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Geometric structure and properties of linear time invariant multivariable systems in the controller canonical form*

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Abstract

In this paper, we analyse some fundamental structural properties of linear time-invariant multivariable systems in the controller canonical form and present a direct method for the computation of bases and associated friends for output-nulling, input-containing and reachability subspaces in terms of the parameters of the system and the invariant zero structure, both in the nondefective and in the defective case. Using this analysis, it is possible to express the solvability conditions of important control and estimation problems in terms of easily checkable conditions on the system matrices.

1 Introduction

Geometric control is a classical tool for the analysis of structural properties of linear and nonlinear systems, and in the solution of fundamental control and estimation problems, such as disturbance decoupling, fault detection, tracking control, unknown-input observation and model matching. For surveys of the extensive literature in this area, we direct the interested reader to the comprehensive monographs [1], [25], [17], [2], see also the recent textbook [8].

The most significant and useful subspaces of the classic geometric theory for linear time-invariant (LTI) systems are the so-called *controlled invariant* and *conditioned invariant* subspaces. The most important types of controlled invariant subspaces are the so-called *output-nulling*, *reachability* ,and *stabilisability* subspaces. Conditioned invariant subspaces are the dual of controlled invariant subspaces. Similarly, *input-containing*, *unobservability* and *detectability* subspaces are the dual of output-nulling, reachability and *stabilisability* subspaces, respectively. In this paper, for the sake of conciseness, these subspaces will be referred to as the *fundamental subspaces*. Controlled invariant, output-nulling, reachability and stabilisability subspaces are used in the solution of control/rejection problems (disturbance decoupling, noninteracting control, and so forth), see e.g. [3]-[5], whereas conditioned invariant, input-containing, unobservability and detectability subspaces are employed in the solution of

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observation/estimation problems (unknown-input observation, fault detection, etc), see e.g. [26]-[29]. Typically, the solvability to these problems is expressed in terms of a condition involving some of these subspaces; this condition (or set of conditions) is constructive, in the sense that the matrices of the controller/filter are usually obtained by computing a so-called *friend* of the subspace used in the solvability condition, which is a matrix that renders that subspace (being it controlled or conditioned invariant) invariant with respect to the closed loop. Therefore, a fundamental problem in geometric control theory is the computation of a friend that enables all the free eigenvalues of the closed-loop to be assigned.

The traditional algorithms for the computation of the largest output-nulling, reachability, input-containing and unobservability subspaces are based on monotonic sequences of subspaces which converge to the desired subspace in a finite number of steps. In the pioneering paper [10], under some unnecessary assumptions, an algorithm was proposed that employs the Rosenbrock system matrix pencil for the calculation of a spanning set of the supremal reachability subspace of a system. A framework for the computation of basis matrices for the aforementioned subspaces of an LTI system was established in [13], which avoided the restrictive assumptions of [10]. In particular, it was shown in [13] that computational methods based on the Rosenbrock system matrix pencil can be used under the same general conditions of the subspace recursion methods in [1] and the special coordinate basis methods in [2]. This procedure was extended in [11] for the case of repeated eigenvalue and invariant zero structure.

In [9], the framework of [10]-[13] was employed for the computation of basis matrices of the fundamental subspaces for single-input single-output (SISO) LTI systems in the controller canonical form and particularly elegant and insightful expressions were obtained in an explicit way. In the SISO case, the supremal output-nulling reachability subspace is the origin, so that the basis matrix for the output-nulling subspace of systems in the controller canonical form depends only on the invariant zero structure. In the same paper, this approach was used to show that it is possible to derive necessary and sufficient conditions for the solvability of the global monotonic tracking control [14] with state feedback in terms of the non-zero entries of the output matrix C.

For multiple-input multiple-output (MIMO) systems [15], [22] in the controller canonical form, the computation of the fundamental subspaces is much more articulated and rich. In [19]-[21], important preliminary results were given in the calculation of output-nulling and reachability subspaces for MIMO LTI systems in the controller canonical form. Of particular importance is the approach taken in [6] for the determination of bases for the supremal output-nulling and reachability subspaces of strictly proper multivariable systems in the controller canonical form. This approach hinges on the Smith canonical form of polynomial matrices and addresses also the defective case. One limitation of [6] is the fact that only the case of double multiplicity of the invariant factors in the Smith canonical form was taken into account. Another major limitation of [6] was the lack of a procedure for the computation of the feedback friends that render the supremal output-nulling and reachability subspace invariant for the closed-loop.

This paper generalises the results of [9], investigating several aspects related to the computation of basis matrices for output-nulling and input-containing subspaces of MIMO LTI systems in the controller canonical form, without the assumption that the system is strictly proper. The second main objective of this paper is to extend the result in [6] to the case of a defective invariant zero structure with arbitrary multiplicities, by exploiting the approach developed in [10]-[13]. The Smith canonical form is used, but two different cases are considered, depending on whether the polynomial matrix containing the invariant factors is zero for an invariant zero or not. If that matrix is zero for an invariant zero, we show that the computation of a basis matrix for output-nulling subspaces can be considerably simplified. The method proposed here allows to derive the explicit structure of the output-nulling or input-containing subspaces at hand, which is very useful in expressing the solvability conditions of a number of control/estimation problems in terms of, for example, the number of minimum-phase invariant zeros of the system, or even more explicitly in terms of the non-zero elements of a matrix of the system, see e.g. [9] for the problem of monotonic tracking. We also show how to compute the associated friends of output-nulling and reachability subspaces; this aspect is crucial, because, as mentioned above, in virtually all control and estimation problems for which a geometric solvability solution is available, the computation of the decoupling filter involves the friends of the output-nulling or input-containing subspaces. Differently from the classical methods for the computation of friends of controlled invariant and output-nulling subspaces, which hinge on state/input decompositions of the system (see e.g. [1, Chapter 4], [17, Theorem 4.18] and [12]) here we show that the explicit structure of output-nulling or input-containing subspaces delivers the corresponding friends of these subspaces in a simple and natural way, with the simultaneous assignment of the free closed-loop eigenstructure. In addition, output-nulling and reachability subspaces are computed for the defective case without the need of any restrictive assumption. This latter aspect is particularly useful for dead-beat control and estimation problems, where the calculation of a friend requires the assignment of repeated eigenvalues at the origin.

The paper is organised as follows. In Section 2 we provide preliminary material for multivariable systems in the controller canonical form and in Section 3 we define the fundamental subspaces of multivariable systems. Section 4 deals with the computation of bases and friends for output-nulling and reachability subspaces and bases for input-containing subspaces for multivariable systems in the controller canonical form. The results are illustrated with numerical examples in Section 5, followed by concluding remarks.

Notation. The origin of a vector space is denoted by $\{0\}$. The image and the kernel of a matrix A are represented by $\operatorname{im} A$ and $\ker A$, respectively. The symbol $\oplus$ stands for the direct sum of subspaces. The symbol $\mathfrak{i}$ represents the imaginary unit, i.e., $\mathfrak{i} = \sqrt{-1}$, while the symbol $\overline{\alpha}$ represents the complex conjugate of $\alpha \in \mathbb{C}$. Given a rational matrix $P(\lambda) \in \mathbb{R}(\lambda)^{m \times n}$, the *normal rank* is defined as the maximum number of its linearly independent row vectors or column vectors and is denoted by $\operatorname{normrank} P$, see e.g. [18]. A $p \times m$ polynomial matrix $T(\lambda)$, can be written as $T(\lambda) = \big[T(\lambda)\big]_c^h \operatorname{diag}\{\lambda^{t_1}, \ldots, \lambda^{t_m}\} + T_c(\lambda)$, where the matrix $\big[T(\lambda)\big]_c^h$ denotes the *highest column* *degree coefficient matrix* and the degree of the j-th column of $T_c(\lambda)$ is lower than t_j, see e.g. [23], [18]. If $\big[T(\lambda)\big]_c^h$ has full rank, then $T(\lambda)$ is called *column proper*, [23], [18]. A polynomial matrix $T(\lambda)$ with $\operatorname{normrank} T(\lambda) = r$ can be decomposed as

$$T(\lambda) = U_L(\lambda)\, E(\lambda)\, U_R(\lambda) = U_L(\lambda) \left[\begin{array}{cc} \operatorname{diag}\{1,\ldots,1,\epsilon_1(\lambda),\ldots,\epsilon_\mu(\lambda)\} & O \\ O & O \end{array}\right] U_R(\lambda),$$

where $U_L(\lambda)$, $U_R(\lambda)$ are, respectively, $p \times p$, $m \times m$ *unimodular* matrices, $E(\lambda)$ is the *Smith canonical form* and $\epsilon_1(\lambda),\ldots,\epsilon_\mu(\lambda)$ denote the *invariant factors* with $\epsilon_1(\lambda)|\ldots|\epsilon_\mu(\lambda)$ and $1 < \deg \epsilon_1(\lambda) \leq \ldots \leq \deg \epsilon_\mu(\lambda)$, $\mu \leq r$, [7], [18]. Given a polynomial matrix $T(\lambda)$, we denote $T(z)^{(i)} = \frac{\mathrm{d}^i}{\mathrm{d}\lambda^i} T(\lambda)\Big|_{\lambda=z}$. Finally, the i-th canonical basis of $\mathbb{R}^m$ is denoted by $\mathbf{e}_i$.

2 Preliminaries

In this paper, the time index set of any signal is denoted by $\mathbb{T}$; this symbol stands for $\mathbb{R}^+$ in the continuous time and $\mathbb{N}$ in the discrete time. Consider a completely reachable MIMO LTI continuous or discrete-time system Σ governed by

$$\begin{aligned} \mathcal{D}\,\mathbf{x}(t) &= A\,\mathbf{x}(t) + B\,\mathbf{u}(t), \\ \mathbf{y}(t) &= C\,\mathbf{x}(t) + D\,\mathbf{u}(t), \end{aligned} \tag{1}$$

where, for all $t \in \mathbb{T}$, $\mathbf{x}(t) \in \mathcal{X} = \mathbb{R}^n$ is the state, $\mathbf{u}(t) \in \mathcal{U} = \mathbb{R}^m$ is the control input, $\mathbf{y}(t) \in \mathcal{Y} = \mathbb{R}^p$ is the output, and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$. We identify the system with the quadruple $\Sigma = (A, B, C, D)$. The operator $\mathcal{D}$ represents the time derivative $\mathcal{D}\,\mathbf{x}(t) = \dot{\mathbf{x}}(t)$ in the continuous time, and the unit time shift $\mathcal{D}\,\mathbf{x}(t) = \mathbf{x}(t+1)$ in the discrete time. Without loss of generality, we assume that $[\begin{array}{cc} C & D \end{array}]$ is full row rank. We also assume that $\operatorname{rank} B = m$ and that Σ is in the so-called *controller canonical form*, see e.g. [24], [23], [18]-[21], [6], i.e., the matrices A, B are in the following form:

$$A = \left[\begin{array}{cccc} A_{1,1} & A_{1,2} & \ldots & A_{1,m} \\ A_{2,1} & A_{2,2} & \ldots & A_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m,1} & A_{m,2} & \ldots & A_{m,m} \end{array}\right], \qquad B = \left[\begin{array}{c} B_1 \\ B_2 \\ \vdots \\ B_m \end{array}\right], \tag{2}$$

where $A_{j,i} \in \mathbb{R}^{\nu_j \times \nu_i}$, $B_j \in \mathbb{R}^{\nu_j \times m}$ for $i \in \{1,\ldots,m\}$, $j \in \{1,\ldots,m\}$ and

$$A_{j,j} = \left[\begin{array}{cccc} 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \\ \alpha_{j,j,0} & \alpha_{j,j,1} & \ldots & \alpha_{j,j,\nu_j-1} \end{array}\right], \quad A_{j,i} = \left[\begin{array}{ccc} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \\ \alpha_{j,i,0} & \ldots & \alpha_{j,i,\nu_j-1} \end{array}\right], \quad B_j = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ \boldsymbol{\beta}_j^\top \end{array}\right],$$

$$\boldsymbol{\beta}_1^\top = [\begin{array}{cccccc} 1 & \beta_{1,2} & \beta_{1,3} & \ldots & \beta_{1,m} \end{array}],\ \boldsymbol{\beta}_2^\top = [\begin{array}{cccccc} 0 & 1 & \beta_{2,3} & \ldots & \beta_{2,m} \end{array}],\ldots,\ \boldsymbol{\beta}_m^\top = [\begin{array}{cccccc} 0 & 0 & 0 & \ldots & 1 \end{array}].$$

Let $\tilde{A} \in \mathbb{R}^{m\times n}$ be the matrix consisting of the $\nu_1, \nu_1+\nu_2, \dots, n$-th rows of A, which are denoted by $\boldsymbol{\alpha}_j^\top$, $j \in \{1,\dots,m\}$, and let $\tilde{B} \in \mathbb{R}^{m\times m}$ be the matrix consisting of the nonzero rows of B, i.e.,

$$\tilde{A} = \begin{bmatrix} \boldsymbol{\alpha}_1^\top \\ \boldsymbol{\alpha}_2^\top \\ \vdots \\ \boldsymbol{\alpha}_m^\top \end{bmatrix} = \begin{bmatrix} \alpha_{1,1,0} & \dots & \alpha_{1,1,\nu_1-1} & \alpha_{1,2,0} & \dots & \alpha_{1,2,\nu_2-1} & \dots & \alpha_{1,m,0} & \dots & \alpha_{1,m,\nu_m-1} \\ \alpha_{2,1,0} & \dots & \alpha_{2,1,\nu_1-1} & \alpha_{2,2,0} & \dots & \alpha_{2,2,\nu_2-1} & \dots & \alpha_{2,m,0} & \dots & \alpha_{2,m,\nu_m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{m,1,0} & \dots & \alpha_{m,1,\nu_1-1} & \alpha_{m,2,0} & \dots & \alpha_{m,2,\nu_2-1} & \dots & \alpha_{m,m,0} & \dots & \alpha_{m,m,\nu_m-1} \end{bmatrix},$$

$$\tilde{B} = \begin{bmatrix} \boldsymbol{\beta}_1^\top \\ \boldsymbol{\beta}_2^\top \\ \vdots \\ \boldsymbol{\beta}_m^\top \end{bmatrix} = \begin{bmatrix} 1 & \beta_{1,2} & \beta_{1,3} & \dots & \beta_{1,m} \\ 0 & 1 & \beta_{2,3} & \dots & \beta_{2,m} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

