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QUASIORDERS, TOLERANCE RELATIONS AND CORRESPONDING "PARTITIONS"

Abstract

The paper deals with a generalization of the notion of partition for wider classes of binary relations than equivalences: for quasiorders and tolerance relations. The counterpart of partition for the quasiorders is based on a generalization of the notion of equivalence class while it is shown that such a generalization does not work in case of tolerances. Some results from [5] are proved in a much more simple way. The third kind of "partition" corresponding to tolerances, not occurring in [5], is introduced.

Keywords: partition, quasiorder, tolerance relation

1. Introduction

The purpose of the paper is to develop a little bit a theory of partitions presented in [5]. We shall consider here the notions of corresponding "partitions" only for two cases: quasiorderings and tolerance relations. The method leading to obtain the class of *quasipartitions*, i.e., the "partitions" corresponding to quasiorders, applied in [5] is quite general, however it does not work for any "well" defined class of binary relations. For example, it cannot be applied just in case of tolerance relations, as we will show it here. The method itself is a little bit complicated and we give up of using it in this paper in case of quasiorderings. The results for that class of relations will be presented directly in a simpler way than in [5]. A completely different but similarly complex way for establishing a notion of partition corresponding to tolerance relations was used in [5]. According to this approach, two different kinds of "partitions" for tolerances were obtained. We shall define here one of them, containing the "partitions" called *tolerance coverings*, also directly, in a simple way. As the class of all equivalence relations is the intersection of two classes: quasiorders and tolerances, so the class of ordinary partitions is the intersection of all the quasipartitions and tolerance coverings. Moreover, a third kind of "partition" corresponding to tolerances will be presented.

2. Quasiorders and quasipartitions

We start from an equivalent definition of quasipartition to the one introduced in [5].

Definition. Let A be any set and $\mathcal{R} \subseteq P(A)$. The family $\mathcal{R}$ is called a *quasipartition* of A iff

(1) for any $a \in A$ there exists the least element in the poset $< \mathcal{R}_a, \subseteq >$, where $\mathcal{R}_a = \{X \in \mathcal{R} : a \in X\}$,

(2) $\forall X \in \mathcal{R}\ \exists a \in A, X = |a|_{\mathcal{R}}$, where for any $b \in A$, $|b|_{\mathcal{R}}$ is the least element in $< \mathcal{R}_b, \subseteq >$. (If $\mathcal{R}$ is clear from a context then the notation $|b|$ is used.)

Notice that the condition (2) implies that for all $X \in \mathcal{R},\ X \neq \emptyset$, while (1) implies that $A \subseteq \bigcup \mathcal{R}$. The empty set ($\mathcal{R} = \emptyset$) is the only quasipartition of empty set ($A = \emptyset$).

The obvious example of a quasipartition of A is any ordinary partition of A. Given a partition $\mathcal{R}$ of A, for any $a \in A,\ \mathcal{R}_a = \{|a|\}$, where $|a|$ is the unique element of the partition to which a belongs. Another example of a quasipartition of A is any closure system $\mathcal{C}$ of A (that is a family of subsets of A closed on any intersection) such that $\mathcal{C} = \{C(\{a\}) : a \in A\}$, where the mapping C is the corresponding closure operation (that is, given any $X \subseteq A,\ C(X) = \bigcap \{Y \in \mathcal{C} : X \subseteq Y\}$). Here for each $a \in A,\ |a| = C(\{a\})$. In particular, the family of all principal filters $[a) = \{x \in A : a \leq x\}$ in a complete lattice $(A, \leq)$ is a quasipartition of A. Here for any $X \subseteq A,\ C(X) = [\inf X)$.

Proposition 2 below shows a general form of a quasipartition. It needs to consider the *quasiorders* (sometimes called *preorders*), that is reflexive and transitive binary relations. The class $QOrd(A)$ of all quasiorders defined on A forms a complete lattice $(QOrd(A), \subseteq)$ such that for any nonempty $\Theta \subseteq QOrd(A)$: $inf\Theta = \bigcap\Theta$ and $sup\Theta = \overline{\bigcup\Theta}$, where for any binary relation $r \subseteq A \times A$, $\overline{r}$ is the transitive closure of r.

For any quasiorder ρ defined on a set A the following obvious fact is useful.

