NORTHERN ILLINOIS UNIVERSITY

The Problem of Adequately Defining Numbers

A Thesis submitted to the
University Honors Program
in Partial Fulfillment of the
Requirements of the Baccalaureate Degree
With Upper Division Honors

Department of Philosophy

by

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May, 1989

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Date: *May 11, 1989*

JOURNAL OF THESIS ABSTRACTS

THESIS SUBMISSION FORM

AUTHOR: D Stephan DeLong

THESIS TITLE: The Problem of Adequately Defining Numbers

THESIS ADVISOR: Dr. James Hudson

ADVISOR'S DEPT: Philosophy

DATE: 5/11/89

HONORS PROGRAM: ____

NAME OF COLLEGE: ____

PAGE LENGTH: 24

BIBLIOGRAPHY (YES OR NO): yes

ILLUSTRATED (YES OR NO): no

COPIES AVAILABLE (HARD COPY,MICROFILM,DISKETTE): Hard Copy

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SUBJECT HEADINGS: (CHOOSE FIVE KEY WORDS)

MATHEMATICAL PHILOSOPHY__ CONCEPT OF NUMBER

ABSTRACT (100-200 words):

Natural numbers, although they pervade much of mathematics, are among the most difficult of entities forwhich to provide definitions. Although it is often overlooked, as the efforts of pure mathematics are directed toward the maximization of rigor, the development of sound definitions for numbers can be viewed as one of the most critical objectives of the discipline.

This paper is an examination and a support for the efforts in this area of the German logician Gottlob Frege, and in particular of his landmark treatise **Die Grundlagen der Arithmetik.** This work marked the first succesful attempt to define numbers through appeal to pure logic alone, and stands as a significant achievement in the history of mathematical philosophy.

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"It is surely a very remarkable thing that despite the range, power, and success of modern mathematics, the concept of the natural number, on which the whole edifice rests, is still something of a mystery."

Friedrich Waissmann

Were we to be approached by a child, and asked to furnish a definition for a number, we might do what is typically done in such a circumstance, namely put pen to paper and make some sort of a mark, such as "3" for example, and then utter "this is the number three."

To a child, such an 'answer' to the inquiry might seem entirely satisfactory, but is it really? If we were to then be reminded that in associating the mark "3" with the phrase "the number three" we have incorporated the definite article, by which we would apparently wish to bestow upon the number three the property of uniqueness, what then would we respond were the child to stand next to a clock whose face bore Roman numerals, point to "III," and consequently say "I thought that **this** was the number three"? How could it be that the number three could be two things which were so completely dissimilar?

Once such an objection were to be lodged, it would become immediately clear to us what was wrong with the definition of the number three which we had provided. The difficulty lies with the fact that "3" and "III" are actually only numerals, or more generally **representations** for the number three, and not really the number three at all.

Recognizing our error, we are left once more where we began, confronted with the problem of providing a definition for a number. It is unfortunate, and even somewhat distressing, that there are few people who are then prepared to advance an alternative to this definition for numbers, or who are even able to identify what sort of an entity numbers might be.

What may be even more distressing is the fact that many people will want to dismiss this disturbing situation as one which is of little importance. They will suggest that to argue the merit of a rigorous definition for numbers is merely to quibble over an insignificant theoretical detail, and then proceed to perform the various tasks involving applications of the numbers, such as arithmetic, without pausing to give the matter a second thought.

However, is it not, as the German logician **Gottlob Frege** once said, a "scandal," that in a discipline such as mathematics, a science whose object is the attainment of absolute rigor, we should encounter entities which are incontestably among the most primitive in the study and yet for which we find it impossible to provide even a semblance of an adequate definition? And if it should turn out that we are unable to define numbers, how in the world are we ever to justify our usage of them in increasingly elaborate mathematical equations?

Could we ever feel secure that the entire structure of mathematics, erected as it is upon the foundation of numbers, is not fundamentally flawed and fallacious? Might it not then be that, when all has been said and done, we shall find mathematics to be nothing more than deft manipulation of empty symbols?

Certainly, it would appear to be well worth the while for us to more intently scrutinize the concept of numbers, so that we might provide the adequate definitions which will ensure the preservation of mathematical rigor.

But how shall we go about defining the numbers? Since they comprise an infinite set, it is immediately clear that to define them by enumeration would be a task which we would have no hope of completing, and since we have already dispelled the notion that we can define them simply through the use of numerals, we would seem to have run out of clear cut places to begin.

