

VARIABLES: WHAT ARE THEY AND WHY ARE THEY IMPORTANT?

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LITERATURE REVIEW

As a student in MATH 412 last semester, I encountered literature that suggested Algebra I students' difficulty with that subject stemmed from their misconceptions concerning the concept of variable. Indeed, the transition from junior high school to high school mathematics is challenging enough and these things that we call variables truly form the base from which the concepts of Algebra develop. A variable can be identified as a special type of mapping, from a set of objects into some number system. This mapping is based upon the measurement of some characteristic of the objects. Frequently, a variable is not named using a word or phrase, but rather, with an abstract symbol (i.e. a letter). Most of the Algebra textbooks I have examined have defined a variable as a letter that stands for a number. This seemingly simplistic treatment of the concept of variable may form the basis for why students' understanding of variables is too narrow.

Sigrid Wagner wrote an article for Mathematics Teacher entitled "What Are These Things Called Variables?" which I found to be extremely helpful in identifying the problem explored in this project. One of his favorite Algebra stories seems to epitomize students' misconceptions concerning variables. "The topic of the lesson was consecutive integer word problems. The teacher is trying to prepare the students for the x , $x+1$, ... literal symbolism by starting with a numerical example. 'What is the next consecutive integer after 17?' she asked. A student replied, '18'. Knowing that the representation for adding one would trouble the students, the teacher asked, 'What do we have to do to 17 to get 18?' 'Add one,' came the reply. 'Now suppose we use x to represent an unknown integer. How can we write the next consecutive integer after x ? That is, how can we represent the number we get when we add 1 to x ?' Without hesitation, the response was, ' y .' "When they are first introduced to literal symbols, often students confuse the linear order of the alphabet and the linear ordering of whole numbers. In Algebra, literal symbols are used in a novel way which may suggest why students are bewildered due to the fact that literal symbols are

relatively easy for elementary students to use. For example, when solving for unknowns in equations, elementary students will see symbols such as boxes, questions marks, or blanks." (Wagner, 474)

According to Wagner's research, literal symbols are easy to use but hard to understand due to the fact that they are like numerals, only they are different and they are like words, only they are different. Indeed, letters are like numerals in several different ways. There is a coincidental ordering of letters and numbers. Even a few letters are actually numerals (i.e. pie and e) because they are standard symbols that do not have simple digital representations. Many times we will see numerals and letters appearing together in mathematical statements, so it is no wonder that letters look as though they should behave just like numerals. Finally, the most evident similarity is that a letter can actually serve as a temporary numeral as evidenced by the fact that it is the symbol one writes until one figures out what the missing number is and can write the real numeral. (Wagner, 475)

In order to truly grasp the concept of variable one must be able to appreciate the differences between letters and numerals. Numerals represent a single number but letters can represent simultaneously, yet individually, many different numbers. When we call literal symbols variables, we are, in essence, referring to this property of simultaneous representation. Due to this property, mathematical language has the ability to make very general statements (definitions, axioms, theorems, formulas) in concise and unambiguous form. Further, both letters and numerals can be used to identify or name things besides numbers. For example, identification numbers are used to label specific, fixed elements of a set, whereas literal symbol names are more often used to identify random, variable elements. In fact, the reason we tend to use letters rather than numerals to represent

generalized elements is because "letters look like abbreviated names and thus are easy even for young students to interpret naively and we know that letters will be used to represent numerical variables later on." (Wagner, 475) In 1979, Marilyn Matz pointed out two other valuable differences between letters and numerals. One is the juxtaposition convention that we use with letters and numerals to indicate multiplication (i.e. $3mn$) as opposed to the place value interpretation that we give to numerals alone (i.e. 347). This works because letters and numerals come from different symbol systems and letters can represent numbers having either single digit or multidigit numerals. The other argument is that the signs attached to literal symbols do not always match their value as they would with numerals. For example, "x" could be negative and "-x" could be positive. This helps us to understand why many students cannot understand the definition of absolute value. (Wagner, 476)

Not only are letters like numerals, but they act like words as well which may enhance students' understanding. Both letters and words can be placeholders in certain expressions. Indeed, students must understand the placeholder property in order to appreciate the generality and flexibility of literal symbols. Letters are often chosen to suggest abbreviations for words as when "n" is used to represent the missing number in $3+n=5$. However, some students want to say that "a" represents apples instead of the number of apples. Both letters and words can mean different things in different contexts. "The meaning of a literal symbol is derived from its role, its domain of values, and its associated truth set." (Wagner, 476)

Although both letters and numerals may assume different meanings in different contexts, they differ with regard to their consistency of meaning throughout a single context. For instance, the value of a literal symbol has to be the

the same wherever the symbol appears in a given context; this is not true for verbal expressions. Also, letters are not associated with fixed sets of meaning the way words are. Another distinguishing factor is that we are free to choose almost any literal symbol to represent a given replacement set. It is important to note that changes in the domain do not yield changes in the literal symbol used to represent it. With regard to verbal expressions, a given word or phrase can automatically restrict the domain set and thus limit the generality of a statement. Further, letters in and of themselves do not have the same connotations that expressions have. However, certain letters do have context-specific connotations which have derived from tradition. There are many interchangeable literal symbols to represent any given domain (i.e. k, k_1, k_2, \dots). Therefore, changes in literal symbols do not necessarily imply changes in the domain. "So we can substitute equivalent expressions to deduce logical relationships among variables without altering either the actual or implied meaning of the literal symbols." (Wagner, 478) On the other hand, changing a verbal expression nearly always leads to some change in its domain.

