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Denominator assignment, invariants and canonical forms under dynamic feedback compensation in linear multivariable systems

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***Abstract*—A result orinally reported by Hammer [6] for linear time invariant (LTI) single input-single output systems and concerning an invariant and a canonical form of the transfer function matrix of the closed loop system under dynamic feedback compensation is generalized for LTI multivariable systems. Based on this result, we characterize the class of transfer function matrices that are obtainable from an open loop transfer function matrix via the use of *proper* dynamic feedback compensators and show that if the closed loop transfer function matrix $P_c(s)$ has a desired denominator polynomial matrix which satisfies a certain sufficient condition, then there exists a proper compensator giving rise to an internally stable closed loop system, whose transfer function matrix is $P_c(s)$.**

***Index Terms*—Decoupling, denominator assignment, Euclidean algorithm, proper feedback compensators.**

I. Introduction

Let Σ be a linear, time invariant (LTI), stabilizable multivariable system characterized by a *strictly proper* transfer function matrix $P(s)$ and consider the transfer function matrix $P_c(s)$ of the closed loop feedback system Σ_c in Fig. 1 where $C(s)$ is the transfer function matrix of a *proper* dynamic compensator. In this paper, using mainly the Euclidean division for polynomial matrices [7],[1],[8], we firstly generalize to the multivariable case a result (originally reported in [6] for single input-single output systems) that concerns an invariant and a canonical form of the transfer function matrix $P_c(s)$ of the closed loop system Σ_c obtained from $P(s)$ via feedback through a proper compensator $C(s)$. This result leads to the characterization of the class $[P(s)]_{\mathcal{R}}$ of closed loop transfer function matrices $P_c(s)$ that are obtainable from $P(s)$ via the use of *proper* dynamic feedback compensators $C(s)$. We next determine the class $\mathcal{P}$ of open loop transfer function matrices $P(s)$ that give rise to an internally stable closed loop system Σ_c with transfer function matrix $P_c(s) \in [P(s)]_{\mathcal{R}}$ via the use of a proper compensator $C(s)$ whose structure depends on $P_c(s)$ and emanates from the above analysis. Based on the above, our main result is then that if $P_c(s) \in [P(s)]_{\mathcal{R}}$ has a desired denominator polynomial matrix $D_{RC}(s)$ which satisfies a certain sufficient condition, then there exists a proper compensator $C(s)$ giving rise to an internally stable closed loop system Σ_c, whose transfer function matrix is $P_c(s)$. Thus, given a $P(s) \in \mathcal{P}$, our results lead to an algorithmic procedure for the computation of a *proper* internally stabilizing and denominator assigning compensator. Finally, the problem of *diagonally decoupling* Σ by the use of proper dynamic output feedback is examined and a new proof of the n. and s. condition presented in [9] is given.

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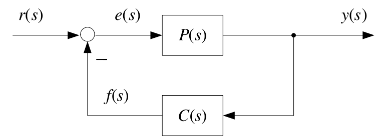

Fig. 1 - The closed loop system Σ_c

In the sequel by $\mathbb{R}[s]^{p\times m}, \mathbb{R}(s)^{p\times m}$, $\mathbb{R}_{pr}(s)^{p\times m}$, $\mathbb{R}_{pr0}(s)^{p\times m}$ we denote the sets of $p\times m$ matrices with elements in the sets respectively of *polynomial, rational*, *proper rational and strictly proper rational* functions, all with coefficients in the field $\mathbb{R}$ of reals and by $\overline{\mathbb{R}}[s]^{m\times m}, \overline{\mathbb{R}}(s)^{m\times m}$, $\overline{\mathbb{R}}_{pr}(s)^{m\times m}$, $\overline{\mathbb{R}}_{pr0}(s)^{m\times m}$ we denote the subsets of the corresponding sets whose elements are *non-singular* (over $\mathbb{R}(s)$) matrices. For a $p(s) = \frac{n(s)}{d(s)} \in \mathbb{R}(s)$ the map $\delta_\infty(\cdot) : \mathbb{R}(s) \to \mathbb{Z}\cup\{\infty\}$ is defined via : $\delta_\infty(p(s)) := \deg d(s) - \deg n(s)$ if $p(s) \neq 0$ and $\delta_\infty(p(s)) := +\infty$, if $p(s) = 0$ and for a $P(s) \in \mathbb{R}(s)^{p\times m}$, $\operatorname{rank}_{\mathbb{R}(s)} P(s) = r$, $\delta_\infty(\cdot) : \mathbb{R}(s)^{p\times m} \to \mathbb{Z}\cup\{\infty\}$ is defined via: $\delta_\infty(P(s)) := \min\{\delta_\infty(\cdot)\}$ among the $\delta_\infty(\cdot)$ of all r order minors of $P(s)$ if $r > 0$ and $\delta_\infty(P(s)) := +\infty$ if $r = 0$. Given a $P(s) \in \mathbb{R}(s)^{p\times m}$ then by $\delta_{\infty ri}(P(s))$ $(\delta_{\infty ci}(P(s)))$ we denote the $\delta_\infty(\cdot)$ of the i row (column) of $P(s)$ which according to the above is the minimum $\delta_\infty(\cdot)$ among the $\delta_\infty(\cdot)$ of its elements [1],[3]. For a $T(s) \in \mathbb{R}[s]^{p\times m}$ the *degree* $\deg T(s)$ of $T(s)$ is defined as maximum degree among the degrees of its maximum order non-zero minors. It simply follows [1] that $\delta_\infty(T(s)) = -\deg T(s)$ and therefore if $T(s) \in \overline{\mathbb{R}}[s]^{m\times m}$ then $\delta_\infty(T(s)) = -\deg|T(s)| \leq 0$ with equality holding iff $T(s)$ is $\mathbb{R}[s]$−unimodular. Given a $T(s) \in \mathbb{R}[s]^{p\times m}$ then by $\deg_{ri} T(s)$ $(\deg_{ci} T(s))$ we denote the degree of the i row (column) of $T(s)$ which is defined as the maximum degree among the degrees of its polynomial elements. The rest of the terminology and notation in the sequel is the standard one found in the literature of the "*polynomial matrix approach*" in books like [10],[11],[12],[7],[1],[8].