Defining the concept of number in general, and the individual numbers in particular, is a problem which has been the object of much philosophic study historically, and is one whose origins can be traced back to the time of the great geometer and philosopher Pythagoras of Samos. Let us examine some of the more prominent views which have arisen on this topic, see why each in turn must be dismissed as being inadequate for our purposes, and come at last to the correct solution.

One suggestion has been that there is much that we could learn from our common usage of the number names (such as one, two, three, etcetera) in our quest for defining the numbers. An example of this is, as we have already seen, the fact that, when speaking of numbers, we employ the definite article, an act which would seem to indicate that we view the numbers as possessing the quality of uniqueness. Let us pursue this line of thought a bit further, and see if there is more information which such an analysis can reveal.

Moritz Cantor, a nineteenth century German philosopher and mathematician, observed the grammatical occurrence of the numbers, and noticed that, when we employ the various number names in context, it is typically in the manner of attributive adjectives that we do so.

If one were to consider the following pair of sentences, for example:

These are two roses.
These are red roses.

it would appear at first glance as though our usage of the words "red" and "two" in these sentences is in some way similar. It was this similarity which Cantor observed, and which led him to suggest that numbers are (in their essence) properties which are held by objects, and that it was as properties that numbers should be defined.

In the sentences above, for example, Cantor would say that there is a property of number, **twoness**, we might call it, which the roses possess, just as they possess a property of color, which we might call **redness**.

However, in spite of the similarity of appearance of the aforementioned sentences, there is a rather significant difference between what is being expressed by each, a difference which is worthy of greater attention.

If we can say that the roses which we observe are red in color, and we then regard each of the roses in turn, it will be the case that each is individually red in color. The property of redness, we would agree, not only belongs to the agglomeration of roses, but also to the individual roses. It makes perfect sense for us to say: These are red roses, therefore each of the roses has the property of redness.

It is immediately apparent that it is not possible for us to do the same sort of thing with regard to the alleged property of twoness. If we were to attempt to do so, we would wind up with: These are two roses, therefore each of the roses has the property of twoness. This is clearly not what one would have liked to happen. If there were any number property which one would have associated with the individual roses, surely it would have (if anything) been "oneness." That would seem to go hand in hand with their being perceived as "individual" roses.

Yet it is not easy to see why we should be able to perform such an analysis with the property of color and yet be unable to do so with number, if indeed it really is the case that they are both properties in the same sense.

But let us not be so easily daunted in our attempt to define numbers as properties of physical objects; it may well be that numbers really are properties of physical objects, and that it is some sort of grammatical oddity which hinders our ability to perform the transformation which failed a moment ago.

The view that numbers can be defined as properties of objects is one which was also held by the noted philosopher **John Stuart Mill,** who said that numbers could be defined through a process of **abstraction** from an "observed matter of fact."

"A number," Mill claimed, "can be viewed as synonymous with the agglomeration of things by which we call the name, and is to be given (to the agglomeration) according to the characteristic manner in which the agglomeration is made up." By "the agglomeration of things by which we call the name," Mill would be referring, in our example, to the roses.

However, before we fully embrace this solution, let us consider in somewhat greater depth Mill's suggestion. In order to abstract a number from the physical condition of the roses in our sight, it is first necessary that we should determine **the** characteristic manner in which the agglomeration is constructed. It is here that the first difficulty is encountered with Mill's suggestion for defining numbers: what is **the** characteristic manner in which this agglomeration is made up?

One might say that it is as two roses, and therefore the number corresponding to the agglomeration should be two. But what if we view the agglomeration not as two roses, but as ten petals? And why not as one pair of flowers? This is a formidable problem for Mill to resolve, and his entire theory for defining numbers hangs in the balance depending upon the choice.

What has gone wrong in this case is that the characteristic manner in which we view an object, in this case the roses, has become a victim of perspective. That the roses are, in fact, red (if that is indeed the case) is by virtue of their surfaces being such that they reflect a certain wavelength of light. This is a condition which endures in spite of any manner in which we should choose to view the roses; it is independent of our point of view.

That the roses possess the quality of twoness depends entirely upon our manner of looking at them. As we have seen, it is quite clearly the case that there are many different numbers which we could ascribe to the physical condition of the roses. Herein lies the difference between the numbers and the properties which can be made predicates of objects.

We might try to avoid this problem by attempting to abstract numbers not from any object which we should come across, but rather restrict our efforts to those things which we can perceive as being indivisible. An effort to do so was made by **Baumann** and **Kopp**, in their attempt to salvage Mill's theory that numbers could be defined in terms of physicality.