In conclusion, Wagner offers invaluable teaching suggestions which all Algebra teachers should employ to give students a better understanding of literal symbols. One of the major points Wagner makes is that there needs to be a greater awareness of the many ways these symbols are used as well as a greater recognition of the particular characteristics they exhibit in various contexts. One needs to alert students to these properties and point out which characteristics are similar to words or numerals and which are unique to literal symbols. These ideas must emerge gradually as different uses of literal symbols appear in the curriculum. For example, in the early grades we can introduce the letter "p" to label a point or the letter "n" to stand for a number and say that these letters are like abbreviations for words. Later, we can use arbitrary letters as labels and explain

that these letters are like names for things. Wagner notes that when we start using arbitrary letters in numerical contexts we should mention that there is no connection between alphabetical order and numerical order. At the Algebra level, students should be made to realize that a letter behaves like a numeral in that it may represent a single number and it may be subject to operations and relations. Students should also know that a letter is like a word in that it can mean different things in different contexts but a letter is different from a word in that it must refer to the same thing throughout a single context. The placeholder analogy leads students to appreciate that letters often represent many different numbers simultaneously. Wagner also states that it is imperative "to contrast the generality of letters with the connotative richness of verbal expressions." (Wagner, 478) Further, teachers must delimit the domain of a literal symbol. Many students memorize that different letters can be used to represent the same thing but they believe that different letters must represent different things. For example, given this problem: $3x + 7 = 15$ and then given $3w + 7 = 15$, students cannot solve the second problem by simply solving the first one. A final suggestion Wagner offers is that when teachers give numerical illustrations of numerical properties (i.e. associative or distributive laws) they could use some examples in which different letters have the same value. (Wagner, 479)

In 1981, the Results from the Second Mathematics Assessment of the National Assessment of Educational Progress provided me with an interesting set of research materials in the area of the concept of variable. The purpose of the Second Assessment was to measure students' ability to work with mathematical variables and relationships. Most of the exercises in the study dealt with algebraic concepts and manipulations, and the rest explored basic variable concepts that

are included in elementary and middle school mathematics curricula. The two studies I was particularly interested in were: (1) variables in equations and inequalities and (2) variables used to represent elements of a number system. The assessment was carried out on 17-year olds and it was an inventory of algebraic skills and understanding retained one to two years after studying elementary algebra. (Carpenter, 56)

It was evident that the students' performance was related to their course background. The exercises assessing students' ability to solve equations included both simple open-sentence problems as well as more complex problems requiring formal knowledge of algebra (i.e. linear equations and inequalities in one unknown, systems of equations and quadratic equations.) Students were also asked to write equations to represent verbal problems. The major results of the study included the fact that although students could solve problems intuitively they could not adapt the more formal procedures. Less than one-half of the students assessed could solve linear equations, two-thirds of those with two years of Algebra could do so. About twenty-five percent of students with a year of Algebra and about forty percent of those students with two years of Algebra could solve systems of equations and quadratic equations by factoring. Performance on Algebra word problems was consistently low. Most of the students did not understand special properties of inequalities and treated inequalities as equations. (Carpenter, 58)

With regards to solving simple equations in one unknown, it appears that open addition sentences were easy because they could be solved through inspection. In contrast, subtraction sentences were more difficult because they involved some sort of transformation. Some additional insights into students' understanding of variables include the fact that the format of the problem significantly affected the students' performance. Surprisingly enough, it seemed that the students were more comfortable using a box to represent a variable as

opposed to a letter. Further, the only type of problem that gave the students difficulty was the one that dealt with nonnumerical coefficients. Indeed, there was relatively little difference in difficulty among the other problems even though they appeared to differ in difficulty and the number of steps required for solution. (Carpenter, 59)

When students were asked to solve systems of equations using the relatively simple method of substitution, even students with two years of Algebra scored low. Quadratic equations have been described as the capstone of traditional first year Algebra courses. According to this assessment; students seemed to be able to identify a quadratic equation; however, even simple quadratic equations were difficult for them to solve. Difficulty has been attributed to an inadequate knowledge of the required subskills. The most troublesome aspect of quadratics appears to have been applying the quadratic formula fo solve these equations. (Carpenter, 61)

Applications of Algebra are derived from the translation of information into algebraic form. This accounted for yet another area of the assessment. Most Algebra students encounter information embedded in various situations and are required to translate it into equations. The foundation for this skill appears at the beginning of the elementary level when students need to write open sentences to represent problem situations. Although Algebra students are relatively successful at solving one-step verbal problems, they had the most difficulty with formulating them to represent problem situations. This lends further credence to the fact that even simple arithmetic relationships cannot be expressed using mathematical symbolism, especially with variables. Thus, it follows that one of the most difficult algebra topics assessed was solving algebra word problems. It was stated that "it was impossible to determine whether students who correctly solved the problem did not write an equation because they were unable to do so or whether they felt it was not necessary to do so to find and answer." (Carpenter, 60) Regardless, one of the major areas of

difficulty for most students was applying algebra skills to solve problems or truly understanding the concept of variable. It was concluded that although a second year of Algebra may improve students' algebraic skills, it will do little to help them learn how to apply these skills. (Carpenter, 60)