The transfer function matrix of the system Σ is equal to $G_\Sigma(\lambda) = C\,(\lambda\,I_n - A)^{-1}\,B + D$. We define the matrices

$$C_\Sigma(\lambda) \stackrel{\text{def}}{=} C\,S(\lambda), \qquad D_\Sigma(\lambda) \stackrel{\text{def}}{=} \tilde{B}^{-1}\,\text{diag}\,\{\lambda^{\nu_1},\dots,\lambda^{\nu_m}\} - \tilde{B}^{-1}\,\tilde{A}\,S(\lambda),$$

where $S(\lambda)$ is the $n \times m$ polynomial matrix defined as

$$S(\lambda) = \begin{bmatrix} \mathbf{s}_1(\lambda) & & & O \\ & \mathbf{s}_2(\lambda) & & \\ & & \ddots & \\ O & & & \mathbf{s}_m(\lambda) \end{bmatrix}, \quad \mathbf{s}_j(\lambda) = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{\nu_j-1} \end{bmatrix}, \quad j \in \{1,\dots,m\},$$

see e.g. [24], [23], [18]-[21], [6]. Notice that $\left[D_\Sigma(\lambda)\right]_c^h = \tilde{B}^{-1}$ is nonsingular, so that $D_\Sigma(\lambda)$ is column proper and its determinant is not the zero polynomial. From the structure theorem of Wolovich and Falb [24], there holds

$$(\lambda\,I_n - A)\,S(\lambda) = B\,D_\Sigma(\lambda), \tag{3}$$

which gives $C\,S(\lambda)\,D_\Sigma^{-1}(\lambda) + D = C\,(\lambda\,I_n - A)^{-1}\,B + D$. Then $G_\Sigma(\lambda) = N_\Sigma(\lambda)\,D_\Sigma^{-1}(\lambda)$ is a right matrix fraction description of the transfer function matrix $G_\Sigma(\lambda)$, where the numerator matrix is $N_\Sigma(\lambda) \stackrel{\text{def}}{=} C_\Sigma(\lambda) + D\,D_\Sigma(\lambda)$, which has full normal rank.[1]

An essential tool used in this paper is the so-called Rosenbrock system matrix pencil $P_\Sigma(\lambda)$, which is defined as $P_\Sigma(\lambda) \stackrel{\text{def}}{=} \left[\begin{smallmatrix} A-\lambda\,I_n & B \\ C & D \end{smallmatrix}\right]$. We recall that the invariant zeros of (A,B,C,D) are the values of λ for which $P_\Sigma(\lambda)$ loses rank with respect to its normal rank.

Lemma 2.1 *Let $P_\Sigma(\lambda)$ be the Rosenbrock system matrix pencil of a MIMO system (A,B,C,D) in the controller canonical form and $N_\Sigma(\lambda) = C_\Sigma(\lambda) + D\,D_\Sigma(\lambda)$. There holds* $\text{normrank}\,P_\Sigma(\lambda) = \text{normrank}\,N_\Sigma(\lambda) + n$.

[1]This follows directly from the fact that $[\;C \;\; D\;]$ has been assumed to be full row rank and $\left[\begin{smallmatrix} S(\lambda) \\ D_\Sigma(\lambda) \end{smallmatrix}\right]$ is full column normal rank by construction.

Proof: From (3), which can be written as $\left[\begin{array}{cc} A-\lambda I_n & B \end{array}\right]\left[\begin{smallmatrix} S(\lambda) \\ D_\Sigma(\lambda) \end{smallmatrix}\right] = O$, it follows

$$P_\Sigma(\lambda)\left[\begin{array}{c} S(\lambda) \\ D_\Sigma(\lambda) \end{array}\right] = \left[\begin{array}{c} O \\ N_\Sigma(\lambda) \end{array}\right] \tag{4}$$

and, from the *Sylvester's rank inequality*,[2] there holds

$$\begin{aligned} \operatorname{normrank} P_\Sigma(\lambda) + \operatorname{normrank}\left[\begin{array}{c} S(\lambda) \\ D_\Sigma(\lambda) \end{array}\right] - n - m &\le \operatorname{normrank}\left(P_\Sigma(\lambda)\left[\begin{array}{c} S(\lambda) \\ D_\Sigma(\lambda) \end{array}\right]\right) \\ &= \operatorname{normrank}\left[\begin{array}{c} O \\ N_\Sigma(\lambda) \end{array}\right] = \operatorname{normrank} N_\Sigma(\lambda). \end{aligned}$$

The normal rank of $\left[\begin{smallmatrix} S(\lambda) \\ D_\Sigma(\lambda) \end{smallmatrix}\right]$ is equal to m for all $\lambda \in \mathbb{C}$ by construction, and therefore

$$\operatorname{normrank} P_\Sigma(\lambda) - n \le \operatorname{normrank} N_\Sigma(\lambda). \tag{5}$$

From the identity $\left[\begin{smallmatrix} A-\lambda I_n & B \\ C & D \end{smallmatrix}\right] = \left[\begin{smallmatrix} I_n & O \\ C\,(A-\lambda I_n)^{-1} & I_p \end{smallmatrix}\right]\left[\begin{smallmatrix} A-\lambda I_n & B \\ O & G_\Sigma(\lambda) \end{smallmatrix}\right]$, we also have that

$$\operatorname{normrank} P_\Sigma(\lambda) = \operatorname{normrank} G_\Sigma(\lambda) + n. \tag{6}$$

Using again the Sylvester's rank inequality, we have

$$\begin{aligned} \operatorname{normrank} G_\Sigma(\lambda) &= \operatorname{normrank}\left(N_\Sigma(\lambda)\, D_\Sigma^{-1}(\lambda)\right) \ge \operatorname{normrank} N_\Sigma(\lambda) + \operatorname{normrank}\, D_\Sigma^{-1}(\lambda) - m \\ &= \operatorname{normrank} N_\Sigma(\lambda) + \operatorname{normrank} D_\Sigma(\lambda) - m = \operatorname{normrank} N_\Sigma(\lambda), \end{aligned} \tag{7}$$

because $\operatorname{normrank}\, D_\Sigma^{-1}(\lambda) = \operatorname{normrank} D_\Sigma(\lambda) = m$, and from (6)-(7), we find

$$\operatorname{normrank} P_\Sigma(\lambda) = \operatorname{normrank} G_\Sigma(\lambda) + n \ge \operatorname{normrank} N_\Sigma(\lambda) + n. \tag{8}$$

From (5) and (8), we obtain $\operatorname{normrank} P_\Sigma(\lambda) = \operatorname{normrank} N_\Sigma(\lambda) + n$. ■

Remark 2.1 If the system is square, i.e., $p = m$, then

$$\begin{aligned} &\det P_\Sigma(\lambda) = \det(A-\lambda I_n)\det\left(D - C\,(A-\lambda I_n)^{-1}B\right) = \det(A-\lambda I_n)\det\left(C\,(\lambda I_n - A)^{-1}B + D\right) \\ &= \det(A-\lambda I_n)\det N_\Sigma(\lambda)/\det D_\Sigma(\lambda) = \det(A-\lambda I_n)\det N_\Sigma(\lambda)/\det(\lambda I_n - A) = (-1)^n \det N_\Sigma(\lambda) \end{aligned}$$

and the invariant zeros are immediately seen to be the roots of the determinant of $N_\Sigma(\lambda)$.

Corollary 2.1 *The invariant zeros of a MIMO system $\Sigma = (A, B, C, D)$ in the controller canonical form are given by the zeros of $N_\Sigma(\lambda)$.*

Proof: In view of Lemma 2.1, $P_\Sigma(\lambda)$ loses rank when $N_\Sigma(\lambda)$ loses rank, i.e., at the zeros of $N_\Sigma(\lambda)$, which are equal to the zeros of the greatest common divisor of all the highest order minors of $N_\Sigma(\lambda)$, see e.g. [18]-[21]. ■

[2] Given two rational matrices $P(\lambda) \in \mathbb{R}(\lambda)^{m\times n}$ and $Q(\lambda) \in \mathbb{R}(\lambda)^{n\times q}$, there holds $\operatorname{normrank}(P(\lambda)\,Q(\lambda)) \ge \operatorname{normrank} P(\lambda) + \operatorname{normrank} Q(\lambda) - n$, [7].

3 Geometric background

We now introduce some concepts from classical geometric control theory that will be used in the sequel. More details can be found for example in [17]. An *output-nulling subspace* $\mathcal{V}$ of $\Sigma = (A, B, C, D)$ is a subspace of $\mathcal{X}$ for which there holds $\left[\begin{smallmatrix} A \\ C \end{smallmatrix}\right] \mathcal{V} \subseteq (\mathcal{V} \oplus \{0\}) + \mathrm{im} \left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$ or, equivalently, for which there exists a real-valued matrix F such that $(A + B\,F)\,\mathcal{V} \subseteq \mathcal{V} \subseteq \ker\,(C + D\,F)$, which is called a *friend* of $\mathcal{V}$. The set of friends of $\mathcal{V}$ is denoted by $\mathfrak{F}(\mathcal{V})$. We denote by $\mathcal{V}^\star$ the largest output-nulling subspace of Σ.

Input-containing subspaces can be defined as the dual of output-nulling subspaces. Indeed, by defining the dual $\Sigma^\top = \left(A^\top, C^\top, B^\top, D^\top\right)$ of Σ, an input-containing subspace $\mathcal{S}$ for Σ can be defined as the orthogonal complement of an output-nulling subspace for $\Sigma^\top$. This is equivalent to saying that an input-containing subspace $\mathcal{S}$ is a subspace of $\mathcal{X}$ satisfying $[\begin{array}{cc} A & B \end{array}]\,((\mathcal{S} \oplus \mathcal{U}) \cap \ker\,[\begin{array}{cc} C & D \end{array}]) \subseteq \mathcal{S}$. We denote by $\mathcal{S}^\star$ the smallest input-containing subspace of Σ.

The so-called *output-nulling reachability* subspace on $\mathcal{V}^\star$, denoted by $\mathcal{R}^\star$, represents the set of initial states which are reachable from the origin and the corresponding output is identically zero and can be computed by $\mathcal{R}^\star = \mathcal{V}^\star \cap \mathcal{S}^\star$. The dual subspace $\mathcal{Q}^\star = \mathcal{V}^\star + \mathcal{S}^\star$ is the so-called *unobservability subspace*. Recall that if $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$ is full column-rank and $[\begin{array}{cc} C & D \end{array}]$ is full row-rank, an LTI system Σ is left-invertible if and only if $\mathcal{R}^\star = \{0\}$ and right-invertible if and only if $\mathcal{Q}^\star = \mathcal{X}$.

Let $F \in \mathfrak{F}(\mathcal{V}^\star)$. The closed-loop spectrum can be partitioned into two parts: **i)** $\sigma(A + B\,F \,|\, \mathcal{V}^\star)$, which is the spectrum of $A + B\,F$ restricted to $\mathcal{V}^\star$; and **ii)** $\sigma(A + B\,F \,|\, \mathcal{X}/\mathcal{V}^\star)$, which is the spectrum of the mapping induced by $A + B\,F$ on the quotient space $\mathcal{X}/\mathcal{V}^\star$. The eigenstructure of $A + B\,F$ restricted to $\mathcal{V}^\star$ can be further split into two sets: the eigenstructure of $\sigma(A + B\,F \,|\, \mathcal{R}^\star)$, which is completely assignable with a suitable choice of F in $\mathfrak{F}(\mathcal{V}^\star)$; and the eigenstructure in $\sigma\,(A + B\,F \,|\, \mathcal{V}^\star/\mathcal{R}^\star)$, which coincides with the *invariant zero structure* of Σ, see e.g. [17, Theorem 7.19] and is fixed for all the choices of F in $\mathfrak{F}(\mathcal{V}^\star)$.

The following two lemmas provide a useful way to compute basis matrices for $\mathcal{R}^\star, \mathcal{V}^\star$, [11]-[13].

Lemma 3.1 *Let $r \stackrel{def}{=} \dim \mathcal{R}^\star$ and let $\lambda_1, \ldots, \lambda_r$ be distinct complex numbers all different from the invariant zeros of the system and such that, if $\lambda_i \in \mathbb{C} \setminus \mathbb{R}$, there exists $j \in \{1, \ldots, r\} \setminus \{i\}$ such that $\lambda_j = \overline{\lambda}_i$. Let $\lambda_1, \ldots, \lambda_r$ be ordered in such a way that the first $2s$ values are complex while the remaining are real and for all odd $k < 2s$ we have $\lambda_{k+1} = \overline{\lambda}_k$. Let $\left[\begin{smallmatrix} V_k \\ W_k \end{smallmatrix}\right]$ be a basis for $\ker P_\Sigma(\lambda_k)$, so that $\left[\begin{smallmatrix} A - \lambda_k I_n & B \\ C & D \end{smallmatrix}\right] \left[\begin{smallmatrix} V_k \\ W_k \end{smallmatrix}\right] = O$, $k \in \{1, \ldots, r\}$ and for all odd $k < 2s$, $\left[\begin{smallmatrix} V_{k+1} \\ W_{k+1} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \overline{V}_k \\ \overline{W}_k \end{smallmatrix}\right]$. Then*

$$\mathcal{R}^\star = \mathrm{im}\,[\begin{array}{ccccccccc} V_1 + V_2 & \mathrm{i}\,(V_2 - V_1) \,|\, & \ldots & |\, V_{2s-1} + V_{2s} & \mathrm{i}\,(V_{2s} - V_{2s-1}) \,|\, V_{2s+1} & \ldots & V_r \end{array}].$$

Remark 3.1 The same result of Lemma 3.1 holds for the computation of $\mathcal{V}^\star$ when we consider $\lambda_1, \ldots, \lambda_r, z_1, \ldots, z_\ell$ distinct complex numbers, where $\lambda_1, \ldots, \lambda_r$ are different from the invariant zeros and $z_1, \ldots, z_\ell$ are the invariant zeros of the system. If, we restrict this operation to the minimum-phase zeros of the system, we obtain the supremal *stabilisability subspace* $\mathcal{V}_g^\star$ of the system, which is the largest output-nulling subspace for which a friend F exists such that all the eigenvalues of $A+B\,F$ are asymptotically stable.