II. Background

Let Σ be a LTI multivariable system characterized by a transfer function matrix $P(s) \in \overline{\mathbb{R}}_{pr0}(s)^{m\times m}$ and let $P(s) = D_L(s)^{-1} N_L(s) = N_R(s) D_R(s)^{-1}$ be respectively *left* and *right coprime matrix fraction descriptions* (MFDs) of $P(s)$ with $D_L(s)$, $N_L(s)$, $D_R(s)$, $N_R(s) \in \overline{\mathbb{R}}[s]^{m\times m}$. Let $P(s)^{-1} \in \overline{\mathbb{R}}(s)^{m\times m}$ and $\widehat{p}_{ij}(s) = \frac{\widehat{n}_{ij}(s)}{\widehat{d}_{ij}(s)}$ its elements. Consider the Euclidean divisions $\widehat{n}_{ij}(s) = q_{ij}(s)\widehat{d}_{ij}(s) + r_{ij}(s)$ where $q_{ij}(s)$ is the quotient and $r_{ij}(s)$ is the remainder and either $r_{ij}(s) = 0$ or $\deg r_{ij}(s) < \deg \widehat{d}_{ij}(s)$ and thus decompose each $\widehat{p}_{ij}(s)$ into *polynomial* and *strictly proper* parts as $\widehat{p}_{ij}(s) = q_{ij}(s) + \frac{r_{ij}(s)}{\widehat{d}_{ij}(s)}$, thus write $P(s)^{-1} = \left[P(s)^{-1}\right]_{pol} + \left[P(s)^{-1}\right]_{sp}$ where $\left[P(s)^{-1}\right]_{pol} \in \mathbb{R}[s]^{m\times m}$ and $\left[P(s)^{-1}\right]_{sp} \in \mathbb{R}_{pr0}(s)^{m\times m}$. Next denote $Q(s) := \left[P(s)^{-1}\right]_{pol} \in \mathbb{R}[s]^{m\times m}$ and by defining the polynomial matrix $R_R(s) := D_R(s) - Q(s) N_R(s)$ obtain the *right* Euclidean polynomial matrix division [7],[5] (theorem 6.3-15), [1] (section 1.9.1): $D_R(s) = Q(s) N_R(s) + R_R(s)$, where $Q(s)$ is the *quotient* of $D_R(s)$ by $N_R(s)$ and $R_R(s)$ the *right* *remainder.* In view of the above

$$P(s)^{-1} = D_R(s)\,N_R(s)^{-1} = Q(s) + R_R(s)\,N_R(s)^{-1} \quad (1)$$

Proposition 1: (i) $Q(s) := \left[P(s)^{-1}\right]_{pol} \in \overline{\mathbb{R}}[s]^{m\times m}$, (ii) $Q(s)^{-1} \in \overline{\mathbb{R}}_{pr0}(s)^{m\times m}$, (iii) $P(s)$ and $Q(s)^{-1}$ have isomorphic zero structures at $s=\infty$, i.e. the Smith-McMillan forms at $s=\infty$ [1],[2] of $P(s)$ and $Q(s)^{-1}$ coincide, i.e. $S^{\infty}_{P(s)} = S^{\infty}_{Q(s)^{-1}}$.

Proof: (i) In view of (1): $P(s)\,P(s)^{-1} = P(s)\,Q(s) + P(s)\left[P(s)^{-1}\right]_{sp} = I_m$ which implies that $P(s)\,Q(s) = I_m - P(s)\left[P(s)^{-1}\right]_{sp} := B_1(s) \in \overline{\mathbb{R}}_{pr}(s)^{m\times m}$ is *biproper* and therefore its inverse: $Q(s)^{-1}P(s)^{-1}$ is biproper, which in turn implies that $Q(s) \in \overline{\mathbb{R}}[s]^{m\times m}$. (ii)&(iii) $P(s)\,Q(s) = B_1(s)$ with $B_1(s) \in \overline{\mathbb{R}}_{pr}(s)^{m\times m}$ *biproper* can be written as $Q(s)^{-1} = B_1(s)^{-1}P(s)$ which implies that $Q(s)^{-1}$ and $P(s)$ are *biproperly equivalent* [1], i.e. $S^{\infty}_{P(s)} = S^{\infty}_{Q(s)^{-1}}$. ■

Proposition 2: Let $Q(s) = Q_q s^q + Q_{q-1}s^{q-1} + \dots + Q_1 s + Q_0$, $Q_i \in \mathbb{R}^{m\times m}$. (i) The strictly polynomial part of $Q(s)$: $Q_+(s) := Q(s) - Q_0 \in \overline{\mathbb{R}}[s]^{m\times m}$, (ii) $Q_+(s)^{-1} \in \overline{\mathbb{R}}_{pr0}(s)^{m\times m}$ (iii) $P(s)$ and $Q_+(s)^{-1}$ have isomorphic zero structures at $s=\infty$, i.e. $S^{\infty}_{P(s)} = S^{\infty}_{Q_+(s)^{-1}}$.

Proof: (i) $Q(s)Q(s)^{-1} = Q_+(s)Q(s)^{-1} + Q_0Q(s)^{-1} = I_m$, which implies that $Q_+(s)Q(s)^{-1} = I_m - Q_0Q(s)^{-1} := B_2(s) \in \overline{\mathbb{R}}_{pr}(s)^{m\times m}$ is *biproper* and therefore its inverse: $Q(s)\,Q_+(s)^{-1}$ is biproper, which in turn implies that $Q_+(s) \in \overline{\mathbb{R}}[s]^{m\times m}$. (ii)&(iii) $Q_+(s)Q(s)^{-1} = B_2(s) \in \overline{\mathbb{R}}_{pr}(s)^{m\times m}$ is *biproper*, therefore $Q_+(s)^{-1} = Q(s)^{-1}B_2(s)^{-1} \in \overline{\mathbb{R}}_{pr0}(s)^{m\times m}$ which, since $B_2(s)^{-1} \in \overline{\mathbb{R}}_{pr}(s)^{m\times m}$ is biproper implies that $Q_+(s)^{-1}$ and $Q(s)^{-1}$ are *biproperly equivalent* [1] which implies that $S^{\infty}_{Q_+(s)^{-1}} = S^{\infty}_{Q(s)^{-1}} = S^{\infty}_{P(s)}$. ■

Proposition 3: In the Euclidean polynomial matrix division $D_R(s) = Q(s)\,N_R(s) + R_R(s)$ the degrees of the corresponding polynomial matrices satisfy

$$\deg D_R(s) = \deg Q(s) + \deg N_R(s) \quad (2)$$

$$\deg R_R(s) < \deg N_R(s) \quad (3)$$

Proof: Let $B_L(s), B_R(s) \in \overline{\mathbb{R}}_{pr}(s)^{m\times m}$ biproper and such that $P(s) = B_L(s)S^{\infty}_{P(s)}B_R(s)$. Then $\delta_\infty(P(s)) = \delta_\infty(B_L(s)) + \delta_\infty\left(S^{\infty}_{P(s)}\right) + \delta_\infty(B_R(s)) = \delta_\infty\left(S^{\infty}_{P(s)}\right)$[1],[3]. But from (iii) in Proposition 1 $S^{\infty}_{P(s)} = S^{\infty}_{Q(s)^{-1}}$, so $\delta_\infty(P(s)) = \delta_\infty\left(S^{\infty}_{P(s)}\right) = \delta_\infty\left(S^{\infty}_{Q(s)^{-1}}\right) = \delta_\infty\left(Q(s)^{-1}\right) = -\delta_\infty(Q(s))$. On the other hand $\delta_\infty(P(s)) = \delta_\infty(N_R(s)) + \delta_\infty\left(D_R(s)^{-1}\right) = \delta_\infty(N_R(s)) - \delta_\infty(D_R(s)) = -\delta_\infty(Q(s))$ (see e.g. Props. 3.80-3.82, section 3.6, chapter 3 in [1]) which, in view of the fact that for any $T(s) \in \mathbb{R}[s]^{p\times m}$: $\delta_\infty(T(s)) = -\deg T(s)$, is the relation (2). Finally $R_R(s)\,N_R(s)^{-1} = \left[P(s)^{-1}\right]_{sp} \in \mathbb{R}_{pr0}(s)^{m\times m}$ implies that $\delta_\infty\left(R_R(s)\,N_R(s)^{-1}\right) > 0$ or $\delta_\infty(R_R(s)) - \delta_\infty(N_R(s)) > 0$ or equivalently $-\deg R_R(s) + \deg N_R(s) > 0$. ■