However, in such an attempt, it would be difficult to find objects which one could truthfully judge to be indivisible, and thereby employ in defining numbers. On inspection, it would seem that it is always the case that we can argue a given object could be dissected into the sum of parts, and thereby considered not as a single object but as a plurality.

It may well be that Baumann and Kopp wished that we should merely accept certain things as being indivisibles, and not attempt to disintegrate them into components. Such an insistence is, of course, entirely unsatisfactory, however, as it is our intent that the definitions of numbers be ones which may be relied upon as being rigorously truthful, and we can only expect that we would be led astray by accepting at the outset premises known to be untrue.

But apart from these criticisms, let us consider an objection to Mill which is far more serious. If we were to accept that numbers can only be defined based upon abstraction from conditions of physicality, what are we to make of a number of substantial size, say, an octillion?

A number such as this must pose such a view a grave problem, indeed, for if it is true, as Mill wrote, that numbers are to be defined only from an "observed matter of fact," how shall we ever extract an octillion from our surroundings? And if we should find, in the end, that we are incapable of doing so, how shall we ever manage to define an octillion?

One might suggest that Mill, in putting forward his claim, was speaking only of the comparatively 'small' numbers, and by this I mean those with which one can readily associate groups of objects lying close at hand, say the numbers one through ten. But if this is the case, what justification can be given for choosing ten as the largest number which may be so defined, and not eleven? And if eleven, why not twelve? For such a question Mill can not possibly provide an answer, for there is no manner in which such a choice could be justified.

Finally, and by far the most damaging to the perspective that numbers are things which we can define as properties which we abstract from that which is physical, consider the undeniable truth that there are many things which we should like to count which have no physical form whatsoever.

If it is the case that the number three, for instance, is something which we abstract from the physical condition of, say, three pencils lying on a table top, how are we to explain our ability to speak of three ideas? Three spoken words? Three abilities? All of these things are not physical, yet we should want our account of numbers to allow for them nonetheless. Were Mill's view to be correct, it would, in fact, be incorrect to speak of anything which was not physical as having any sort of number associated with it.

Let us, therefore, dismiss the notion that numbers may be defined as being properties of that which is physical, and turn to the other significant historical point of view on the problem of defining number. This view is one which may be judged to be the extreme opposite of the preceding point of view, as the contention here is that number is not to be defined in a manner which is in the least physical, but rather as an idea.

Gottfried von Leibniz was among the leading figures who rejected the view that number was something which could be defined through an abstraction from the physical world. Quite on the contrary, it was his contention that the definitions of the numbers had nothing whatever to do with physical objects, but rather were definable only as something which was entirely subjective.

This was a view which was also held by **George Berkeley**, who claimed that "number is nothing which can be fixed and settled, (drawn from) things themselves. It is entirely a creature of the mind, considering, either an idea by itself, or any combination of ideas to which it gives one name."

This point of view would seem to have a number of immediate advantages over that which was proposed by Mill and his followers, and we might at first think that we have come upon the solution to the problem. For one thing, in accepting such a view we would seem to have regained our ability to place with those things which are "number-able" all of the non-material things which we had lost when we had attempted to define numbers as properties of objects. Also, it would appear that the lack of certainty which we discovered to be present regarding the number, stemming from the problem of shifting perspective would vanish if it were up to us to mentally define the numbers.

However, in achieving these gains, it will be seen that we must pay a high price indeed. Recall our earlier insistence that (as our usage of the number names would indicate) the numbers themselves be unique. If we allow that the numbers can only be defined subjectively, varying necessarily from individual to individual, it would seem that this quality of numbers would be undone.

No longer would there be a single number two, for example. Rather, at any given time, there would be well over four billion number twos in the world, one for each person, and there would be no way in which we could ever be certain that any of them were the same. If I were to mention the number two in conversation, whose two would I mean? Mine? Yours? How are we ever to know if we are speaking of the same two?

And what then would we make of knowledge which has been passed down through history? Could we possibly have any reason for believing that Newton was referring to <u>our</u> two when he employed the symbol '2' in his calculations? That Newton's two and our two might happen to agree with respect to some of their properties and applications would be no guarantee that they were one and the same.

The answer to the question posed above is, of course, no. As is the case with all forms of communication, without mutual understanding there can be no exchange of information, and as a result, were we to accept that numbers are something which can only be defined subjectively, mathematics must grind to a halt. This is the price which we incur for accepting Leibniz's alternative to Mill's point of view.