FACT 1. *$\forall x, y \in A$: $x\rho y$ iff $(x] \subseteq (y]$, where for any $a \in A$, $(a] = \{x \in A : x\rho a\}$.*

PROPOSITION 2. *For any reflexive and transitive relation ρ on a set A, the family $A/\rho = \{(x] : x \in A\}$ is a quasipartition of A in which for each $a \in A$, $|a| = (a]$.*

PROOF. Notice that for any $a \in A$: $(A/\rho)_a = \{(x] : a\rho x\}$. So from reflexivity of ρ we have $(a] \in (A/\rho)_a$, while from transitivity it follows that $(a] \subseteq (x]$ for each $x \in A$ such that $a\rho x$. Hence $(a]$ is the least element in $(A/\rho)_a$, that is the condition (1) of the definition of quasipartition is satisfied. Obviously, the condition (2) follows from the very definition of (A/ρ). □

Given a quasiorder ρ, the quasipartition A/ρ will be called a *quotient set of A with respect to the quasiorder ρ.*

Now one may show that for any quasipartition $\mathcal{R}$ of A, the relation $\rho_{\mathcal{R}}$ defined on A by

$$x\rho_{\mathcal{R}}y \text{ iff } \mathcal{R}_y \subseteq \mathcal{R}_x \text{ iff } \forall X \in \mathcal{R}(y \in X \Rightarrow x \in X),$$

is reflexive and transitive. The relation $\rho_{\mathcal{R}}$ will be called a *quasiorder determined by the quasipartition $\mathcal{R}$.*

LEMMA 3. *For any quasipartition $\mathcal{R}$ of A, for any $x, y \in A$:*
(i) $x\rho_{\mathcal{R}}y$ iff $|x| \subseteq |y|$,
(ii) $(x] = |x|$, where $(x] = \{a \in A : a\rho_{\mathcal{R}}x\}$.

PROOF. For (i). $(\Rightarrow)$: Suppose that $\mathcal{R}_y \subseteq \mathcal{R}_x$. Then $|y| \in \mathcal{R}_x$, therefore $|x| \subseteq |y|$.

$(\Leftarrow)$: Assume that $|x| \subseteq |y|$ and choose a $Z \in \mathcal{R}_y$. So $Z \in \mathcal{R}$ and $|y| \subseteq Z$. Hence and from the assumption it follows that $|x| \subseteq Z$, that is $x \in Z$ and finally $Z \in \mathcal{R}_x$.

For (ii). $(\subseteq)$: Let $a \in (x]$. Then $|a| \subseteq |x|$ by (i) so $a \in |x|$.

$(\supseteq)$: Let $a \in |x|$. Then $|x| \in \mathcal{R}_a$, therefore $|a| \subseteq |x|$ which implies that $a \in (x]$ by (i). $\Box$

Lemma 4. *Any quasiorder ρ on a set A is determined by its quotient set: $\rho = \rho_{(A/\rho)}$. Any quasipartition $\mathcal{R}$ of a set A is the quotient set of A with respect to the quasiorder determined by itself: $\mathcal{R} = A/\rho_{\mathcal{R}}$.*

Proof. Assume that ρ is any reflexive and transitive relation on A. From Lemma 3(i) it follows that $x\rho_{(A/\rho)}y$ iff $|x|_{(A/\rho)} \subseteq |y|_{(A/\rho)}$. However, in view of Proposition 2 we have $|a|_{(A/\rho)} = (a]$, where $(a] = \{u \in A : u\rho a\}$. Thus, $x\rho_{(A/\rho)}y$ iff $(x] \subseteq (y]$ iff $x\rho y$, by Fact 1.

Now, let $\mathcal{R}$ be any quasipartition of A. We show that $\mathcal{R} = \{(a] : a \in A\}$, where for each $a \in A$: $(a] = \{x \in A : x\rho_{\mathcal{R}}a\}$.

$(\subseteq)$: Let $Z \in \mathcal{R}$. Then by condition (2) of the definition of a quasipartition, there exists an $a \in A$ such that $Z = |a|$. So from Lemma 3(ii) it follows that $Z = (a]$.

$(\supseteq)$: Due to Lemma 3(ii), any set $(a] \in A/\rho_{\mathcal{R}}$ is the least element in $\mathcal{R}_a$, so it belongs to $\mathcal{R}$. $\Box$

Definition. For any quasipartitions $\mathcal{R}_1$ and $\mathcal{R}_2$ of a set A, we say that $\mathcal{R}_1$ is a *refinement* of $\mathcal{R}_2$, in symbols, $\mathcal{R}_1 \leq \mathcal{R}_2$ iff for all $a \in A$, $|a|_{\mathcal{R}_1} \subseteq |a|_{\mathcal{R}_2}$.

It is easy to show that the relation $\leq$ is a partial ordering on the family $QPart(A)$ of all quasipartitions of A.

Proposition 5. *The mapping $\phi : QOrd(A) \longrightarrow QPart(A)$ defined by $\phi(\rho) = A/\rho$, is an isomorphism of the complete lattices $(QOrd(A), \subseteq)$, $(QPart(A), \leq)$.*

Proof. Consider the function $\psi : QPart(A) \longrightarrow QOrd(A)$ defined by $\psi(\mathcal{R}) = \rho_{\mathcal{R}}$. Then, according to Lemma 4, $(\phi \circ \psi)(\rho) = \psi(\phi(\rho)) = \psi(A/\rho) = \rho_{(A/\rho)} = \rho$ and $(\psi \circ \phi)(\mathcal{R}) = \phi(\psi(\mathcal{R})) = \phi(\rho_{\mathcal{R}}) = A/\rho_{\mathcal{R}} = \mathcal{R}$. This proves that ϕ is a bijection. Now one needs to prove that for all $\rho_1, \rho_2 \in QOrd(A)$: $\rho_1 \subseteq \rho_2$ iff $A/\rho_1 \leq A/\rho_2$.