III. An invariant and a canonical form under dynamic feedback compensation

Considering the input-output transfer function matrix $P_c(s)$ of the feedback closed loop system Σ_c in Fig. 1, with $C(s) \in \mathbb{R}_{pr}(s)^{m\times m}$ and $P_c(s) = [I_m + P(s)\,C(s)]^{-1}P(s) \in \overline{\mathbb{R}}_{pr0}(s)^{m\times m}$ we see that $P_c(s)^{-1} \in \overline{\mathbb{R}}(s)^{m\times m}$ is given by

$$P_c(s)^{-1} = P(s)^{-1} + C(s) \quad (4)$$

Definition 4: [6],[13]$(P_1(s), P_2(s)) \in \overline{\mathbb{R}}_{pr0}(s)^{m\times m} \times \overline{\mathbb{R}}_{pr0}(s)^{m\times m}$ are called dynamically feedback equivalent (via $C(s)$) if $P_1(s)^{-1} - P_2(s)^{-1} = C(s) \in \mathbb{R}_{pr}(s)^{m\times m}$.

The *dynamic feedback equivalence class* of a $P(s) \in \overline{\mathbb{R}}_{pr0}(s)^{m\times m}$ of the *dynamic feedback equivalence relation* $\mathcal{R}$ we denote by $[P(s)]_{\mathcal{R}}$. In view of (1), eq. (4) can be written as $P_c(s)^{-1} = Q(s) + R_R(s)\,N_R(s)^{-1} + C(s)$ and by decomposing $C(s)$ into its *constant* $K \in \mathbb{R}^{m\times m}$ and $C_{sp}(s) \in \mathbb{R}_{pr0}(s)^{m\times m}$ *strictly proper* parts as $C(s) = K + C_{sp}(s)$ we obtain

$$P_c(s)^{-1} = Q_+(s) + (Q_0 + K) + R_R(s)\,N_R(s)^{-1} + C_{sp}(s) \quad (5)$$

which is the decomposition of $P_c(s)^{-1} \in \overline{\mathbb{R}}(s)^{m\times m}$ into its *polynomial part* $Q_+(s) + Q_0 + K$ and *strictly proper part* $R_R(s)\,N_R(s)^{-1} + C_{sp}(s)$. By comparing (5) with (1) we have the following

Proposition 5: Let $f : \overline{\mathbb{R}}_{pr0}(s)^{m\times m} \to \overline{\mathbb{R}}[s]^{m\times m}$ be the map: $P(s) \mapsto Q_+(s)$. Then $(P(s), P_c(s)) \in \mathcal{R} \Leftrightarrow fP(s) = fP_c(s)$, i.e. the strictly polynomial part of $P(s)^{-1}$: $Q_+(s)$ is a complete invariant of the dynamic feedback equivalence relation $\mathcal{R}$.

After Kalman [4] we call $Q_+(s)$ the "atom" of $P(s)$. Going back to (5) we see that if we choose $C(s) := -Q_0 - R_R(s)\,N_R(s)^{-1} \in \mathbb{R}_{pr}(s)^{m\times m}$, then the transfer function matrix $P_c(s)$ of Σ_c is given by $P_c(s) = Q_+(s)^{-1} \in \overline{\mathbb{R}}_{pr0}(s)^{m\times m}$ and we can state

Proposition 6: Let $g : \overline{\mathbb{R}}_{pr0}(s)^{m\times m} \to \overline{\mathbb{R}}_{pr0}(s)^{m\times m}$ be the map: $P(s) \mapsto Q_+(s)^{-1}$. Then (i) $\left(P(s), Q_+(s)^{-1}\right) \in \mathcal{R}$ and (ii) $(P_1(s), P_2(s)) \in \mathcal{R} \Leftrightarrow gP_1(s) = Q_+(s)^{-1} = gP_2(s)$, i.e. $Q_+(s)^{-1} \in \overline{\mathbb{R}}_{pr0}(s)^{m\times m}$ is a *canonical form* for the dynamic feedback equivalence relation $\mathcal{R}$.

Combining assertions (iii) in Proposition 2 and (ii) in Proposition 6 we obtain

Corollary 7: The zero structure at $s=\infty$ of a transfer function matrix $P(s) \in \overline{\mathbb{R}}_{pr0}(s)^{m\times m}$ of a LTI multivariable system Σ is a complete invariant under dynamic feedback compensation through a proper compensator $C(s)$, i.e. for every $C(s) \in \mathbb{R}_{pr}(s)^{m\times m}$ the Smith-McMillan forms of $P(s)$ and $P_c(s)$ at $s=\infty$: $S^{\infty}_{P(s)}$ and $S^{\infty}_{P_c(s)}$ [1] coincide, equivalently $(P(s), P_c(s)) \in \mathcal{R} \Leftrightarrow S^{\infty}_{P(s)} = S^{\infty}_{P_c(s)} = S^{\infty}_{Q(s)^{-1}} = S^{\infty}_{Q_+(s)^{-1}}$.

IV. Synthesis

A. Denominator assignment

Considering equation (4) an obvious question one can pose is the following. Given a stabilizable system Σ with transfer function matrix $P(s)$, can one choose a desired closed loop transfer function matrix $P_c(s) \in [P(s)]_{\mathcal{R}}$ having a desired pole structure in $\mathbb{C}^-$ and then obtain from (4) the dynamic feedback proper compensator $C(s)$ which gives rise to $P_c(s)$, guaranteeing also that the closed loop feedback system Σ_c is internally stable? Let us assume that we choose a desired $P_c(s) \in [P(s)]_{\mathcal{R}}$ by requiring that $P_c(s)^{-1} = Q_+(s) + R_{RC}(s) N_{RC}(s)^{-1} = [Q_+(s) N_{RC}(s) + R_{RC}(s)] N_{RC}(s)^{-1}$ where $R_{RC}(s)$, $N_{RC}(s)$ are (yet unknown) $m \times m$ polynomial matrices such that $N_{RC}(s)^{-1}$ exists and $R_{RC}(s) N_{RC}(s)^{-1} \in \mathbb{R}_{pr}(s)^{m\times m}$ or equivalently such that $P_c(s) = N_{RC}(s) [Q_+(s) N_{RC}(s) + R_{RC}(s)]^{-1} \in \overline{\mathbb{R}}_{pr0}(s)^{m\times m}$ and has desired poles in $\mathbb{C}^-$, i.e. such that the closed loop right *denominator* polynomial matrix $D_{RC}(s)$ of $P_c(s)$:

