sum_{i=1}^{N} V^{\text{walls}}(\boldsymbol{r}_i, \gamma_i) + H^{\text{rad}}_{0,\Lambda} \end{aligned} \tag{10}$$

with $v(\boldsymbol{r}_i - \boldsymbol{r}_j)$ the static Coulomb potential

$$v(\boldsymbol{r}_i - \boldsymbol{r}_j) = \frac{1}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}. \tag{11}$$

As is common in atomic physics when matter is nonrelativistic and high-energy processes are neglected, we use the Coulomb gauge and electrostatic Gaussian units [30]. The Coulomb gauge has the advantage of clearly disentangling electrostatic and magnetic couplings in the Hamiltonian. The divergence-free vector potential $\boldsymbol{A}(\boldsymbol{r})$ is supposed to satisfy periodic boundary conditions on the sides of the box Λ. Its expansion in Fourier modes $\boldsymbol{\kappa}$ is given by

$$\boldsymbol{A}(\boldsymbol{r}) = \left(\frac{4\pi\hbar c^2}{\Lambda} \right)^{1/2} \sum_{\boldsymbol{\kappa},\lambda} g(\boldsymbol{\kappa}) \frac{\boldsymbol{e}_{\boldsymbol{\kappa},\lambda}}{\sqrt{2\omega_{\boldsymbol{\kappa}}}} \big(a^*_{\boldsymbol{\kappa},\lambda} \mathrm{e}^{-i\boldsymbol{\kappa}\cdot\boldsymbol{r}} + a_{\boldsymbol{\kappa},\lambda} \mathrm{e}^{i\boldsymbol{\kappa}\cdot\boldsymbol{r}} \big), \tag{12}$$

where $a^*_{\boldsymbol{\kappa},\lambda}, a_{\boldsymbol{\kappa},\lambda}$ are the creation and annihilation operators for the mode $\boldsymbol{\kappa}, \lambda$ of frequency $\omega_{\boldsymbol{\kappa}} = c|\boldsymbol{\kappa}|$ with commutation relations $[a_{\boldsymbol{\kappa},\lambda}, a^*_{\boldsymbol{\kappa}',\lambda'}] = \delta_{\boldsymbol{\kappa},\boldsymbol{\kappa}'}\delta_{\lambda,\lambda'}$; $\boldsymbol{e}_{\boldsymbol{\kappa},\lambda}, \lambda = 1, 2$ are the polarization vectors; $g(\boldsymbol{\kappa})$, $g(\boldsymbol{0}) = 1$, is a real, spherically-symmetric, and smooth form factor taking care of ultraviolet divergencies. It is supposed to decay rapidly to 0 beyond the characteristic wavenumber $\kappa_{\text{cut}} \equiv \frac{2\pi}{\lambda_{\text{cut}}} = \frac{\bar{m}}{\hbar c}$ where $\bar{m}$ is an average particle mass. The term $H^{\text{rad}}_{0,\Lambda}$ in (10) is the free field Hamiltonian

$$H^{\text{rad}}_{0,\Lambda} = \sum_{\boldsymbol{\kappa},\lambda} \hbar\omega_{\boldsymbol{\kappa}}\ a^*_{\boldsymbol{\kappa},\lambda} a_{\boldsymbol{\kappa},\lambda}. \tag{13}$$

The wall potential $V^{\text{walls}}(\boldsymbol{r}_i, \gamma_i)$ confines the particles either to slab A or to slab B_d, depending on whether γ_i designates a species in A

or B_d. Note that we neglect spin–field couplings in this model (see comments in the Concluding remarks).

The states of this system of particles and field are supposed to be thermalised at the inverse temperature $\beta = (k_{\rm B}T)^{-1}$, and statistical averages, denoted by $\langle \ldots \rangle$, are taken with the usual Gibbs weight $\mathrm{e}^{-\beta H_{\Lambda,L}}$. We introduce the finite-volume grand-canonical potential of the full system

$$\Phi_{\Lambda,L,d} = -k_{\rm B}T \ln \mathrm{Tr}\ \mathrm{e}^{-\beta(H_{\Lambda,L} - \boldsymbol{\mu}\cdot\boldsymbol{N})} \tag{14}$$

where the trace $\mathrm{Tr} = \mathrm{Tr}_{\rm mat}\mathrm{Tr}_{\rm rad}$ is carried over particles' and field's degrees of freedom. Here $\boldsymbol{\mu} = \{\mu_{\gamma_a}, \mu_{\gamma_b}\}$ is the collection of chemical potentials that fix the average particle densities in each of the plates and $\boldsymbol{N} = \{N_{\gamma_a}, N_{\gamma_b}\}$ are the corresponding particle numbers. In (14) $\mathrm{Tr}_{\rm mat}$ is carried out only on neutral configurations in each plate. The average force by unit surface exerted between infinitely extended plates immersed in the electromagnetic field is then defined by Formula (8), the temperature and chemical potentials being fixed.

In addition to the separation d, there is a number of other characteristic lengths in the system, in particular, the thermal wavelength of photon $\lambda_{\rm ph} = \beta\hbar c$ and of particles $\lambda_{\rm mat} = \hbar\sqrt{\beta/\bar{m}}$. Moreover, the plates are assumed to be conducting, and therefore characterized by a screening length $\lambda_{\rm screen}$. Our derivation holds for the following hierarchy of lengths:

$$\lambda_{\rm cut} = \frac{\lambda_{\rm mat}}{\sqrt{\beta mc^2}} \ll \lambda_{\rm mat} \ll \lambda_{\rm ph} = \sqrt{\beta mc^2}\lambda_{\rm mat} \ll d, \tag{15}$$

$$\lambda_{\rm screen} \ll a, b \ll d. \tag{16}$$

The first set of inequalities is necessary for the consistency of the nonrelativistic treatment of matter, which requires $\beta\bar{m}c^2 \gg 1$ (the thermal energy is much smaller than the rest mass energy of the particles). Inequality (16) means that the plates' thickness should be large enough for allowing the screening mechanisms to take place inside the conductors. Finally, the conductors will be assumed to be invariant under translations and rotations in the plate directions.

Loop formalism

Our analysis relies on the formalism developed in [29], based on a joint functional representation of both matter and field. In this formalism, the field degrees of freedom can be integrated out exactly. Then, the particle variables live in an auxiliary classical-like phase space whose elements are loops of random shape (for the statistical mechanics of

charged loops, see the review [31], Chap. V and references therein). A loop $\mathcal{L} = (\boldsymbol{r}, \chi)$ is specified by a position $\boldsymbol{r}$ in space and a number of internal degrees of freedom $\chi = (\gamma, p, \boldsymbol{X}(\cdot))$ consisting of a species index γ, a charge number $p \in 1, 2, 3, ...$ and a closed Brownian path $s \mapsto \boldsymbol{X}(s)$, $s \in [0, p]$, $\boldsymbol{X}(0) = \boldsymbol{X}(p)$. The loop's shape $\boldsymbol{X}(s)$ is a Gaussian stochastic process (Brownian bridge) whose functional integral has unit normalization, zero mean, and covariance given by

$$\int \mathrm{D}(\boldsymbol{X}) X^\mu(s) X^\nu(s') = \delta_{\mu\nu}\ p \left(\min\left\{ \frac{s}{p}, \frac{s'}{p'} \right\} - \frac{s}{p}\frac{s'}{p'} \right). \tag{17}$$

The loop's path is

$$\boldsymbol{r}^{[s]} \equiv \boldsymbol{r} + \lambda_\gamma \boldsymbol{X}(s), \quad 0 \leq s \leq p \tag{18}$$

where $\lambda_\gamma = \hbar\sqrt{\beta/m_\gamma}$ is the de Broglie thermal wavelength. The occurrence of the Brownian path results from the Feynman–Kac path integral and λ_γ gives the extension of the quantum particle's fluctuation. The number p accounts for the quantum statistics of the species γ. It corresponds to grouping together p particles that are permuted accordingly to a cyclic permutation of length p.

The pairwise interaction $e_{\gamma_i} e_{\gamma_j} V(\boldsymbol{i}, \boldsymbol{j})$ between two loops $\boldsymbol{i} \equiv \mathcal{L}_i$ and $\boldsymbol{j} \equiv \mathcal{L}_j$ is the sum of two contributions

$$e_{\gamma_i} e_{\gamma_j} V(\boldsymbol{i}, \boldsymbol{j}) = e_{\gamma_i} e_{\gamma_j} \big[V^{\mathrm{c}}(\boldsymbol{i}, \boldsymbol{j}) + \mathcal{W}^{\mathrm{m}}(\boldsymbol{i}, \boldsymbol{j}) \big]. \tag{19}$$

The first contribution, inherited from the Coulomb potential, is

$$V^{\mathrm{c}}(\boldsymbol{i}, \boldsymbol{j}) = \int_0^{p_i} \mathrm{d}s_i \int_0^{p_j} \mathrm{d}s_j \delta(\widetilde{s_i} - \widetilde{s_j})\ \frac{1}{\big|\boldsymbol{r}_i^{[s_i]} - \boldsymbol{r}_j^{[s_j]}\big|}, \tag{20}$$

where $\tilde{s} = s \bmod 1$ and $\delta(\widetilde{s_i} - \widetilde{s_j})$ takes into account the equal-time constraint imposed by the Feynman–Kac formula. The second contribution is the effective potential resulting from the elimination of the field's degrees of freedom. We call it the magnetic potential. It is given in Fourier representation by Formula (66)-(67) of [29]:

$$\mathcal{W}^{\mathrm{m}}(\boldsymbol{i}, \boldsymbol{j}) = \int \frac{\mathrm{d}\boldsymbol{K}}{(2\pi)^3}\, \mathrm{e}^{i\boldsymbol{K}\cdot(\boldsymbol{r}_i - \boldsymbol{r}_j)}\ \mathcal{W}^{\mathrm{m}}(\chi_i, \chi_j, \boldsymbol{K}), \tag{21}$$

$$\begin{aligned} \mathcal{W}^{\mathrm{m}}(\chi_i, \chi_j, \boldsymbol{K}) =& \frac{1}{\beta\sqrt{m_{\gamma_i} m_{\gamma_j}} c^2} \int_0^{p_i} \mathrm{d}X_i^\mu(s_i)\, \mathrm{e}^{i\boldsymbol{K}\cdot\lambda_{\gamma_i}\boldsymbol{X}_i(s_i)} \\ & \times \int_0^{p_j} \mathrm{d}X_j^\nu(s_j)\, \mathrm{e}^{-i\boldsymbol{K}\cdot\lambda_{\gamma_j}\boldsymbol{X}_j(s_j)} \frac{4\pi g^2(K)}{K^2} \delta_{\mu\nu}^{\mathrm{tr}}(\boldsymbol{K}) \mathcal{Q}(K, \tilde{s_i} - \tilde{s_j}), \end{aligned}$$

with

$$\mathcal{Q}(K, \tilde{s_i} - \tilde{s_j}) \equiv \frac{\lambda_{\mathrm{ph}} K}{2\sinh(\lambda_{\mathrm{ph}} K/2)} \cosh[\lambda_{\mathrm{ph}} K(|\tilde{s_i} - \tilde{s_j}| - 1/2)], \tag{22}$$

and

$$\delta^{\rm tr}_{\mu\nu}(\boldsymbol{K}) \equiv \delta_{\mu\nu} - \frac{k_\mu k_\nu}{K^2} \qquad (\boldsymbol{K} = \{k_\mu\}_{\mu=1}^3,\ K = |\boldsymbol{K}|) \tag{23}$$

is the transverse Kronecker function. In (21), $\int_0^p \mathrm{d}X^\mu(s)$ are stochastic line integrals along the loop shape. The function $\mathcal{Q}$, depending only on $\lambda_{\rm ph}$, is the manifestation of the quantum photon field. This formula holds when the field region Λ has been extended to infinity, replacing the discrete sum on Fourier modes by an integral.