$(\Rightarrow)$: Assume that $\rho_1 \subseteq \rho_2$. Then obviously, for any $a \in A$, $(a]_{\rho_1} \subseteq (a]_{\rho_2}$, so $A/\rho_1 \leq A/\rho_2$ due to Proposition 2.

$(\Leftarrow)$: Assume that $A/\rho_1 \leq A/\rho_2$ and let $x\rho_1 y$. Then $x \in (y]_{\rho_1}$. From the assumption and Proposition 2 it follows that $(y]_{\rho_1} \subseteq (y]_{\rho_2}$, therefore $x \in (y]_{\rho_2}$, so $x\rho_2 y$. $\Box$

Now we may formulate a counterpart of the well-known fact for equivalence relations.

Proposition 6. *Let $(B, \leq)$ be any poset, A be any set and $f : A \longrightarrow B$ be any mapping from A onto B. Then*

(1) a relation ρ, defined on A by $a\rho b$ iff $f(a) \leq f(b)$, is a quasiorder,

(2) a mapping $g : A/\rho \longrightarrow B$ of the form $g((a]) = f(a)$ is well defined and is an isomorphism of the quasipartition $(A/\rho, \subseteq)$ ordered by inclusion and the poset $(B, \leq)$,

(3) $f = k_\rho \circ g$, where the mapping $k_\rho : A \longrightarrow A/\rho$ is defined by $k_\rho(a) = (a]$.

Proof. We show only condition (2). Suppose that $(a] = (b]$. Then by Fact 1, $a\rho b$ and $b\rho a$, therefore $f(a) = f(b)$, that is the mapping g is well defined. In order to show that g is 1-1 assume that $g((a]) = g((b])$. So $f(a) = f(b)$ and hence $a\rho b$ and $b\rho a$, thus $(a] = (b]$ due to Fact 1. Furthermore, g is onto for f is onto. Finally, the preservation of the suitable orderings by the map g follows: $(a] \subseteq (b]$ iff $a\rho b$ iff $f(a) \leq f(b)$ iff $g((a]) \leq g((b])$. □

In the case of equivalence relations, not only any mapping f from a set A to a set B defines an equivalence relation θ on A ($a\theta b$ iff $f(a) = f(b)$) but also conversely, any equivalence relation θ on A is induced by a mapping: $a\theta b$ iff $k_\theta(a) = k_\theta(b)$. The analogous fact which is simply a generalization of that one, holds for quasiorders. Namely, not only any mapping from a set to a poset defines a quasiorder (Proposition 6(1)) but also conversely, any quasiorder ρ on a set A is induced by a mapping from A to a poset:

for any $a, b \in A$, $a\rho b$ iff $k_\rho(a) \subseteq k_\rho(b)$,

due to Fact 1. The last statement is well-known in case a quasiorder ρ is a partial ordering $\leq$. Then it simply expresses the condition of preservation of partial orderings by isomorphism $k_\leq$ in the representation theorem for posets saying that a poset $(A, \leq)$ is isomorphic with the poset $(A/\leq, \subseteq)$ (one may derive this theorem from Proposition 6(2) in case $(B, \leq) = (A, \leq)$ and f is the identity function of A). This representation theorem may be a little bit generalized in the following way.

Proposition 7. *Let $\rho \in QOrd(A)$. Then ρ is a partial ordering on A iff k_ρ is an isomorphism of the relational systems (A, ρ), $(A/\rho, \subseteq)$.*

Proof. (⇒): This implication is the representation theorem.
(⇐): It is enough to assume that k_ρ is 1-1. Suppose that $a\rho b$ and $b\rho a$. Then, according to Fact 1, $(a] = (b]$ so $k_\rho(a) = k_\rho(b)$ and finally, $a = b$. □

3. Tolerance relations and tolerance coverings

In case of tolerance relations the role of a partition of a set A on which the relation is defined, is played by a family of subsets of A that will be called *tolerance covering* of A (the same family but defined in other way, is called *τ-covering* in [1] and [8], comp. also [5]).

Definition. Given a set A, any family $\mathcal{S}$ of subsets of A satisfying the following conditions

(1) $\mathcal{S}$ is an antichain in the poset $(\wp(A), \subseteq)$ of all subsets of A,
(2) $\forall Z \subseteq A (\forall x, y \in Z\ \exists X \in \mathcal{S} (x \in X\ \&\ y \in X) \Rightarrow \exists X \in \mathcal{S}, Z \subseteq X)$,
(3) $\bigcup \mathcal{S} = A$,

will be called a *tolerance covering of the set* A.

Let $\Sigma \subseteq \wp(\wp(A))$ be the set of all tolerance coverings of A. One may equip the set Σ with the partial ordering $\leq$ defined by $\mathcal{S}_1 \leq \mathcal{S}_2$ iff $\forall X \in \mathcal{S}_1 \exists Y \in \mathcal{S}_2, X \subseteq Y$.