$$D_{RC}(s) := Q_+(s) N_{RC}(s) + R_{RC}(s) \in \overline{\mathbb{R}}[s]^{m\times m} \quad (6)$$

has desired zeros in $\mathbb{C}^-$. Assuming that the polynomial matrix $Q_+(s)$ is computed from $P(s)$ as in (1) and $D_{RC}(s)$ is chosen appropriately so that $N_{RC}(s)^{-1}$ exists and $R_{RC}(s) N_{RC}(s)^{-1} \in \mathbb{R}_{pr}(s)^{m\times m}$ one can interpret (6) as a (left) Euclidean polynomial matrix division of $D_{RC}(s)$ by $Q_+(s)$ where $N_{RC}(s)$ is the *quotient* and $R_{RC}(s)$ the *left remainder* so that $Q_+(s)^{-1} R_{RC}(s) \in \mathbb{R}_{pr0}(s)^{m\times m}$ [7],[1]. In such a case equation (6) can be written as $Q_+(s)^{-1} D_{RC}(s) = N_{RC}(s) + Q_+(s)^{-1} R_{RC}(s)$ which is the decomposition of $T(s) := Q_+(s)^{-1} D_{RC}(s)$ to its polynomial part: $N_{RC}(s)$ and strictly proper part: $Q_+(s)^{-1} R_{RC}(s)$. With these considerations it follows that a right MFD of the dynamic feedback compensator $C(s)$ giving rise to a closed loop system Σ_c with transfer function $P_c(s) = N_{RC}(s) D_{RC}(s)^{-1}$ is given by

$$\begin{aligned} C(s) &= P_c(s)^{-1} - P(s)^{-1} \\ &= D_{RC}(s) N_{RC}(s)^{-1} - D_R(s) N_R(s)^{-1} \\ &= -Q_0 + R_{RC}(s) N_{RC}(s)^{-1} - R_R(s) N_R(s)^{-1} \\ &=: Y(s) X(s)^{-1} \end{aligned} \quad (7)$$

where $Y(s) := -Q_0 N_{RC}(s) \overline{N}_R(s) + R_{RC}(s) \overline{N}_R(s) - R_R(s) \overline{N}_{RC}(s)$, $X(s) := N_R(s) \overline{N}_{RC}(s)$ and $\overline{N}_R(s) \in \overline{\mathbb{R}}[s]^{m\times m}$ is such that if $N_{RC}(s)^{-1} N_R(s)$ constitutes a *left coprime MFD* then there exists a $\overline{N}_{RC}(s) \in \overline{\mathbb{R}}[s]^{m\times m}$, such that $\overline{N}_R(s) \overline{N}_{RC}(s)^{-1}$ constitutes a *right coprime MFD*, i.e. $N_{RC}(s)^{-1} N_R(s) = \overline{N}_R(s) \overline{N}_{RC}(s)^{-1}$. Substituting the expression for $C(s)$ in (7) into the characteristic polynomial matrix of Σ_c we obtain

$$D_L(s) X(s) + N_L(s) Y(s) = N_L(s) D_{RC}(s) \overline{N}_R(s) \quad (8)$$

(see Appendix a). These considerations lead to

Proposition 8: With the dynamic feedback compensator $C(s)$ given by (7), the closed loop system Σ_c with transfer function $P_c(s) = N_{RC}(s) D_{RC}(s)^{-1}$ is internally stable iff the characteristic polynomial matrix of Σ_c in (8) or equivalently the polynomial matrices $N_L(s)$, $D_{RC}(s)$, $\overline{N}_R(s)$ have all their zeros in $\mathbb{C}^-$.

Since the desired denominator $D_{RC}(s)$ of $P_c(s)$ can be assigned having all its zeros in $\mathbb{C}^-$ via the choice of a proper compensator $C(s)$ as in (7), we have

Proposition 9: The class $\mathcal{P}$ of transfer function matrices $P(s)$ that under proper feedback compensators as in (7) give rise to *internally stable* closed loop systems Σ_c with transfer function matrix $P_c(s) \in [P(s)]_{\mathcal{R}}$ is the class of transfer function matrices $P(s) \in \overline{\mathbb{R}}_{pr0}(s)^{m\times m}$ with no *zeros* in $\overline{\mathbb{C}}^+$, i.e. such that $|N_L(s_i)| \neq 0, |N_R(s_i)| \neq 0, \forall s_i \in \overline{\mathbb{C}}^+$.

The above analysis gives rise to our main result which we state in the sequel as Theorem 12. In order to illuminate Theorem 12 and for ease of reference to the above analysis, we state it firstly together with a proof for the scalar (s.i.s.o.) case maintaining the notation, as

Theorem 10: Let Σ be a LTI stabilizable s.i.s.o. (scalar) system with input-output transfer function $P(s) = N_R(s) D_R(s)^{-1} = D_L(s)^{-1} N_L(s) \in \overline{\mathbb{R}}_{pr0}(s)$ and let $Q(s) \in \overline{\mathbb{R}}[s]$ be the quotient in the Euclidean division of $D_R(s)$ by $N_R(s)$. Let $P_c(s) := N_{RC}(s) D_{RC}(s)^{-1} \in \mathbb{R}_{pr0}(s)$ where $D_{RC}(s) \in \overline{\mathbb{R}}[s]$ is such that it has desired zeros in $\mathbb{C}^-$ and $N_{RC}(s) \in \overline{\mathbb{R}}[s]$ is the quotient in the division of $D_{RC}(s)$ by $Q_+(s) := Q(s) - Q_0$. If $P(s) \in \mathcal{P}$, then the feedback compensator $C(s) := P_c(s)^{-1} - P(s)^{-1}$ is proper and gives rise to an internally stable closed loop system Σ_c with transfer function matrix $P_c(s) = N_{RC}(s) D_{RC}(s)^{-1} \in [P(s)]_{\mathcal{R}}$ if

$$\deg D_{RC}(s) \geq 2 \deg Q_+(s) - 1 \quad (9)$$

Proof: Choose $D_{RC}(s)$ so that (9) is satisfied and having desired zeros in $\mathbb{C}^-$ and consider the Euclidean division: $D_{RC}(s) = Q_+(s) N_{RC}(s) + R_{RC}(s)$ where (by Proposition 3)

$$\deg D_{RC}(s) = \deg Q_+(s) + \deg N_{RC}(s) \quad (10)$$

$$\deg R_{RC}(s) < \deg Q_+(s) \quad (11)$$

so that (9) due to (10) can be written as $\deg Q_+(s) + \deg N_{RC}(s) \geq 2 \deg Q_+(s) - 1$ or $\deg N_{RC}(s) \geq \deg Q_+(s) - 1$ which due to (11) implies that $\deg N_{RC}(s) > \deg R_{RC}(s) - 1$ or that $\deg N_{RC}(s) + 1 > \deg R_{RC}(s)$ or that

$$\deg N_{RC}(s) \geq \deg R_{RC}(s) \quad (12)$$

Consider now the compensator in (7) where since in the scalar case $\overline{N}_R(s) = N_R(s)$, $\overline{N}_{RC}(s) = N_{RC}(s)$