Another feature of this assessment that was particularly interesting to me was the part that explored operations with variables. This included manipulation of algebraic expressions and the use of variables to represent basic mathematical concepts. Evidently, students' success in simplifying algebraic expressions was directly correlated to their understanding of the arithmetic concepts represented by the algebraic expressions. It also was obvious that many younger students had difficulty using variables to express mathematical relations, but most of the older students could use variables to express most simple mathematical relations. With respect to simplifying algebraic expressions, no matter what the age level of the student, their performance decreased as the expressions became more complex. Further, the students demonstrated little ability to manipulate relational expressions. It was particularly noted that the students did not recognize that the processes appropriate for adding fractions also apply to rational expressions involving variables. "Students' ability to operate with variables was dependent on the complexity of the operations involved. In general, students had significantly more difficulty simplifying expressions that included variables or using variables to express mathematical relations than they did when only numerical operations were involved." (Carpenter, 63)

In conclusion, the last area of the assessment that I found particularly helpful for my exploration of the concept of variable was the part that used variables to express mathematical concepts. There is no doubt that variables

provide a convenient notation to express general mathematical properties and relationships. Indeed, in order to understand formulas or to solve problems depends on the ability to use variables to describe situations and express specific relations between terms. As would be expected, students' ability to represent particular relations using variables depended on their understanding of the relations involved and on the amount of algebra they had taken. For, example the highest success rate on the following question was about sixty-five percent: What expression expresses the idea that when a number is multiplied by 0, the result is 0? Also, algebra notation made identifying equivalent fractions a much more difficult task. (Carpenter, 70)

One of the most important questions I explored in my interviews was the one inspired by Peter Rosnick's article for Mathematics Teacher: "Some Misconceptions Concerning the Concept of Variable." Rosnick's convictions include that although there exist many compelling arguments concerning the causes of difficulty with mathematics which includes cultural, political and psychological factors, we must not neglect the inherent difficulty of the subject itself. He points out that as the curriculum becomes more abstract, the symbols become more obscure. According to Rosnick, "unfamiliarity with mathematical symbols and the abstract concepts to which they refer builds contempt for mathematics." (Rosnick, 418) The question that this article deals with is an extension of research done by the Cognitive Development Project at the University of Massachusetts that focused on students' ability to translate English sentences into algebraic expressions and vice versa. Evidently, even Calculus students and first-year engineering students have a difficult time with these translations. Research has confirmed that the difficulty is derived from misconceptions surrounding the use of letters in equations. The research was based on the classic Students and Professors problem: Write an equation, using the variables S and

P to represent the following statement: "At this university there are six times as many students as professors." (Rosnick, 419)

Surprisingly enough, thirty-seven percent of entering engineering students at the University of Massachusetts were unable to write the correct equation, $S=6P$. The most common mistake was that students reversed the equation, $P=6S$. Based upon student interviews the reason why students reverse the equation is attributed to the fact that students believe that S is a label standing for students rather than a variable standing for number of students. They interpret $6S=P$ as six students for every one professor. The fact that variables stand for numbers may sound obvious but it is not at all to students. Further, over forty percent of the college students interviewed could not correctly identify that "P" stood for the number of professors. This supports the hypothesis that students seem to view the use of letters in equations as labels that refer to concrete entities. "These results support a conclusion that we have believed for some time: that the tendency on the part of many students to write the reversed equation is not only a common one but is one that is deeply entrenched." (Rosnick, 420) Indeed, most students who believed that "S" stood for "professor" also believed that six students equals one professor ($6S=P$) was the correct equation. They believe this so profusely that when given $S=6P$, they assume that the meanings of the letters have been switched. (Rosnick, 420)

The implications of this study seem relatively clear. Students need to develop a better understanding of the basic concepts of variable and equation. They need to be able to distinguish between different ways in which letters can be used in equations. "They should learn to distinguish when letters are used as labels referring to concrete entities or, alternately, as variables standing abstractly for some number or numbers of things." (Rosnick, 420) We need to be aware of the fact that many high school and college students have not yet reached the necessary level of intellectual development to be able to make that distinction. Teachers also need to be exceedingly careful about defining

variables. It is important that teachers keep updated about the difficulties students have in trying to understand labels, variables, constants, parameters, and all the other uses of letters. (Rosnick, 421)

Gary Perlman wrote an article for Mathematics Teacher entitled "Making Mathematical Notation More Meaningful" which provided helpful ideas that will make variables meaningful. In order to simplify the task of communicating mathematical ideas with symbols, Perlman suggests making symbols familiar as well as making them mnemonics. Further, he stresses that notations should lend themselves to generalizations. Indeed, a notation should not simply be employed to facilitate conciseness, but rather, it should convey meaning in and of itself. Perhaps Alfred Whitehead, in "An Introduction to Mathematics", expressed it best when he said: 'By the aid of symbolism, we can make transitions in reasoning almost mechanically by the eye which otherwise would call upon higher faculties of the brain. By relieving the brain of all unnecessary work, a good notation sets it free to concentrate on more advanced problems.' One of the reasons it is important to make notation meaningful is because many advances in mathematics have been due to suggestive notations. Therefore, Perlman explains that it is necessary for notations to be as concise as possible. Hence, by packing a lot of information into a small space, more information is available at a glance, which allows one to observe relationships among ideas. Although there exists a risk whenever we introduce new notation, we can assuage some of it by ensuring that commonly denoted things have their own names. Obviously, it requires effort to learn the meaning of new symbols. Perlman suggests that if a concept is used frequently, then a notation specifically for it is appropriate; but symbolism just for the sake of using symbols can be an unnecessary burden on memory. (Perlman, 463)