Notice that any partition of A that is, a family $\mathcal{S}$ of nonempty pairwise disjoint subsets of A, fulfilling the condition (3), is a tolerance covering of A.

By a tolerance relation on A we mean any reflexive and symmetric relation defined on A ([7], cf. [6] to discover the significance of tolerances). Let $\mathcal{T}$ be the family of all tolerance relations defined on A. Obviously, the poset $(\mathcal{T}, \subseteq)$ is a complete lattice such that for any $T \subseteq \mathcal{T}$, $\sup T = \bigcup T$ and $\inf T = \bigcap T$ if $T \neq \emptyset$ ($\inf \emptyset = A^2$).

In the sequel, the following simple general fact dealing with any binary relation will be useful.

Fact 8. *Given a binary relation ρ defined on a set A consider the family $\{Z \subseteq A : Z^2 \subseteq \rho\}$ ordered by inclusion. Then any element of the family is included in a maximal element of the family.*

Proof. By application of Kuratowski-Zorn lemma. □

For any binary relation ρ defined on a set A let $Max(\rho)$ be the family of all the maximal elements in the poset $(\{Z \subseteq A : Z^2 \subseteq \rho\}, \subseteq)$.

Consider the following two mappings $\phi : \mathcal{T} \longrightarrow \wp(\wp(A))$, $\psi : \Sigma \longrightarrow \wp(A^2)$. For each $\rho \in \mathcal{T}$, $\phi(\rho) = Max(\rho)$ and for any $\mathcal{S} \in \Sigma$, $\psi(\mathcal{S}) = \bigcup\{X^2 : X \in \mathcal{S}\}$.

Lemma 9. *For each tolerance relation ρ on A, $\phi(\rho) \in \Sigma$ and for each tolerance covering $\mathcal{S}$ of A, $\psi(\mathcal{S}) \in \mathcal{T}$.*

Proof. Let ρ be a tolerance relation on A. Then condition (1) of the definition of a tolerance covering is satisfied by $Max(\rho)$ in an obvious way. In order to show (2) suppose that for a given $Z \subseteq A$ for any elements $x, y \in Z$ there is an $X \in Max(\rho)$ such that $x, y \in X$. Now take any $a, b \in Z$ into consideration. Then $\{a, b\}^2 \subseteq X^2 \subseteq \rho$, for some $X \in Max(\rho)$ so $(a, b) \in \rho$. In this way $Z^2 \subseteq \rho$, thus there exists an $X \in Max(\rho)$ such that $Z \subseteq X$ due to Fact 8. Now, the condition (3) is satisfied due to reflexivity of ρ and Fact 8. Finally, $Max(\rho)$ is a tolerance covering of A. The second part of lemma is obvious. □

Lemma 10. *For each tolerance relation ρ on A, $\psi(\phi(\rho)) = \rho$ that is, $\rho = \bigcup\{X^2 : X \in Max(\rho)\}$, and for any tolerance covering $\mathcal{S}$ of A, $\phi(\psi(\mathcal{S})) = \mathcal{S}$ that is, $\mathcal{S} = Max(\bigcup\{X^2 : X \in \mathcal{S}\})$.*

Proof. First we show that given a tolerance relation ρ, for any $a, b \in A$, $(a, b) \in \rho$ iff $a, b \in X$ for some $X \in Max(\rho)$. Given $a, b \in A$ the implication $(\Leftarrow)$ is obvious and the inverse one follows from Fact 8. Now we show the second part of the lemma.

$(\supseteq)$: Let Z be a maximal set such that $Z^2 \subseteq \bigcup\{Y^2 : Y \in \mathcal{S}\}$. Therefore, for any $a, b \in Z$ there is a $Y \in \mathcal{S}$ such that $a \in Y$ & $b \in Y$. Hence and from (2) it follows that $Z \subseteq X$ for some $X \in \mathcal{S}$. Since $X^2 \subseteq \bigcup\{Y^2 : Y \in \mathcal{S}\}$ so $Z = X$ due to maximality of Z. Finally, $Z \in \mathcal{S}$.

$(\subseteq)$: Let $Z \in \mathcal{S}$. Then it is obvious that $Z^2 \subseteq \bigcup\{Y^2 : Y \in \mathcal{S}\}$. In order to show that Z is a maximal element in the family $\{X \subseteq A : X^2 \subseteq \bigcup\{Y^2 : Y \in \mathcal{S}\}\}$ suppose conversely that $Z \subseteq X_0$, $Z \neq X_0$ and $X_0^2 \subseteq \bigcup\{Y^2 : Y \in \mathcal{S}\}$. Hence and from (2) it follows that $X_0 \subseteq Y_0$ for some $Y_0 \in \mathcal{S}$. Thus, $Z \subseteq Y_0$ and $Z \neq Y_0$ which is impossible for $\mathcal{S}$ is an antichain. □

It is seen that any tolerance relation ρ is completely determined by the family of sets $Max(\rho)$. Furthermore, any tolerance covering of the set A is uniquely determined by corresponding tolerance relation.