We now consider the defective case. Let Λ be the Jordan structure that we wish to associate to the mapping $A+B\,F\,|\,\mathcal{R}^\star$, where $\Lambda=\text{blkdiag}\,\{J(\lambda_1),\ldots,J(\lambda_\nu)\}$, and $\mathcal{L}=\{\lambda_1,\ldots,\lambda_\nu\}\subset\mathbb{C}$ be self-conjugate; we denote by μ_i the multiplicity of λ_i, so that $\mu_1+\ldots+\mu_\nu=\dim\mathcal{R}^\star$, and $\mu_i=\mu_j$ whenever $\lambda_i=\overline{\lambda}_j$. In Λ, each $J(\lambda_i)$ is a Jordan matrix for λ_i of order μ_i, and may be composed of up to g_i mini-blocks $J(\lambda_i)=\text{blkdiag}\,\{J_1(\lambda_i),\ldots,J_{g_i}(\lambda_i)\}$, where $1\le g_i$. Let $p_{i,k}$, $k\in\{1,\ldots,g_i\}$, $i\in\{1,\ldots,\nu\}$ denote the order of each Jordan mini-block $J_k(\lambda_i)$, so that $p_{i,k}=p_{j,k}$ whenever $\lambda_i=\overline{\lambda}_j$ and $\mu_i=p_{i,1}+\ldots+p_{i,g_i}$. We denote $\mathcal{P}\stackrel{\text{def}}{=}\{p_{i,k}\},\ i\in\{1,\ldots,\nu\}$, $k\in\{1,\ldots,g_i\}$ the partial multiplicities of the eigenvalues $\lambda_1,\ldots,\lambda_\nu$ in Λ. Thus, $\mathcal{L}$ and $\mathcal{P}$ univocally identify Λ up to the order of the Jordan blocks. The possible mini-block orders $p_{i,k}$ of the Jordan structure of $A+B\,F\,|\,\mathcal{R}^\star$ are constrained by the conditions of the Rosenbrock theorem, [16]. If $\mathcal{L}$ and $\mathcal{P}$ satisfy such conditions, we say that the pair $(\mathcal{L},\mathcal{P})$ defines an *assignable Jordan structure* for $A+B\,F\,|\,\mathcal{R}^\star$, see also [12].

Lemma 3.2 *Let $(\mathcal{L},\mathcal{P})$ comprise an assignable Jordan structure for $A+B\,F\,|\,\mathcal{R}^\star$. For all odd $i<2s$ and for $i\in\{2s+1,\ldots,\nu\}$ and $j\in\{1,\ldots,g_i\}$, there exist vectors $\begin{bmatrix}\mathbf{v}_{i,j,k}\\ \mathbf{w}_{i,j,k}\end{bmatrix}$, such that $\begin{bmatrix}A-\lambda_i I_n & B\\ C & D\end{bmatrix}\begin{bmatrix}\mathbf{v}_{i,j,k}\\ \mathbf{w}_{i,j,k}\end{bmatrix}=\begin{bmatrix}\mathbf{v}_{i,j,k-1}\\ \mathbf{0}\end{bmatrix}$, $k\in\{2,\ldots,p_{i,j}\}$, where $\begin{bmatrix}\mathbf{v}_{i,j,1}\\ \mathbf{w}_{i,j,1}\end{bmatrix}$ is a basis for $\ker P_\Sigma(\lambda_i)$ and for all odd $i<2s$, $\begin{bmatrix}\mathbf{v}_{i+1,j,k}\\ \mathbf{w}_{i+1,j,k}\end{bmatrix}=\begin{bmatrix}\overline{\mathbf{v}}_{i,j,k}\\ \overline{\mathbf{w}}_{i,j,k}\end{bmatrix}$. Let $\begin{bmatrix}V_{i,j}\\ W_{i,j}\end{bmatrix}=\begin{bmatrix}\mathbf{v}_{i,j,1} & \cdots & \mathbf{v}_{i,j,p_{i,j}}\\ \mathbf{w}_{i,j,1} & \cdots & \mathbf{w}_{i,j,p_{i,j}}\end{bmatrix}$, $\begin{bmatrix}V_i\\ W_i\end{bmatrix}=\begin{bmatrix}V_{i,1} & \cdots & V_{i,g_i}\\ W_{i,1} & \cdots & W_{i,g_i}\end{bmatrix}$. Then*

$$\mathcal{R}^\star=\text{im}[\ V_1+V_2\ \ \mathrm{i}\,(V_2-V_1)\ |\ \ldots\ |\ V_{2s-1}+V_{2s}\ \ \mathrm{i}\,(V_{2s}-V_{2s-1})\ |\ V_{2s+1}\ \ \ldots\ \ V_\nu\ ].$$

Remark 3.2 For $\mathcal{V}^\star$ the same result holds, but $(\mathcal{L},\mathcal{P})$ must also contain the invariant zero structure. Likewise, for $\mathcal{V}_g^\star$ we must have that $(\mathcal{L},\mathcal{P})$ contains the minimum-phase invariant zero structure.

In [10, Proposition 5], it is shown that $\mathcal{V}^\star$ is made up of two parts, which may have nontrivial intersection: $\mathcal{V}_z^\star$ which is linked to $\ker P_\Sigma(\lambda)$ when λ is an invariant zero, and $\mathcal{R}^\star$ which is linked to $\ker P_\Sigma(\lambda)$ when λ is not an invariant zero. In [6], it is shown how to compute $\mathcal{V}^\star$, considering a strictly proper system (A,B,C) in the controller canonical form. However, the theorem is proved considering the particular case of invariant zeros with double multiplicity. In the SISO case it was proved that $\mathcal{R}^\star=\{0\}$, but this result does not hold true in general in the MIMO case. The output-nulling reachability subspace $\mathcal{R}^\star$ is the origin in the case of MIMO systems in the controller canonical form if the kernel of $C_\Sigma(\lambda)$ is the origin for all λ different from the invariant zeros. If $p\ge m$, then $\text{normrank}\,C_\Sigma(\lambda)=m$ and the nullity of $C_\Sigma(\lambda)$, which is equal to $m-\text{normrank}\,C_\Sigma(\lambda)$, is zero.

We now consider the dual of Lemma 3.1.

Lemma 3.3 *Let $q\stackrel{def}{=}\dim\mathcal{Q}^\star$ and let $\lambda_1,\ldots,\lambda_q$ be distinct complex numbers such that, if $\lambda_i\in\mathbb{C}\setminus\mathbb{R}$, there exists $j\in\{1,\ldots,q\}\setminus\{i\}$ such that $\lambda_j=\overline{\lambda}_i$. Let $\lambda_1,\ldots,\lambda_q$ be ordered in such a way that the first $2s$ values are complex while the remaining are real and for all odd $k<2s$ we have $\lambda_{k+1}=\overline{\lambda}_k$. Let*

$\left[\begin{array}{cc} Q_k & P_k \end{array}\right]$ *a basis for the left null-space of* $P_\Sigma(\lambda_k)$*, so that* $\left[\begin{array}{cc} Q_k & P_k \end{array}\right]\left[\begin{smallmatrix} A-\lambda_k I_n & B \\ C & D \end{smallmatrix}\right] = O,\ k \in \{1,\ldots,q\}$. *and for all odd* $k < 2s$, $\left[\begin{array}{cc} Q_{k+1} & P_{k+1} \end{array}\right] = \left[\begin{array}{cc} \overline{Q}_k & \overline{P}_k \end{array}\right]$*. Then*

$$\mathcal{Q}^\star = \ker\left(\left[\begin{array}{cc|c|cc|ccc} (Q_1+Q_2)^\top & \mathfrak{i}\,(Q_2-Q_1)^\top & \ldots & (Q_{2s-1}+Q_{2s})^\top & \mathfrak{i}\,(Q_{2s}-Q_{2s-1})^\top & Q_{2s+1}^\top & \ldots & Q_q^\top \end{array}\right]^\top\right).$$

The case of nontrivial Jordan structure can be stated analogously, and will be omitted for the sake of brevity.

4 Fundamental subspaces and the controller canonical form

4.1 Output-nulling and reachability subspaces

In this section, we consider nonstrictly proper multivariable systems in the controller canonical form. The following theorem shows how to compute $\mathcal{R}^\star$ and construct the associated friend.

Theorem 4.1 *Consider a MIMO system* $\Sigma = (A,B,C,D)$ *in the controller canonical form and choose distinct and complex* $\lambda_1,\ldots,\lambda_{2s},\lambda_{2s+1},\ldots,\lambda_r$ *different from the invariant zeros, ordered in such a way that for all odd* $i < 2s$ *we have* $\lambda_{i+1} = \overline{\lambda}_i$ *and* λ_i *are real for all* $i \geq 2s+1$*. Then* $\mathcal{R}^\star$ *is computed by* $\mathcal{R}^\star = \operatorname{im} V$*, where*

$$V = \left[\begin{array}{cc|c|cc|ccc} \mathfrak{Re}\,\{V(\lambda_1)\} & \mathfrak{Im}\,\{V(\lambda_1)\} & \ldots & \mathfrak{Re}\,\{V(\lambda_{2s-1})\} & \mathfrak{Im}\,\{V(\lambda_{2s-1})\} & V(\lambda_{2s+1}) & \ldots & V(\lambda_r) \end{array}\right], \tag{9}$$

$V(\lambda_i) = S(\lambda_i)\,\tilde{V}(\lambda_i)$ *and* $\tilde{V}(\lambda_i)$ *is a basis matrix for* $\ker N_\Sigma(\lambda_i)$, $i \in \{1,\ldots,r\}$.

Proof: Multiplying both sides of (4) on the right by a basis polynomial matrix $\tilde{V}(\lambda)$ of $\ker N_\Sigma(\lambda)$ gives

$$P_\Sigma(\lambda)\left[\begin{array}{c} S(\lambda)\,\tilde{V}(\lambda) \\ D_\Sigma(\lambda)\,\tilde{V}(\lambda) \end{array}\right] = O.$$

Denoting $V(\lambda) \stackrel{\text{def}}{=} S(\lambda)\,\tilde{V}(\lambda)$, we compute $V(\lambda_i)$ for $i \in \{1,\ldots,r\}$, such that for all odd $i < 2s, V(\lambda_{i+1}) = \overline{V(\lambda_i)}$. For all odd $i < 2s$, we construct s pairs of real vectors

$$\left[\begin{array}{cc} V(\lambda_i)+V(\lambda_{i+1}) & \mathfrak{i}\,(V(\lambda_{i+1})-V(\lambda_i)) \end{array}\right] = \left[\begin{array}{cc} 2\,\mathfrak{Re}\{V(\lambda_i)\} & 2\,\mathfrak{Im}\{V(\lambda_i)\} \end{array}\right]$$

and a basis for $\mathcal{R}^\star$ is given by the image of V in (9). ■

In order to construct a friend of $\mathcal{R}^\star$ such that $\sigma(A+B\,F\,|\,\mathcal{R}^\star) = \{\lambda_1,\ldots,\lambda_r\}$, first choose one vector $\mathbf{v}_i$ from each matrix $V(\lambda_i)$ in such a way that for all $i < 2s$ the vector $\mathbf{v}_{i+1}$ chosen from $V(\lambda_{i+1})$ is the complex conjugate of $\mathbf{v}_i$ and the r vectors $\mathbf{v}_i,\ i \in \{1,\ldots,r\}$ are linearly independent, and define the full column-rank, real matrix $X \stackrel{\text{def}}{=} [\,\mathbf{v}_1{+}\mathbf{v}_2\ \ \mathfrak{i}\,(\mathbf{v}_2{-}\mathbf{v}_1)\,|\,\ldots\,|\,\mathbf{v}_{2s-1}{+}\mathbf{v}_{2s}\ \ \mathfrak{i}\,(\mathbf{v}_{2s}{-}\mathbf{v}_{2s-1})\,|\,\mathbf{v}_{2s+1}\ \ldots\ \mathbf{v}_r\,]$. Next, denoting $W(\lambda) \stackrel{\text{def}}{=} D_\Sigma(\lambda)\,\tilde{V}(\lambda)$ and computing $W(\lambda_i)$ for $i \in \{1,\ldots,r\}$, select the r vectors $\mathbf{w}_i$ from the matrix $[\,W(\lambda_1)\ \ \ldots\ \ W(\lambda_{2s})\ \ W(\lambda_{2s+1})\ \ \ldots\ \ W(\lambda_r)\,]$ which correspond to the chosen $\mathbf{v}_i$, and define the real matrix $Y \stackrel{\text{def}}{=} [\, \mathbf{w}_1+\mathbf{w}_2 \;\; \mathrm{i}\,(\mathbf{w}_2-\mathbf{w}_1) \,|\, \ldots \,|\, \mathbf{w}_{2s-1}+\mathbf{w}_{2s} \;\; \mathrm{i}\,(\mathbf{w}_{2s}-\mathbf{w}_{2s-1}) \,|\, \mathbf{w}_{2s+1} \;\; \ldots \;\; \mathbf{w}_r \,]$. A friend of $\mathcal{R}^\star$ is constructed as $F = Y\,X^\dagger$, see [13].

We now remove the assumption that $\lambda_1,\ldots,\lambda_r$ are distinct and assume that they are real for simplicity.