Another point that Perlman makes is that a notation should be precise and promote generalizations. Hence, similar ideas should have similar notations. "How one idea is represented should be consistent with how related ideas are

represented and it should provide clues about how they are represented." (Perlman, 463) Perlman purports that symbols should be mnemonics. Arbitrarily chosen symbols will not promote memory and so will be harder to learn than ones well chosen. Thus, a symbol should have something in common with the object it represents. Whether a mnemonic has anything to do with the concept it denotes is not as important as having a relation to some existing attribute such as its name. To help students learn notation, we should give a rationale whenever new notation is introduced. Otherwise, a symbol's mnemonic value may not be picked up on by the student. Teachers should always avoid symbols which are obscure. That is what makes variables so handy. Students are used to dealing with letters. As other research has confirmed, it may be beneficial to point out the similarities between letters and words. (Perlman, 465)

Further, Perlman suggests that symbols and concepts should be unique or in a one-to-one correspondence within a given mathematical topic. For example, the same symbol should not be used to denote more than one object. However, it is common for one symbol to have more than one meaning. These multiple meanings for symbols usually do not present a precision problem if the student is aware there can be more than one meaning for symbols and is able to determine which meanings are meant by the contexts in which they appear. In addition, existing notation should be maintained. In conclusion, Perlman discusses the strong analogy between learning mathematics' language and one's mother tongue. Vocabulary and grammar must be learned prior to meaningful communication. When students begin learning an area of mathematics, a great deal of effort should be devoted to teaching notation. Finally, by the time students get to more advanced mathematics, the time devoted to introducing notation gets shorter, but no less crucial. (Perlman, 467)

We teach equation solving all the time, but are the students really learning? It is clearly evident that we need to construct meaning for the concept of equation which in essence means building the meaning for the concept of variable. According to Nicolas Herscovics, even the simpler "think of a number" approach to introducing the concept will be far too complex for those students who cannot accept the representation of a number by a letter. No one will be able to construct the meaning for equation if one cannot even understand the symbols they manipulate. For example, nonconservers are those students who do not realize that the value of an unknown is independent of the letter used. Further, when students solve a problem such as $2 \times ? = 18$, we presume that they have some understanding of the concept of an unknown and of equation and thus can handle Algebra. In reality, students may be just guessing or using mental arithmetic. Herscovics suggests starting from arithmetic to confirm the concept of equation, thus allowing the student to comprehend the concept intuitively before it becomes formalized symbolically. There appears to be a consensus on the pedagogical value in handling solvable equations first and leaving those that have no solution to a later time. (Herscovics, 574)

After the essence of the concept of equation is understood (i.e. that the equal sign indicates that the operations on each side yield identical values or arithmetic identities). Herscovics suggests taking an arithmetic identity and covering up one of the numbers with a finger. Thus, he purports defining an equation as an arithmetic identity with a hidden number. Now, it is comfortable for the students to work with more practical representations such as empty boxes. Again, these are stepping stones in the gradual process for developing meaning for the new mathematical form variables. Hence, Herscovics feels that it is a rather short leap to replace the box with a letter of the alphabet.

Students will be able to justify the letter being called an unknown because it corresponds closely to the idea of a hidden number. It is essential at this stage that the students be encouraged to choose many different letters. Also, it is important for students to realize that a given arithmetic identity can lead to the construction of many different equations. This kind of variety in building equations prevents students from putting unnecessary restrictions on the concept of equation. In order to avoid confusion, it is suggested that we point out that the convention that the same letter could be used in an equation more than once as long as it was used to hide the same number, otherwise, one must use two different letters.

In the process of solving an equation, students have to transform expressions involving the unknown. Studies have proven that many students experience a great deal of difficulty in operating on these expressions. These difficulties may be due to the fact that the students have a hard time thinking of a letter as representing a number. In conclusion, if we stress the method that solving an equation is analogous to "unwrapping a present", which handles simultaneously the arithmetic identity with the boxed-in number and the corresponding equation, the student is constantly reminded that the letter stands for a number.

(Herscovics, 579)

In 1981, Narode and Rosnick conducted a study which focused on intuitive misconceptions in algebra as a source of math anxiety entitled, "Focus on Learning Problems in Mathematics." Granted, many students have difficulty with simplifying or solving algebraic equations, but it is the difficulties which pertain to formulating or interpreting the meaning of algebraic equations that the paper addresses. Even science-oriented college students make errors that result from mistranslations of meaning in formulating or interpreting equations. We must be concerned about the value of algebra if it does not hold meaning for students. I was interested in the number of common misconceptions found concerning the

meaning of variables and equations. The misconceptions seemed to be derived from "stable conceptual schemata." which seem convincingly logical to the student. (Narode, 46)

One of the most prevalent equation formulation errors that he investigated was the reversal error; as evidenced by the "Students and Professors" problem addressed earlier. A possible cause for this error may be what is called "word order matching" where the student will write an equation in a way that parallels the order of the words in an English sentence. Even when students demonstrated their understanding of the relative sizes of the groups of students and professors by drawing pictures, by rewording the problem or by constructing data tables and graphs, they still reversed. A second behavior pattern observed was when students attempted to solve certain word problems involving ratios in terms of "lots". Students were given the following problem: At Mindy's restaurant, for every four people who order cheesecake, there are five people who order strudel. Let C represent the number of cheesecakes and S represent the number of strudels. Despite the fact that many students will write the reversed equation, some students use this incorrect expression to correctly find one numerical quantity given the other. Thus, they think of the quantities by corresponding sets of "lots" where groups of four cheesecakes are placed in one-to-one correspondence with groups of five strudels. It may be concluded that the students understand the numerical relationship described in the problem. Further, the students have misinterpreted the meaning of the variables C and S . The students must be shown why their solution is wrong. (Narode, 51)