LEMMA 11. *For any tolerance relations ρ_1, ρ_2, $\rho_1 \subseteq \rho_2$ iff $\phi(\rho_1) \leq \phi(\rho_2)$.*

PROOF. ($\Rightarrow$): Let $\rho_1 \subseteq \rho_2$. We should show that for any $X \in Max(\rho_1)$ there is a $Y \in Max(\rho_2)$ such that $X \subseteq Y$. However this is obvious since when $X^2 \subseteq \rho_1$, then $X^2 \subseteq \rho_2$ so X is contained in some $Y \in Max(\rho_2)$ due to Fact 8.

($\Leftarrow$): Assume that for any $X \in Max(\rho_1)$ there is a $Y \in Max(\rho_2)$ such that $X \subseteq Y$. Let $(x, y) \in \rho_1$. Then $\{x, y\}^2 \subseteq \rho_1$. Hence and from Fact 8 it follows that $\{x, y\} \subseteq X$ for some $X \in Max(\rho_1)$. So from the assumption we have $\{x, y\} \subseteq Y$ for some $Y \in Max(\rho_2)$ that is, $(x, y) \in \rho_2$. □

COROLLARY 12. *The complete lattice $(\mathcal{T}, \subseteq)$ of all tolerance relations defined on a set A and the complete lattice $(\Sigma, \leq)$ of all tolerance coverings of A are isomorphic.*

PROOF. Obvious due to Lemmas 10 and 11. □

As we see, the role of partition corresponding to a given tolerance relation ρ defined on A is played by the family $Max(\rho)$ of all maximal subsets X of A such that $X^2 \subseteq \rho$. In general, this family does not coincide with the family A/ρ of "tolerance" classes of $\rho:\ \{(x] : x \in A\}$, where for any $x \in A, (x] = \{y \in A : (y, x) \in \rho\}$ (in several papers, e.g., [3], [4], the elements from $Max(\rho)$ are just called tolerance classes while in the others, for example [2], these elements are said to be blocks of the tolerance ρ). The connections between both families are given in the following proposition.

PROPOSITION 13. *Let ρ be any tolerance relation defined on a set A. Then*
(1) for each subset X of A the following conditions are equivalent:
(i) $X \in Max(\rho)$,
(ii) $\forall y \in A\ (y \in X$ iff $\forall x \in X, (y, x) \in \rho)$,
(iii) $X = \bigcap\{(x] : x \in X\}$,
(2) for any $a \in A$, $(a] = \bigcup\{X \in Max(\rho) : a \in X\}$.

PROOF. For (1). We show only that $(i) \Leftrightarrow (ii)$ for (iii) is a notation variant of (ii).

$(i) \Rightarrow (ii)$: Assume (i) and take any $y \in A$. Then the implication $y \in X \;\Rightarrow\; \forall x \in X, (y,x) \in \rho$ is obvious due to (i). In order to show the inverse one suppose that for all $x \in X, (y,x) \in \rho$ but $y \notin X$. Hence from (i) it follows that $(X \cup \{y\})^2 \subseteq \rho$ which implies, contrary to (i) that X is not maximal in the family $\{Y \subseteq A : Y^2 \subseteq \rho\}$.

$(ii) \Rightarrow (i)$: Assume (ii). Then the fact $X^2 \subseteq \rho$ follows directly. In order to prove that X is maximal suppose that $X \subseteq Y$ and $Y^2 \subseteq \rho$ for some $Y \subseteq A$. Consider any $a \in Y$. Then for all $x \in X, (a,x) \in \rho$ so $a \in X$ due to (ii). Thus $Y \subseteq X$ and finally $X = Y$.

For (2). Consider any $a \in A$. Then applying Fact 8 we have $b \in (a]$ iff $b\rho a$ iff $\{a,b\}^2 \subseteq \rho$ iff there is an $X \in Max(\rho)$ such that $\{a,b\} \subseteq X$ iff $b \in \bigcup\{X \in Max(\rho) : a \in X\}$. □

Due to condition (2) of Proposition 13, one may show that given a tolerance relation ρ on A, $Max(\rho) = A/\rho$ iff ρ is an equivalence relation of A (notice that the assumption $Max(\rho) = A/\rho$ and (2) of Proposition 13 imply that for any $a \in A$, $\{X \in Max(\rho) : a \in X\} = \{(a]\}$).