And, as if this weren't enough, consider now the problem which arises from the fact that the numbers form an infinite set (there is no such thing as a largest number). Within our finite minds, how are we ever to come to grasp an infinite set? And if we can possess knowledge of only finitely many numbers, is it nonsensical to speak of extremely large numbers, those which we can not define intuitively?

Clearly, to accept the proposition that the numbers may be defined as being entirely subjective entities is to step upon a slippery slope indeed. In so doing, we seem to make inevitable the collapse of much of mathematics, and make impossible the communication of mathematical ideas.

Now, before proceeding to introduce that view which I consider to be the solution to the problem of defining numbers, let us pause to recapitulate what we have deduced to this point. It is, as we have seen, not possible to define numbers as properties abstracted from physical objects, as Mill and his followers mistakenly believed, just as it is not possible for us to attempt to define the numbers subjectively, calling upon intuition, as Leibniz and Berkeley sought to do.

Perhaps the way to move beyond these views is to ask what it is that is flawed with each position. While it seemed at first that each manner of defining the numbers had a measure of promise, what was it about those views which allowed us to so easily discount them?

The first to recognize and exhibit what was happening in each case was Frege, in his landmark treatise **Die Grundlagen der Arithmetik,** which was devoted to resolving precisely the problem which has been taken up here. It was Frege's contention that in each of the previous suggestions for defining numbers there was at work a vicious underlying circularity of definition, and that this circularity was the center of the difficulty which was preventing the discovery of number definitions.

Consider for a moment the manner in which Leibniz would have us define numbers, that is to say, subjectively. He would say that (in some manner) the notion of number is one which is intuitive, and is in a sense a priori. We all, according to Leibniz, possess an innate understanding of the numbers.

But how do we come to view (in our mind's eye) something as having a certain number of members or elements? If our knowledge of number is subjective, it would seem that some mental process must be at work which allows us to understand the individual numbers and to associate them with collections of entities.

If someone were to ask us to visualize a set with two entities, such as the much belabored example of the roses, this would not seem to be a difficult thing to do. However, having done this, what if we were then to be asked, how do you know that you have two roses in your set?

The answer which leaps to mind is, naturally, that we think of the roses, mentally count up how many there are, and that is the number which we have (two, in this case). But now we have placed ourselves in a somewhat uncomfortable position, as the process of counting is one which presupposes for its meaningfulness that we have in our hands firm knowledge and definitions of the numbers. When we count, what we do is to establish the existence of a one to one relationship between the numbers (1,2,...,n) and the elements of a set having n members. It remains to define what the numbers (1,2,...,n) in fact are.

A moment's consideration reveals this condition to exist in Mill's proposal, as well. This would seem to explain (at least partly) what might have been wrong with the preceding points of view toward defining the numbers. If we are to hold any hope at all of providing definitions of the numbers, therefore, we must find some way of speaking about them without falling back unconsciously upon the principle of counting.

Let us recall one of the problems which we encountered with regard to the pair of roses, specifically how we found that the number with which the roses were to be associated changed as we altered the way in which we chose to look at the roses. Our situation might be improved if we could develop an understanding as to why this came about.

We had noted that the property of redness belonged not only to the pair of roses, but to the individual roses as well, and our inability to say the same with regard to twoness was part of our motivation for discarding the notion that number was a property in the manner of color.

Frege considered this situation, and came to the following conclusion: when we say that the number to be associated with the roses is two, it is because we include in the statement the term 'roses.' On the other hand, when we look at the roses and determine that the number to be associated with these objects is ten, it is because we have included in the statement the term 'petals.' It is in the absence of these additional terms that we encounter uncertainty with regard to the number to be associated with the roses.

Thus the reason that the number of the roses changed was due to the fact that we altered our way of looking at them through the introduction of these other terms. Because we were able to do so, it would seem reasonable to conclude that the number does not, as Mill and Cantor had believed, belong to the roses at all, either individually or as an agglomeration. Rather, we would expect the number to have more to do with the terms which affected our way of viewing the roses.

This, said Frege, is at the root of the difficulty. Our problems in providing a definition of number originate in our intuitive tendency to attribute the number to the wrong entity, and it will only be after we have "returned number to its rightful owner" that we will be able to begin to make headway in any attempt to define the numbers.

To what, then, does the number rightfully belong? In looking at the example of the roses, we saw that the number two, for instance, is derived when we consider the pair of roses, and consider the quantity of roses which are present. The number ten, on the other hand, comes to us when we consider the quantity of petals which are present. Similarly, the number one comes to us when we consider the quantity of flower pairs which are present.