Another misconception which often occurs is the spontaneous inclusion of a new quantity - the total. It appears that, for example, the number of professors and students only make sense if some kind of sum is included in the equation. Students may also symbolize the relationship between two quantities in a fractional ratio problem by considering their sum. "This is an instance where a problem has triggered the activity of more cognitive structures than students need to

solve it. It is encouraging that they understand the problem situation well enough to see relationships that are not explicitly stated." (Narode, 53)

The totals equations themselves are often wrong because the students' synthesis of relationships does not work. We cannot blame the students because this is a natural consequence of the way the student's ideas are structured. In conclusion, Narode points out that these students often evidence a clear conception of the problem situation sufficient to produce numerical answers. The student proceeds to use algebraic symbols to symbolize the relationship in his conception of the problem, but he does so in an overly associative manner. "His inductive attempts at symbolization come into direct conflict with the standard interpretation of variables and equations." (Narode, 56) It is imperative for teachers to note that a student who evidences these characteristics may not respond to teaching by admonishment and demonstration. Believing that he understands something and being told otherwise by a teacher can place a student in a paradoxical situation which could produce math anxiety which, in turn, will cause a student to doubt his own mathematical reasoning skills. "Avoiding this all too frequent outcome is perhaps the most compelling motive for paying attention to the intuitive ideas expressed by students. Misconceptions which deal with meaning and understanding are easy to overlook. The curriculum focuses only on the rules of symbol manipulation. It appears that more attention to equation formulation skills is called for to complement the existing emphasis on equation manipulation skills." (Narode, 57)

The intrinsic value of a student interview was made clearly evident to me after reading an article by Robert Davis entitled "Cognitive Processes Involved in Solving Simple Algebraic Equations." The interview was conducted with Henry, a gifted, seventh grade Algebra student. He had much less experience with algebraic notation that he would have had in a standard ninth-grade Algebra course, but in a mathematical sense he apparently mastered all of the essential concepts

and techniques. "The fact that what was mathematically necessary for the solution of the equation $3/x = 6/(3x + 1)$ differed considerably from what was cognitively necessary and the details of how they differed, constitute the main value of the fifteen-minute interview." (Davis, 9) Davis suggests that the slow progression by minute steps through Algebra aids in obscuring the major themes of Algebra. Further, he supports the conviction that student response to the use of heuristics has been disappointing. Students want us to tell them what to do and how to do it or, in other words, how to play the game. There may be too much emphasis on effective heuristic methods of problem analysis.

Here are some of the problems Henry had in tackling the problem. First of all, he was confused about whether $3/x$ meant $x/3$ or $3/x$. Henry proceeded to use a form of the distributive law that does not exist (i.e. $x/3 = 6/3x + 6/1$). Henry was convinced this was not correct after Davis gave him a few numerical examples which proved "his" law did not work. Henry's goal was to get all of the x 's on one side and all the numbers on the other side. The interviewer pointed out that that was one way of approaching the problem but there could be others. Henry, after a suggestion was made to clear the fractions to make the problem easier, responded by saying that the fractions did not bother him but having the unknown in the denominator did. Davis pointed out that he needed to multiply through by x and avoided any long digression by questioning if x were 0. There seemed to be no problem in obtaining 3 on the left hand side, but when asked what do we get on the right hand side, Henry responded by asking how can we multiply by x when we do not know what x is? Clearly, Henry was not prepared to accept $6x$ as a name for the answer of "what you get when you multiply 6 by x ," but only as a statement of the task that 6 is supposed to be multiplied by x . Davis finally gets Henry to accept that we obtain $3 = 6x/(3x + 1)$ but how he does this is left out. In order to get rid of the other fraction, Henry did not see a parallel and simply wanted to multiply by x or by 3. Evidently,

Henry was utilizing two different interpretations of the equal sign. Henry was trying to "seperate" the x from the "numbers", hence, evidence of a strong misconception of the concept of variable. Henry was finally at the point of $3(3x + 1) = 6x / (3x + 1) (3x + 1)$. Henry continued by incorrectly multiplying the left hand side to get $3(3x + 1)$ or $9x + 1$. After pointing out the correct use of the distributive law as opposed to the associative law Henry was using, Davis put him back on the right track. For dealing with the right hand side, the interviewer tried to use the analogy with $2/5 (3) = 2 (3)/5$ to justify $[6x(3x + 1)] / (3x + 1)$ and then the analogy with a problem recalling the cancellation law. Henry tried this by canceling the x in the denominator with the first x in the numerator. It took some convincing that $9x + 3 = 6x$ did not equal $3 + 3$ and finally a solution was obtained. (Davis, 27)

One might infer that what is cognitively necessary to solve such equations is quite different from what is mathematically necessary. For example, Henry could make appropriate replacements of the variables in order to use the distributive law. "Many major cognitive adjustments - accommodations rather than assimilations are required if one is to do the necessary mental flip-flop and start seeing the equal sign in new ways and even seeing $3/x$ as an answer instead of a problem. This may not be acquired by the gentle accumulation of small increments." (Davis, 29)

PROCEDURES

In this segment of my Honors Capstone Project I would like to discuss the procedures I went through in order to prepare me for the student interviews I conducted. Since I had no prior experience with interviewing, I followed the suggestions pertaining to interviewing techniques I acquired from my advisor Dr. Nancy Mack and the "Algebra Learning Project" conducted by Sigrid Wagner, Robert Jensen and Sid Rachlin.