In general, there is no an isomorphism from the complete lattice $(\mathcal{T}, \subseteq)$ to a poset composed of the quotient sets A/ρ, $\rho \in \mathcal{T}$. The reason consists in that the mapping $\mathcal{T} \ni \rho \longmapsto A/\rho$ is not one-to-one. This fact holds for any set A with cardinality ≥ 4. To show it consider the different four elements $a, b, c, d \in A$ and an equivalence relation ρ on A such that $\{a,b,c,d\}^2 \subseteq \rho$. Next consider the following tolerance relations

$$\rho_1 = \rho - \{(a,b),(b,a),(c,d),(d,c)\},$$

$$\rho_2 = \rho - \{(a,c),(c,a),(b,d),(d,b)\}.$$

Then we have

$$(a]_{\rho_1} = (a]_\rho - \{b\} = (d]_{\rho_2},$$

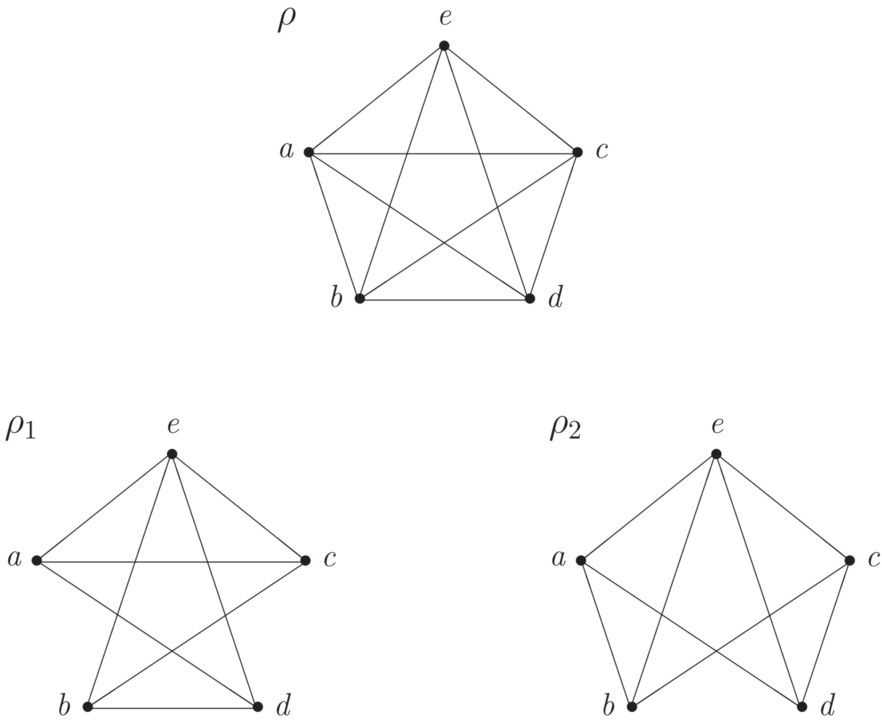
$$(b]_{\rho_1} = (b]_\rho - \{a\} = (c]_{\rho_2},$$

$$(c]_{\rho_1} = (c]_\rho - \{d\} = (b]_{\rho_2},$$

$$(d]_{\rho_1} = (d]_\rho - \{c\} = (a]_{\rho_2} \text{ and}$$

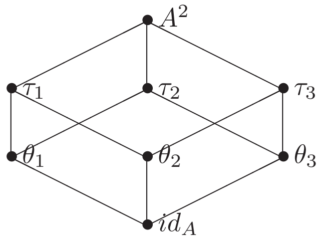
$$(x]_{\rho_1} = (x]_\rho = (x]_{\rho_2} \text{ for any } x \in A - \{a,b,c,d\}.$$

So $A/\rho_1 = A/\rho_2$, however, $\rho_1 \neq \rho_2$. It is convenient to illustrate the relations ρ, ρ_1, ρ_2 for example in the form of the following graphs.



It would be interesting to mention that in case the cardinality of A is 3, let us say, $A = \{a, b, c\}$, the mapping $\mathcal{T} \ni \rho \longmapsto A/\rho$ is one-to-one, however it differs from the isomorphism $\mathcal{T} \ni \rho \longmapsto Max(\rho)$. On the figure below the lattice of all tolerances of A is presented.

$id_A = \{(a,a),(b,b),(c,c)\}$
$\theta_1 = \{(a,b),(b,a)\} \cup id_A$
$\theta_2 = \{(a,c),(c,a)\} \cup id_A$
$\theta_3 = \{(b,c),(c,b)\} \cup id_A$
$\tau_1 = \{(a,b),(b,a),(a,c),(c,a)\} \cup id_A$
$\tau_2 = \{(a,b),(b,a),(b,c),(c,b)\} \cup id_A$
$\tau_3 = \{(a,c),(c,a),(b,c),(c,b)\} \cup id_A$



Here $A/\theta_i = Max(\theta_i)$, $i = 1,2,3$, $A/id_A = Max(id_A)$, $A/A^2 = Max(A^2)$ and $A/\tau_i = Max(\tau_i) \cup \{A\}$, $i = 1,2,3$.