Theorem 4.2 *Let $\Sigma = (A,B,C,D)$ be a MIMO system in the controller canonical form and let $r = \dim \mathcal{R}^\star$. Let $\lambda_1,\ldots,\lambda_\nu$ be real numbers different from the invariant zeros, with multiplicities $\mu_1,\ldots,\mu_\nu$, respectively, such that $\mu_1+\ldots+\mu_\nu = r$. Then a basis matrix for $\mathcal{R}^\star$ is given by $\mathcal{R}^\star = \mathrm{im}\,[\, V_1 \;\; V_2 \;\; \ldots \;\; V_\nu \,]$, where for all $i \in \{1,\ldots,\nu\}$*

$$V_i = \left[\, V(\lambda_i) \quad V(\lambda_i)^{(1)} \quad \ldots \quad \frac{V(\lambda_i)^{(\mu_i-1)}}{(\mu_i-1)!} \,\right], \quad V(\lambda_i) = S(\lambda_i)\,\tilde{V}(\lambda_i) \tag{10}$$

and $\tilde{V}(\lambda_i)$ is a basis matrix for $\ker N_\Sigma(\lambda_i)$*.*

Proof: We multiply (4) both sides on the right by a basis polynomial vector $\tilde{\mathbf{v}}(\lambda)$ of $\ker N_\Sigma(\lambda)$ and obtain

$$\begin{bmatrix} A-\lambda I_n & B \\ C & D \end{bmatrix} \begin{bmatrix} V(\lambda) \\ W(\lambda) \end{bmatrix} = O, \tag{11}$$

where $V(\lambda) = S(\lambda)\,\tilde{V}(\lambda),\; W(\lambda) = D_\Sigma(\lambda)\,\tilde{V}(\lambda)$. Taking the first derivative of (11), we have

$$\begin{bmatrix} A-\lambda I_n & B \\ C & D \end{bmatrix} \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}\lambda} V(\lambda) \\ \frac{\mathrm{d}}{\mathrm{d}\lambda} W(\lambda) \end{bmatrix} = \begin{bmatrix} V(\lambda) \\ O \end{bmatrix}.$$

We prove by induction that

$$\begin{bmatrix} A-\lambda I_n & B \\ C & D \end{bmatrix} \begin{bmatrix} \frac{1}{\kappa!}\frac{\mathrm{d}^\kappa}{\mathrm{d}\lambda^\kappa} V(\lambda) \\ \frac{1}{\kappa!}\frac{\mathrm{d}^\kappa}{\mathrm{d}\lambda^\kappa} W(\lambda) \end{bmatrix} = \begin{bmatrix} \frac{1}{(\kappa-1)!}\frac{\mathrm{d}^{\kappa-1}}{\mathrm{d}\lambda^{\kappa-1}} V(\lambda) \\ O \end{bmatrix}. \tag{12}$$

We have proved that it holds true for $\kappa = 1$ and assume that it holds true for $\kappa-1$, i.e., let us assume that

$$\begin{bmatrix} A-\lambda I_n & B \\ C & D \end{bmatrix} \begin{bmatrix} \frac{1}{(\kappa-1)!}\frac{\mathrm{d}^{\kappa-1}}{\mathrm{d}\lambda^{\kappa-1}} V(\lambda) \\ \frac{1}{(\kappa-1)!}\frac{\mathrm{d}^{\kappa-1}}{\mathrm{d}\lambda^{\kappa-1}} W(\lambda) \end{bmatrix} = \begin{bmatrix} \frac{1}{(\kappa-2)!}\frac{\mathrm{d}^{\kappa-2}}{\mathrm{d}\lambda^{\kappa-2}} V(\lambda) \\ O \end{bmatrix}$$

and differentiating

$$\begin{bmatrix} A-\lambda I_n & B \\ C & D \end{bmatrix} \begin{bmatrix} \frac{1}{(\kappa-1)!}\frac{\mathrm{d}^{\kappa}}{\mathrm{d}\lambda^{\kappa}} V(\lambda) \\ \frac{1}{(\kappa-1)!}\frac{\mathrm{d}^{\kappa}}{\mathrm{d}\lambda^{\kappa}} W(\lambda) \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{(\kappa-1)!}+\frac{1}{(\kappa-2)!}\right)\frac{\mathrm{d}^{\kappa-1}}{\mathrm{d}\lambda^{\kappa-1}} V(\lambda) \\ O \end{bmatrix} = \begin{bmatrix} \frac{\kappa}{(\kappa-1)!}\frac{\mathrm{d}^{\kappa-1}}{\mathrm{d}\lambda^{\kappa-1}} V(\lambda) \\ O \end{bmatrix},$$

we obtain (12). Computing $\frac{1}{\kappa!}\frac{\mathrm{d}^\kappa}{\mathrm{d}\lambda^\kappa} V(\lambda)$ for each λ_i and $\kappa \in \{1,\ldots,\mu_i-1\},\; i \in \{1,\ldots,\nu\}$, we obtain V_i as in (10) and a basis matrix for $\mathcal{R}^\star$ is given by the image of $V = [\, V_1 \;\; V_2 \;\; \ldots \;\; V_\nu \,]$. ∎

Theorem 4.2 does not discuss the construction of the friend F. On the other hand, one can easily proceed as outlined above by taking into account the constraints imposed by the Rosenbrock Theorem on the dimensions of the Jordan mini-blocks of the mapping $A+B\,F\,|\,\mathcal{R}^\star$ as detailed in [12].

Now we focus our attention on the subspace $\mathcal{V}_z^\star$ associated to the invariant zeros of a system. For this reason we assume that $\mathcal{R}^\star=\{0\}$, i.e., the case where the system is left-invertible. The following theorem provides a structure for a basis matrix of $\mathcal{V}^\star=\mathcal{V}_z^\star$ in the case where the invariant zeros are distinct and complex.

Theorem 4.3 *Consider a left-invertible MIMO system $\Sigma=(A,B,C,D)$ in the controller canonical form with ℓ distinct and complex invariant zeros $z_1,\ldots,z_{2\sigma},z_{2\sigma+1},\ldots,z_\ell$, ordered in such a way that for all odd $i<2\sigma$ we have $z_{i+1}=\overline{z}_i$ and z_i are real for all $i\geq 2\sigma+1$. Then a basis matrix for $\mathcal{V}^\star$ is given by $\mathcal{V}^\star=\mathcal{V}_z^\star=\operatorname{im}V$, where*

$$V=[\;\mathfrak{Re}\,\{\mathbf{v}_1\}\;\;\mathfrak{Im}\,\{\mathbf{v}_1\}\;|\ldots|\;\mathfrak{Re}\,\{\mathbf{v}_{2\sigma-1}\}\;\;\mathfrak{Im}\,\{\mathbf{v}_{2\sigma-1}\}\;|\;\mathbf{v}_{2\sigma+1}\;\;\ldots\;\;\mathbf{v}_\ell\;], \tag{13}$$

$\mathbf{v}_i=S(z_i)\,\tilde{\mathbf{v}}_i$ and $\tilde{\mathbf{v}}_i$ is a basis vector for $\ker N_\Sigma(z_i)$ for $i\in\{1,\ldots,\ell\}$. The matrix W for the computation of an associated friend $F=W\,V^\dagger$ that assigns the invariant zero structure of Σ is given by

$$W=[\;\mathfrak{Re}\,\{\mathbf{w}_1\}\;\;\mathfrak{Im}\,\{\mathbf{w}_1\}\;|\ldots|\;\mathfrak{Re}\,\{\mathbf{w}_{2\sigma-1}\}\;\;\mathfrak{Im}\,\{\mathbf{w}_{2\sigma-1}\}\;|\;\mathbf{w}_{2\sigma+1}\;\;\ldots\;\;\mathbf{w}_\ell\;], \tag{14}$$

where $\mathbf{w}_i=D_\Sigma(z_i)\,\tilde{\mathbf{v}}_i,\;i\in\{1,\ldots,\ell\}$.

Proof: Since the system has distinct invariant zeros, the Smith canonical form of $P_\Sigma(z_i)$ has one invariant factor $\epsilon_1(\lambda)=(\lambda-z_1)\ldots(\lambda-z_\ell)$, which implies that the kernel of $P_\Sigma(z_i)$ is 1-dimensional. From

$$P_\Sigma(z_i)\left[\begin{array}{c} S(z_i)\\ D_\Sigma(z_i)\end{array}\right]=\left[\begin{array}{c} O\\ N_\Sigma(z_i)\end{array}\right], \tag{15}$$

we have that a basis vector for $\ker P_\Sigma(z_i)$ is given by

$$\left[\begin{array}{c}\mathbf{v}_i\\ \mathbf{w}_i\end{array}\right]\stackrel{\text{def}}{=}\left[\begin{array}{c} S(z_i)\\ D_\Sigma(z_i)\end{array}\right]\tilde{\mathbf{v}}_i,$$

where $\tilde{\mathbf{v}}_i\in\ker N_\Sigma(z_i)$. Since the invariant zeros are distinct, the Smith canonical form of $N_\Sigma(\lambda)$ is $\left[\begin{smallmatrix}\operatorname{diag}\{1,\ldots,1,\epsilon_1(\lambda)\}\\ O\end{smallmatrix}\right]$. Consequently, the dimension of $\ker N(z_i)$ is 1. For each invariant zero, we compute vectors $\mathbf{v}_i=S(z_i)\,\tilde{\mathbf{v}}_i$, where $\tilde{\mathbf{v}}_i$ is a basis vector for $\ker N_\Sigma(z_i)$, such that for all odd $i<2s$, $\mathbf{v}_{i+1}=\overline{\mathbf{v}}_i$. In view of Lemma 3.1 and Remark 3.1, we construct σ pairs of real vectors

$$\left[\;\mathbf{v}_i+\mathbf{v}_{i+1}\;\;\mathfrak{i}\,(\mathbf{v}_{i+1}-\mathbf{v}_i)\;\right]=\left[\;2\,\mathfrak{Re}\{\mathbf{v}_i\}\;\;2\,\mathfrak{Im}\{\mathbf{v}_i\}\;\right]$$

for all odd $i<2\sigma$ and a basis for $\mathcal{V}^\star$ is given by (13). We also construct σ pairs of real vectors for the complex $\mathbf{w}_i$ and W is constructed as in (14). ■

Remark 4.1 If a MIMO system (A,B,C,D) in the controller canonical form is square, i.e., $p=m$, and D is nonsingular, then it has n invariant zeros and thus $\mathcal{V}^\star=\mathcal{X}$.

Remark 4.2 For a SISO system (A,B,C) in the controller canonical form with ℓ invariant zeros ordered as in Theorem 4.3, the transfer function is given by $G_\Sigma(\lambda)=c_\Sigma(\lambda)/d_\Sigma(\lambda)$. Since $z_1,\ldots,z_\ell$ are the roots of $c_\Sigma(\lambda)$, the nullity of $c_\Sigma(z_i)$ is 1, so that

$$\mathcal{V}^\star=\operatorname{im}\left[\begin{array}{cc|c|cc|ccc} \mathfrak{Re}\{1\} & \mathfrak{Im}\{1\} & \ldots & \mathfrak{Re}\{1\} & \mathfrak{Im}\{1\} & 1 & \ldots & 1\\ \mathfrak{Re}\{z_1\} & \mathfrak{Im}\{z_1\} & \ldots & \mathfrak{Re}\{z_{2s-1}\} & \mathfrak{Im}\{z_{2s-1}\} & z_{2s+1} & \ldots & z_\ell\\ \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots\\ \mathfrak{Re}\{z_1^{n-1}\} & \mathfrak{Im}\{z_1^{n-1}\} & \ldots & \mathfrak{Re}\{z_{2s-1}^{n-1}\} & \mathfrak{Im}\{z_{2s-1}^{n-1}\} & z_{2s+1}^{n-1} & \ldots & z_\ell^{n-1} \end{array}\right],$$

see also [9].

The following theorem concerns the case of invariant zeros with multiplicity greater than 1. We assume, for the sake of simplicity of exposition, that the system has one invariant zero with multiplicity ℓ and the multiplicities of z as root of the $\mu\le m$ invariant factors in the Smith canonical form of $N_\Sigma(\lambda)$ are $r_1, r_2,\ldots,r_\mu$, so that $r_1+r_2+\ldots+r_\mu=\ell$. If a system has more than one repeated invariant zeros, the same procedure may be followed for each of them.

Theorem 4.4 *Let $\Sigma=(A,B,C,D)$ be a left-invertible MIMO system in the controller canonical form that has a real invariant zero z with multiplicity ℓ. Decompose $N_\Sigma(\lambda)=C_\Sigma(\lambda)+D\,D_\Sigma(\lambda)$ as $N_\Sigma(\lambda)=U_L(\lambda)\,E(\lambda)\,U_R(\lambda)$, where $E(\lambda)$ is the Smith canonical form of $N_\Sigma(\lambda)$ with invariant factors $\epsilon_j(\lambda)$ and $\deg\epsilon_j(\lambda)=r_j$, $j\in\{1,\ldots,\mu\}$. Then $\mathcal{V}^\star=\mathcal{V}_z^\star=\operatorname{im}V=\operatorname{im}[\,V_1\ \ V_2\ \ \ldots\ \ V_\mu\,]$, where*

$$V_j=\left[\begin{array}{c|c|c|c} S(z)\,\tilde{\mathbf{v}}_{0,j} & S(z)^{(1)}\,\tilde{\mathbf{v}}_{0,j}+S(z)\,\tilde{\mathbf{v}}_{1,j} & \ldots & \frac{S(z)^{(r_j-1)}}{(r_j-1)!}\,\tilde{\mathbf{v}}_{0,j}+\sum_{\kappa=1}^{r_j-1}\frac{S(z)^{(r_j-\kappa-1)}}{(r_j-\kappa-1)!}\,\tilde{\mathbf{v}}_{\kappa,j}\end{array}\right], \quad (16)$$

and

$$\begin{aligned}\tilde{\mathbf{v}}_{0,j}&=U_R^{-1}(z)\,\mathbf{e}_{m-\mu+j},\\ \tilde{\mathbf{v}}_{\kappa,j}&=-U_R^{-1}(z)\sum_{l=0}^{\kappa-1}\frac{U_R(z)^{(\kappa-l)}}{(\kappa-l)!}\,\tilde{\mathbf{v}}_{l,j},\qquad \kappa\in\{1,\ldots,r_j-1\},\ j\in\{1,\ldots,\mu\}.\end{aligned}$$

The matrix W for the computation of an associated friend $F=W\,V^\dagger$ that assigns the invariant zero structure of Σ is given by $W=[\,W_1\ \ W_2\ \ \ldots\ \ W_\mu\,]$, where for all $j\in\{1,\ldots,\mu\}$

$$W_j=\left[\begin{array}{c|c|c|c} D_\Sigma(z)\,\tilde{\mathbf{v}}_{0,j} & D_\Sigma(z)^{(1)}\,\tilde{\mathbf{v}}_{0,j}+D_\Sigma(z)\,\tilde{\mathbf{v}}_{1,j} & \ldots & \frac{D_\Sigma(z)^{(r_j-1)}}{(r_j-1)!}\,\tilde{\mathbf{v}}_{0,j}+\sum_{\kappa=1}^{r_j-1}\frac{D_\Sigma(z)^{(r_j-\kappa-1)}}{(r_j-\kappa-1)!}\,\tilde{\mathbf{v}}_{\kappa,j}\end{array}\right].$$

Proof: If we multiply $P_\Sigma(z)\begin{bmatrix} S(z)\\ D_\Sigma(z)\end{bmatrix}=\begin{bmatrix} O\\ N_\Sigma(z)\end{bmatrix}$ on both sides by a basis for $\ker N_\Sigma(z)$, which is denoted by $\tilde{V}_0$, we have that $P_\Sigma(z)\begin{bmatrix} S(z)\,\tilde{V}_0\\ D_\Sigma(z)\,\tilde{V}_0\end{bmatrix}=O$ and $\begin{bmatrix} V_0\\ W_0\end{bmatrix}\stackrel{\text{def}}{=}\begin{bmatrix} S(z)\,\tilde{V}_0\\ D_\Sigma(z)\,\tilde{V}_0\end{bmatrix}$ is a basis for $\ker P_\Sigma(z)$.