Written in terms of loop variables, the grand-canonical partition function of the full system, normalised by that of the free radiation field, has a classical structure:

$$\begin{aligned}\Xi_{L,d} &= \lim_{\Lambda\to\mathbb{R}^3} \frac{\mathrm{Tr}\ \mathrm{e}^{-\beta(H_{\Lambda,L}-\boldsymbol{\mu}\cdot\boldsymbol{N})}}{\mathrm{Tr}\ \mathrm{e}^{-\beta H^{\rm rad}_{0,\Lambda}}} \\ &= \sum_{n_A=0}^{\infty}\frac{1}{n_A!}\sum_{n_B=0}^{\infty}\frac{1}{n_B!}\int_A\prod_a^{n_A}\mathrm{d}\mathcal{L}_a\, z(\mathcal{L}_a)\int_{B_d}\prod_b^{n_B}\mathrm{d}\mathcal{L}_b\, z(\mathcal{L}_b)\,\mathrm{e}^{-\beta U(\{\mathcal{L}_a\},\{\mathcal{L}_b\})}.\end{aligned} \tag{24}$$

In (24), the loop integration $\int_A \mathrm{d}\mathcal{L}_a = \int_A \mathrm{d}\boldsymbol{r}_a\,\mathrm{d}\chi_a = \int_A \mathrm{d}\boldsymbol{r}_a \sum_{\gamma_a}\sum_{p_a}\mathrm{D}(\boldsymbol{X}_a)$ is carried over paths $\boldsymbol{r}_a^{[s]}$ entirely contained in slab A, respectively, $\int_{B_d}\mathrm{d}\mathcal{L}_b$ over paths in slab B_d. This corresponds to choosing hard walls on the faces of the slabs, *i.e.*, Dirichlet boundary conditions for the particle wavefunctions. In (24) the sums run only on neutral configurations of loops in each slab.

The total loop energy U can be separated into intra and interplate contributions:

$$U = U_A + U_{B_d} + U_{AB_d} \tag{25}$$

where

$$U_A = \sum_{\boldsymbol{i},\boldsymbol{j}\in A} e_{\gamma_i} e_{\gamma_j} V(\boldsymbol{i},\boldsymbol{j}) \tag{26}$$

is the sum of interactions occurring among loops confined into slab A (likewise for U_{B_d} in slab B_d), and

$$U_{AB_d} = \sum_{\boldsymbol{i}\in A}\sum_{\boldsymbol{j}\in B_d} e_{\gamma_i} e_{\gamma_j} V(\boldsymbol{i},\boldsymbol{j}) \tag{27}$$

is the interaction energy between the two plates. Moreover, each loop is equipped with an effective activity $z(\mathcal{L})$ containing the loop self-energy $e_\gamma^2 V(\mathcal{L},\mathcal{L})$:

$$z(\mathcal{L}) = \frac{(2s_\gamma+1)(\eta_\gamma)^{p-1}}{p}\frac{(\mathrm{e}^{\beta\mu_\gamma})^p}{(2\pi p\lambda_\gamma^2)^{3/2}}\mathrm{e}^{-\beta\frac{e_\gamma^2}{2}V(\mathcal{L},\mathcal{L})}. \tag{28}$$

The factor $(2s_\gamma + 1)$ accounts for the spin-degeneracy of the energy levels, μ_γ is the chemical potential of species γ and $\eta_\gamma = \pm 1$ for bosonic/fermionic species.

We stress that although the loop partition function (24) has a classical form, it is a mathematically exact representation of the original grand-canonical partition function of the system of quantum charges and photons as defined by the Hamiltonian (10).

3 The Casimir force

In view of (14)-(8) and the fact that the free-field partition function does not depend on d, the Casimir force expressed with the help of the loop partition function (24) reads

$$\begin{aligned} f(d) &= \lim_{L\to\infty} \lim_{\Lambda\to\mathbb{R}^3} \frac{k_{\rm B}T}{L^2} \frac{\partial}{\partial d} \Big[\ln \mathrm{Tr}\ \mathrm{e}^{-\beta(H_{\Lambda,L} - \boldsymbol{\mu}\cdot\boldsymbol{N})} - \ln \mathrm{Tr}\ \mathrm{e}^{-\beta H^{\rm rad}_{0,\Lambda}} \Big] \\ &= \lim_{L\to\infty} \frac{k_{\rm B}T}{L^2}\ \frac{\frac{\partial}{\partial d}\Xi_{L,d}}{\Xi_{L,d}}. \end{aligned} \tag{29}$$

The dependence upon d in the partition function (24) occurs only in the confinement of the loops in the slab B_d. For the rest of the paper, it is convenient to shift the positional integration variable $x_b \in [d, b+d]$ of a loop in B_d to $x_b - d$, so that slab B_d is moved to the fixed region $B = [0,b]\times L^2$. Then, the d-dependence is transfered to the interaction potential between slabs:

$$V(\mathcal{L}_a, \mathcal{L}_b), \quad x_b \in [d, b+d] \quad \mapsto \quad V(\mathcal{L}_a, \mathcal{L}_b + d), \quad x_b \in [0,b], \tag{30}$$

where $\mathcal{L}_b + d$ is the loop $\mathcal{L}_b$ shifted along the x-axis from x_b to $x_b + d$. This amounts to measure the positions in slab B_d from its inner face. To abbreviate the notation, we set

$$V_{AB}(\mathcal{L}_a, \mathcal{L}_b) \equiv V(\mathcal{L}_a, \mathcal{L}_b + d). \tag{31}$$

From now on it will be implicitly understood that $V_{AB}(\mathcal{L}_a, \mathcal{L}_b)$ depends on d according to (31) and that in forthcoming integrals the path of the loop $\mathcal{L}_a$ is restricted to the fixed slab A of volume $[-a, 0] \times L^2$ while that of $\mathcal{L}_b$ to the fixed slab B of volume $[0, b] \times L^2$.

Differentiating with respect to d in (29) is equivalent to differentiating the potential $V(\mathcal{L}_a, \mathcal{L}_b + d)$ with respect to x_b, or with respect to $-x_a$ (since the dependence on the x components is $x_a - x_b - d$). This brings in the average force along x

$$f(d) = \lim_{L\to\infty} \frac{1}{L^2} \left\langle \sum_a^A \sum_b^B e_{\gamma_a} e_{\gamma_b} (\partial_{x_a} V^{\rm c}_{AB} + \partial_{x_a} \mathcal{W}^{\rm m}_{AB})(\mathcal{L}_a, \mathcal{L}_b) \right\rangle_{\rm loops}. \tag{32}$$

The bracket $\langle\cdots\rangle_{\rm loops}$ denotes the grand-canonical statistical average in the phase space of loops with activities (28) and with respect to the Gibbs weight $\mathrm{e}^{-\beta U}$ associated to the loop potential energy (25). Since the force is a two-body observable, its thermal average can be expressed as an integral over the two-loop correlation $\rho_L^{(2)}$ between a loop in A and a loop in B:[3]

$$f(d) = \lim_{L\to\infty} \frac{1}{L^2} \int_A \mathrm{d}\mathbf{1} \int_B \mathrm{d}\mathbf{2}\, e_{\gamma_1} e_{\gamma_2} (\partial_{x_1} V^{\rm c}_{AB} + \partial_{x_1} \mathcal{W}^{\rm m}_{AB})(\mathbf{1},\mathbf{2})\, \rho^{(2)}_{AB,L}(\mathbf{1},\mathbf{2}), \tag{33}$$

where we have followed the notation (31),

$$\rho^{(2)}_{AB,L}(\mathcal{L}_1,\mathcal{L}_2) \equiv \rho^{(2)}_L(\mathcal{L}_1,\mathcal{L}_2+d), \tag{34}$$

and the $\mathrm{d}\mathbf{1}$ integration is carried on loops in A, respectively, the $\mathrm{d}\mathbf{2}$ integration on loops in B.

At this stage we take the limit $L\to\infty$ of infinite plate surfaces. Since the conductors are assumed to become homogeneous in the $\boldsymbol{y} = (y,z)$ plane of the plates, the two-loop correlation function tends to a function $\rho^{(2)}_{AB}(1,2,|\boldsymbol{y}_1-\boldsymbol{y}_2|)$. In this geometry, it is convenient to decompose $\boldsymbol{r}_1 = (x_1,\boldsymbol{y}_1)$, and

$$\mathbf{1} = (1,\boldsymbol{y}_1), \quad \mathrm{d}\mathbf{1} = \mathrm{d}1\ \mathrm{d}\boldsymbol{y}_1, \quad \mathrm{d}1 = \mathrm{d}x_1 \mathrm{d}\chi_1 \tag{35}$$

where $1 = (x_1,\chi_1)$ denotes the position along x of the loop $\mathcal{L}_1$ and its internal degrees of freedom (likewise for 2). In the limit, the factor $1/L^2$ cancels with one of the $\boldsymbol{y}$-integral in (33), yielding

$$f(d) = \int_A \mathrm{d}1 \int_B \mathrm{d}2 \int \mathrm{d}\boldsymbol{y}\ e_{\gamma_1} e_{\gamma_2} (\partial_{x_1} V^{\rm c}_{AB} + \partial_{x_1} \mathcal{W}^{\rm m}_{AB})(1,2,\boldsymbol{y}) \rho^{(2)}_{AB}(1,2,\boldsymbol{y}). \tag{36}$$

We introduce in (36) the loop Ursell function $h(\mathbf{1},\mathbf{2})$ defined in the usual way by

$$\rho(\mathbf{1})\rho(\mathbf{2})h(\mathbf{1},\mathbf{2}) \equiv \rho^{(2)}(\mathbf{1},\mathbf{2}) - \rho(\mathbf{1})\rho(\mathbf{2}), \tag{37}$$

with $\rho(\mathcal{L})$ the loop density, so that

$$\begin{aligned} f(d) = \int_A \mathrm{d}1 \int_B \mathrm{d}2 \int \mathrm{d}\boldsymbol{y}\ & e_{\gamma_1} e_{\gamma_2} (\partial_{x_1} V^{\rm c}_{AB} + \partial_{x_1} \mathcal{W}^{\rm m}_{AB})(1,2,\boldsymbol{y}) \\ & \times \rho_A(1)\rho_B(2) h_{AB}(1,2,\boldsymbol{y}) + f_{\rm cap}(d), \end{aligned} \tag{38}$$

[3]Loop correlation functions are defined similarly to particle density correlation functions, see [31], Chap. V.

where again $h_{AB}(\mathcal{L}_1,\mathcal{L}_2)=h(\mathcal{L}_1,\mathcal{L}_2+d)$ and $\rho_A(\mathcal{L}_1)=\rho(\mathcal{L}_1),\rho_B(\mathcal{L}_2)=\rho(\mathcal{L}_2+d)$ according to the notation (31), (34). The capacitor force $f_{\rm cap}(d)=f^{\rm c}_{\rm cap}+f^{\rm m}_{\rm cap}$ comes from the subtracted product of the plates' density in (37). We show in Appendix A that the electrostatic part $f^{\rm c}_{\rm cap}(d)$ due to the term $\partial_{x_1}V^{\rm c}_{AB}$ reduces to

$$f^{\rm c}_{\rm cap}(d)=2\pi\left[\int_{-a}^{0}{\rm d}x_1\,c_A(x_1)\right]\left[\int_{0}^{b}{\rm d}x_2\,c_B(x_2)\right], \tag{39}$$

where $c_A(x_1)$ and $c_B(x_2)$ are the mean charge densities in plate A and B. It corresponds to the standard force (in Gaussian units) between a capacitor's plates whose surface charge densities are respectively $\int_{-a}^{0}{\rm d}x_1c_A(x_1)$ and $\int_0^b{\rm d}x_2c_B(x_2)$. In this work, we assume strict neutrality in virtue of (9), so that $f^{\rm c}_{\rm cap}(d)\equiv 0$. Moreover, we also show in Appendix A that the magnetic contribution $f^{\rm m}_{\rm cap}(d)$ decays faster than any inverse power of d. In the sequel, we will thus drop the capacitor force and focus only on the first term of (38), which is the proper Casimir force generated by fluctuations.