Now we will show that any tolerance relation on a set A is defined by a mapping from A to a semilattice with a least element. This is a counterpart of the fact that any quasiorder on A is defined by a mapping from A to a poset. So let $(B, \wedge, 0)$ be a (lower) semilattice with zero. Consider any mapping $f : A \longrightarrow B - \{0\}$. Then the binary relation ρ_f defined on A by

$$(x,y) \in \rho_f \quad \text{iff} \quad f(x) \wedge f(y) \neq 0,$$

is a tolerance relation on A. Conversely, we have

Proposition 14. *For any tolerance relation ρ defined on a set A there is a similattice $(B, \wedge, 0)$ and a mapping $f : A \longrightarrow B - \{0\}$ such that $\rho = \rho_f$.*

Proof. Suppose that ρ is any tolerance relation on A. Consider the semilattice $(\wp(\wp(A)), \cap, \emptyset)$ and the mapping $f_\rho : A \longrightarrow \wp(\wp(A))$ defined by $f_\rho(a) = \{X \in Max(\rho) : a \in X\}$. Obviously, for any $a \in A$ there is an $X \in Max(\rho)$ such that $a \in X$ (due to Fact 8 or Lemma 10) so $f_\rho(a) \neq \emptyset$. Now notice that the following four conditions are equivalent (first two due to Lemma 10):

$(a,b) \in \rho$,
there is an $X \in Max(\rho)$ such that $a, b \in X$,
$\{X \in Max(\rho) : a \in X\} \cap \{X \in Max(\rho) : b \in X\} \neq \emptyset$,
$f_\rho(a) \cap f_\rho(b) \neq \emptyset$.

Finally, $\rho = \rho_f$. $\square$

To the end let us consider another possible counterpart of partition corresponding to tolerance relation. Consider the class $\wp(\wp(A)-\{\emptyset\})$ of the all families $\mathcal{R} \subseteq \wp(A)$ such that $\emptyset \notin \mathcal{R}$. In the next lemma we assume that $\leq$ is any binary relation defined on $\wp(\wp(A)-\{\emptyset\})$ satisfying the following condition:

$$(\leq) \quad \text{for any } \mathcal{R}, \mathcal{S} \subseteq \wp(A) - \{\emptyset\},\ \mathcal{R} \leq S \ \text{ iff } \ \text{for each } a \in A,\ \mathcal{R}_a \leq \mathcal{S}_a,$$

where, as before, given $\mathcal{R} \subseteq \wp(A)$, $\mathcal{R}_a = \{X \in \mathcal{R} : a \in X\}$. For example, the ordinary set inclusion fulfils the condition $(\leq)$. Also, the relation defined by $\mathcal{R} \leq S$ iff $\forall X \in \mathcal{R} \exists Y \in \mathcal{S}, X \subseteq Y$.

Lemma 15. *Let* $\mathsf{R} \subseteq \wp(\wp(A)-\{\emptyset\})$ *be such that the relation* $\leq$ *restricted to* R *is a partial ordering. Then the relation* $\sqsubseteq$ *defined on the class* $\{\{\mathcal{R}_a : a \in A\} : \mathcal{R} \in \mathsf{R}\}$ *by* $\{\mathcal{R}_a : a \in A\} \sqsubseteq \{\mathcal{S}_a : a \in A\}$ *iff for all* $a \in A$, $\mathcal{R}_a \leq \mathcal{S}_a$, *is also a partial ordering and the mapping* $\Phi : \mathsf{R} \longrightarrow \{\{\mathcal{R}_a : a \in A\} : \mathcal{R} \in \mathsf{R}\}$ *defined by* $\Phi(\mathcal{R}) = \{\mathcal{R}_a : a \in A\}$ *is an isomorphism of the posets* $(\mathsf{R}, \leq)$, $\{\{\mathcal{R}_a : a \in A\} : \mathcal{R} \in \mathsf{R}\}, \sqsubseteq)$.

Proof. First we show that Φ is one-to-one. So suppose that $\{\mathcal{R}_a : a \in A\} = \{\mathcal{S}_a : a \in A\}$ for $\mathcal{R}, \mathcal{S} \in \mathsf{R}$. In order to show the inclusion $\mathcal{R} \subseteq \mathcal{S}$ assume that $X \in \mathcal{R}$. Then $X \in \mathcal{R}_a$ for some $a \in A$ since $X \neq \emptyset$. Thus from the assumption it follows that $X \in \mathcal{S}_b$ for some $b \in A$. So $X \in \mathcal{S}$. The inverse inclusion is proved in the same way. Finally, $\mathcal{R} = \mathcal{S}$. Now suppose that the relation $\leq$ restricted to R is a partial ordering. In order to show the reflexivity of $\sqsubseteq$ assume that $\{\mathcal{R}_a : a \in A\} = \{\mathcal{S}_a : a \in A\}$ for some $\mathcal{R}, \mathcal{S} \in \mathsf{R}$. Then $\mathcal{R} = \mathcal{S}$ as it has been just shown so $\mathcal{R} \leq \mathcal{S}$ and consequently $\{\mathcal{R}_a : a \in A\} \sqsubseteq \{\mathcal{S}_a : a \in A\}$ by definition of $\sqsubseteq$ and the condition $(\leq)$. The transitivity and antisymmetry of $\sqsubseteq$ follow from the definition of $\sqsubseteq$ and $(\leq)$ immediately. Finally, the condition $\mathcal{R} \leq \mathcal{S}$ iff $\{\mathcal{R}_a : a \in A\} \sqsubseteq \{\mathcal{S}_a : a \in A\}$ for any $\mathcal{R}, \mathcal{S} \in \mathsf{R}$, follows also by definition of $\sqsubseteq$ and $(\leq)$. $\square$