There holds $\tilde{V}_0=U_R^{-1}(z)\,[\,\mathbf{e}_{m-\mu+1}\ \ \ldots\ \ \mathbf{e}_m\,]$ with $U_R^{-1}(z)\,\mathbf{e}_{m-\mu+j}$ corresponding to the invariant factor $\epsilon_j(\lambda)$. Denoting by $\tilde{\mathbf{v}}_{0,j}$ the column vector of $\tilde{V}_0$ corresponding to the invariant factor $\epsilon_j(\lambda)$, we have $\tilde{\mathbf{v}}_{0,j}=U_R^{-1}(z)\,\mathbf{e}_{m-\mu+j}$, $j\in\{1,\ldots,\mu\}$.

For every invariant factor $\epsilon_j(\lambda)$, we will find a matrix $V_j = [\ \mathbf{v}_{0,j} \quad \mathbf{v}_{1,j} \quad \ldots \quad \mathbf{v}_{r_j-1,j}\ ]$ belonging to $\mathcal{V}_z^\star$. The first vector is $\mathbf{v}_{0,j} = S(z)\,\tilde{\mathbf{v}}_{0,j}$ and we will compute $\mathbf{v}_{\kappa,j}$, $\kappa \in \{1,\ldots,r_j-1\}$, which must satisfy

$$\begin{bmatrix} A - z\,I_n & B \\ C & D \end{bmatrix} \begin{bmatrix} \mathbf{v}_{\kappa,j} \\ \mathbf{w}_{\kappa,j} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{\kappa-1,j} \\ \mathbf{0} \end{bmatrix}.$$

Taking the first derivative of (4), we have

$$\begin{bmatrix} -I_n & O \\ O & O \end{bmatrix} \begin{bmatrix} S(\lambda) \\ D_\Sigma(\lambda) \end{bmatrix} + \begin{bmatrix} A - \lambda\,I_n & B \\ C & D \end{bmatrix} \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}\lambda} S(\lambda) \\ \frac{\mathrm{d}}{\mathrm{d}\lambda} D_\Sigma(\lambda) \end{bmatrix} = \begin{bmatrix} O \\ \frac{\mathrm{d}}{\mathrm{d}\lambda} N_\Sigma(\lambda) \end{bmatrix}$$

or, equivalently,

$$\begin{bmatrix} A - \lambda\,I_n & B \\ C & D \end{bmatrix} \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}\lambda} S(\lambda) \\ \frac{\mathrm{d}}{\mathrm{d}\lambda} D_\Sigma(\lambda) \end{bmatrix} = \begin{bmatrix} S(\lambda) \\ \frac{\mathrm{d}}{\mathrm{d}\lambda} N_\Sigma(\lambda) \end{bmatrix}$$

and if we take higher derivatives, we have

$$\begin{bmatrix} A - \lambda\,I_n & B \\ C & D \end{bmatrix} \begin{bmatrix} \frac{1}{\kappa!}\frac{\mathrm{d}^\kappa}{\mathrm{d}\lambda^\kappa} S(\lambda) \\ \frac{1}{\kappa!}\frac{\mathrm{d}^\kappa}{\mathrm{d}\lambda^\kappa} D_\Sigma(\lambda) \end{bmatrix} = \begin{bmatrix} \frac{1}{(\kappa-1)!}\frac{\mathrm{d}^{\kappa-1}}{\mathrm{d}\lambda^{\kappa-1}} S(\lambda) \\ \frac{1}{\kappa!}\frac{\mathrm{d}^\kappa}{\mathrm{d}\lambda^\kappa} N_\Sigma(\lambda) \end{bmatrix},$$

so that

$$\begin{bmatrix} A - z\,I_n & B \\ C & D \end{bmatrix} \begin{bmatrix} \frac{S(z)^{(\kappa)}}{\kappa!} \\ \frac{D_\Sigma(z)^{(\kappa)}}{\kappa!} \end{bmatrix} = \begin{bmatrix} \frac{S(z)^{(\kappa-1)}}{(\kappa-1)!} \\ \frac{N_\Sigma(z)^{(\kappa)}}{\kappa!} \end{bmatrix}, \ \kappa \in \{1,\ldots,r_j-1\}. \tag{17}$$

The second vector of V_j will satisfy

$$\begin{bmatrix} A - z\,I_n & B \\ C & D \end{bmatrix} \begin{bmatrix} \mathbf{v}_{1,j} \\ \mathbf{w}_{1,j} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{0,j} \\ \mathbf{0} \end{bmatrix}.$$

Consider (17) for $\kappa = 1$ and multiply both sides on the right by $\tilde{\mathbf{v}}_{0,j}$. Then we have

$$\begin{bmatrix} A - z\,I_n & B \\ C & D \end{bmatrix} \begin{bmatrix} S(z)^{(1)}\,\tilde{\mathbf{v}}_{0,j} \\ D_\Sigma(z)^{(1)}\,\tilde{\mathbf{v}}_{0,j} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{0,j} \\ N_\Sigma(z)^{(1)}\,\tilde{\mathbf{v}}_{0,j} \end{bmatrix} \tag{18}$$

and we add equation (18) with

$$\begin{bmatrix} A - z\,I_n & B \\ C & D \end{bmatrix} \begin{bmatrix} S(z)\,\tilde{\mathbf{v}}_{1,j} \\ D_\Sigma(z)\,\tilde{\mathbf{v}}_{1,j} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ N_\Sigma(z)\,\tilde{\mathbf{v}}_{1,j} \end{bmatrix},$$

so that

$$\begin{bmatrix} A - z\,I_n & B \\ C & D \end{bmatrix} \begin{bmatrix} S(z)^{(1)}\,\tilde{\mathbf{v}}_{0,j} + S(z)\,\tilde{\mathbf{v}}_{1,j} \\ D_\Sigma(z)^{(1)}\,\tilde{\mathbf{v}}_{0,j} + D_\Sigma(z)\,\tilde{\mathbf{v}}_{1,j} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{0,j} \\ N_\Sigma(z)^{(1)}\,\tilde{\mathbf{v}}_{0,j} + N_\Sigma(z)\,\tilde{\mathbf{v}}_{1,j} \end{bmatrix}.$$

If $\tilde{\mathbf{v}}_{1,j}$ is such that $N_\Sigma(z)^{(1)}\,\tilde{\mathbf{v}}_{0,j} + N_\Sigma(z)\,\tilde{\mathbf{v}}_{1,j} = \mathbf{0}$, then

$$\begin{aligned}\mathbf{v}_{1,j} &= S(z)^{(1)}\,\tilde{\mathbf{v}}_{0,j} + S(z)\,\tilde{\mathbf{v}}_{1,j},\\ \mathbf{w}_{1,j} &= D_\Sigma(z)^{(1)}\,\tilde{\mathbf{v}}_{0,j} + D_\Sigma(z)\,\tilde{\mathbf{v}}_{1,j}.\end{aligned}$$

To compute the third vector of V_j, the following must be satisfied

$$\begin{bmatrix} A - z\,I_n & B \\ C & D \end{bmatrix}\begin{bmatrix} \mathbf{v}_{2,j} \\ \mathbf{w}_{2,j} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{1,j} \\ \mathbf{0} \end{bmatrix}.$$

Adding

$$\begin{bmatrix} A - z\,I_n & B \\ C & D \end{bmatrix}\begin{bmatrix} \frac{S(z)^{(2)}\,\tilde{\mathbf{v}}_{0,j}}{2!} \\ \frac{D_\Sigma(z)^{(2)}\,\tilde{\mathbf{v}}_{0,j}}{2!} \end{bmatrix} = \begin{bmatrix} S(z)^{(1)}\,\tilde{\mathbf{v}}_{0,j} \\ \frac{N_\Sigma(z)^{(2)}\,\tilde{\mathbf{v}}_{0,j}}{2!} \end{bmatrix}$$

to

$$\begin{bmatrix} A - z\,I_n & B \\ C & D \end{bmatrix}\begin{bmatrix} S(z)^{(1)}\,\tilde{\mathbf{v}}_{1,j} \\ D_\Sigma(z)^{(1)}\,\tilde{\mathbf{v}}_{1,j} \end{bmatrix} = \begin{bmatrix} S(z)\,\tilde{\mathbf{v}}_{1,j} \\ N_\Sigma(z)^{(1)}\,\tilde{\mathbf{v}}_{1,j} \end{bmatrix},$$

yields

$$\begin{bmatrix} A - z\,I_n & B \\ C & D \end{bmatrix}\begin{bmatrix} \frac{S(z)^{(2)}}{2!}\,\tilde{\mathbf{v}}_{0,j} + S(z)^{(1)}\,\tilde{\mathbf{v}}_{1,j} \\ \frac{D_\Sigma(z)^{(2)}}{2!}\,\tilde{\mathbf{v}}_{0,j} + D_\Sigma(z)^{(1)}\,\tilde{\mathbf{v}}_{1,j} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{1,j} \\ \frac{N_\Sigma(z)^{(2)}}{2!}\,\tilde{\mathbf{v}}_{0,j} + N_\Sigma(z)^{(1)}\,\tilde{\mathbf{v}}_{1,j} \end{bmatrix} \tag{19}$$

and $\frac{N_\Sigma(z)^{(2)}}{2!}\,\tilde{\mathbf{v}}_{0,j} + N_\Sigma(z)^{(1)}\,\tilde{\mathbf{v}}_{1,j}$ may not be equal to $\mathbf{0}$. We add equation (19) with

$$\begin{bmatrix} A - z\,I_n & B \\ C & D \end{bmatrix}\begin{bmatrix} S(z)\,\tilde{\mathbf{v}}_{2,j} \\ D_\Sigma(z)\,\tilde{\mathbf{v}}_{2,j} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ N_\Sigma(z)\,\tilde{\mathbf{v}}_{2,j} \end{bmatrix},$$

so that

$$\begin{aligned}&\begin{bmatrix} A - z\,I_n & B \\ C & D \end{bmatrix}\begin{bmatrix} \frac{S(z)^{(2)}}{2!}\,\tilde{\mathbf{v}}_{0,j} + S(z)^{(1)}\,\tilde{\mathbf{v}}_{1,j} + S(z)\,\tilde{\mathbf{v}}_{2,j} \\ \frac{D_\Sigma(z)^{(2)}}{2!}\,\tilde{\mathbf{v}}_{0,j} + D_\Sigma(z)^{(1)}\,\tilde{\mathbf{v}}_{1,j} + D_\Sigma(z)\,\tilde{\mathbf{v}}_{2,j} \end{bmatrix}\\ &\qquad = \begin{bmatrix} \mathbf{v}_{1,j} \\ \frac{N_\Sigma(z)^{(2)}}{2!}\,\tilde{\mathbf{v}}_{0,j} + N_\Sigma(z)^{(1)}\,\tilde{\mathbf{v}}_{1,j} + N_\Sigma(z)\,\tilde{\mathbf{v}}_{2,j} \end{bmatrix}.\end{aligned}$$

If $\tilde{\mathbf{v}}_{2,j}$ is such that $\frac{N_\Sigma(z)^{(2)}}{2!}\,\tilde{\mathbf{v}}_{0,j} + N_\Sigma(z)^{(1)}\,\tilde{\mathbf{v}}_{1,j} + N_\Sigma(z)\,\tilde{\mathbf{v}}_{2,j} = \mathbf{0}$, then

$$\begin{aligned}\mathbf{v}_{2,j} &= \frac{S(z)^{(2)}}{2!}\,\tilde{\mathbf{v}}_{0,j} + S(z)^{(1)}\,\tilde{\mathbf{v}}_{1,j} + S(z)\,\tilde{\mathbf{v}}_{2,j},\\ \mathbf{w}_{2,j} &= \frac{D_\Sigma(z)^{(2)}}{2!}\,\tilde{\mathbf{v}}_{0,j} + D_\Sigma(z)^{(1)}\,\tilde{\mathbf{v}}_{1,j} + D_\Sigma(z)\,\tilde{\mathbf{v}}_{2,j}.\end{aligned}$$

If we continue with the same procedure, we find

$$\begin{aligned}\mathbf{v}_{\kappa,j} &= \frac{S(z)^{(\kappa)}}{\kappa!}\,\tilde{\mathbf{v}}_{0,j} + \ldots + S(z)^{(1)}\,\tilde{\mathbf{v}}_{\kappa-1,j} + S(z)\,\tilde{\mathbf{v}}_{\kappa,j},\\ \mathbf{w}_{\kappa,j} &= \frac{D_\Sigma(z)^{(\kappa)}}{\kappa!}\,\tilde{\mathbf{v}}_{0,j} + \ldots + D_\Sigma(z)^{(1)}\,\tilde{\mathbf{v}}_{\kappa-1,j} + D_\Sigma(z)\,\tilde{\mathbf{v}}_{\kappa,j}\end{aligned}$$

and $\tilde{\mathbf{v}}_{\kappa,j}$, $\kappa \in \{1,\ldots,r_j-1\}$ are such that the following equations are satisfied

$$\begin{aligned}
\mathbf{0} &= N_\Sigma(z)^{(1)}\,\tilde{\mathbf{v}}_{0,j} + N_\Sigma(z)\,\tilde{\mathbf{v}}_{1,j},\\
\mathbf{0} &= \frac{N_\Sigma(z)^{(2)}}{2!}\,\tilde{\mathbf{v}}_{0,j} + N_\Sigma(z)^{(1)}\,\tilde{\mathbf{v}}_{1,j} + N_\Sigma(z)\,\tilde{\mathbf{v}}_{2,j},\\
&\vdots\\
\mathbf{0} &= \frac{N_\Sigma(z)^{(r_j-1)}}{(r_j-1)!}\,\tilde{\mathbf{v}}_{0,j} + \ldots + N_\Sigma(z)^{(1)}\,\tilde{\mathbf{v}}_{r_j-2,j} + N_\Sigma(z)\,\tilde{\mathbf{v}}_{r_j-1,j},
\end{aligned} \tag{20}$$

where $\tilde{\mathbf{v}}_{0,j} = U_R^{-1}(z)\,\mathbf{e}_{m-\mu+j}$.