My primary goal was to learn about the students' thinking processes. If the student was silent for a time, I would try to ask what she was thinking. It was beneficial for me, when the student was writing but not talking, to ask them to tell me what they were writing. I learned that I was never to interrupt a student while talking in order to paraphrase; but rather to wait until they were finished and then to ask them to repeat or paraphrase. Surprisingly enough, it is important to ask a student to explain a procedure even if we "know" what was done. Further, asking neutral questions avoids asking leading questions which might suggest a particular answer or process. With respect to comments, I learned they should be as neutral and nondirective as possible so as to be general and not task-specific.

One of the goals of any interviewing session is to ensure that the students have a positive experience. Hence, it is essential to be uniformly positive with all the students. For example, reinforcement should be given wherever possible as opposed to utilizing phrases such as "why you are confused", "hanging you up", "causing you trouble", and "why you are having difficulty". Non-evaluative comments should also be avoided whether a method is different, whether something is obvious, easy, or similar. One of the hardest aspects of interviewing for me to learn was not to bug the students unnecessarily. There exists a fine line between giving them a chance to think and constantly asking them what they

are thinking. Therefore, it is important to wait quite a while before concluding a student is stuck and giving hints. The justification for this is because if hints come too readily, not only will students feel that they cannot solve problems themselves, but they will begin relying on the interviewer for solutions.

The underlying assumption of the interviewing process is that we are investigating the students' ability to generalize structures and processes and to reverse processes. One must also be concerned with how close the student is to understanding certain ideas. In other words, we must note the effect that hints have upon the students' ability to generalize and reverse. I learned the general strategy for conducting interviews; namely, if the student has difficulty understanding the problem or knowing what to do at any subsequent critical step in solving the problem, one should employ standard problem solving heuristics to provide hints. With regard to specific strategies, foremost in the mind of the interviewer should be to try to help the student understand. This may be facilitated by asking the student to restate the problem in his/her own words, asking them what the problem is asking for or by asking them what they are given in the problem. If necessary, one should be capable of making the problem more understandable through generalization or reversibility. Generalization allows us to ask students if they have seen other problems similar to the one they are faced with. Further, it allows us to ask the students how we could make the problem simpler as well as allowing us to suggest a simpler form. Reversibility allows us to ask the students what their answer should look like as well as asking them what they need to know in order to get the answer.

The last important aspect of interviewing I had to learn was how to handle errors. Errors may occur in understanding the problem or at any subsequent step in the solution process. In any given task, we treat errors in pre-

requisite knowledge differently from errors which are critical to the task on hand. Thus, it is important to use judgment in gauging individual differences in students' understanding of prerequisite knowledge. For example, some students will benefit from a redevelopment of certain topics while others will only need a quick reminder. As interviewers, we do not want to interrupt a students' train of thought unnecessarily. Some of the guidelines I followed are given below. If a student obtains a correct solution to a problem using a correct process and seems satisfied, it is customary to go on to the next problem. However, if a student obtains a correct solution to a problem using a correct process and wonders if it is correct, then, the interviewer should suggest checking. An incorrect process that accidentally leads to a correct solution is treated as a critical error. If a student makes a simple error involving prerequisite knowledge that does not essentially change either the problem or the solution and seems satisfied, then we should go on to the next problem. Again, if the student wonders if he/she is right then we should point out the small error, indicate how it would affect the answer, and reassure the student that such a mistake is unimportant. If a student makes an error involving prerequisite knowledge that simplifies or complicates the problem and proceeds to a wrong solution, we should ask the student to check the answer. If the solution checks, presumably due to a repeat of the error, we should pose a new problem by correcting the incorrect step. If the solution does not check, either the student may discover the error or the interviewer can point it out and the student can work the corrected problem. On the other hand, if the student gets completely stuck, we should point out the error without elaborate teaching. Further, if a student misunderstands terminology, we should clarify it as soon as possible without interrupting the student. Misunderstanding the problem is treated as a critical error.

Finally, if a student makes a critical error, regardless of how it affects the problem or solution, and proceeds on to a wrong solution, we should ask the student to check the answer. In most instances, it is necessary to go back to

the original wording of the problem for the error to manifest itself. If the student does not discover the error, it may become necessary to pose a simpler version of the step containing the error. At this point, one should explore the students' ability to generalize by suggesting still other versions of that step. We may even want the student to suppose other instances. Then we should refer back to the original task and ask the student if anything needs to be changed. If the student still does not discover the error, then we can point out the step in which the mistake occurred. If necessary, we can make the correction and ask the student to finish the problem. At this point, if the procedure does not work, we may indicate that we can return to the problem at the end of the session and go on to the next problem. On the other hand, if the student gets stuck at a specific step in the procedure, we can use judgment as to whether the trouble involves prerequisite knowledge or critical knowledge and handle the situation either by quickly correcting it or by diverting and developing the idea. However, if the student does not understand the problem, then we should give hints, moving from general to specific. For example, in general, we might ask the student what the problem is asking for or ask them to paraphrase the question. Specifically, we can ask the student if they have seen similar problems, what would make this problem easier or what the answer should look like. These hints proved invaluable to me as I started my interviews. It was one thing to read through these guidelines, but it was another to actually put them into practice. I found that when I was "stuck" during the interviews I turned to these interviewing techniques to guide me towards my next question.