$$\begin{aligned} \deg X(s) &= \deg [N_{RC}(s) N_R(s)] \\ &= \deg N_{RC}(s) + \deg N_R(s) \end{aligned} \quad (13)$$

$$\begin{aligned} \deg Y(s) &= \deg \begin{bmatrix} -Q_0 N_{RC}(s) N_R(s) \\ +R_{RC}(s) N_R(s) \\ -R_R(s) N_{RC}(s) \end{bmatrix} \\ &\leq \max \left\{ \begin{array}{c} \deg[-Q_0 N_{RC}(s) N_R(s)], \\ \deg[R_{RC}(s) N_R(s)], \\ \deg[R_R(s) N_{RC}(s)] \end{array} \right\} \\ &= \max \left\{ \begin{array}{c} \deg N_{RC}(s) + \deg N_R(s), \\ \deg R_{RC}(s) + \deg N_R(s), \\ \deg R_R(s) + \deg N_{RC}(s) \end{array} \right\} \\ &\overset{(12),(3)}{=} \deg N_{RC}(s) + \deg N_R(s) \\ &\overset{(13)}{=} \deg X(s) \end{aligned}$$

i.e. $\deg Y(s) \leq \deg X(s) \Rightarrow C(s) \in \mathbb{R}_{pr}(s)$. ■

Remark 11: If the numerator $N_R(s)$ of the plant $P(s) = N_R(s) D_R(s)^{-1}$ is a constant: $N_R(s) = c_0 \in \mathbb{R}$ then $\deg D_R(s) = \deg Q(s) = \deg Q_+(s)$ and $R_R(s) = 0$. In such a case condition (9) reduces to the classical result that for *any* closed loop characteristic polynomial $D_{RC}(s)$ there exists a *proper* compensator $C(s) = \frac{Y(s)}{X(s)}$ such that $X(s)D_R(s) + Y(s)N_R(s) = D_{RC}(s)$ if $\deg D_{RC}(s) \geq 2\deg D_R(s) - 1$, e.g. [14].

The next Theorem is our main result and constitutes the generalization of Theorem 10 to the multivariable case.

Theorem 12: Let Σ be a LTI stabilizable multivariable system with input-output transfer function $P(s) = N_R(s) D_R(s)^{-1} \in \overline{\mathbb{R}}_{pr0}(s)^{m\times m}$ and let $Q(s) \in \overline{\mathbb{R}}[s]^{m\times m}$ be the quotient in the division of $D_R(s)$ by $N_R(s)$. Let $U(s) \in \overline{\mathbb{R}}[s]^{m\times m}$ be $\mathbb{R}[s]$-unimodular and such that $\overline{Q}_+(s) := U(s) Q_+(s)$ is *row proper*. Let $\overline{D}_{RC}(s) = diag\left[\overline{D}_1(s), \overline{D}_2(s), \ldots, \overline{D}_m(s)\right] \in \overline{\mathbb{R}}[s]^{m\times m}$ where $\overline{D}_i(s)$, $i = 1, ..., m$ monic polynomials with desired zeros in $\mathbb{C}^-$ and let $D_{RC}(s) = U(s)^{-1} \overline{D}_{RC}(s)$. Let $N_{RC}(s)$ be the quotient in the division of $\overline{D}_{RC}(s)$ by $\overline{Q}_+(s)$ or of $D_{RC}(s)$ by $Q_+(s)$. Let $P_c(s) := N_{RC}(s) D_{RC}(s)^{-1}$. If $P(s) \in \mathcal{P}$, then the feedback compensator $C(s) := P_c(s)^{-1} - P(s)^{-1} \in \mathbb{R}_{pr}(s)^{m\times m}$ and gives rise to an internally stable closed loop system Σ_c with transfer function matrix $P_c(s) \in [P(s)]_{\mathcal{R}}$ if

$$\deg \overline{D}_i(s) \geq \deg_{ri} \overline{Q}_+(s) + \max_{j,k=1,\ldots,m} \{\deg q_{jk}(s)\} - 1 \quad (14)$$

$i = 1, ..., m$, where $\max_{j,k=1,\ldots,m} \{\deg q_{jk}(s)\}$ is the maximum degree among the degrees of the elements $q_{jk}(s)$ of $Q_+(s)$.

Proof: see the Appendix b. ■

The above theorem gives rise to the following.

1) Denominator assignment algorithm: Given a LTI stabilizable multivariable system Σ characterized by a $m \times m$ non-singular strictly proper transfer function matrix $P(s) \in \mathcal{P}$, then the algorithm to compute a *proper* dynamic compensator $C(s)$ such that the resulting closed loop feedback system Σ_c in Fig. 1 is internally stable and its input-output transfer function matrix $P_c(s)$ is strictly proper and has a right MFD with a desired denominator matrix $D_{RC}(s)$ is

1) Compute: $P(s)^{-1}$, $\left[P(s)^{-1}\right]_{pol} =: Q(s)$, $\left[P(s)^{-1}\right]_{sp}$.
2) Compute $Q_+(s) := Q(s) - Q_0$ and choose a $U(s) \in \overline{\mathbb{R}}[s]^{m\times m}$ and $\mathbb{R}[s]$-unimodular such that $\overline{Q}_+(s) := U(s) Q_+(s)$ is row proper.
3) Choose $\overline{D}_{RC}(s) = diag\left[\overline{D}_1(s), \ldots, \overline{D}_m(s)\right] \in \overline{\mathbb{R}}[s]^{m\times m}$, $\overline{D}_i(s) \in \mathbb{R}[s]$, $i = 1, ..., m$ with desired zeros in $\mathbb{C}^-$ so that (14) is satisfied.
4) Compute: $T(s) := \overline{Q}_+(s)^{-1} \overline{D}_{RC}(s) \in \overline{\mathbb{R}}[s]^{m\times m}$, $[T(s)]_{pol} =: N_{RC}(s)$ and $[T(s)]_{sp}$.
5) Compute $D_{RC}(s) = U(s)^{-1} \overline{D}_{RC}(s) \in \overline{\mathbb{R}}[s]^{m\times m}$, $P_c(s) := N_{RC}(s) D_{RC}(s)^{-1} \in \overline{\mathbb{R}}_{pr0}(s)^{m\times m}$ and $C(s) = P_c(s)^{-1} - P(s)^{-1} \in \mathbb{R}_{pr}(s)^{m\times m}$.