There is a noteworthy simplification in the electrostatic part due to $\partial_{x_1}V^{\rm c}_{AB}(1,2,\boldsymbol{y})$ in the Casimir force. Namely, one can omit all multipolar contributions in the loop Coulomb force $\partial_{x_1}e_{\gamma_1}e_{\gamma_2}V^{\rm c}(\mathbf{1},\mathbf{2})$ (see (20)) replacing it by its pure monopole term $\partial_{x_1}p_1p_2e_{\gamma_1}e_{\gamma_2}\frac{1}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}=p_1p_2e_{\gamma_1}e_{\gamma_2}\partial_{x_1}v(\boldsymbol{r}_1-\boldsymbol{r}_2)$. The underlying reason is that the electric force expressed in terms of the original two-point particle correlation function involves the average standard Coulomb force $\partial_{x_1}v(\boldsymbol{r}_1-\boldsymbol{r}_2)$ between point charges. The equivalence with the present formulation in terms of loops is given in Appendix B.

Finally, we represent the remaining $\boldsymbol{y}$-integral in the two-dimensional (transverse) Fourier space $\boldsymbol{k}$ and introduce the dimensionless variable $\boldsymbol{q}=\boldsymbol{k}d$:

$$\begin{aligned}f(d)=&\frac{1}{d^2}\int_A{\rm d}1\int_B{\rm d}2\int\frac{{\rm d}\boldsymbol{q}}{(2\pi)^2}e_{\gamma_1}e_{\gamma_2}\left(p_1p_2\,\partial_{x_1}v_{AB}+\partial_{x_1}\mathcal{W}^{\rm m}_{AB}\right)(1,2,\tfrac{\boldsymbol{q}}{d})\\&\times\rho_A(1)\rho_B(2)h_{AB}(1,2,\tfrac{\boldsymbol{q}}{d}),\end{aligned} \tag{40}$$

where

$$\partial_{x_1}v_{AB}(1,2,\tfrac{\boldsymbol{q}}{d})=\partial_{x_1}\int{\rm d}\boldsymbol{y}\;\frac{{\rm e}^{i\frac{\boldsymbol{q}}{d}\cdot\boldsymbol{y}}}{\sqrt{(x_1-x_2-d)^2+\boldsymbol{y}^2}}=2\pi\,{\rm e}^{-q}\,{\rm e}^{-q(x_2-x_1)/d}. \tag{41}$$

The general formula (40) is an expression structurally similar to the one developed in the purely classical model [26, Form. (29)]. It reduces to it when charges are classical and the field is switched off. The main purpose is now to extract the d-dependence of the Ursell function $h_{AB}(1,2,\frac{\boldsymbol{q}}{d})$, which embodies all correlations between the two plates.

4 Screening of the electrostatic interaction

The Ursell function can be conveniently analysed by performing a Mayer expansion in the phase space of loops. The Mayer bonds $f(\boldsymbol{i},\boldsymbol{j}) = \mathrm{e}^{-\beta e_{\gamma_i} e_{\gamma_j} V(\boldsymbol{i},\boldsymbol{j})} - 1$ are built from the basic loop-loop interaction (19). The Coulombic part of (19) decays as r^{-1} so that the Mayer bond is not integrable. In order to remedy to this nonintegrability, it is necessary to take screening effects into account. To this end, we first make the following observation. From the Feynman–Kac formula the potential (20) inherits the quantum-mechanical equal-time constraint: *i.e.*, every element of charge $e_{\gamma_i}\lambda_{\gamma_i}\mathrm{d}\boldsymbol{X}_i(s_i)$ of the first loop does not interact with every other element $e_{\gamma_j}\lambda_{\gamma_j}\mathrm{d}\boldsymbol{X}_j(s_j)$ as would be the case in classical physics, but the interaction takes place only if $s_1 = s_2$. It is therefore of interest to split V^{c} into $V^{\mathrm{el}} + \mathcal{W}^{\mathrm{c}}$, where

$$V^{\mathrm{el}}(\boldsymbol{i},\boldsymbol{j}) = \int_0^{p_i}\mathrm{d}s_i \int_0^{p_j}\mathrm{d}s_j \frac{1}{\left|\boldsymbol{r}_i^{[s_i]} - \boldsymbol{r}_j^{[s_j]}\right|}, \tag{42}$$

$$\mathcal{W}^{\mathrm{c}}(\boldsymbol{i},\boldsymbol{j}) = \int_0^{p_i}\mathrm{d}s_i \int_0^{p_j}\mathrm{d}s_j \big(\delta(\widetilde{s_i}-\widetilde{s_j})-1\big) \frac{1}{\left|\boldsymbol{r}_i^{[s_i]} - \boldsymbol{r}_j^{[s_j]}\right|}. \tag{43}$$

The contribution V^{el} is the genuine classical Coulomb interaction between two uniformly charged wires of shapes $\boldsymbol{r}_i^{[s_i]}$ and $\boldsymbol{r}_j^{[s_j]}$, whereas the quantum-mechanical constraint appears in $\mathcal{W}^{\mathrm{c}}$.

Now, the complete two-loop potential (19) reads

$$V = V^{\mathrm{el}} + \mathcal{W}, \quad \text{with } \mathcal{W} = \mathcal{W}^{\mathrm{c}} + \mathcal{W}^{\mathrm{m}}. \tag{44}$$

It is known that $\mathcal{W}^{\mathrm{c}}$ and $\mathcal{W}^{\mathrm{m}}$ have a dipolar r^{-3} decay at large distance, see Section VI of [29]. One can therefore view the system of loops as behaving like classical random charged wires (interacting with V^{el}) with additional electric and magnetic multipolar interaction $\mathcal{W}$.

We deal with the screening effect generated by the classical Coulombic part V^{el} by the standard Debye–Hückel method. This amounts to introduce the effective screened potential Φ corresponding to the chain-resummation of the linear part $-\beta e_{\gamma_i} e_{\gamma_j} V^{\mathrm{el}}(\boldsymbol{i},\boldsymbol{j})$ of the bond $f(\boldsymbol{i},\boldsymbol{j})$: Φ satisfies the integral equation

$$\Phi(\boldsymbol{i},\boldsymbol{j}) = V^{\mathrm{el}}(\boldsymbol{i},\boldsymbol{j}) - \int \mathrm{d}\mathbf{1} \frac{\kappa^2(1)}{4\pi} V^{\mathrm{el}}(\boldsymbol{i},\mathbf{1})\Phi(\mathbf{1},\boldsymbol{j}), \tag{45}$$

where

$$\kappa^{-1}(1) = [4\pi\beta e_{\gamma_1}^2 \rho(1)]^{-1/2} \tag{46}$$

defines a local screening length in the system of loops. This potential is now short-range in the sense that it is integrable on the $\boldsymbol{y}$-direction along the plates (see Appendix C), implying

$$\lim_{\boldsymbol{k}\to\boldsymbol{0}} |\Phi(1,2,\boldsymbol{k})| < \infty. \tag{47}$$

The Mayer series is reorganised by the Abbe–Meeron resummation process ([31], Chap. V and references therein) into so-called "prototype" graphs Π with integrable bonds $F(\boldsymbol{i},\boldsymbol{j})$ and $F^{\mathrm{R}}(\boldsymbol{i},\boldsymbol{j})$[4] given by

$$F(\boldsymbol{i},\boldsymbol{j}) = -\beta e_{\gamma_i} e_{\gamma_j} \Phi(\boldsymbol{i},\boldsymbol{j}), \tag{48}$$

$$F^{\mathrm{R}}(\boldsymbol{i},\boldsymbol{j}) = \mathrm{e}^{-\beta e_{\gamma_i} e_{\gamma_j} (\Phi+\mathcal{W})(\boldsymbol{i},\boldsymbol{j})} - 1 + \beta e_{\gamma_i} e_{\gamma_j} \Phi(\boldsymbol{i},\boldsymbol{j}). \tag{49}$$

The resummed Mayer graph series of the Ursell function reads

$$h(\mathbf{1},\mathbf{2}) = \sum_{\Pi} \frac{1}{S_\Pi} \int \mathrm{d}\mathbf{3}\, \rho(3) \cdots \int \mathrm{d}\boldsymbol{m}\, \rho(m) \prod_{\{\boldsymbol{i},\boldsymbol{j}\}\in\Pi} \mathcal{F}(\boldsymbol{i},\boldsymbol{j}), \tag{50}$$

where $\mathcal{F} \in \{F, F^{\mathrm{R}}\}$. The diagrams Π have two root points and $m-2$ internal circles (m=2,3,...), and a symmetry number S_Π. In (50), the weights of the integrated points are the density, so that the graphs contain no articulation points. Prototype graphs are subject to an important rule: convolution chains of bonds F are forbidden to avoid double counting of the original Mayer graphs.

Perfect screening sum rules

On the microscopic level, the conducting behaviour of a system at equilibrium is characterized by the fulfilment of the "perfect screening sum rule" [32]: a fixed charge in the system is neutralized by the mean charge density surrounding it. This property is expressed by the following constraint on the two-particle Ursell function:

$$\sum_{\gamma_1} \int \mathrm{d}\boldsymbol{r}_1\; e_{\gamma_1}\, \rho(\boldsymbol{r}_1,\gamma_1) h(\boldsymbol{r}_1,\gamma_1;\boldsymbol{r}_2,\gamma_2) = -e_{\gamma_2}. \tag{51}$$

It turns out that the same perfect screening sum rule holds in the auxiliary system of loops

$$\int \mathrm{d}\mathbf{1}\; p_1\, e_{\gamma_1}\, \rho(\mathbf{1}) h(\mathbf{1},\mathbf{2}) = -p_2\, e_{\gamma_2}. \tag{52}$$

The interpretation is the same: the fixed loop $\mathbf{2}$ with charge $p_2 e_{\gamma_2}$ is surrounded by a screening cloud of loops with opposite total charge.

[4] At large distance, $F^{\mathrm{R}}(\boldsymbol{i},\boldsymbol{j}) \sim -\beta e_{\gamma_i} e_{\gamma_j} \mathcal{W}(\boldsymbol{i},\boldsymbol{j}) \sim r^{-3}$ is at the border of integrability. Hence, some care has to be exercised as it is the case in dipole gases.