In order to be truly prepared to enhance students' understanding of the concept of variable, I had to be sure that I had a concrete grasp of the

structure behind the concept. A task analysis provided me with the prerequisite knowledge an Algebra student would need in order to appreciate some of the underlying aspects of the concept of variable. In light of the seemingly simplistic treatment of variables found in the Algebra books in use today, I was prepared to assist the students I interviewed in building their understanding of variables. In fact, I did not expect the students to give me a reasonable working definition of variable when I first began the interviewing process. As discussed later on in this project, I was pleased to learn that the students I interviewed had a fairly sophisticated grasp of the concept. Nevertheless, it was very worthwhile for me to contemplate what a student would really need to know in order to understand variables.

One encounters variables in several contexts. For example, we use variables in formulas ($A=LW$), in equations ($40=5x$), in identities ($\sin x = \cos x \tan x$), in properties ($1=n(1/n)$), and in equations of functions of direct variation ($y=kx$). These different situations reflect different uses to which the idea of variable is placed, however, all require the same operating notion of the concept. In reference to formulas, variables have the feel of knowns (i.e. some quantity). Equations tend to lead us to think of variables as unknowns; whereas, in identities, variables serve as arguments of functions. Properties, unlike the others, generalize arithmetic patterns and variables identify certain instances of the pattern. Further, only with functions of direct variation do we acquire the feel of "variability" from which the term variable arose. (Usiskin, 9)

Conceptions of variables have changed over time. In a text of the 1950's, the word variable is not mentioned until the discussion of systems and then it is described as a "changing number." Today, the introduction of variables emerges much earlier as evidenced through the use of formulas in textbooks. For example, texts will refer to variables as a literal number that may have two or more

values during a particular discussion. In the early 1960's, variables were defined as symbols for which one substituted names for some objects, usually a number in algebra. These objects are called values of the variable. Presently, there is a tendency to avoid the "name-object" distinction and to think of a variable simply as a symbol for which things from a particular replacement set can be substituted. According to Kramer, there is an alternative viewpoint possible for variables. It appears that at the early part of this century, the formalist school of mathematics considered variables and all other mathematics' symbols merely as marks on paper related to each other by assumed or derived properties that are also marks on paper. "Ironically, modern day computers can operate as both experienced and inexperienced users of algebra do operate, blindly manipulating variables without any concern or knowledge of what they represent." (Usiskin, 10)

It seems that several misconceptions concerning variables derive from the fact that many students think all variables are letters that stand for numbers. Students need to be exposed to the fact that even in high school mathematics, the values a variable takes are not always numbers. For example, in geometry, variables often represent points. In logic, variables often stand for propositions; in analysis, variables often stand for functions; in linear algebra, a variable may stand for a matrix; and in higher algebra a variable could stand for an operation. The last of these examples demonstrates that variables need not be represented by letters. "In summary, variables have many possible definitions, referents, and symbols. Trying to fit the idea of variable into a single conception oversimplifies the idea and in turn distorts the purposes of algebra." (Usiskin, 11)

Perhaps the major issue surrounding the teaching of algebra in schools today has to do with the extent to which students should be required to be able to do various manipulative skills by hand. We must take into account; however, that in the future students may not have to do much algebraic manipulation. A second issue relating to the algebra curriculum is the question of the role of functions and the timing of their introduction. It has been argued that functions should be used as the major vehicle through which variables and algebra are introduced. Usiskin purports that the purposes we have for teaching algebra, the conceptions we have of the subject, and the uses of variables are inextricably related. "Purposes for algebra are determined by, or are related to, different conceptions of algebra, which correlate with the different relative importance given to various uses of variables." (Usiskin, 11)

To facilitate understanding of variables, we may introduce algebra as generalized arithmetic. In this conception, it is natural to think of variables as pattern generalizers in dealing with generalizing properties. Also, at a more advanced level, the notion of variable as pattern generalizer is fundamental in mathematical modeling when dealing with relations between numbers. The key instructions for the student in this conception of algebra are translate and generalize. These are important skills for both algebra and arithmetic. Usiskin goes as far as stating that it is impossible to adequately study arithmetic without explicitly or implicitly dealing with variables. "Historically, the invention of algebraic notation in 1564 by Francois Viète had immediate effects. Within fifty years, analytic geometry had been invented and brought to an advanced form. Within a hundred years, there was calculus. Such is the power of algebra as generalized arithmetic." (Usiskin, 12)

Under this conception of algebra as a generalizer of patterns, we do not have unknowns. Hence, an alternate approach may be a conception of algebra as

a study of procedures. In this conception of algebra, variables are either unknowns or constants. The key instructions in this conception are simplify and solve, which are more similar than they are made out to be. Still another approach for building meaning for the concept of variable is to view algebra as the study of relationships among quantities. For instance, when we use variables in formulas, we do not get the feel of unknowns because we are not solving for anything. The crucial distinction between this and the previous conceptions is that here variables "vary". Due to its intrinsic algebraic nature, some mathematics educators believe that algebra should first be introduced through this use of variable. Further, Fey and Good see the following as the key questions on which to base the study of algebra: For a given function $f(x)$, find 1) $f(x)$ for $x=a$, 2) x so that $f(x)=a$, 3) x so that maximum or minimum values of $f(x)$ occur, 4) the rate of change in f near $x=a$, 5) the average value of f over the interval (a,b) . Under this conception, a variable is an argument (i.e. stands for a domain value of a function) or a parameter (i.e. stands for a number on which other numbers depend). Only in this conception do the notions of independent and dependent variable exist. A final conception towards approaching the concept of variable is to introduce algebra as the study of structures. Indeed, when we study algebra at the college level it involves structures such as groups, rings, integral domains, fields and vector spaces. "It seems to bear little resemblance to the study of algebra at the high school level, although the fields of real numbers and complex numbers and the various rings of polynomials underlie the theory of algebra and properties of integral domains and groups explain why certain equations can be solved and others not." (Usiskin, 15) When we ascribe properties to operations on real numbers and polynomials, we are, in essence, recognizing algebra as the study of structures. We want students to have