Each of these notions (quantity of roses present, quantity of rose petals present, and quantity of flower pairs) Frege described as **concepts**, and it is these concepts to which any statement of number must belong. If we include in our contemplation of the roses the concept 'quantity of petals,' we find that the number with which we associate the concept is ten; this judgement is not subject to the variance of perspective, it merely is or is not the case.

Bertrand Russell, in his analysis of Frege's **Grundlagen**, considered the role of concepts in determining numbers at greater length, and came to the conclusion that, when we consider, for example, the roses, we may say that the quantity of roses which we have are an instance of the number two, which is then an instance of number. What is not the case is that the roses themselves are an instance of number, and it is here that Mill and Cantor were led astray.

The number two, Russell said, is independent of physicality, and is rather something which the concept "quantity of roses" has in common with all other pairs of entities which we may care to consider, and which distinguishes pairs from all other types of sets, such as trios and quartets.

Thus a statement of number is revealed as something which is objective, a trait belonging not to individual objects, but rather to sets of objects together with their distinguishing concepts, and that trait is precisely the feature that each set possesses the same quantity of members.

The problem, now that we have restored numbers to their rightful possessors, was, as Frege saw it, how we might determine that two sets possess an equal quantity of members, not once relying upon the principle of counting to do so. If this could be done, it would be possible to define numbers according to these collections of sets (all of which have an equal quantity of members) by associating a number with each collection. In so doing, we would manage to define numbers in a non-circular manner, and our task will have been completed.

In his **Grundlagen**, Frege proposed that the manner in which this circularity could be avoided was by invoking a concept which will be familiar to those who have dabbled in mathematics and logic, the **one to one correspondence.**

Let us digress for a moment from the task at hand, and expand upon this notion for the benefit of the uninitiated, in order that the utility of the concept may come to light. If we were to come across a collection of cups and saucers lying upon a table before us, so arranged that beneath every cup there was a saucer, and that upon every saucer stood a cup, we could say, without worry, that we were in the presence of an equal quantity of cups and saucers. We could say so in spite of the fact that we might have no idea what the number of the cups and saucers actually was, for the cups and saucers would stand in a one to one correspondence with one another.

The way in which Frege referred to such a condition was to say that the set of cups and the set of saucers were **equinumerous.** Using this terminology, we can say that all sets which are equinumerous share the property of having the same quantity of members, and hence number will be the one characteristic common to equinumerous sets.

What we need in order to progress to the definitions of the individual numbers, then, is the introduction of what we may describe as **characteristic sets,** sets against which any other set may be compared. With each characteristic set **A** we will associate a number **a,** and then any set which is found to be equinumerous with **A** will, by definition, have **a** members.

As Frege pointed out, the assignment of number-names to the various characteristic sets is an act which is purely arbitrary. We shall, for convenience sake, make our assignment of number-names in such a manner that their familiar usage will not be impaired.

Consider the set A, whose membership is defined by agreement with the concept "not identical with oneself." That is, let

$$A = (x : x \text{ is not identical with } x)$$

As every object possesses the quality of self-identity, it is immediately apparent that the set A possesses no members. If we then choose to assign the number 0 to the concept "not equal with oneself," then every set found to be equinumerous with the set A will, by definition, be said to have 0 members.

Of course, any set having no members would have served equally as well for the characteristic set of the number 0, which is evident from the fact that every set which has no members is equinumerous with A.

Following the definition given for the number 0, we proceed immediately to the number one: let B be the set given by

$$B = (x : x \text{ is identical with } 0)$$

Clearly, (since, as we have already pointed out, every object is identical with itself) 0 is in this set, but the set contains nothing else. This follows since nothing satisfies the condition of being both identical with zero and yet not identical with zero, and therefore the only member of the set B is 0. We then define any set which is equinumerous with the set B to have 1 member.

In principle, this process of defining characteristic sets then proceeds ad infinitum, defining next the characteristic set of two as

C = (x,y : x is identical with 0, y identical with 1)

whose only members are 0 and 1, and so on. Clearly, as we are adopting the practice of associating the familiar arabic numeral representative with each characteristic set, the characteristic set with which the number **n** shall be associated will be given by:

$$N = (0, 1, \dots , n-1).$$

Now, as the numbers have it in their nature to be arranged in a typical ordering, what is necessary at this point is to show that we can define the relationship which each number bears to the number which precedes it. Although it is beyond the scope of this effort to demonstrate, from this relationship it will be possible to develop the familiar Peano postulates, with which one can construct the arithmetic properties which we commonly associate with the numbers, and thereby give further justification for acceptance of the Fregean definitions.