The sum rule (52) holds in great generality for infinitely extended conductors, in particular for slab geometries. A justification of this sum rule is easily given when the loop Ursell correlation h is replaced by the single bond F. The equation (45) written in the transverse Fourier space reads

$$\Phi(i,j,\boldsymbol{k}) = V^{\rm el}(i,j,\boldsymbol{k}) - \int \mathrm{d}1 \frac{\kappa^2(1)}{4\pi} V^{\rm el}(i,1,\boldsymbol{k})\Phi(1,j,\boldsymbol{k}), \tag{53}$$

where from (42)

$$V^{\rm el}(i,j,\boldsymbol{k}) = \int_0^{p_i} \mathrm{d}s_i \int_0^{p_j} \mathrm{d}s_j \; \mathrm{e}^{i\boldsymbol{k}\cdot[\lambda_{\gamma_i}\boldsymbol{Y}_i(s_i) - \lambda_{\gamma_j}\boldsymbol{Y}_j(s_j)]} \, \tfrac{2\pi}{k} \mathrm{e}^{-k\left|x_i^{[s_i]} - x_j^{[s_j]}\right|}. \tag{54}$$

Here $X(s)$ and $\boldsymbol{Y}(s)$ are the components of $\boldsymbol{X}(s)$ along x and in the $\boldsymbol{y}$ plane and $x^{[s]} = x + \lambda_\gamma X(s)$ is the component along x of $\boldsymbol{r}^{[s]}$ (18). We divide both members of (53) by $V^{\rm el}(i,j,\boldsymbol{k})$ and let $\boldsymbol{k} \to 0$. In view of the fact that $\Phi(i,j,\boldsymbol{k})$ remains finite (see (47)), that $\lim_{\boldsymbol{k}\to\boldsymbol{0}} V^{\rm el}(i,1,\boldsymbol{k})/V^{\rm el}(i,j,\boldsymbol{k}) = p_1/p_j$, and from the definition (46), one obtains

$$\int \mathrm{d}1 \, p_1 \, e_{\gamma_1} \, \rho(1) \, F(1,j,\boldsymbol{k}=\boldsymbol{0}) = -p_j e_{\gamma_j} \ , \tag{55}$$

which is the same as (52) with F replacing h.

In fact, it can be shown that the general case (52) is a consequence of (55), by using the same dressing argument as that presented in (63)-(68) of [26, Sec. 5].

5 Asymptotic Casimir force

To analyse the asymptotic d-dependence of the force (40), we need to extract that of the electrostatic part $\partial_{x_1} v_{AB}$ and of the magnetic part $\partial_{x_1} \mathcal{W}^{\rm m}_{AB}$ together with that of the Ursell correlation h_{AB}. It is immediate from (41) that $\partial_{x_1} v_{AB}(1,2,\frac{\boldsymbol{q}}{d})$ has the limit

$$\partial_{x_1} v_{AB}(1,2,\tfrac{\boldsymbol{q}}{d}) \to 2\pi \mathrm{e}^{-q} = \mathrm{O}(1) \tag{56}$$

as $d \to \infty$. We will establish in Section 6 the following facts:

$$\partial_{x_1} \mathcal{W}^{\rm m}_{AB}(1,2,\tfrac{\boldsymbol{q}}{d}) = \mathrm{O}(d^{-2}), \tag{57}$$

$$h_{AB}(1,2,\tfrac{\boldsymbol{q}}{d}) = \mathrm{O}(d^{-1}). \tag{58}$$

As a consequence, the average of the magnetic part $\partial_{x_1} \mathcal{W}^{\rm m}_{AB}(1,2,\frac{\boldsymbol{q}}{d})$ does not contribute to the Casimir force at leading order since it is

$\mathrm{O}(d^{-5})$, whereas the electrostatic part of the force is $\mathrm{O}(d^{-3})$. To calculate the coefficient of this $\propto d^{-3}$ dominant contribution, the exact structure of h_{AB} at $\mathrm{O}(d^{-1})$ is needed. The latter is analysed in detail in Subsection 6.3. In short, both bonds F_{AB} (48) and F^{R}_{AB} (49) are of order d^{-1} and the diagrams contributing to $h_{AB}(1,2,\frac{\boldsymbol{q}}{d})$ at this order comprise only one of these AB links. Those having a single F_{AB} bond sum up to the factorized expression

$$-\frac{1}{\beta d}\frac{q}{4\pi\sinh q}\frac{G^0_A(1,0,\mathbf{0})}{e_{\alpha_0}}\frac{G^0_B(0,2,\mathbf{0})}{e_{\beta_0}}, \tag{59}$$

where

$$\begin{aligned} G^0_A(1,0,\mathbf{0}) &= h^0_A(1,0,\mathbf{0}) \\ &\quad - \int \mathrm{d}i\, \rho^0_A(i)\left[F^0_A(1,i,\mathbf{0}) + \tfrac{\delta(1,i)}{\rho^0_A(i)}\right](h^0_A)^{\mathrm{nn}}(i,0,\mathbf{0}) \end{aligned} \tag{60}$$

comprises internal correlations occurring in slab A. The superscript "0" qualifies statistical-mechanical quantities characterizing the system governed by the same Hamiltonian (10) but where V^{walls} confines particles in a single slab (either A or B). $h^0_A(1,0,\mathbf{0}) = h^0_A(1,0,\boldsymbol{k}=\mathbf{0})$ is the Ursell correlation between a loop "1" in slab A and a classical charge e_{α_0} located at its right border, denoted by the loop argument

$$0 \equiv \big(x{=}0,\, \alpha_0,\, p{=}1,\, \boldsymbol{X}(\cdot)\equiv\mathbf{0}\big). \tag{61}$$

The structure of G^0_A is determined by the excluded convolution rule applied to F_{AB}. The partial Ursell function $(h^0_A)^{\mathrm{nn}}$ occurring in the right hand side of (60) is defined in Subsection 6.3, see (84). The same notations and definitions apply to the plate B.

The diagrams having a single F^{R}_{AB} bond sum up to the expression

$$\begin{aligned} &\int \mathrm{d}i\, \rho^0_A(i) \int \mathrm{d}j\, \rho^0_B(j)\left[h^0_A(1,i,\mathbf{0}) + \tfrac{\delta(1,i)}{\rho^0_A(i)}\right] \\ &\quad \times (-\beta e_{\gamma_i} e_{\gamma_j})\mathcal{W}_{AB}(i,j,\tfrac{\boldsymbol{q}}{d})\left[h^0_B(j,2,\mathbf{0}) + \tfrac{\delta(j,2)}{\rho^0_B(j)}\right] \end{aligned} \tag{62}$$

where again h^0_A and h^0_B are the Ursell functions of the single plate systems A and B.

The rest of the analysis relies on the application of the perfect sum rule (52) for loops. Indeed, introducing the contribution (62) into the force (40), one builds the integral

$$\int \mathrm{d}1\; p_1 e_{\gamma_1} \rho^0_A(1)\left[h^0_A(1,i,\mathbf{0}) + \tfrac{\delta(1,i)}{\rho^0_A(i)}\right] = 0 \tag{63}$$

which vanishes by (52).

Introducing now the contribution (59) into the force (40), we see that the integrals on the two slabs factorize as

$$f(d) \overset{d\to\infty}{\sim} -\frac{1}{4\pi\beta d^3}\int_0^\infty \mathrm{d}q\ \frac{q^2\mathrm{e}^{-q}}{\sinh q} \times \left[\int \mathrm{d}1\, p_1\, e_{\gamma_1}\, \rho_A^0(1)\tfrac{G_A^0(1,0,\mathbf{0})}{e_{\alpha_0}}\right]\left[\int \mathrm{d}2\, p_2\, e_{\gamma_2}\, \rho_B^0(2)\tfrac{G_B^0(0,2,\mathbf{0})}{e_{\beta_0}}\right]. \tag{64}$$

From (60), we have

$$\int \mathrm{d}1\, p_1\, e_{\gamma_1}\, \rho_A^0(1)\tfrac{G_A^0(1,0,\mathbf{0})}{e_{\alpha_0}} = \int \mathrm{d}1\, p_1\, e_{\gamma_1}\, \rho_A^0(1)\tfrac{h_A^0(1,0,\mathbf{0})}{e_{\alpha_0}} = -1. \tag{65}$$

The first equality follows from the sum rule (55) for the F bond in the single plate A. The second equality is again a consequence of the perfect screening for loops (52). Perfect screening in plate B implies similar identities for the second bracket in (64).

Noticing that the q-integral provides the constant $\zeta(3)/2$, the Casimir force at large separation is

$$f(d) \overset{d\to\infty}{\sim} -\frac{\zeta(3)}{8\pi\beta d^3}, \tag{66}$$

which is the main result of this paper.

6 Asymptotic correlations between the two slabs

To extract the asymptotic large-separation behaviour of the Ursell correlation $h_{AB}(1,2,\frac{\boldsymbol{q}}{d})$, we select the class of prototype graphs that give the dominant contribution by analysing them one by one, as done in [26, App. C].

It is important to distinguish situations where arguments $\mathcal{L}_i, \mathcal{L}_j$ both lie in the same plate or in the two different plates. As done before (see (31), (34)) we index any quantity with arguments $\mathcal{L}_i \in A = [0,a]\times\mathbb{R}^2$ and $\mathcal{L}_j \in B = [0,b]\times\mathbb{R}^2$ with an index AB. We introduce a similar notation for interactions and correlations internal to a given plate, using the index AA, BB when loops lie in the same slab, *e.g.*,

$$\begin{aligned} &F_{AA}(\mathcal{L}_i,\mathcal{L}_j) = F(\mathcal{L}_i,\mathcal{L}_j), \quad F_{BB}(\mathcal{L}_i,\mathcal{L}_j) = F(\mathcal{L}_i+d,\mathcal{L}_j+d),\\ &h_{AA}(\mathcal{L}_i,\mathcal{L}_j) = h(\mathcal{L}_i,\mathcal{L}_j), \quad h_{BB}(\mathcal{L}_i,\mathcal{L}_j) = h(\mathcal{L}_i+d,\mathcal{L}_j+d). \end{aligned} \tag{67}$$

In the limit $d \to \infty$, the plates will have no mutual interaction anymore: AB-correlations are expected to vanish whereas AA and BB quantities will tend to those pertaining to the system constituted by a single plate. Using the superscript "0" to qualify the statistical mechanical description of the single plates, one will have in particular

$$\begin{aligned} \rho_A(1) &\overset{d\to\infty}{\longrightarrow} \rho_A^0(1), \qquad & \rho_B(2) &\overset{d\to\infty}{\longrightarrow} \rho_B^0(2), \\ F_{AA}(\boldsymbol{i},\boldsymbol{j}) &\to F_A^0(\boldsymbol{i},\boldsymbol{j}), \qquad & F_{BB}(\boldsymbol{i},\boldsymbol{j}) &\to F_B^0(\boldsymbol{i},\boldsymbol{j}), \\ h_{AA}(\boldsymbol{i},\boldsymbol{j}) &\to h_A^0(\boldsymbol{i},\boldsymbol{j}), \qquad & h_{BB}(\boldsymbol{i},\boldsymbol{j}) &\to h_B^0(\boldsymbol{i},\boldsymbol{j}). \end{aligned} \tag{68}$$

In the next subsections, we analyse in more detail the behaviour of $h_{AB}(1, 2, \frac{\boldsymbol{q}}{d})$ occurring in the force formula (40) at large d.