Example 13: Let $P(s) = \begin{bmatrix} \frac{s+1}{s(s-2)} & \frac{1}{s(s-1)} \\ \frac{s+2}{(s-1)(s-2)(s-3)} & 0 \end{bmatrix}$ with a right MFD: $P(s) = N_R(s) D_R(s)^{-1}$ where

$$D_R(s) = \begin{bmatrix} (s-1)(s-2)(s-3) & 0 \\ -3(s-1) & s(s-1) \end{bmatrix}$$

$$N_R(s) = \begin{bmatrix} s^2 - 3s - 1 & 1 \\ s+2 & 0 \end{bmatrix}$$

Since $|N_R(s)| = -(s+2)$, $P(s) \in \mathcal{P}$. Now

$$P(s)^{-1} = \begin{bmatrix} 0 & \frac{(s-1)(s-2)(s-3)}{s+2} \\ s(s-1) & \frac{-(s+1)(s-3)(s-1)^2}{s+2} \end{bmatrix}$$

$$Q_+(s) := Q(s) - Q_0 = \begin{bmatrix} 0 & s^2 - 8s \\ s(s-1) & -s^3 + 6s^2 - 14s \end{bmatrix}$$

is not row proper, while

$$\overline{Q}_+(s) := U(s) Q_+(s) = \begin{bmatrix} 0 & s(s-8) \\ s(s-1) & -2s(s+7) \end{bmatrix}$$

is row proper with $\deg_{r1} \overline{Q}_+(s) = 2$ and $\deg_{r2} \overline{Q}_+(s) = 2$. From condition (14) we have that the degrees of the diagonal elements of $\overline{D}_{RC}(s)$ must satisfy the inequalities $\deg \overline{D}_1(s) \geq 2 + 3 - 1 = 4$, $\deg \overline{D}_2(s) \geq 2 + 3 - 1 = 4$. If we choose $\overline{D}_{RC}(s) = diag\left[(s+1)^4, (s+3)^4\right]$, then

$$T(s) := \overline{Q}_+(s)^{-1} \overline{D}_{RC}(s) = \begin{bmatrix} \frac{(2s+14)(s+1)^4}{s(s^2-9s+8)} & \frac{(s+3)^4}{s(s-1)} \\ \frac{(s+1)^4}{s(s-8)} & 0 \end{bmatrix}$$

$$N_{RC}(s) := [T(s)]_{pol} = \begin{bmatrix} 2s^2 + 40s + 412 & s^2 + 13s + 67 \\ s^2 + 12s + 102 & 0 \end{bmatrix}$$

$$D_{RC}(s) = U(s)^{-1} \overline{D}_{RC}(s) = \begin{bmatrix} (s+1)^4 & 0 \\ -s(s+1)^4 & (s+3)^4 \end{bmatrix}$$

$$P_c(s) = N_{RC}(s) D_{RC}(s)^{-1} = \begin{bmatrix} p_c(1,1) & \frac{s^2+13s+67}{(s+3)^4} \\ \frac{s^2+12s+102}{(s+1)^4} & 0 \end{bmatrix}$$

$$p_c(1,1) := \frac{s^7+19s^6+189s^5+1350s^4+7775s^3+27\,011s^2+47\,803s+33\,372}{(s^2+4s+3)^4}$$

$$\begin{aligned} C(s) &= P_c(s)^{-1} - P(s)^{-1} \\ &= \begin{bmatrix} 0 & \frac{-27s^3+502s^2-1041s+614}{s^3+14s^2+126s+204} \\ \frac{175s+81}{s^2+13s+67} & c(2,2) \end{bmatrix} \end{aligned}$$

$$c(2,2) := \frac{-2\left(422s^5+5707s^4+32\,206s^3+40\,306s^2+54\,016s+43\,623\right)}{s^5+27s^4+375s^3+2780s^2+11\,094s+13\,668}$$

and the feedback compensator $C(s)$ is proper and gives rise to an *internally stable* closed loop system Σ_c with the transfer function matrix $P_c(s) \in \mathcal{R}$ whose right MFD has denominator the polynomial matrix $D_{RC}(s)$, which has as zeros (closed loop poles) the desired zeros of $\overline{D}_{RC}(s)$.

B. Decoupling

The problem of *diagonally decoupling* Σ by the use of dynamic output feedback is to determine conditions under which there exists a proper feedback compensator $C(s)$ such that, apart from internally stabilizing the closed loop system Σ_c, it also gives rise to a closed loop transfer function matrix $P_c(s) \in [P(s)]_{\mathcal{R}}$ which is diagonal, non-singular and has desired poles in $\mathbb{C}^-$. If such a compensator exists, then we say that Σ is decouplable. This problem was originally examined in [9] where, using different tools, the necessary and sufficient condition for its solution was established. Our approach, which relies on Theorem 12, arrives to the same n. and s. condition which we state in

Theorem 14: Let Σ be a LTI stabilizable multivariable system with input-output transfer function $P(s) \in \overline{\mathbb{R}}_{pr0}(s)^{m\times m}$ and let $Q_+(s) : Q(s) - Q_0 \in \overline{\mathbb{R}}[s]^{m\times m}$ be the strictly polynomial part of $P(s)^{-1}$. If $P(s) \in \mathcal{P}$, then Σ is decouplable iff $Q_+(s)$ is diagonal.

Proof: ($\Rightarrow$) If $Q_+(s)$ is diagonal then it is *row proper* and coincides with $\overline{Q}_+(s)$ in Theorem 12. If $\overline{D}_{RC}(s)$ is chosen diagonal satisfying conditions (14) and with desired zeros in $\mathbb{C}^-$, then $D_{RC}(s) = \overline{D}_{RC}(s)$ and $T(s) := Q_+(s)^{-1} D_{RC}(s) \in \overline{\mathbb{R}}(s)^{m\times m}$ is diagonal and $[T(s)]_{pol} =: N_{RC}(s) \in \overline{\mathbb{R}}[s]^{m\times m}$ is also diagonal and therefore $P_c(s) = N_{RC}(s) D_{RC}(s)^{-1} \in [P(s)]_{\mathcal{R}}$ is diagonal.

($\Leftarrow$) If $P_c(s) \in [P(s)]_{\mathcal{R}}$ is diagonal then $P_c(s)^{-1}$ is diagonal and so from (5) its polynomial part: $\left[P_c(s)^{-1}\right]_{pol} = Q_+(s)+Q_0+K = Q_q s^q + Q_{q-1}s^{q-1} + ... + Q_1 s + (Q_0+K)$ is diagonal and hence Q_i, $i = 1, ..., q$ and Q_0+K are diagonal, which implies that $Q_+(s)$ is diagonal. ■

V. Conclusions

In this paper we have established an invariant and a canonical form of linear multivariable systems under the action of proper dynamic compensation in the feedback path. Through these results two classes of strictly proper transfer function matrices have been characterized. These are: (i) the class $[P(s)]_{\mathcal{R}}$ of closed loop transfer function matrices that are obtainable from an open loop strictly proper transfer function matrix $P(s)$ and (ii) the class $\mathcal{P}$ of open loop transfer function matrices that, under a proper feedback compensator $C(s)$, whose structure emanates from our analysis, give rise to an internally stable closed loop system with transfer function matrix $P_c(s) \in [P(s)]_{\mathcal{R}}$. Based on these results and for systems characterized by transfer function matrices in the class $\mathcal{P}$, a simple algorithmic procedure has been derived for the computation of a proper internally stabilizing and denominator assigning feedback compensator. It has been suggested by one of the reviewers that similar results have been settled in [15] via the "geometric approach". Both the "geometric" or the "polynomial matrix" approaches to the denominator assignment algorithm presented here, heavily rely on some demanding computations using: real matrices, real polynomial matrices or real rational matrices. The numerical problems involved in such computations, especially in applying the proposed algorithm to real systems, are extremely delicate and demanding and can not be ignored. The numerical issues associated with the above computations and involved in the proposed algorithm have not been addressed here and need further investigation. It has also been pointed out by another reviewer that since the relationships $P_c(s) = [I_m + P(s)C(s)]^{-1} P(s) = P(s)[I_m + C(s)P(s)]^{-1}$ imply that the effect that any feedback compensator $C(s)$ has on $P(s)$ can be viewed as an open loop precompensation on $P(s)$ via $K(s) := [I_m + C(s)P(s)]^{-1} \in \overline{\mathbb{R}}_{pr}(s)^{m\times m}$, the *dynamic feedback equivalence relation* $\mathcal{R}$ and its complete invariant polynomial matrix $Q_+(s)$ studied here must be closely related to the *dynamic precompensation equivalence relation*, which was studied in [16], and its invariant known as the "interactor". We believe that these considerations need further investigation.