To introduce this method for ordering the numbers, all that we need do is to generalize a relationship which our definition of a given number bears to that of its predecessor, and use that relationship as the criterion for succession. It will be immediately apparent that, owing to the manner in which the numbers have been defined, the statement

> There is a concept F and an object x falling under F such that the number (as defined) which applies to the concept F is n, and the number which applies to the concept "falling under F but

not identical with x" is m

is such a generalization, and so can be taken as synonymous with the statement

n immediately follows m in the series of natural numbers

That this is so can be seen if we consider an example of the numbers as they have been defined. In the case of the number one, the general concept F is "equal to zero," the object x falling under F is 0 itself, the number n applying to the concept F is 1, and the number applying to the concept "falling under F but not identical with x" is 0. Thus 1 immediately follows 0 in the series of natural numbers, by definition.

Having completed the definitions of the individual numbers, which was our goal from the outset, let us examine the superiority of this suggestion over those which have been previously given. In so doing, we shall justify the claim that the offered definition truly is an improvement over those which we might have accepted before, and be confident that we have overcome those shortcomings which caused us to dismiss those views as incorrect.

First of all, in the view of Mill, our definitions of the numbers had required the observation of a "matter of fact," and then a consideration of the characteristic manner in which the agglomeration comprising that observed matter was made up. We have revealed the weaknesses of such a contention, and now point out that the Fregean definitions require no actual, observed, matter of fact for their validity.

The importance of this improvement is especially noticeable when we consider how the definitions as given by Mill faltered when considering numbers of substantial size, such as an octillion. With the requirement that numbers be defined by abstraction from an observed matter of fact, our ability to define such large numbers was cast into great doubt.

However, now that we have freed ourselves from the encumbrance of physicality by associating numbers with characteristic sets, we need not have any concern over whether or not there are actual sets consisting of an octillion members, other than the characteristic set of an octillion which would be (in theory) generated by our process of definition. It is enough that we maintain that any set which **could** be placed into a one to one correspondence with the characteristic set of an octillion would, by definition, be said to contain an octillion members.

Also, the obstacles which arose from the necessity of observing objects in their characteristic manner have been overcome by defining the numbers through the use of sets, whose membership is determined by agreement with specified traits. Each member of the set is then considered as one entity falling under the specifying trait of the set, which removes the possibility of considering individual entities as a plurality.

When we had looked at the offering of Leibniz and Berkeley, we had discovered that their contention (that numbers could only be defined subjectively) destroyed the notion that numbers were unique objects of study, made impossible the communication of mathematical ideas, and called for the containment of an infinite set of numbers within our all-too-finite minds.

From the Fregean point of view, we find that numbers have again had their objective certainty restored. The principle of equinumerosity which is called upon in defining the numbers is one which assures us of unanimous accord when speaking of the numbers, as the characteristic sets by which the numbers have been defined are the same for everyone. As a consequence, when someone is heard to utter "These are two roses," for example, it is only with the characteristic set {0,1} that the number two is affiliated, and thus there is no room for misunderstanding. Hence the ambiguity which had existed with Leibniz's subjective view is eliminated.

By defining numbers through the use of characteristic sets which stand apart from individuals we have also restored communicability to mathematical ideas, for much the same reasons as those which were just described. When I use the number two, for example, in calculation, there is no longer the worry that the two which I employ might not be the same as that of someone else, as the objectivity of the definitions ensures our agreement.

Finally, since the definitions of the numbers have been established as objective constructions exterior to our minds, the problem which arose from attempting to confine the infinite set of numbers within our finite minds is avoided entirely.

Thus, having demonstrated that the Fregean definitions for numbers, based as they are upon the notion of one to one correspondence with characteristic sets, are not subject to the criticisms which caused us to reject those views which were given previously, and having shown that by so defining the numbers we find agreement with the familiar qualities of numbers, we can feel confident that our definitions for the numbers are satisfactory.

We recall that one of the motivations for seeking an adequate definition for the numbers was the assurance of a solid foundation for the theoretical structure of mathematics. As our definitions for the numbers are based upon the simple logical relation of the one to one correspondence and the familiar notion of a set, we have established a certainty for our knowledge of the numbers which originates from the truths of logic, and stands apart from intuition. Consequently, the uncertainty which was present when we began our efforts has been overcome, and hence the rigor of mathematics as based upon the numbers assured.

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