6.1 Large-distance behaviour of the screened potential Φ_{AB}

The main fact to be established in this subsection is the factorization of the screened potential $\Phi_{AB}(i, j, \frac{\boldsymbol{q}}{d})$, at order d^{-1}, into two independent parts pertaining to the individual slabs A and B. We extend the arguments developed for a system of classical charges presented in Section 3.2.3 of Ref. [33]. One observes first that this factorization is already present in the bare Coulomb potential $V^{\rm el}(i, j, \boldsymbol{k})$ (54). Indeed taking into account the shift (30) of the loops' positions as well as $x_i^{[s_i]} < 0 < x_j^{[s_j]}$ for all s_i, s_j, one can write

$$\begin{aligned} V_{AB}^{\rm el}(i,j,\boldsymbol{k}) &= \frac{k\mathrm{e}^{-kd}}{2\pi} \left[\int_0^{p_i} \mathrm{d}s_i \ \mathrm{e}^{i\boldsymbol{k}\cdot\lambda_{\gamma_i}\boldsymbol{Y}_i(s_i)} \tfrac{2\pi}{k} \mathrm{e}^{-k\left|x_i^{[s_i]}\right|} \right] \\ &\quad \times \left[\int_0^{p_j} \mathrm{d}s_j \ \mathrm{e}^{-i\boldsymbol{k}\cdot\lambda_{\gamma_j}\boldsymbol{Y}_j(s_j)} \tfrac{2\pi}{k} \mathrm{e}^{-k\left|x_j^{[s_j]}\right|} \right] \\ &\equiv \frac{k\mathrm{e}^{-kd}}{2\pi} \ V_{AA}^{\rm el}(i,0,\boldsymbol{k}) \ V_{BB}^{\rm el}(0,j,\boldsymbol{k}). \end{aligned} \tag{69}$$

We have identified the first bracket to the Coulomb potential inside A between a loop i and a loop variable denoted 0, corresponding to an (arbitrary) classical charge situated on the inner side of the slab (see (61)), and likewise for the second bracket. We show below that the factorization extends to the screened potential as $d \to \infty$ in the form

$$\Phi_{AB}(i, j, \tfrac{\boldsymbol{q}}{d}) \overset{d\to\infty}{\sim} \frac{1}{d} \frac{q}{4\pi \sinh q} \ \Phi_A^0(i, 0, \boldsymbol{0}) \, \Phi_B^0(0, j, \boldsymbol{0}), \tag{70}$$

where $\Phi_A^0(i, 0, \boldsymbol{0})$ is the screened potential at $\boldsymbol{k} = \boldsymbol{0}$ inside the single plate A between a loop i and a classical charge 0 at its right boundary, and likewise for $\Phi_B^0(0, j, \boldsymbol{0})$.

In the chain summation of $V^{\rm el}$ bonds that constitutes Φ_{AB}, obtained by iterating (53), we keep only the dominant chains, which turn out to be of order d^{-1}. We follow the steps performed in the classical situation in [33]. We split every integral on internal convolution points into an A and a B contribution. Using again the notation (67), i.e. specifying $V^{\rm el}$ as $V^{\rm el}_{AA}(i,j,\boldsymbol{k}), V^{\rm el}_{BB}(i,j,\boldsymbol{k}), V^{\rm el}_{AB}(i,j,\boldsymbol{k})$ according to the location of its argument in A or B, $V^{\rm el}$ chains are expanded into chains made of $V^{\rm el}_{AA}, V^{\rm el}_{BB}$ and $V^{\rm el}_{AB}$ bonds. One notes that $V^{\rm el}_{AA}(i,j,\boldsymbol{k}) = V^{\rm el}_{BB}(i,j,\boldsymbol{k})$, and that $V^{\rm el}_{BA}(i,j,\boldsymbol{k}) = V^{\rm el}_{AB}(j,i,-\boldsymbol{k})$ by space inversion in the $\boldsymbol{y}$ plane. We call $V^{\rm el}_{AB}(i,j,\boldsymbol{k})$ a traversing bond, and chains that link A with B, traversing chains. Clearly, traversing chains that contribute to $\Phi_{AB}(i,j,\frac{\boldsymbol{q}}{d})$ have necessarily an odd number of traversing bonds $V^{\rm el}_{AB}$. Let $\Phi^{(2n+1)}_{AB}$ be the sum of chains containing exactly $2n+1$ traversing bonds. The contribution $\Phi^{(1)}_{AB}$ is a sum of convolution chains of the type $\cdots V^{\rm el}_{AA} * V^{\rm el}_{AA} * V^{\rm el}_{AB} * V^{\rm el}_{BB} * V^{\rm el}_{BB} \cdots$. Using the factorization (69) of $V^{\rm el}_{AB}$, one can resum on eiter side of $V^{\rm el}_{AB}$ convolution chains of $V^{\rm el}_{AA}$ and $V^{\rm el}_{BB}$ into quantities $\widetilde{\Phi}_{AA}(i,0,\frac{\boldsymbol{q}}{d})$ and $\widetilde{\Phi}_{BB}(0,j,\frac{\boldsymbol{q}}{d})$; $\widetilde{\Phi}_{AA}$ differs from the screened potential Φ_{AA} by the omission of traversing chains starting in plate A and returning to it, which describe part of the electrical influence of B on A (likewise for $\widetilde{\Phi}_{BB}$). But for large d, these traversing chains do not contribute anymore and by (68), $\widetilde{\Phi}_{AA}(i,0,\frac{\boldsymbol{q}}{d})$ and $\widetilde{\Phi}_{BB}(0,j,\frac{\boldsymbol{q}}{d})$ also tend as $d\to\infty$ to the screened loop potentials $\Phi^0_A(i,0,\boldsymbol{0})$ and $\Phi^0_B(0,j,\boldsymbol{0})$ of the single plates systems. Hence,

$$\Phi^{(1)}_{AB}(i,j,\tfrac{\boldsymbol{q}}{d}) \overset{d\to\infty}{\sim} \frac{q\mathrm{e}^{-q}}{2\pi d}\ \Phi^0_A(i,0,\boldsymbol{0})\ \Phi^0_B(0,j,\boldsymbol{0}) \tag{71}$$

is of order d^{-1} with a factorized coefficient.

The contribution $\Phi^{(3)}_{AB}$ is then formed by convolution chains of the type $\cdots V^{\rm el}_{AA} * V^{\rm el}_{AB} * \Phi^{(1)}_{BA} * V^{\rm el}_{AB} * V^{\rm el}_{BB} \cdots$. Using (69), (71) and resumming again the convolution chains of $V^{\rm el}_{AA}$ and $V^{\rm el}_{BB}$ on either extremity into $\widetilde{\Phi}_{AA}$ and $\widetilde{\Phi}_{BB}$, one obtains

$$\begin{aligned}\Phi^{(3)}_{AB}(i,j,\tfrac{\boldsymbol{q}}{d}) = \left(\frac{q\mathrm{e}^{-q}}{2\pi d}\right)^3 \widetilde{\Phi}_{AA}(i,0,\tfrac{\boldsymbol{q}}{d}) \left[-\int \mathrm{d}1 \frac{\kappa^2_B(1)}{4\pi} V^{\rm el}_{BB}(0,1,\tfrac{\boldsymbol{q}}{d}) \widetilde{\Phi}_{BB}(1,0,\tfrac{\boldsymbol{q}}{d})\right] \\ \times \left[-\int \mathrm{d}2 \frac{\kappa^2_A(2)}{4\pi} \widetilde{\Phi}_{AA}(0,2,\tfrac{\boldsymbol{q}}{d}) V^{\rm el}_{AA}(2,0,\tfrac{\boldsymbol{q}}{d})\right] \widetilde{\Phi}_{BB}(0,j,\tfrac{\boldsymbol{q}}{d}).\end{aligned} \tag{72}$$

By definition of $\widetilde{\Phi}_{AA}$ and $\widetilde{\Phi}_{BB}$, these quantities satisfy the integral relation (53) relative to A and B with inverse screening lengths $\kappa^2_A(1)$

and $\kappa_B^2(1)$ in place of $\kappa^2(1)$. The brackets in (72) thus reduce to

$$\widetilde{\Phi}_{BB}(0,0,\tfrac{\boldsymbol{q}}{d}) - V_{BB}^{\rm el}(0,0,\tfrac{\boldsymbol{q}}{d}) = -\frac{2\pi d}{q} + \mathrm{O}(1), \tag{73}$$

$$\widetilde{\Phi}_{AA}(0,0,\tfrac{\boldsymbol{q}}{d}) - V_{AA}^{\rm el}(0,0,\tfrac{\boldsymbol{q}}{d}) = -\frac{2\pi d}{q} + \mathrm{O}(1). \tag{74}$$

On the right hand side of (73) and (74), the dominant terms come from the Coulomb potentials (see (54)), while the estimates O(1) reflect the fact that $\widetilde{\Phi}_{AA}(i,j,\boldsymbol{k})$ and $\widetilde{\Phi}_{BB}(i,j,\boldsymbol{k})$ are bounded in $\boldsymbol{k}$ (see (47) and Appendix C) . This yields

$$\Phi_{AB}^{(3)}(i,j,\tfrac{\boldsymbol{q}}{d}) \overset{d\to\infty}{\sim} \frac{q\mathrm{e}^{-q}}{2\pi d}\,\mathrm{e}^{-2q}\Phi_A^0(i,0,\boldsymbol{0})\Phi_B^0(0,j,\boldsymbol{0}). \tag{75}$$

By induction on n, one easily sees that $\Phi_{AB}^{(2n+1)}(i,j,\frac{\boldsymbol{q}}{d})$ receives instead a prefactor $\frac{q\mathrm{e}^{-q}}{2\pi d}\mathrm{e}^{-2nq}$. Summing over $n = 1,2,3,...$ gives the result (70). It is interesting to see that the screened electrostatic interaction between the plates, at order d^{-1}, involves only particles located close to the inner faces of the slabs.

For later convenience we write the equivalent factorized form of the bond F (48),

$$F_{AB}(i,j,\tfrac{\boldsymbol{q}}{d}) \overset{d\to\infty}{\sim} -\frac{1}{\beta d}\frac{q}{4\pi\sinh q}\,\frac{F_A^0(i,0,\boldsymbol{0})}{e_{\alpha_0}}\,\frac{F_B^0(0,j,\boldsymbol{0})}{e_{\beta_0}}, \tag{76}$$

where e_{α_0} and e_{β_0} are two charges located at the inner boundary of the slabs.