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VI. Appendix A

Eqs. $P(s) = D_L(s)^{-1} N_L(s) = N_R(s) D_R(s)^{-1}$ and $N_{RC}(s)^{-1} N_R(s) = \overline{N}_R(s)\overline{N}_{RC}(s)^{-1}$ can be written as

$$D_L(s) N_R(s) = N_L(s) D_R(s) \tag{15}$$

$$N_R(s)\overline{N}_{RC}(s) = N_{RC}(s)\overline{N}_R(s) \tag{16}$$

$$\begin{aligned}
& D_L(s) X(s) + N_L(s) Y(s) \\
& = D_L(s) N_R(s)\overline{N}_{RC}(s) - N_L(s) Q_0 N_{RC}(s)\overline{N}_R(s) \\
& + N_L(s) R_{RC}(s)\overline{N}_R(s) - N_L(s) R_R(s)\overline{N}_{RC}(s) \\
& \overset{(15)}{=} N_L(s) D_R(s)\overline{N}_{RC}(s) - N_L(s) Q_0 N_{RC}(s)\overline{N}_R(s) \\
& + N_L(s) R_{RC}(s)\overline{N}_R(s) - N_L(s) R_R(s)\overline{N}_{RC}(s) \\
& = N_L(s) Q(s) N_R(s)\overline{N}_{RC}(s) + N_L(s) R_{RC}(s)\overline{N}_R(s) \\
& - N_L(s) Q_0 N_{RC}(s)\overline{N}_R(s) \\
& \overset{(16)}{=} N_L(s) Q(s) N_{RC}(s)\overline{N}_R(s) + N_L(s) R_{RC}(s)\overline{N}_R(s) \\
& - N_L(s) Q_0 N_{RC}(s)\overline{N}_R(s) \\
& = N_L(s)[Q_+(s) N_{RC}(s) + R_{RC}(s)]\overline{N}_R(s) \\
& = N_L(s) D_{RC}(s)\overline{N}_R(s)
\end{aligned}$$

VII. Appendix B

Lemma 15: Let $A(s) \in \overline{\mathbb{R}}[s]^{m\times m}$, $\operatorname{rank}_{\mathbb{R}(s)} A(s) = m$ be row (column) proper. Then $A(s)^{-1} \in \overline{\mathbb{R}}_{pr}(s)^{m\times m}$ is column (row) reduced at $s = \infty$ [1] and $\delta_{\infty ci} A(s)^{-1} = \deg_{ri} A(s)$ $\left(\delta_{\infty ri}\left(A(s)^{-1}\right) = \deg_{ci} A(s)\right)$.

Proof: Write $A(s)$ as $A(s) = diag\left[s^{r_1}, ..., s^{r_m}\right][A(s)]_r^h + A_r(s)$ where $r_i := \deg_{ri} A(s) \geq 0$, $[A(s)]_r^h \in \mathbb{R}^{m\times m}$ is the highest row degree coefficient matrix of $A(s)$ and $A_r(s) \in \mathbb{R}[s]^{m\times m}$ with $\deg_{ri} A_r(s) < r_i$. Write $A(s)^{-1}$ as $A(s)^{-1} = \left[A(s)^{-1}\right]_c^l diag\left[s^{-q_\infty^1}, ..., s^{-q_\infty^m}\right] + A_c^{-1}(s)$, where $q_\infty^i := \delta_{\infty ci}\left(A(s)^{-1}\right)$, $\left[A(s)^{-1}\right]_c^l \in \mathbb{R}^{m\times m}$ is the least column valuation coefficient matrix of $A(s)^{-1}$ and $A_c^{-1}(s) \in \mathbb{R}_{pr}(s)^{m\times m}$ with $\delta_{\infty ci}\left(A_c^{-1}(s)\right) > q_\infty^i$ [1]. Then $A(s)^{-1}A(s) = I_m$ gives $\left(\left[A(s)^{-1}\right]_c^l diag\left[s^{-q_\infty^1}, ..., s^{-q_\infty^m}\right] + A_c^{-1}(s)\right) \times \left(diag\left[s^{r_1}, ..., s^{r_m}\right][A(s)]_r^h + A_r(s)\right) = I_m$. After some calculations and if we take limits, we have

$$\lim_{s\to\infty}\left\{\begin{array}{c}
\left[A(s)^{-1}\right]_c^l diag\left[s^{r_1-q_\infty^1}, ..., s^{r_m-q_\infty^m}\right][A(s)]_r^h \\
+\left[A(s)^{-1}\right]_c^l diag\left[s^{-q_\infty^1}, ..., s^{-q_\infty^m}\right] A_r(s) \\
+A_c^{-1}(s) diag\left[s^{r_1}, ..., s^{r_m}\right][A(s)]_r^h \\
+A_c^{-1}(s) A_r(s)
\end{array}\right\}$$

$$= I_m \tag{17}$$

Since the right hand side of (17) is finite, a necessary condition for the left hand side to be finite is that $r_i - q_\infty^i \leq 0$ or $r_i \leq q_\infty^i, i = 1, ..., m$. The inequalities $\deg_{ri} A_r(s) < r_i \leq q_\infty^i$, $i = 1, ..., m$ imply that $\lim_{s\to\infty} \left\{ diag \left[s^{-q_\infty^1}, ..., s^{-q_\infty^m} \right] A_r(s) \right\} = 0$ and the inequalities $\delta_{\infty ci}(A_c^{-1}(s)) > q_\infty^i \geq r_i \geq 0$, $i = 1, ..., m$ imply that $A_c^{-1}(s) \in \mathbb{R}_{pr0}(s)^{m\times m}$ and therefore that $\lim_{s\to\infty} \left\{ A_c^{-1}(s) diag \left[s^{r_1}, ..., s^{r_m} \right] \right\} = 0, \lim_{s\to\infty} \left\{ A_c^{-1}(s) A_r(s) \right\} = 0$. Thus from (17)

$$\lim_{s\to\infty} \left\{ \left[A(s)^{-1} \right]_c^l diag \left[s^{r_1 - q_\infty^1}, ..., s^{r_m - q_\infty^m} \right] \times [A(s)]_r^h \right\} = I_m \tag{18}$$

and if $r_i < q_\infty^i$, $i = 1, ..., m$ then the left hand side of (18) tends to the zero matrix while the right hand side is the identity, hence it must be $r_i = q_\infty^i$, $i = 1, ..., m$ and $\left[A(s)^{-1} \right]_c^l [A(s)]_r^h = I_m$ so that $[A(s)^{-1}]_c^l = \left([A(s)]_r^h \right)^{-1}$ is non-singular and thus $A(s)^{-1}$ is column reduced at $s = \infty$. The assertion in the brackets follows similarly. ■