To complete the proof, we show how to compute $\tilde{\mathbf{v}}_{\kappa,j}$, $\kappa \in \{1,\ldots,r_j-1\}$. The first equation of (20) can be written as

$$\begin{aligned}
\mathbf{0} &= U_L(z)^{(1)}E(z)U_R(z)\tilde{\mathbf{v}}_{0,j} + U_L(z)E(z)^{(1)}U_R(z)\tilde{\mathbf{v}}_{0,j} + U_L(z)E(z)U_R(z)^{(1)}\tilde{\mathbf{v}}_{0,j} + U_L(z)E(z)U_R(z)\tilde{\mathbf{v}}_{1,j}\\
&= U_L(z)\,E(z)\,\big(U_R(z)^{(1)}\,\tilde{\mathbf{v}}_{0,j} + U_R(z)\,\tilde{\mathbf{v}}_{1,j}\big),
\end{aligned}$$

since $E(z)\,U_R(z)\,\tilde{\mathbf{v}}_{0,j} = \mathbf{0}$ by construction and $E(z)^{(1)}\,U_R(z)\,\tilde{\mathbf{v}}_{0,j} = \mathbf{0}$ as z is a repeated zero. In this case, $U_R(z)^{(1)}\,\tilde{\mathbf{v}}_{0,j} + U_R(z)\,\tilde{\mathbf{v}}_{1,j}$ must be equal to $\mathbf{0}$, so that $\tilde{\mathbf{v}}_{1,j} = -U_R(z)^{-1}\,U_R(z)^{(1)}\,\tilde{\mathbf{v}}_{0,j}$. We will prove by induction that $\tilde{\mathbf{v}}_{\kappa,j} = -U_R^{-1}(z)\sum_{l=0}^{\kappa-1}\frac{U_R(z)^{(\kappa-l)}}{(\kappa-l)!}\,\tilde{\mathbf{v}}_{l,j}$, $\kappa \in \{1,\ldots,r_j-1\}$. To do so, let

$$\begin{aligned}
\boldsymbol{\psi}_1 &= U_R(z)^{(1)}\,\tilde{\mathbf{v}}_{0,j} + U_R(z)\,\tilde{\mathbf{v}}_{1,j},\\
\boldsymbol{\psi}_2 &= \frac{U_R(z)^{(2)}}{2!}\,\tilde{\mathbf{v}}_{0,j} + U_R(z)^{(1)}\,\tilde{\mathbf{v}}_{1,j} + U_R(z)\,\tilde{\mathbf{v}}_{2,j},\\
&\vdots\\
\boldsymbol{\psi}_{r_j-1} &= \sum_{l=0}^{r_j-1}\frac{U_R(z)^{(r_j-1-l)}}{(r_j-1-l)!}\,\tilde{\mathbf{v}}_{l,j}.
\end{aligned}$$

We prove that $\boldsymbol{\psi}_\kappa = \mathbf{0}$, $\kappa \in \{1,\ldots,r_j-1\}$. We have proved that $\boldsymbol{\psi}_1 = \mathbf{0}$. Let us assume that $\boldsymbol{\psi}_2 = \ldots = \boldsymbol{\psi}_{r_j-2} = \mathbf{0}$, so that

$$\begin{aligned}
\tilde{\mathbf{v}}_{2,j} &= -U_R^{-1}(z)\left(\frac{U_R(z)^{(2)}}{2!}\,\tilde{\mathbf{v}}_{0,j} + U_R(z)^{(1)}\,\tilde{\mathbf{v}}_{1,j}\right),\\
&\vdots\\
\tilde{\mathbf{v}}_{r_j-2,j} &= -U_R^{-1}(z)\sum_{l=0}^{r_j-3}\frac{U_R(z)^{(r_j-2-l)}}{(r_j-2-l)!}\,\tilde{\mathbf{v}}_{l,j},
\end{aligned}$$

and consider the last equation of (20). If it is written explicitly and taking into account that $E(z)\,U_R(z)\,\tilde{\mathbf{v}}_{0,j} = \mathbf{0}$ by construction and $E(z)^{(\kappa)}\,U_R(z)\,\tilde{\mathbf{v}}_{0,j} = \mathbf{0}$, $\kappa \in \{1,\ldots,r_j-1\}$ as z is a repeated zero, it follows that

$$\mathbf{0} = U_L(z)\,E(z)\,\boldsymbol{\psi}_{r_j-1} + \sum_{k=1}^{r_j-2}\left(\sum_{i=0}^{k}\frac{U_L(z)^{(i)}\,E(z)^{(k-i)}}{i!\,(k-i)!}\right)\boldsymbol{\psi}_{r_j-1-k}.$$

Since $\boldsymbol{\psi}_2 = \ldots = \boldsymbol{\psi}_{r_j-2} = \mathbf{0}$, then $\boldsymbol{\psi}_{r_j-1}$ must be equal to $\mathbf{0}$ to ensure that $U_L(z)\,E(z)\,\boldsymbol{\psi}_{r_j-1} = \mathbf{0}$, regardless of $E(z)$ being zero or not. Therefore

$$\tilde{\mathbf{v}}_{r_j-1,j} = -U_R^{-1}(z)\sum_{l=0}^{r_j-2}\frac{U_R(z)^{(r_j-1-l)}}{(r_j-1-l)!}\,\tilde{\mathbf{v}}_{l,j}.$$

Consequently, the r_j vectors of $\mathcal{V}_z^\star$ corresponding to the invariant factor $\epsilon_j(\lambda)$ are given by V_j as in (16). The same follows for every invariant factor $\epsilon_j(\lambda)$, $j \in \{1,\dots,\mu\}$ and a basis for $\mathcal{V}^\star$ is given by $\mathcal{V}^\star = \operatorname{im}[\, V_1 \;\; V_2 \;\; \dots \;\; V_\mu \,]$. Computing the vectors $\mathbf{w}_{\kappa,j}$, $\kappa \in \{1,\dots,r_j-1\}$, $j \in \{1,\dots,\mu\}$, the matrix W is constructed accordingly.

■

The following corollary shows that there is an alternative way to compute a basis for $\mathcal{V}^\star = \mathcal{V}_z^\star$ if $N_\Sigma(z) = O$.

Corollary 4.1 *If $N_\Sigma(z) = O$, then $\mathcal{V}^\star = \mathcal{V}_z^\star = \operatorname{im} V = \operatorname{im}[\, V_1 \;\; V_2 \;\; \dots \;\; V_m \,]$, where*

$$V_j = \begin{bmatrix} S(z)\,\tilde{\mathbf{v}}_{0,j} & S(z)^{(1)}\,\tilde{\mathbf{v}}_{0,j} & \dots & \frac{S(z)^{(r_j-1)}}{(r_j-1)!}\,\tilde{\mathbf{v}}_{0,j} \end{bmatrix}, \tag{21}$$

$$\tilde{\mathbf{v}}_{0,j} = U_R^{-1}(z)\,\mathbf{e}_j, \quad j \in \{1,\dots,m\}.$$

The matrix W for the computation of an associated friend $F = W\,V^\dagger$ that assigns the invariant zero structure of Σ is given by $W = [\, W_1 \;\; W_2 \;\; \dots \;\; W_m \,]$, where for all $j \in \{1,\dots,m\}$

$$W_j = \begin{bmatrix} D_\Sigma(z)\,\tilde{\mathbf{v}}_{0,j} & D_\Sigma(z)^{(1)}\,\tilde{\mathbf{v}}_{0,j} & \dots & \frac{D_\Sigma(z)^{(r_j-1)}}{(r_j-1)!}\,\tilde{\mathbf{v}}_{0,j} \end{bmatrix}.$$

Proof: If $N_\Sigma(z) = O$, then $E(z) = O$ and we have m invariant factors and z is a root of all of them. Thus, a basis matrix for the kernel of $E(z)$ is the identity matrix, so that the kernel of $N_\Sigma(z)$ is $U_R^{-1}(z)$ with $U_R^{-1}(z)\,\mathbf{e}_j$ corresponding to the invariant factor $\epsilon_j(\lambda)$. Consequently, $\tilde{\mathbf{v}}_{0,j} = U_R^{-1}(z)\,\mathbf{e}_j$, $j \in \{1,\dots,m\}$. The equation $\mathbf{0} = N_\Sigma(z)^{(1)}\,\tilde{\mathbf{v}}_{0,j} + N_\Sigma(z)\,\tilde{\mathbf{v}}_{1,j} = U_L(z)\,E(z)\left(U_R(z)^{(1)}\,\tilde{\mathbf{v}}_{0,j} + U_R(z)\,\tilde{\mathbf{v}}_{1,j}\right)$ holds true for every $\tilde{\mathbf{v}}_{1,j}$, because $E(z) = O$ and thus $\tilde{\mathbf{v}}_{1,j}$ can be taken equal to $\mathbf{0}$. Using the same arguments for the subsequent equations of (20), we can take $\tilde{\mathbf{v}}_{2,j} = \dots = \tilde{\mathbf{v}}_{r_j-1,j} = \mathbf{0}$ and it follows that the r_j vectors of $\mathcal{V}^\star$ corresponding to the invariant factor $\epsilon_j(\lambda)$ are given by V_j as in (21). The same follows for every invariant factor $\epsilon_j(\lambda)$, $j \in \{1,\dots,m\}$ and a basis for $\mathcal{V}^\star$ is given by $\mathcal{V}^\star = \operatorname{im}[\, V_1 \;\; V_2 \;\; \dots \;\; V_m \,]$. The matrix W is constructed accordingly. ■

Remark 4.3 For a SISO system (A,B,C) in the controller canonical form that has one repeated invariant zero z with multiplicity ℓ, we have the case of $E(z) = 0$ and $\mathcal{V}^\star = \mathcal{V}_z^\star = \operatorname{im} V_1$, where

$$V_1 = \begin{bmatrix} S(z) & S(z)^{(1)} & \dots & \frac{S(z)^{(\ell-1)}}{(\ell-1)!} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ z & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ z^{\ell-1} & (\ell-1)\,z^{\ell-2} & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ z^{n-1} & (n-1)\,z^{n-2} & \dots & \binom{n-1}{\ell-1}\,z^{n-\ell} \end{bmatrix},$$

see also [9].

4.2 Input-containing subspaces

In this section, we generalise Theorem 4 in [9] to the multivariable case. Since input-containing subspaces are the dual of output-nulling subspaces, we will first assume that the system is right-invertible, so that $\mathcal{Q}^\star = \mathcal{X}$. If $p \leq m$, so that $N_\Sigma(\lambda)$ is full row-rank, then the left null-space of $N_\Sigma(\lambda)$ is the origin and $\mathcal{Q}^\star = \mathcal{X}$.

Let

$$\begin{aligned}
\tilde{A} &= [\ \tilde{\boldsymbol{\alpha}}_{1,0}\ \ \dots\ \ \tilde{\boldsymbol{\alpha}}_{1,\nu_1-1}\ \ \tilde{\boldsymbol{\alpha}}_{2,0}\ \ \dots\ \ \tilde{\boldsymbol{\alpha}}_{2,\nu_2-1}\ \ \dots\ \ \tilde{\boldsymbol{\alpha}}_{m,0}\ \ \dots\ \ \tilde{\boldsymbol{\alpha}}_{m,\nu_m-1}\ ],\\
\tilde{B}^{-1} &= [\ \tilde{\boldsymbol{\beta}}_1\ \ \tilde{\boldsymbol{\beta}}_2\ \ \dots\ \ \tilde{\boldsymbol{\beta}}_m\ ],\\
C &= [\ \boldsymbol{\gamma}_{1,0}\ \ \dots\ \ \boldsymbol{\gamma}_{1,\nu_1-1}\ \ \boldsymbol{\gamma}_{2,0}\ \ \dots\ \ \boldsymbol{\gamma}_{2,\nu_2-1}\ \ \dots\ \ \boldsymbol{\gamma}_{m,0}\ \ \dots\ \ \boldsymbol{\gamma}_{m,\nu_m-1}\ ],
\end{aligned}$$

where $\tilde{\boldsymbol{\alpha}}_{j,k}$, $\tilde{\boldsymbol{\beta}}_j$, $\boldsymbol{\gamma}_{j,k}$ represent the columns of $\tilde{A}$, $\tilde{B}^{-1}$, C respectively.