6.2 Large-distance behaviour of the dipolar potential $\mathcal{W}_{AB}$

The partial Fourier transform $\mathcal{W}_{AB}(1,2,\boldsymbol{k})$ is related to the three-dimensional Fourier transform $\mathcal{W}(\chi_1,\chi_2,\boldsymbol{\kappa})$ ($\boldsymbol{\kappa} = (k_1,\boldsymbol{k})$) of (44) by

$$\mathcal{W}_{AB}(1,2,\boldsymbol{k}) = \int\frac{\mathrm{d}k_1}{2\pi}\mathrm{e}^{ik_1(x_1-x_2-d)}\mathcal{W}(\chi_1,\chi_2,\boldsymbol{\kappa}), \tag{77}$$

remembering that $\mathcal{W}_{AB}(\mathcal{L}_1,\mathcal{L}_2) \equiv \mathcal{W}(\mathcal{L}_1,\mathcal{L}_2+d)$. Changing $k_1 \mapsto q_1/d$ and setting $\boldsymbol{k} = \boldsymbol{q}/d$,

$$\mathcal{W}_{AB}(1,2,\tfrac{\boldsymbol{q}}{d}) = \frac{1}{d}\int\frac{\mathrm{d}q_1}{2\pi}\mathrm{e}^{iq_1\frac{x_1-x_2}{d}}\mathrm{e}^{-iq_1}\mathcal{W}(\chi_1,\chi_2,\tfrac{q_1}{d},\tfrac{\boldsymbol{q}}{d}), \tag{78}$$

which shows that $\mathcal{W}_{AB}(1,2,\frac{\boldsymbol{q}}{d}) = \mathrm{O}(d^{-1})$ provided the integral has a limit as $d\to\infty$. The analysis of $\mathcal{W}(\chi_1,\chi_2,\boldsymbol{\kappa})$ at small $\boldsymbol{\kappa}$ has

been carried out in [29]. It was observed that the dipolar electric part $\mathcal{W}^{\rm c}$ was screened by thermalized photons at large distance. As a consequence $\mathcal{W}(\chi_1,\chi_2,\boldsymbol{K})$ behaves as $W^{\rm m}(\chi_1,\chi_2,\boldsymbol{K})$ when $\boldsymbol{K}\to 0$, where $W^{\rm m}$ is the magnetic potential corresponding to a classical electromagnetic field (*i.e.*, setting $\lambda_{\rm ph}\equiv 0$ in $\mathcal{W}^{\rm m}$). The behaviour of $W^{\rm m}(\chi_1,\chi_2,\boldsymbol{K})$ itself was worked out in [34, Form. (25)]:

$$W^{\rm m}(\chi_1,\chi_2,\boldsymbol{K})\sim \frac{\lambda_{\gamma_1}\lambda_{\gamma_2}}{\beta\sqrt{m_{\gamma_1}m_{\gamma_2}}c^2}\int_0^{p_1}\mathrm{d}X_1^\mu(s_1)\int_0^{p_2}\mathrm{d}X_2^\mu(s_2) \times[\boldsymbol{K}\cdot\boldsymbol{X}_1(s_1)][\boldsymbol{K}\cdot\boldsymbol{X}_2(s_2)]\frac{4\pi}{K^2}\delta^{\rm tr}_{\mu\nu}(\boldsymbol{K}),\quad \boldsymbol{K}\to\boldsymbol{0}. \quad (79)$$

It is analogous to dipolar magnetic interaction between two classical current loops of shape $\boldsymbol{X}_1(\cdot)$ and $\boldsymbol{X}_2(\cdot)$. Since this interaction only depends on the unit vector $\hat{\boldsymbol{K}}=\boldsymbol{K}/K$, $\mathcal{W}(\chi_1,\chi_2,\frac{q_1}{d},\frac{\boldsymbol{q}}{d})$ is asymptotically independent of d, implying

$$\mathcal{W}_{AB}(1,2,\tfrac{\boldsymbol{q}}{d})=\mathrm{O}(d^{-1}). \quad (80)$$

An explicit expression for the asymptotic form of (78) can be found in Appendix D.

The potentials $\mathcal{W}^{\rm c}_{AB}$ and $\mathcal{W}^{\rm m}_{AB}$ could as well be separately analysed in the same way. One sees on (78) that $\partial_{x_1}\mathcal{W}^{\rm m}_{AB}(1,2,\frac{\boldsymbol{q}}{d})$ has an additional d^{-1} factor, so that

$$\mathcal{W}^{\rm m}_{AB}(1,2,\tfrac{\boldsymbol{q}}{d})=\mathrm{O}(d^{-1}),\quad \partial_{x_1}\mathcal{W}^{\rm m}_{AB}(1,2,\tfrac{\boldsymbol{q}}{d})=\mathrm{O}(d^{-2}). \quad (81)$$

6.3 The Ursell function at order $\mathrm{O}(d^{-1})$

From (76), (80) and the definition of $F^{\rm R}_{AB}$ (49), one sees that $F_{AB}(i,j,\frac{\boldsymbol{q}}{d})=\mathrm{O}(d^{-1})$ and $F^{\rm R}_{AB}(i,j,\frac{\boldsymbol{q}}{d})=\mathrm{O}(d^{-1})$. It is clear that the decay rate of a prototype graph will depend on the number of its traversing bonds. A rough counting gives $d^{-n_{AB}}d^{-n^{\rm R}_{AB}}$ where n_{AB} is the number of F_{AB} bonds and $n^{\rm R}_{AB}$ the number of $F^{\rm R}_{AB}$ bonds. A closer inspection shows that this decay can be even faster, at least as

$$d^{-2I}d^{-n_{AB}}d^{-n^{\rm R}_{AB}},\qquad d\to\infty, \quad (82)$$

where I, $0\le I\le n_{AB}+n^{\rm R}_{AB}-1$, depends on the topology of the specific diagram and $I=0$ if there is a single traversing bond. Formula (82) can be established repeating the same steps as Appendix C of [26] (with integrals over loop degrees of freedom).

From (82), the slowest decaying Mayer diagrams have either $n_{AB}=1, n^{\rm R}_{AB}=0$ or $n_{AB}=0, n^{\rm R}_{AB}=1$, namely only one F_{AB} or one $F^{\rm R}_{AB}$ bond. In forming the complete correlation function of the two-slab

system, these bonds have to be dressed at their extremities by appropriate internal correlations of the individual slabs in conformity with the diagrammatic rules. Thus, the complete expression of the Ursell function $h_{AB}(1,2,\frac{\boldsymbol{q}}{d})$ at order $\mathrm{O}(d^{-1})$ is

$$h_{AB} \overset{d\to\infty}{\sim} D_{AA} * F_{AB} * D_{BB} + D^R_{AA} * F^{\mathrm{R}}_{AB} * D^R_{BB}. \tag{83}$$

The formation of the dressing function D differs from that of D^R because of the excluded convolution rule in prototype graphs: no F_{AA} or F_{BB} bond can be attached alone to the extremities of F_{AB} whereas there is no such restriction for F^{R}_{AB}.

The dressing function D^R_{AA} (D^R_{BB}) of F^{R}_{AB} consists of all possible AA (BB) internal graphs. According to the discussion at the beginning of the section (see (68)), it tends to the Ursell function h^0_A (h^0_B) of the individual plate. The corresponding contribution to $h_{AB}(1,2,\frac{\boldsymbol{q}}{d})$ is thus given at $\mathrm{O}(d^{-1})$ by (62). The delta terms in (62) account for the situation where no bonds are attached to the extremities of F^{R}_{AB}.

To deal with the excluded convolution rule when forming the dressing function D_{AA}, we introduce the function $h^{nn}_{AA}(\boldsymbol{i},\boldsymbol{j})$ defined by the sum of all prototype graphs that do not begin nor end with a F link alone. Its relation to the Ursell function is

$$h = F + h^{nn} * F + F * h^{nn} * F + F * h^{nn} + h^{nn}. \tag{84}$$

Then $D_{AA} = h^{nn}_{AA} + F_{AA} * h^{nn}_{AA} + \delta/\rho_A$ (likewise for D_{BB}). With the factorization (76) of the link F_{AB}, one sees that $(D_{AA} * F_{AB} * D_{BB})(1,2,\frac{\boldsymbol{q}}{d})$ has the factorized form (59) with

$$G^0_A = \left(\frac{\delta}{\rho^0_A} + (h^0_A)^{nn} + F^0_A * (h^0_A)^{nn}\right) * F^0_A. \tag{85}$$

Since the bond F_{AB} is already $\mathrm{O}(d^{-1})$, all other quantities have been evaluated for single plate systems according to (68). The final form (60) of G^0_A follows by noticing that the first three terms of (84) build G^0_A as given in (85).

7 Concluding remarks

In this paper the large-separation asymptotics (6) of the Casimir force between two conducting plates has been derived exactly from the principles of quantum electrodynamics and statistical mechanics for any fixed nonzero temperature, taking all microscopic degrees of freedom of matter and field into account. This does not give a direct proof

that the TE mode reflexion coefficient does not contribute at zero frequency, but a strong evidence for it. The derivation applies to any model of conductor consisting of mobile quantum charges. The latter can be negative and positive charge carriers (like ions and anions in electrolytes), or, *e.g.*, form the one-component electron gas in the jellium model of a metal, the central common point to all these systems being the screening mechanisms and the perfect screening sum rules.

Let us note that there can be no contradiction between the behaviour (6) and the Nernst heat theorem. As mentioned in the introduction, it has been argued, and controversially debated, that the use of the Drude expression of the dielectric function (yielding (6)) was not consistent with the Nernst postulate which requires that the entropy of the total system vanishes at zero temperature. In our setting the question arises in different terms. The asymptotic formula (6) is definitely true, whereas the Nernst theorem is the separate affirmation that the QED Hamiltonian (10) has a unique (or not extensively degenerate) ground state, a nontrivial and uncorrelated mathematical problem.

A number of questions deserve further studies. We have disregarded paramagnetic forces due to the Pauli coupling $-\mu\boldsymbol{\sigma}\cdot\boldsymbol{B}$ of electronic and nuclear magnetic moments $\mu\boldsymbol{\sigma}$ to the magnetic field $\boldsymbol{B}$. Preliminary investigations using spin coherent states functional integrals indicate that such interactions result in an additional effective dipolar potential which, as the orbital diamagnetic part $\mathcal{W}^{\mathrm{m}}$, will not contribute to the asymptotic force.

We have kept the thickness a and b of the plates finite while the separation $d \to \infty$. Then, because of perfect screening, the asymptotic force is universal as well as independent of a and b. This corresponds to the present experimental situation where only thin coats of metal of order of 50 nm are deposited on a substrate [35]. Compared to separations ranging from 0.5 to 3 μm, the regime is clearly $a, b \ll d$. The opposite situation of thick plates $a, b \gg d$, namely taking here $a = b = \infty$ at the very beginning, requires a careful analysis since the magnetic potential $\mathcal{W}^{\mathrm{m}}(1, 2, \frac{\boldsymbol{q}}{d})$ looses integrability as $d \to \infty$ (see the factor $\exp(i\, q_1\, [x_1 - x_2]/d)$ in (78)). Then x-integrals have to be performed before taking the limit $d \to \infty$ which appears to lead to a modification of the d^{-3} coefficient with small (nonuniversal) terms of order $\mathrm{O}\big((\beta mc^2)^{-1}\big)$.

Expression (66) is the first term of an expansion in inverse powers of d whose terms will be of the form $A_n/d^n,\ n \geq 4$. The amplitudes $A_n(\rho, T)$ are no more universal. They will depend on the thermodynamic and geometric parameters of the plates (temperature, densities ρ, thickness a, b) as well as their microscopic characteristics (particle

masses and charges). Looking at the form of the electrostatic and magnetic dipole potentials $\mathcal{W}^{\rm m}$ and $\mathcal{W}^{\rm c}$, the expansion can be cast in terms of dimensionless quantities including $(\lambda_{\rm mat}/d)^n$, $(\lambda_{\rm ph}/d)^n$, where $\lambda_{\rm mat}$ and $\lambda_{\rm ph}$ are the matter and photon thermal lengths. This expansion is therefore only meaningful when the condition (15) is met. Of great interest would be the calculation of the first subdominant amplitude A_4 that includes corrections from an imperfectly-conducting metal and to compare it with the predictions of the Lifshitz theory [6]. Also the effect of the capacitor force, which cannot be completely turned off in experiment, can be estimated by analysing the term (39) at large separation.

Finally, an open question is the understanding of the crossover to the zero-temperature Casimir force $f \sim -\pi^2\hbar c/240d^4$ due to pure quantum fluctuations. In the framework of Lifshitz theory, it was shown that plasmon modes at the surfaces of the plates combine with photonic modes to build the above usual zero-temperature Casimir force calculated as if the plates were inert [36]. Notice that we have not added to the Hamiltonian (10) the vacuum energy $\frac{1}{2}\sum_{K,\lambda}\hbar\omega_K$. In fact this (infinite) constant plays no role since it will anyway not appear in the force formula (8) (it is independent of d in our setting). To study the zero-temperature case one cannot rely on the above-mentioned expansion since $\lambda_{\rm mat}, \lambda_{\rm ph} \to \infty$ as $T \to 0$ and condition (15) does not hold anymore. One has to reconsider the whole analysis by first evaluating the force (40) in the zero-temperature limit at fixed d and then let $d \to \infty$. In other words, the limits $T \to 0$ and $d \to \infty$ are not permutable, and the issue is about obtaining a simultaneous control of the force jointly for T near zero and d large. This will be the subject of forthcoming work.