Proof: Of Theorem 12. Let $\overline{D}_{RC}(s) = diag \left[\overline{D}_1(s), \ldots, \overline{D}_m(s) \right]$ so that condition (14) is satisfied. Then the Euclidean division $\overline{D}_{RC}(s) = \overline{Q}_+(s) N_{RC}(s) + \overline{R}_{RC}(s)$ gives

$$\overline{Q}_+(s)^{-1} \overline{D}_{RC}(s) = N_{RC}(s) + \overline{Q}_+(s)^{-1} \overline{R}_{RC}(s) \tag{19}$$

Denoting by $d_i := \deg \overline{D}_i(s), q_\infty^i := \delta_{\infty ci} \left(\overline{Q}_+(s)^{-1} \right)$, $r_i := \deg_{ri} \overline{Q}_+(s), n_{ci} := \deg_{ci} N_{RC}(s)$ and in view of Lemma 15, eq. (19) can be written as

$$\begin{aligned}
&\left(\left[\overline{Q}_+(s)^{-1} \right]_c^l diag \left[s^{-q_\infty^1}, \ldots, s^{-q_\infty^m} \right] + \overline{Q}_c^{-1}(s) \right) \times \\
&\times \left(diag \left[s^{d_1}, \ldots, s^{d_m} \right] + \overline{D}_{RC}^r(s) \right) \\
&= [N_{RC}(s)]_c^h diag \left[s^{n_{c1}}, \ldots, s^{n_{cm}} \right] + N_{RC}^c(s) \\
&+ \overline{Q}_+(s)^{-1} \overline{R}_{RC}(s) \\
&\left[\overline{Q}_+(s)^{-1} \right]_c^l diag \left[s^{d_1 - q_\infty^1}, \ldots, s^{d_m - q_\infty^m} \right] \\
&+ \left[\overline{Q}_+(s)^{-1} \right]_c^l diag \left[s^{-q_\infty^1}, \ldots, s^{-q_\infty^m} \right] \overline{D}_{RC}^r(s) \\
&+ \overline{Q}_c^{-1}(s) diag \left[s^{d_1}, \ldots, s^{d_m} \right] + \overline{Q}_c^{-1}(s) \overline{D}_{RC}^r(s) \\
&= [N_{RC}(s)]_c^h diag \left[s^{n_{c1}}, \ldots, s^{n_{cm}} \right] + N_{RC}^c(s) \\
&+ \overline{Q}_+(s)^{-1} \overline{R}_{RC}(s)
\end{aligned}$$

which implies that

$$\begin{aligned}
&\left[\overline{Q}_+(s)^{-1} \right]_c^l diag \left[s^{d_1 - q_\infty^1}, \ldots, s^{d_m - q_\infty^m} \right] \\
&= [N_{RC}(s)]_c^h diag \left[s^{n_{c1}}, \ldots, s^{n_{cm}} \right]
\end{aligned} \tag{20}$$

or that $d_i - q_\infty^i = n_{ci} \Rightarrow d_i = q_\infty^i + n_{ci} = r_i + n_{ci}$, or

$$\deg \overline{D}_i(s) = \deg_{ri} \overline{Q}_+(s) + \deg_{ci} N_{RC}(s) \tag{21}$$

and therefore, from (20), $\left[\overline{Q}_+(s)^{-1} \right]_c^l = [N_{RC}(s)]_c^h$ which implies that $\det [N_{RC}(s)]_c^h \neq 0$, i.e. that $N_{RC}(s)$ is column proper. Now in view of (21) condition (14) can be written as $\deg_{ri} \overline{Q}_+(s) + \deg_{ci} N_{RC}(s) \geq \deg_{ri} \overline{Q}_+(s) + \max_{j,k=1,\ldots,m} \{\deg q_{jk}(s)\} - 1$, which implies that

$$\deg_{ci} N_{RC}(s) \geq \max_{j,k=1,\ldots,m} \{\deg q_{jk}(s)\} - 1 \tag{22}$$

Let $[T(s)]_{sp} =: \left[t_{ij}^{sp}(s) \right]$ and $t_{ij}^{sp}(s) = \frac{t_{ij}(s)}{T_{ij}(s)}$, $\deg t_{ij}(s) < \deg T_{ij}(s)$, $Q_+(s) = [q_{ij}(s)]$, $R_{RC}(s) = [r_{ij}^c(s)]$, $i, j = 1, ..., m$. From $[T(s)]_{sp} = Q_+(s)^{-1} R_{RC}(s) \in \mathbb{R}_{pr0}(s)^{m\times m}$ we have that $R_{RC}(s) := Q_+(s) [T(s)]_{sp}$ so that $r_{ji}^c(s) = \sum_{k=1}^m q_{jk}(s) t_{ki}^{sp}(s)$ and $\delta_\infty(r_{ji}^c(s)) \geq \min_{k=1,\ldots,m} \{\delta_\infty(q_{jk}(s)) + \delta_\infty(t_{ki}^{sp}(s))\}$ which, since $\delta_\infty(t_{ki}^{sp}(s)) > 0$ implies that $\delta_\infty(r_{ji}^c(s)) > \min_{k=1,\ldots,m} \{\delta_\infty(q_{jk}(s))\}$ or equivalently that $\deg r_{ji}^c(s) < \max_{k=1,\ldots,m} \{\deg q_{jk}(s)\} = \deg_{rj} Q_+(s)$ or that $\deg_{ci} R_{RC}(s) = \max_{j=1,\ldots,m} \{\deg r_{ji}^c(s)\} < \max_{j,k=1,\ldots,m} \{\deg q_{jk}(s)\}$, which and due to (22) implies that $\deg_{ci} R_{RC}(s) \leq \max_{j,k=1,\ldots,m} \{\deg q_{jk}(s)\} - 1 \overset{(22)}{\leq} \deg_{ci} N_{RC}(s)$. Now $\deg_{ci} R_{RC}(s) \leq \deg_{ci} N_{RC}(s)$, together with the fact that $N_{RC}(s)$ is column proper, implies that $R_{RC}(s) N_{RC}(s)^{-1} \in \mathbb{R}_{pr}(s)^{m\times m}$ and in view of (1) $C(s) := P_c(s)^{-1} - P(s)^{-1} = -Q_0 + R_{RC}(s) N_{RC}(s)^{-1} - R_R(s) N_R(s)^{-1} \in \mathbb{R}_{pr}(s)^{m\times m}$. ■

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