Now, put $\mathsf{R} = \Sigma = \{Max(\rho) : \rho \in \mathcal{T}\}$ and for any $\mathcal{R}, \mathcal{S} \subseteq \wp(A) - \{\emptyset\}$, $\mathcal{R} \leq \mathcal{S}$ iff $\forall X \in \mathcal{R}\ \exists Y \in \mathcal{S}, X \subseteq Y$. Then the required condition $(\leq)$ is satisfied:

$$Max(\rho_1) \leq Max(\rho_2) \text{ iff for each } a \in A, Max(\rho_1)_a \leq Max(\rho_2)_a$$

and $\leq$ is a partial ordering on Σ. So using the map f_ρ from the proof of Proposition 14 and applying Lemma 15 and Corollary 12 one may state the following corollary.

Corollary 16. *The mapping Φ assigning to each tolerance covering $Max(\rho)$ the family $\{f_\rho(a) : a \in A\}$ where for each $a \in A$, $f_\rho(a) = \{X \in Max(\rho) : a \in X\}$ is an isomorphism of the posets $(\{Max(\rho) : \rho \in \mathcal{T}\}, \leq)$, $\{\{f_\rho(a) : a \in A\} : \rho \in \mathcal{T}\}, \sqsubseteq)$. Consequently, the complete lattices $(\mathcal{T}, \subseteq)$, $(\{\{f_\rho(a) : a \in A\} : \rho \in \mathcal{T}\}, \sqsubseteq)$ are isomorphic.*

4. Discussions

The counterpart of partition for a quasiorder ρ defined on a set A is a natural extension of the ordinary partition associated with an equivalence relation. Simply it is the set A/ρ of all "equivalence" classes $(a]_\rho = \{x \in A : x\rho a\}$, $a \in A$. While the main counterpart of partition (established in the paper) for a tolerance ρ defined on A is the set $Max(\rho)$ of all maximal subsets X of A such that $X^2 \subseteq \rho$. Both sets, A/ρ and $Max(\rho)$, coincide iff ρ is an equivalence relation. The reason that the quotient sets A/ρ, ρ is a tolerance on A, do not form "partitions" of A is that the correspondence $\rho \longmapsto A/\rho$ is not one-to-one whenever the cardinality of A is greater than 3. This correspondence is obviouly one-to-one when it is restricted to equivalence relations on A. In case it concerns the class of all tolerances, it is one-to-one iff A is the cardinality less than 4. So, in particular, one may treat the classes $\{a, b, c\}/\rho$, ρ is a tolerance on 3-element set $\{a, b, c\}$, as the counterparts of partitions. In case ρ is a tolerance not being an equivalence, the set $\{a, b, c\}/\rho$ is not a partition of $\{a, b, c\}$ in the usual sense.

In the paper one may also find another kind of "partitions" for tolerances (cf. Corollary 16). Namely, given a tolerance relation ρ defined on A, the elements of such a "partition" associated with ρ are the families $\{X \in Max(\rho) : a \in X\}$, $a \in A$. In case ρ is an equivalence relation, such a "partition" is of the form: $\{\{[a]\} : a \in A\}$, where given $a \in A$, $[a]$ is the equivalence class determined by a. However, one could have some justified doubts to call such a family a counterpart of partition of A. Most likely a counterpart of partition of a given set should be a family of its subsets.

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References

[1] I. Chajda, J. Niederle, B. Zelinka, *On existence conditions for compatible tolerances*, **Czechoslovak Math. J.** 26 (1976), pp. 304–311.
[2] G. Czédli, *Factor lattices by tolerances*, **Acta Scientiarum Mathematicarum** 44 (1982), pp. 35–42.
[3] S. N. Gerasin, V. V. Shlyakhov, S. V. Yakovlev, *Set coverings and tolerance relations*, **Cybernetics and System Analysis** 44 (2008), pp. 333–340.
[4] A. I. Krivoruchko, *Tolerance classes*, **Cybernetics and System Analysis** 20 (1984), pp. 6–11.
[5] M. Nowak, *On some generalization of the concept of partition*, **Studia Logica** 102 (2014), pp. 93–116.
[6] J. Pogonowski, **Tolerance spaces with application to linguistics**, Adam Mickiewicz University Press, Poznań, 1981.
[7] E. C. Zeeman, *The Topology of the Brain and Visual Perception*, [in:] M. K. Fort (ed.), **The Topology of 3-Manifolds and Related Topics**, 1962, pp. 240–256.
[8] B. Zelinka, *A remark on systems of maximal cliques of a graph*, **Czechoslovak Math. J.** 27 (1977), pp. 617–618.

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