A Capacitor force

Given that in the electrostatic part of the total force (36), only the monopolar part $p_1p_2\partial_{x_1}v_{AB}$ of the loop Coulomb force $\partial_{x_1}V^{\rm c}_{AB}$ contributes (see Appendix B), the electrostatic capacitor force $f^{\rm c}_{\rm cap}(d)$ is

$$f^{\rm c}_{\rm cap}(d) = \int_A {\rm d}1 \int_B {\rm d}2 \int {\rm d}\boldsymbol{y}\, e_{\gamma_1} e_{\gamma_2}\, p_1 p_2\, \partial_{x_1} v_{AB}(1,2,\boldsymbol{y}) \rho_A(1)\rho_B(2). \quad (86)$$

The loop densities $\rho_A(1)$ and $\rho_B(2)$ are independent of $\boldsymbol{y}$ by space homogeneity in the plates' directions and $\int {\rm d}\boldsymbol{y}\ \partial_{x_1} v_{AB}(1,2,\boldsymbol{y}) = 2\pi$ (set $\boldsymbol{q} = \boldsymbol{0}$ in (41)). The remaining integrals factorize, and yield the particle densities $\rho_A(x_1,\gamma_1)$ and $\rho_B(x_2,\gamma_2)$ in plate A and B by means

of the identity

$$\rho(\boldsymbol{r},\gamma)=\sum_{p=1}^{\infty}p\int\mathrm{D}(\boldsymbol{X})\;\rho(\mathcal{L}) \tag{87}$$

(see [37, Appendix D]). Introducing the charge density $c(x)=\sum_\gamma e_\gamma\rho(x,\gamma)$ in plate A and B, $f^{\mathrm{c}}_{\mathrm{cap}}(d)$ is thus given by Formula (39). Note that the charge densities $c_A(x)$ and $c_B(x)$ are still subject to the mutual interaction between the slabs, thus depend on the separation d.

The magnetic part of the capacitor force is

$$f^{\mathrm{m}}_{\mathrm{cap}}(d)=\int_A\mathrm{d}1\int_B\mathrm{d}2\int\mathrm{d}\boldsymbol{y}\,e_{\gamma_1}e_{\gamma_2}\partial_{x_1}\mathcal{W}^{\mathrm{m}}_{AB}(1,2,\boldsymbol{y})\rho_A(1)\rho_B(2). \tag{88}$$

We show hereafter that

$$\int\mathrm{d}\boldsymbol{y}\,\partial_{x_1}\mathcal{W}^{\mathrm{m}}_{AB}(1,2,\boldsymbol{y})=\int\frac{\mathrm{d}k_1}{2\pi}\,\mathrm{e}^{ik_1(x_1-x_2-d)}\,ik_1\,\mathcal{W}^{\mathrm{m}}(\chi_1,\chi_2,k_1,\boldsymbol{k}=\boldsymbol{0}) \tag{89}$$

decays faster than any inverse power of (x_1-x_2-d) as $d\to\infty$. This ensures that the decay of $f^{\mathrm{m}}_{\mathrm{cap}}(d)$ with the plates' separation has no power-law tail in view of (88).

The dipolar decay of $\mathcal{W}^{\mathrm{m}}(\boldsymbol{1},\boldsymbol{2})$ (21) at large distance, responsible for the power-law estimates (81), is generated by the nonanalyticity $k_\mu k_\nu/{K}^2$ due to $\delta^{\mathrm{tr}}_{\mu\nu}(\boldsymbol{\mathcal{K}})$ in the transverse Coulomb potential $4\pi\delta^{\mathrm{tr}}_{\mu\nu}(\boldsymbol{\mathcal{K}})/{K}^2$ ($\boldsymbol{\mathcal{K}}=(k_1,\boldsymbol{k})$). However, setting $\boldsymbol{k}=\boldsymbol{0}$ eliminates the k_1 dependency in the transverse Kronecker symbol:

$$\delta^{\mathrm{tr}}_{\mu\nu}(k_1,\boldsymbol{k}=\boldsymbol{0})=\begin{cases}\delta_{\mu\nu} & \text{if } \mu,\nu\neq 1,\\ 0 & \text{if } \mu=\nu=1.\end{cases} \tag{90}$$

Any nonanalyticity is thus removed in $\mathcal{W}^{\mathrm{m}}(\chi_1,\chi_2,k_1,\boldsymbol{k}=\boldsymbol{0})$ around $k_1=0$, ensuring the fast decay of (89). Indeed, in $\mathcal{W}^{\mathrm{m}}(\chi_1,\chi_2,k_1,\boldsymbol{k}=\boldsymbol{0})$, one is left with the stochastic integrals

$$\int_0^{p_1}\mathrm{d}\boldsymbol{Y}_1(s_1)\cdot\int_0^{p_2}\mathrm{d}\boldsymbol{Y}_2(s_2)\frac{4\pi g^2(k_1)}{k_1^2}\mathcal{Q}(k_1,\widetilde{s_1}-\widetilde{s_2})\mathrm{e}^{ik_1\lambda_{\gamma_1}X_1(s_1)}\mathrm{e}^{-ik_1\lambda_{\gamma_2}X_2(s_2)}.$$

Both $g^2(k_1)$ and $\mathcal{Q}(k_1,\widetilde{s_1}-\widetilde{s_2})$ are analytic functions of k_1 expandable as $1+\mathrm{O}(k_1^2)$. One sees by expanding the integrant around $k_1=0$ that the only singular terms are functions of only s_1 or s_2. Their stochastic integration identically vanishes by Itô's lemma, stating that $\int_0^p\mathrm{d}\boldsymbol{X}(s)\equiv 0$.

B Electrostatic force

In Formula (32), the Casimir force has an electrostatic part $f^{\mathrm{c}}(d)$ due to $\partial_x V^{\mathrm{c}}_{AB}$ and a magnetic part $f^{\mathrm{m}}(d)$ resulting from differentiating the magnetic potential. One could write the average electrostatic force between the two slabs directly by summing the Coulomb forces between the point charges:

$$\begin{aligned} f^{\mathrm{c}}(d) &= \lim_{L\to\infty} \frac{1}{L^2} \left\langle \sum_a \sum_b e_{\gamma_a} e_{\gamma_b} \partial_x v_{AB}(\boldsymbol{r}_a, \boldsymbol{r}_b) \right\rangle \\ &= \lim_{L\to\infty} \frac{1}{L^2} \int_A \mathrm{d}\boldsymbol{r}_1 \int_B \mathrm{d}\boldsymbol{r}_2 \sum_{\gamma_1} \sum_{\gamma_2} e_{\gamma_1} e_{\gamma_2} \partial_x v_{AB}(\boldsymbol{r}_1, \boldsymbol{r}_2) \\ &\quad \times \rho^{(2)}_{AB,L}(\boldsymbol{r}_1, \gamma_1; \boldsymbol{r}_2, \gamma_2), \end{aligned} \tag{91}$$

where $v_{AB}(\boldsymbol{r}_a, \boldsymbol{r}_b)$ is the Coulomb potential (11), $\rho^{(2)}_L$ is the particle density correlation function, and the same notation (31), (34) translating positions from slab B_d to slab B is used. Going to the phase space of loops by means of the identity

$$\rho^{(2)}_{AB,L}(\boldsymbol{r}_1, \gamma_1; \boldsymbol{r}_2, \gamma_2) = \sum_{p_1=1}^{\infty} \sum_{p_2=1}^{\infty} p_1 p_2 \int \mathrm{D}(\boldsymbol{X}_1) \int \mathrm{D}(\boldsymbol{X}_2) \rho^{(2)}_{AB,L}(\mathbf{1}, \mathbf{2}) \tag{92}$$

(see [37, Appendix D]) yields

$$f^{\mathrm{c}}(d) = \lim_{L\to\infty} \frac{1}{L^2} \int_A \mathrm{d}\mathbf{1} \int_B \mathrm{d}\mathbf{2}\ e_{\gamma_1} e_{\gamma_2} \big(p_1\, p_2\, \partial_{x_1} v_{AB}\big)(\mathbf{1}, \mathbf{2}) \rho^{(2)}_{AB,L}(\mathbf{1}, \mathbf{2}). \tag{93}$$

In Formula (93), multipolar contributions of the Coulomb force are not present, in contrast to the electrostatic part of (33). The strict equivalence of these formulae relies on an invariance property regarding the choice of a reference point for a loop's position. Clearly, the loop

$$\mathcal{L}^{[u]} \equiv \big(\boldsymbol{r}^{[u]}, \gamma, p, \boldsymbol{X}^{[u]}(\cdot)\big) \quad \text{with} \quad \boldsymbol{X}^{[u]}(s) \equiv \boldsymbol{X}(s+u) - \boldsymbol{X}(u) \tag{94}$$

describes the same path as the loop $\mathcal{L} = \big(\boldsymbol{r}, \gamma, p, \boldsymbol{X}(\cdot)\big)$ but has its origin $\boldsymbol{r}^{[u]} = \boldsymbol{r} + \lambda_\gamma \boldsymbol{X}(u)$ shifted by the vector $\lambda_\gamma \boldsymbol{X}(u)$ (the time parameter is shifted by u). Such a shift does not affect the loop density:

$$\rho(\mathcal{L}^{[u]}) = \rho(\mathcal{L}) \quad \forall u, \tag{95}$$

whereas the two-loop correlation function satisfies

$$\rho^{(2)}_L(\mathcal{L}_1^{[u_1]}, \mathcal{L}_2^{[u_2]}) = \rho^{(2)}_L(\mathcal{L}_1, \mathcal{L}_2) \quad \text{if} \quad u_1 - u_2 \in \mathbb{Z} \tag{96}$$

(see below). In the electrostatic part of (33),

$$(\partial_{x_1} V^{\mathrm{c}}_{AB})(\mathbf{1},\mathbf{2}) = \int_0^{p_1}\mathrm{d}s_1 \int_0^{p_2}\mathrm{d}s_2 \delta(\widetilde{s_1}-\widetilde{s_2})\; \partial_{x_1} v_{AB}(\boldsymbol{r}_1^{[s_1]}, \boldsymbol{r}_2^{[s_2]}) \qquad (97)$$

(see (20)) and one can replace $\rho^{(2)}_{AB,L}(\mathcal{L}_1,\mathcal{L}_2)$ by $\rho^{(2)}_{AB,L}(\mathcal{L}_1^{[s_1]},\mathcal{L}_2^{[s_2]})$ (at fixed p_1,p_2,s_1,s_2) because of the equal-time constraint in (97) that forces s_1-s_2 to be integer. Performing first the changes of variable $\boldsymbol{r}_1 \mapsto \boldsymbol{r}_1^{[s_1]}$, $\boldsymbol{r}_2 \mapsto \boldsymbol{r}_2^{[s_2]}$ and then

$$\boldsymbol{X}_1(\cdot) \mapsto \boldsymbol{X}_1^{[s_1]}(\cdot), \quad \boldsymbol{X}_2(\cdot) \mapsto \boldsymbol{X}_2^{[s_2]}(\cdot), \qquad (98)$$

one obtains the electrostatic force

$$\lim_{L\to\infty} \frac{1}{L^2} \int_A \mathrm{d}\mathbf{1} \int_B \mathrm{d}\mathbf{2}\, e_{\gamma_1} e_{\gamma_2} \int_0^{p_1}\mathrm{d}s_1 \int_0^{p_2}\mathrm{d}s_2 \delta(\widetilde{s_1}-\widetilde{s_2})\, \partial_{x_1} v_{AB}(\boldsymbol{r}_1,\boldsymbol{r}_2) \rho^{(2)}_{AB,L}(\mathbf{1},\mathbf{2}). \qquad (99)$$

Indeed, the Jacobian of the transformations (98) is equal to 1: the random process $\boldsymbol{X}^{[u]}(\cdot)$ is still Gaussian with unit normalization, zero mean, and same covariance (17) as $\boldsymbol{X}(\cdot)$, so that the Gaussian measure is unchanged: $\mathrm{D}(\boldsymbol{X}^{[u]}) = \mathrm{D}(\boldsymbol{X})$ (see [37, Lemma 1] or [38, Lemma 2]). Formula (99) eventually reduces to (91) since $\int_0^{p_1}\mathrm{d}s_1 \int_0^{p_2}\mathrm{d}s_2 \delta(\widetilde{s_1}-\widetilde{s_2}) = p_1\, p_2$.