Theorem 4.5 *Consider a right-invertible MIMO system Σ in the controller canonical form with ℓ distinct invariant zeros. Then $\mathcal{S}^\star = \ker Q$, where*

$$Q = \begin{bmatrix} \boldsymbol{\xi}_1^\top \\ \boldsymbol{\xi}_2^\top \\ \vdots \\ \boldsymbol{\xi}_\ell^\top \end{bmatrix} = \begin{bmatrix} \xi_{1,1,0} & \cdots & \xi_{1,1,\nu_1-1} & \xi_{1,2,0} & \cdots & \xi_{1,2,\nu_2-1} & \cdots & \xi_{1,m,0} & \cdots & \xi_{1,m,\nu_m-1} \\ \xi_{2,1,0} & \cdots & \xi_{2,1,\nu_1-1} & \xi_{2,2,0} & \cdots & \xi_{2,2,\nu_2-1} & \cdots & \xi_{2,m,0} & \cdots & \xi_{2,m,\nu_m-1} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \xi_{\ell,1,0} & \cdots & \xi_{\ell,1,\nu_1-1} & \xi_{\ell,2,0} & \cdots & \xi_{\ell,2,\nu_2-1} & \cdots & \xi_{\ell,m,0} & \cdots & \xi_{\ell,m,\nu_m-1} \end{bmatrix},$$

$$\begin{aligned}
&\xi_{i,j,k} = -\boldsymbol{\chi}_i^\top\Big(D\,\tilde{\boldsymbol{\beta}}_j\,z_i^{\nu_j-k-1} + \textstyle\sum_{l=k+1}^{\nu_j-1}\big(\boldsymbol{\gamma}_{j,l} - D\,\tilde{B}^{-1}\,\tilde{\boldsymbol{\alpha}}_{j,l}\big)z_i^{l-k-1}\Big), \quad && k = 0,\dots,\nu_j-2\\
&\xi_{i,j,\nu_j-1} = -\boldsymbol{\chi}_i^\top\,D\,\tilde{\boldsymbol{\beta}}_j, && j \in \{1,\dots,m\}
\end{aligned} \tag{22}$$

and $\boldsymbol{\chi}_i^\top$ is a basis row vector for the left null-space of $N_\Sigma(z_i)$.

Proof: Since the invariant zeros are distinct, let us consider a vector

$$[\ \boldsymbol{\xi}_i^\top\ \ \boldsymbol{\chi}_i^\top\ ] = [\ \xi_{i,1,0}\ \ \cdots\ \ \xi_{i,1,\nu_1-1}\ \ \xi_{i,2,0}\ \ \cdots\ \ \xi_{i,2,\nu_2-1}\ \ \cdots\ \ \xi_{i,m,0}\ \ \cdots\ \ \xi_{i,m,\nu_m-1}\ \ \boldsymbol{\chi}_i^\top\ ]$$

in the left null-space of $P_\Sigma(z_i)$. Imposing

$$[\ \boldsymbol{\xi}_i^\top\ \ \boldsymbol{\chi}_i^\top\ ]\,P_\Sigma(z_i) = \mathbf{0}^\top, \tag{23}$$

it follows for $j \in \{1,\dots,m\}$

$$\begin{aligned}
\xi_{i,j,\nu_j-1} &= -\boldsymbol{\chi}_i^\top\,D\,\tilde{\boldsymbol{\beta}}_j,\\
\xi_{i,j,\nu_j-2} &= -\boldsymbol{\chi}_i^\top\,\big(D\,\tilde{\boldsymbol{\beta}}_j\,z_i + \boldsymbol{\gamma}_{j,\nu_j-1} - D\,\tilde{B}^{-1}\,\tilde{\boldsymbol{\alpha}}_{j,\nu_j-1}\big),\\
&\vdots\\
\xi_{i,j,0} &= -\boldsymbol{\chi}_i^\top\left(D\,\tilde{\boldsymbol{\beta}}_j\,z_i^{\nu_j-1} + \sum_{l=1}^{\nu_j-1}\big(\boldsymbol{\gamma}_{j,l} - D\,\tilde{B}^{-1}\,\tilde{\boldsymbol{\alpha}}_{j,l}\big)z_i^{l-1}\right),\\
0 &= -z_i\,\xi_{i,j,0} + [\ \xi_{i,1,\nu_1-1}\ \ \dots\ \ \xi_{i,m,\nu_m-1}\ ]\,\tilde{\boldsymbol{\alpha}}_{j,0} + \boldsymbol{\chi}_i^\top\,\boldsymbol{\gamma}_{j,0}.
\end{aligned}$$

From (23), we also have that $\left[\ \boldsymbol{\xi}_i^\top \ \ \boldsymbol{\chi}_i^\top\ \right] P_\Sigma(z_i) \begin{bmatrix} S(z_i) \\ D_\Sigma(z_i) \end{bmatrix} = \mathbf{0}^\top$ and from (15), we obtain $\boldsymbol{\chi}_i^\top N_\Sigma(z_i) = \mathbf{0}^\top$. Computing im $\boldsymbol{\chi}_i^\top = \left(\ker\ N_\Sigma(z_i)^\top\right)^\top$ and $\boldsymbol{\xi}_i^\top$, $i \in \{1,\ldots,\ell\}$, using (22), we obtain

$$[\ Q \ \ P\ ] = \left[\begin{array}{c|c} \boldsymbol{\xi}_1^\top & \boldsymbol{\chi}_1^\top \\ \boldsymbol{\xi}_2^\top & \boldsymbol{\chi}_2^\top \\ \vdots & \vdots \\ \boldsymbol{\xi}_\ell^\top & \boldsymbol{\chi}_\ell^\top \end{array}\right] = \left[\begin{array}{ccccccc|c} \xi_{1,1,0} & \ldots & \xi_{1,1,\nu_1-1} & \ldots & \xi_{1,m,0} & \ldots & \xi_{1,m,\nu_m-1} & \boldsymbol{\chi}_1^\top \\ \xi_{2,1,0} & \ldots & \xi_{2,1,\nu_1-1} & \ldots & \xi_{2,m,0} & \ldots & \xi_{2,m,\nu_m-1} & \boldsymbol{\chi}_2^\top \\ \vdots & \ldots & \vdots & \ldots & \vdots & \ldots & \vdots & \vdots \\ \xi_{\ell,1,0} & \ldots & \xi_{\ell,1,\nu_1-1} & \ldots & \xi_{\ell,m,0} & \ldots & \xi_{\ell,m,\nu_m-1} & \boldsymbol{\chi}_\ell^\top \end{array}\right]$$

and therefore $\mathcal{S}^\star = \ker Q$.

■

Remark 4.4 If D is full row-rank and since $D_\Sigma(\lambda)$ is column proper, then the system has n invariant zeros and Q is square and nonsingular, so that $\mathcal{S}^\star = \{0\}$. If $D \neq O$ and not full row-rank and if $N_\Sigma(\lambda)$ has column degrees $n_1,\, n_2, \ldots, n_m$, then $n_j \le \nu_j$ and $\xi_{i,j,n_j} = \ldots = \xi_{i,j,\nu_j-1} = 0$, $i \in \{1,\ldots,\ell\}$, $j \in \{1,\ldots,m\}$. If $N_\Sigma(\lambda)$ is column proper, then $n_1 + \ldots + n_m = \ell$ and the determinant of the matrix with entries $\xi_{i,j,0},\ldots,\xi_{i,j,n_j-1}$, $i \in \{1,\ldots,\ell\}$, $j \in \{1,\ldots,m\}$ is nonzero. Consequently,

$$\mathcal{S}^\star = \text{im}\left(\text{diag}\left\{\begin{bmatrix} O \\ I_{\nu_1-n_1} \end{bmatrix},\ldots,\begin{bmatrix} O \\ I_{\nu_m-n_m} \end{bmatrix}\right\}\right).$$

If $N_\Sigma(\lambda)$ is not column proper, then

$$\mathcal{S}^\star = \ker Q = \ker \begin{bmatrix} \xi_{1,1,0} & \ldots & \xi_{1,1,n_1-1} & 0 & \ldots & 0 & \ldots & \xi_{1,m,0} & \ldots & \xi_{1,m,n_m-1} & 0 & \ldots & 0 \\ \xi_{2,1,0} & \ldots & \xi_{2,1,n_1-1} & 0 & \ldots & 0 & \ldots & \xi_{2,m,0} & \ldots & \xi_{2,m,n_m-1} & 0 & \ldots & 0 \\ \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \ldots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\ \xi_{\ell,1,0} & \ldots & \xi_{\ell,1,n_1-1} & 0 & \ldots & 0 & \ldots & \xi_{\ell,m,0} & \ldots & \xi_{\ell,m,n_m-1} & 0 & \ldots & 0 \end{bmatrix}.$$

Remark 4.5 For a strictly proper MIMO system $\Sigma = (A,B,C)$, and if $C_\Sigma(\lambda)$ has column degrees $c_1,\ldots,c_m$, there holds $c_j < \nu_j$ and $\xi_{i,j,c_j} = \ldots = \xi_{i,j,\nu_j-1} = 0$, $i \in \{1,\ldots,\ell\}$, $j \in \{1,\ldots,m\}$. If $C_\Sigma(\lambda)$ is column proper, then $c_1 + \ldots + c_m = \ell$ and the determinant of the matrix with entries the nonzero $\xi_{i,j,0},\ldots,\xi_{i,j,c_j-1}$ is nonzero. Consequently,

$$\mathcal{S}^\star = \text{im}\left(\text{diag}\left\{\begin{bmatrix} O \\ I_{\nu_1-c_1} \end{bmatrix},\ldots,\begin{bmatrix} O \\ I_{\nu_m-c_m} \end{bmatrix}\right\}\right).$$

If $C_\Sigma(\lambda)$ is not column proper, then

$$\mathcal{S}^\star = \ker Q = \ker \begin{bmatrix} \xi_{1,1,0} & \ldots & \xi_{1,1,c_1-1} & 0 & \ldots & 0 & \ldots & \xi_{1,m,0} & \ldots & \xi_{1,m,c_m-1} & 0 & \ldots & 0 \\ \xi_{2,1,0} & \ldots & \xi_{2,1,c_1-1} & 0 & \ldots & 0 & \ldots & \xi_{2,m,0} & \ldots & \xi_{2,m,c_m-1} & 0 & \ldots & 0 \\ \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \ldots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\ \xi_{\ell,1,0} & \ldots & \xi_{\ell,1,c_1-1} & 0 & \ldots & 0 & \ldots & \xi_{\ell,m,0} & \ldots & \xi_{\ell,m,c_m-1} & 0 & \ldots & 0 \end{bmatrix},$$

where for all $i \in \{1,\ldots,\ell\}$, $k \in \{0,\ldots,c_j-1\}$, $j \in \{1,\ldots,m\}$

$$\xi_{i,j,k} = -\boldsymbol{\chi}_i^\top \sum_{l=k+1}^{c_j} \boldsymbol{\gamma}_{j,l}\, z_i^{l-k-1},\ \text{im}\ \boldsymbol{\chi}_i^\top = \left(\ker\ C_\Sigma(z_i)^\top\right)^\top.$$

There is an alternative way to compute $\xi_{i,j,k}$. Notice that (23) becomes $\boldsymbol{\xi}_i^\top (A - z_i\, I_n) + \boldsymbol{\chi}_i^\top C = \mathbf{0}^\top$, $\boldsymbol{\xi}_i^\top B = \mathbf{0}^\top$. From the second equation, it follows that $[\ \xi_{i,1,\nu_1-1} \ \ \dots \ \ \xi_{i,m,\nu_m-1}\ ]\, \tilde{B} = \mathbf{0}^\top$, which implies $\xi_{i,1,\nu_1-1} = \ldots = \xi_{i,m,\nu_m-1} = 0$, so that

$$[\ \xi_{i,1,0} \ \ \dots \ \ \xi_{i,1,\nu_1-2} \ \ \dots \ \ \xi_{i,m,0} \ \ \dots \ \ \xi_{i,m,\nu_m-2}\ ] = -\boldsymbol{\chi}_i^\top C\, P(z_i)^\dagger,$$

where $P(z_i)$ is the full row-rank matrix which contains the remaining rows of $A - z_i\, I_n$ if we remove the ν_1, $\nu_1 + \nu_2, \ldots, n$-th rows.

Remark 4.6 For a SISO system (A, B, C, D) with ℓ invariant zeros, if $D \neq 0$, then $\mathcal{S}^\star = \{0\}$ and if $D = 0$, then $\mathcal{S}^\star = \operatorname{im} \left[\begin{smallmatrix} O \\ I_{n-\ell} \end{smallmatrix} \right]$, see also [9].