The properties (95) and (96) both follow from the fact that the loop's self-energy in the activity (28) is invariant under a shift of origin and the loop pairwise interaction $V(\mathcal{L}_i,\mathcal{L}_j)$ (19) is invariant when the loops $\mathcal{L}_i$ and $\mathcal{L}_j$ have their origin shifted to $\mathcal{L}_i^{[u_1]}$ and $\mathcal{L}_j^{[u_2]}$ with $u_1-u_2 \in \mathbb{Z}$. The restriction $u_1-u_2\in\mathbb{Z}$ is the manifestation of the Feynman–Kac equal-time constraint in V^{c} and of the quantum nature of the photon field in $\mathcal{W}^{\mathrm{m}}$, occurring through the function $\mathcal{Q}(K,\widetilde{s_1}-\widetilde{s_2})$ (22): $\mathcal{W}^{\mathrm{m}}$ is unchanged because $\mathcal{Q}\big(K,\widetilde{(s_1+u_1)}-\widetilde{(s_2+u_2)}\big) = \mathcal{Q}(K,\widetilde{s_1}-\widetilde{s_2})$ when $u_1-u_2\in\mathbb{Z}$ by periodicity of the function $s\mapsto \mathcal{Q}(K,s)$.

C Screening of the resummed interaction Φ

The classical Debye–Hückel potential $\Phi^{\mathrm{class}}(x_1,x_2,\boldsymbol{y})$ for slab geometry has been extensively studied in [26]. Because of the wall constraint on the screening clouds, this potential does not decay exponentially fast as would be the case in the bulk, but still has an integrable tail

$\sim y^{-3}$ along the wall.[5] This implies that its transverse Fourier transform $\Phi^{\rm class}(x_1, x_2, \boldsymbol{k})$ is finite at $\boldsymbol{k} = \boldsymbol{0}$, see Formula (A.11) of [26]. In the sequel, we infer that the same property remains true for the screened potential between loops $\Phi(1, 2, \boldsymbol{k})$ defined by (45). The only difference between the Coulomb potential $v(\boldsymbol{r}_1 - \boldsymbol{r}_2)$ for point charges and $V^{\rm el}(\boldsymbol{1}, \boldsymbol{2})$ is the extension of the loops that generates additional multipole interactions. To disentangle the monopole interaction from the multipole contributions, we proceed with the same method as in Sec. V.B.2 of [31] and only sketch the arguments. Introducing the multipole operator

$$\mathcal{M}_i = \int_0^{p_i} \mathrm{d}s \, \sum_{l=1}^{\infty} \frac{\left[\lambda_{\gamma_i} \boldsymbol{X}_i(s) \cdot \nabla_{\boldsymbol{r}_i}\right]^l}{l!}, \quad i = 1, 2, \tag{100}$$

the loop interaction is decomposed into its charge–charge (cc), charge–multipole (cm, mc) or multipole–multipole (mm) components: $V^{\rm el} = V^{\rm el}_{\rm cc} + V^{\rm el}_{\rm cm} + V^{\rm el}_{\rm mc} + V^{\rm el}_{\rm mm}$, where

$$\begin{aligned} V^{\rm el}_{\rm cc}(\boldsymbol{1}, \boldsymbol{2}) &= p_1 p_2 \; v(\boldsymbol{r}_1 - \boldsymbol{r}_2), \qquad V^{\rm el}_{\rm cm}(\boldsymbol{1}, \boldsymbol{2}) = p_1 \, \mathcal{M}_2 \; v(\boldsymbol{r}_1 - \boldsymbol{r}_2), \\ V^{\rm el}_{\rm mc}(\boldsymbol{1}, \boldsymbol{2}) &= \mathcal{M}_1 \, p_2 \; v(\boldsymbol{r}_1 - \boldsymbol{r}_2), \quad V^{\rm el}_{\rm mm}(\boldsymbol{1}, \boldsymbol{2}) = \mathcal{M}_1 \, \mathcal{M}_2 \; v(\boldsymbol{r}_1 - \boldsymbol{r}_2). \end{aligned} \tag{101}$$

Summing the chains of $V^{\rm el}$ to form Φ amounts to summing all possible chains with bonds $V^{\rm el}_{\rm cc}$, $V^{\rm el}_{\rm cm}$, $V^{\rm el}_{\rm mc}$, and $V^{\rm el}_{\rm mm}$. Summing first the pure $V^{\rm el}_{\rm cc}$ chains builds the classical Debye–Hückel potential $\Phi^{\rm class}$ with screening length $\kappa^{-1}(x) = [4\pi\beta \sum_\gamma \sum_p \int \mathrm{D}(\boldsymbol{X}) p^2 e_\gamma^2 \rho(x, \chi)]^{-1/2}$.[6] One is then left with the screened bonds

$$\begin{aligned} F_{\rm cc} &= -\beta e_{\gamma_1} e_{\gamma_2} \, p_1 p_2 \; \Phi^{\rm class}, \qquad F_{\rm cm} = -\beta e_{\gamma_1} e_{\gamma_2} \; p_1 \, \mathcal{M}_2 \; \Phi^{\rm class}, \\ F_{\rm mc} &= -\beta e_{\gamma_1} e_{\gamma_2} \, \mathcal{M}_1 \, p_2 \; \Phi^{\rm class}, \; F_{\rm mm} = -\beta e_{\gamma_1} e_{\gamma_2} \, \mathcal{M}_1 \, \mathcal{M}_2 \; \Phi^{\rm class}. \end{aligned} \tag{102}$$

Finally, Φ is built from convolution chains of these screened bonds subject to excluded convolution rules with respect to $F_{\rm cc}$. One sees that

$$F_{\rm cm}(\boldsymbol{1}, \boldsymbol{2}) = -\beta e_{\gamma_1} e_{\gamma_2} p_1 \int_0^{p_2} \mathrm{d}s_2 \left[\Phi^{\rm class}(\boldsymbol{r}_1, \boldsymbol{r}_2 + \lambda_{\gamma_2} \boldsymbol{X}_2(s_2)) - \Phi^{\rm class}(\boldsymbol{r}_1, \boldsymbol{r}_2)\right] \tag{103}$$

is integrable in the $\boldsymbol{y}$ direction and so has a finite transverse Fourier transform at $\boldsymbol{k} = \boldsymbol{0}$. The same holds for the other screened bonds and

[5] This was noticed long ago in Ref. [39], see also [32, Sec. III.C.2].

[6] This screening length reduces to the classical expression $[4\pi\beta \sum_\gamma e_\gamma^2 \rho(x, \gamma)]^{-1/2}$ when exchange effects are disregarded [31, Sec. V.B.2].

their chain convolutions, hence the result (47). These considerations apply to the various types of screened potentials considered in this work, *e.g.*, the screened potential of the single plate systems Φ_A^0, Φ_B^0 and the potentials $\widetilde{\Phi}_{AA}$, $\widetilde{\Phi}_{BB}$ occurring in Subsection 6.1.

D Dipole potential $\mathcal{W}_{AB}$ at order $\mathrm{O}(d^{-1})$

From (78) and (79), the asymptotic form of $\mathcal{W}_{AB}(1,2,\frac{\boldsymbol{q}}{d})$ is

$$\begin{aligned}
&\mathcal{W}_{AB}(1,2,\frac{\boldsymbol{q}}{d}) \overset{d\to\infty}{\sim} \frac{1}{d}\int\frac{\mathrm{d}q_1}{2\pi}\mathrm{e}^{-iq_1x}\mathcal{W}(\chi_1,\chi_2,\tfrac{q_1}{d},\tfrac{\boldsymbol{q}}{d})\Big|_{x=1}\\
&=\frac{1}{d}\frac{\lambda_{\gamma_1}\lambda_{\gamma_2}}{\beta\sqrt{m_{\gamma_1}m_{\gamma_2}}c^2}\int_0^{p_1}\mathrm{d}X_1^\mu(s_1)\int_0^{p_2}\mathrm{d}X_2^\mu(s_2)\int\frac{\mathrm{d}q_1}{2\pi}\mathrm{e}^{-iq_1x}\\
&\big[q_1X_1(s_1)+\boldsymbol{q}\cdot\boldsymbol{Y}_1(s_1)\big]\big[q_1X_2(s_2)+\boldsymbol{q}\cdot\boldsymbol{Y}_2(s_2)\big]\frac{4\pi}{q_1^2+q^2}\delta_{\mu\nu}^{\mathrm{tr}}(q_1,\boldsymbol{q})\Big|_{x=1}\\
&=\frac{1}{d}\frac{\lambda_{\gamma_1}\lambda_{\gamma_2}}{\beta\sqrt{m_{\gamma_1}m_{\gamma_2}}c^2}\int_0^{p_1}\mathrm{d}X_1^\mu(s_1)\int_0^{p_2}\mathrm{d}X_2^\mu(s_2)\\
&\big[iX_1(s_1)\tfrac{\partial}{\partial x}+\boldsymbol{q}\cdot\boldsymbol{Y}_1(s_1)\big]\big[iX_2(s_2)\tfrac{\partial}{\partial x}+\boldsymbol{q}\cdot\boldsymbol{Y}_2(s_2)\big]v^{\mu\nu}(x,\boldsymbol{q})\Big|_{x=1}
\end{aligned}\tag{104}$$

where

$$\begin{aligned}
v^{\mu\nu}(x,\boldsymbol{q}) &= \int\frac{\mathrm{d}q_1}{2\pi}\mathrm{e}^{iq_1x}\frac{4\pi}{q_1^2+q^2}\delta_{\mu\nu}^{\mathrm{tr}}(q_1,\boldsymbol{q})\\
&= \frac{\pi}{q}\mathrm{e}^{-q|x|}\times\begin{cases}\delta_{\mu\nu}+q|x|, & \nu=\mu=1,\\ \delta_{\mu\nu}-iq_\mu x, & \mu\neq1,\nu=1,\\ 2\delta_{\mu\nu}-(1+q|x|)\frac{q_\mu q_\nu}{q^2}, & \mu\neq1,\nu\neq1\end{cases}
\end{aligned}\tag{105}$$

is the partial Fourier transform of the transverse Coulomb potential. The final result (Formula (5.88) in [40]) is obtained by working out the derivatives in (104).

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