5 Numerical examples

Example 5.1 Consider the controller canonical form

$$A = \left[\begin{array}{cccc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ -6 & -5 & -4 & -3 & -2 & -1 \end{array} \right], \qquad B = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ \hline 0 & 0 \\ 0 & 1 \end{array} \right].$$

We have $\nu_1 = 4$, $\nu_2 = 2$, so that

$$S(\lambda) = \left[\begin{array}{c|c} 1 & 0 \\ \lambda & 0 \\ \lambda^2 & 0 \\ \lambda^3 & 0 \\ \hline 0 & 1 \\ 0 & \lambda \end{array} \right], \qquad D_\Sigma(\lambda) = \left[\begin{array}{cc} \lambda^4 - 4\lambda^3 - 3\lambda^2 - 2\lambda - 1 & -6\lambda - 5 \\ 3\lambda^3 + 4\lambda^2 + 5\lambda + 6 & \lambda^2 + \lambda + 2 \end{array} \right].$$

(i) Let

$$C = \left[\begin{array}{cccc|cc} 1 & 2 & 1 & 1 & 1 & 1 \\ \hline -1 & -2 & -1 & 1 & 1 & 1 \end{array} \right].$$

We have $C_\Sigma(\lambda) = C\, S(\lambda) = \left[\begin{smallmatrix} \lambda^3+\lambda^2+2\lambda+1 & \lambda+1 \\ \lambda^3-\lambda^2-2\lambda-1 & \lambda+1 \end{smallmatrix} \right] = \left[\begin{smallmatrix} 1 & 0 \\ 2\lambda^2+4\lambda+3 & 1 \end{smallmatrix} \right] \left[\begin{smallmatrix} 1 & 0 \\ 0 & (\lambda+1)^3 \end{smallmatrix} \right] \left[\begin{smallmatrix} \lambda^3+\lambda^2+2\lambda+1 & \lambda+1 \\ -2(\lambda^2+2) & -2 \end{smallmatrix} \right] = U_L(\lambda)\, E(\lambda)\, U_R(\lambda)$. The system has one invariant zero $z = -1$ with multiplicity 3 (one invariant factor with degree 3). We compute

$$\tilde{\mathbf{v}}_0 = U_R^{-1}(-1)\, \mathbf{e}_2 = \left[\begin{array}{cc} -1 & 0 \\ 3 & -1/2 \end{array} \right] \left[\begin{array}{c} 0 \\ 1 \end{array} \right] = \left[\begin{array}{c} 0 \\ 1 \end{array} \right], \qquad \tilde{\mathbf{v}}_1 = -U_R^{-1}(-1)\, U_R(z)^{(1)}\, \tilde{\mathbf{v}}_0 = \left[\begin{array}{c} 1 \\ -3 \end{array} \right],$$

$$\tilde{\mathbf{v}}_2 = -U_R^{-1}(-1) \left(\frac{U_R(z)^{(2)}}{2!}\, \tilde{\mathbf{v}}_0 + U_R(z)^{(1)}\, \tilde{\mathbf{v}}_1 \right) = \left[\begin{array}{c} 0 \\ 2 \end{array} \right],$$

so that

$$\begin{bmatrix} \mathbf{v}_0 \\ \mathbf{w}_0 \end{bmatrix} = \begin{bmatrix} S(-1)\,\tilde{\mathbf{v}}_0 \\ D_\Sigma(-1)\,\tilde{\mathbf{v}}_0 \end{bmatrix}, \quad \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{w}_1 \end{bmatrix} = \begin{bmatrix} S(-1)^{(1)}\,\tilde{\mathbf{v}}_0 + S(-1)\,\tilde{\mathbf{v}}_1 \\ D_\Sigma(-1)^{(1)}\,\tilde{\mathbf{v}}_0 + D_\Sigma(-1)\,\tilde{\mathbf{v}}_1 \end{bmatrix},$$

$$\begin{bmatrix} \mathbf{v}_2 \\ \mathbf{w}_2 \end{bmatrix} = \begin{bmatrix} \frac{S(-1)^{(2)}}{2!}\,\tilde{\mathbf{v}}_0 + S(-1)^{(1)}\,\tilde{\mathbf{v}}_1 + S(-1)\,\tilde{\mathbf{v}}_2 \\ \frac{D_\Sigma(-1)^{(2)}}{2!}\,\tilde{\mathbf{v}}_0 + D_\Sigma(-1)^{(1)}\,\tilde{\mathbf{v}}_1 + D_\Sigma(-1)\,\tilde{\mathbf{v}}_2 \end{bmatrix}$$

and

$$\left[\begin{array}{c} V \\ \hline W \end{array}\right] = \left[\begin{array}{ccc} \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_2 \\ \hline \mathbf{w}_0 & \mathbf{w}_1 & \mathbf{w}_2 \end{array}\right] = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 3 \\ 1 & -3 & 2 \\ -1 & 4 & -5 \\ \hline 1 & -6 & 8 \\ 2 & -5 & 14 \end{array}\right].$$

Then $\mathcal{V}^\star = \operatorname{im} V$ and a friend is computed by $F = W\,V^\dagger = \frac{1}{54}\begin{bmatrix} -25 & 28 & -31 & 34 & 10 & -44 \\ 179 & -86 & -7 & 100 & 4 & -104 \end{bmatrix}$.
(ii) Let

$$C = \left[\begin{array}{cccc|cc} 1 & 3 & 3 & 1 & 1 & 1 \\ \hline -1 & -3 & -3 & -1 & 1 & 1 \end{array}\right].$$

We have $C_\Sigma(\lambda) = C\,S(\lambda) = \begin{bmatrix} (\lambda+1)^3 & \lambda+1 \\ -(\lambda+1)^3 & \lambda+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda+1 & 0 \\ 0 & (\lambda+1)^3 \end{bmatrix}\begin{bmatrix} (\lambda+1)^2 & 1 \\ -2 & 0 \end{bmatrix} = U_L(\lambda)\,E(\lambda)\,U_R(\lambda)$ with $U_R^{-1}(-1) = \begin{bmatrix} 0 & -1/2 \\ 1 & 0 \end{bmatrix}$. We compute $\tilde{\mathbf{v}}_{0,1} = U_R^{-1}(-1)\,\mathbf{e}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\tilde{\mathbf{v}}_{0,2} = U_R^{-1}(-1)\,\mathbf{e}_2 = \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}$ and therefore

$$\left[\begin{array}{c} V \\ \hline W \end{array}\right] = \left[\begin{array}{c|ccc} S(-1)\,\tilde{\mathbf{v}}_{0,1} & S(-1)\,\tilde{\mathbf{v}}_{0,2} & S(-1)^{(1)}\,\tilde{\mathbf{v}}_{0,2} & \frac{S(-1)^{(2)}}{2!}\,\tilde{\mathbf{v}}_{0,2} \\ \hline D_\Sigma(-1)\,\tilde{\mathbf{v}}_{0,1} & D_\Sigma(-1)\,\tilde{\mathbf{v}}_{0,2} & D_\Sigma(-1)^{(1)}\,\tilde{\mathbf{v}}_{0,2} & \frac{D_\Sigma(-1)^{(2)}}{2!}\,\tilde{\mathbf{v}}_{0,2} \end{array}\right] = \left[\begin{array}{c|ccc} 0 & -1/2 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & -1/2 & 1 & -1/2 \\ 0 & 1/2 & -3/2 & 3/2 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 1 & -3/2 & 6 & -15/2 \\ 2 & -1 & -3 & 5/2 \end{array}\right].$$

Then, $\mathcal{V}^\star = \operatorname{im} V$ and a friend is computed by $F = W\,V^\dagger = \begin{bmatrix} 3/4 & 9/4 & -3/4 & -21/4 & 1/2 & -1/2 \\ 21/5 & -2/5 & -7/5 & 6/5 & 1 & -1 \end{bmatrix}$.
(iii) Let $C = [\,1 \;\; 2 \;\; 1 \;\; 1 \;\; 1 \;\; 1\,]$. The system has no invariant zeros, which implies that $\mathcal{V}_z^\star$ is the origin, and the dimension of $\mathcal{V}^\star = \mathcal{R}^\star$ is 5. A basis vector for $\ker C_\Sigma(\lambda)$ is $\tilde{\mathbf{v}}(\lambda) = \begin{bmatrix} -\lambda-1 \\ \lambda^3+\lambda^2+2\lambda+1 \end{bmatrix}$, so that $\mathbf{v}(\lambda) = S(\lambda)\,\tilde{\mathbf{v}}(\lambda) = [\,-\lambda-1 \;\; -\lambda\,(\lambda+1) \;\; -\lambda^2\,(\lambda+1) \;\; -\lambda^3\,(\lambda+1) \;\; \lambda^3+\lambda^2+2\lambda+1 \;\; \lambda\,(\lambda^3+\lambda^2+2\lambda+1)\,]^\top$, $\mathbf{w}(\lambda) = D_\Sigma(\lambda)\,\tilde{\mathbf{v}}(\lambda) = \begin{bmatrix} -\lambda^5-3\lambda^4-4\lambda^3-12\lambda^2-13\lambda-4 \\ \lambda^5-\lambda^4-2\lambda^3-4\lambda^2-6\lambda-4 \end{bmatrix}$. Assume that we are interested in constructing a friend for $\mathcal{V}^\star = \mathcal{R}^\star$ that assigns one repeated eigenvalue $\lambda = 0$. Then we compute

$$\left[\begin{array}{c} V \\ \hline W \end{array}\right] = \left[\begin{array}{ccccc} \mathbf{v}(0) & \mathbf{v}(0)^{(1)} & \frac{\mathbf{v}(0)^{(2)}}{2!} & \frac{\mathbf{v}(0)^{(3)}}{3!} & \frac{\mathbf{v}(0)^{(4)}}{4!} \\ \hline \mathbf{w}(0) & \mathbf{w}(0)^{(1)} & \frac{\mathbf{w}(0)^{(2)}}{2!} & \frac{\mathbf{w}(0)^{(3)}}{3!} & \frac{\mathbf{w}(0)^{(4)}}{4!} \end{array}\right] = \left[\begin{array}{ccccc} -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 1 \\ \hline -4 & -13 & -12 & -4 & -3 \\ -4 & -6 & -4 & -2 & -1 \end{array}\right].$$

A basis matrix for $\mathcal{V}^\star = \mathcal{R}^\star$ is $\operatorname{im} V$ and an associated friend is computed by $F = W\,V^\dagger = \left[\begin{smallmatrix} 2 & 3 & -1 & -1 & -2 & -4 \\ 10/3 & -1/3 & 1/3 & -2/3 & -2/3 & -5/3 \end{smallmatrix}\right]$.

Example 5.2 Consider the system $\Sigma = (A, B, C, D)$ in the controller canonical form

$$A = \left[\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -5 & -4 & -3 & -2 & -1 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -7 & -6 \end{array}\right], \qquad B = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ \hline 0 & 0 \\ 0 & 1 \end{array}\right],$$

$$C = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & -1 & 0 \\ -1 & 1 & 1 & 1 & 1 \end{array}\right] = \left[\, \boldsymbol{\gamma}_{1,0} \;\; \boldsymbol{\gamma}_{1,1} \;\; \boldsymbol{\gamma}_{1,2} \mid \boldsymbol{\gamma}_{2,0} \;\; \boldsymbol{\gamma}_{2,1} \,\right], \qquad D = \left[\begin{array}{cc} 0 & 0 \\ 0 & 3 \end{array}\right],$$

We have $\nu_1 = 3$, $\nu_2 = 2$, so that

$$S(\lambda) = \left[\begin{array}{c|c} 1 & 0 \\ \lambda & 0 \\ \lambda^2 & 0 \\ \hline 0 & 1 \\ 0 & \lambda \end{array}\right], \qquad D_\Sigma(\lambda) = \left[\begin{array}{cc} \lambda^3 + 3\lambda^2 + 4\lambda + 5 & \lambda + 2 \\ 0 & \lambda^2 + 6\lambda + 7 \end{array}\right].$$

Let also

$$\tilde{A} = \left[\begin{array}{ccccc} -5 & -4 & -3 & -2 & -1 \\ 0 & 0 & 0 & -7 & -6 \end{array}\right] = \left[\, \tilde{\boldsymbol{\alpha}}_{1,0} \;\; \tilde{\boldsymbol{\alpha}}_{1,1} \;\; \tilde{\boldsymbol{\alpha}}_{1,2} \;\; \tilde{\boldsymbol{\alpha}}_{2,0} \;\; \tilde{\boldsymbol{\alpha}}_{2,1} \,\right], \quad \tilde{B}^{-1} = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right] = \left[\, \tilde{\boldsymbol{\beta}}_1 \;\; \tilde{\boldsymbol{\beta}}_2 \,\right].$$

We compute $N_\Sigma(\lambda) = C_\Sigma(\lambda) + D\,D_\Sigma(\lambda) = \left[\begin{smallmatrix} 1 & -1 \\ \lambda^2+\lambda-1 & 3\lambda^2+19\lambda+22 \end{smallmatrix}\right]$, which is not column proper. The invariant zeros of the system are given by the roots of $\det N_\Sigma(\lambda)$, i.e., $z_1 = -3/2$ and $z_2 = -7/2$.

We compute $\operatorname{im} \boldsymbol{\chi}_1^\top = \left(\ker N_\Sigma(z_1)^\top\right)^\top = \operatorname{im} [\, 1 \;\; 4 \,]$, $\operatorname{im} \boldsymbol{\chi}_2^\top = \left(\ker N_\Sigma(z_2)^\top\right)^\top = \operatorname{im} [\, -31/4 \;\; 1 \,]$ and $Q = \left[\begin{smallmatrix} \boldsymbol{\xi}_1^\top \\ \boldsymbol{\xi}_2^\top \end{smallmatrix}\right] = \left[\begin{smallmatrix} \xi_{1,1,0} & \xi_{1,1,1} & \xi_{1,1,2} & \xi_{1,2,0} & \xi_{1,2,1} \\ \xi_{2,1,0} & \xi_{2,1,1} & \xi_{2,1,2} & \xi_{2,2,0} & \xi_{2,2,1} \end{smallmatrix}\right]$ using (22). Consequently,

$$\mathcal{S}^\star = \ker Q = \ker \left[\begin{array}{ccccc} 2 & -4 & 0 & -58 & -12 \\ 5/2 & -1 & 0 & -17/2 & -3 \end{array}\right] = \operatorname{im} \left[\begin{smallmatrix} -15 & 0 & 0 \\ -8 & -3 & 0 \\ 0 & 0 & 1 \\ 5 & 0 & 0 \\ -24 & 1 & 0 \end{smallmatrix}\right].$$

Concluding remarks

In this paper, we generalised the results of [9] for proper multivariable systems and employed the framework of [10]-[13], based on the Rosenbrock matrix system pencil, for the computation of the output-nulling, reachability, stabilisability subspaces and their duals. It was shown that this technique enables us to obtain particularly elegant and useful expressions for the basis matrices for the aforementioned subspaces.

We also showed how the computation of output-nulling subspaces can be simplified if the numerator matrix $N_\Sigma(\lambda)$ is the zero matrix for an invariant zero. The computation of output-nulling reachability subspaces was shown explicitly and considering the defective case as well. The results were exploited for the computation of associated friends of output-nulling and reachability subspaces.

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