the referents (usually real numbers) for variables in mind as they use the variables. However, we also want students to be able to operate on the variables without always having to go to the level of the referent as when we ask students to check factoring problems by multiplying the binomial. We must instill faith in the properties of variables, in relationships between x's and y's whether they be addends, factors, bases, or exponents. Thus the variable becomes an arbitrary object in a structure related by certain properties as in abstract algebra. It is ironic that the two manifestations of the use of variable-theory and manipulation are often opposing views in policy setting toward the algebra curriculum even though they come from the same view of variable. (Usiskin, 16)

Herscovics provides one way to assist students create meaning for the concept on the basis of their existing knowledge. The "teaching outline" introduces algebraic expressions as answers to problems with easy visual representations. Herscovics recommends introducing the use of letters cautiously at first by letting them represent hidden quantities and then using them to stand for specific unknown quantities. This "geometric" approach is used systematically to help students construct meaning for expressions involving one unknown and one operation to expressions with several unknowns and multiple operations. Students should also be encouraged to reverse the process by generating problems corresponding to given expressions. Herscovics accounted for four cognitive obstacles in the learning of algebraic expressions that had been identified in prior research when he designed his teaching outline. The first was the students' lack of a numerical referent for the letter used. Indeed, if the student does not view letters as representing numbers, then performing arithmetic operations with them is a meaningless task. Another obstacle concerning algebraic expressions is their perception as incomplete statements as evidenced by students' inability to accept the lack of closure. Another cognitive obstacle is what is referred to as the

name-process dilemma that distinguishes algebra from arithmetic. The final obstacle is that of the different meaning associated with concatenation in algebra. In arithmetic, the juxtaposition of two numbers denotes addition; whereas in algebra, concatenation denotes multiplication. (Herscovics, 34)

If time would have permitted I would have enjoyed utilizing Herscovics' teaching outline in order to build students' understanding of variable. At this time, I would like to briefly describe his teaching outline as it would enhance any introductory algebra course and perhaps curtail future problems. Herscovics employs three types of problems due to their easy geometric representation. The first type of problem dealt with determining the total number of dots in a rectangular array for which only one dimension was shown; the second type was finding the length of a line segment in which the number part was hidden; the third type involved finding the area of a rectangle with only one dimension shown. The hiding was done by using a cardboard cover. (Herscovics, 34)

Herscovics first goal seems to be to move the students from placeholder to letter representation while keeping in mind that the introduction to algebraic expressions should be as simple as possible. Thus, students are initially taught to use the familiar box as a placeholder for the partially hidden dimension. It appears that the use of the box was not troublesome for students; they even correctly perceived the box as representing a total number or an undetermined quantity that was partially hidden. After the initial work with the box as a placeholder, Herscovics reintroduced the problems with the only change being a request that the students use a letter of their choice instead of a box to represent the hidden quantity. It is inferred that this approach will convey the arbitrary nature of the symbol. Again, it appeared that the students accepted the use of a letter and were willing to understand that the letter stood

for a number. "The presentation used here provided a situation where the letter became a natural extension of the students' use of the box as a placeholder, and both were explicitly linked to a numerical referent." (Herscovics, 36)

Further, the students did not seem bothered by a number multiplying a letter; so it was concluded that no cognitive dilemma existed.

The second aspect of Herscovics teaching outline proceeds to move the students' understanding from letters representing hidden quantities to letters representing unknown quantities. Due to the fact that the problem types were the same in this lesson, it was presumed that the use of a letter representing an unknown would be just an easy extension of its previous use in representing a hidden quantity. However, when students were asked to give the area of a rectangle with length "a" and width "8" , they thought they had to provide a numerical answer. Herscovics offers a possible explanation for this difficulty as an inability to accept the lack of closure or the inability to accept the algebraic expression by itself. Students seemed to be willing to accept the expression in the context of an equation which is referred to as the name-process dilemma. So as to maintain the numerical referent for the literal symbol, the students were asked to measure the length of the unknown dimension in the area problem. The measurement was followed by the substitution of the numerical value in the algebraic expression representing its area. (Herscovics, 37)

Herscovics seems especially concerned about the introduction of concatenation which apparently is underestimated in terms of students problems with it. The students were given instruction so that they were able to know that when a letter is attached to a number, there is a hidden multiplication there. Further, they were given instruction so that they clearly understood why this was not the case in arithmetic. Also, two more conventions were stressed to the

students. The first was that of multiplying a letter by the number one or writing $1x$ as x . The second convention was that the product of a number and a letter is always written with the number preceding the letter. It seemed that the students accepted these notational conventions without any difficulty. (Herscovics, 38)

The third lesson in the teaching outline deals with algebraic expressions involving multiple terms. The problems in this lesson involve both multiplication and addition. Results from this lesson confirm the possibility of students' transferring the concatenation convention from algebra to arithmetic. An interesting outcome found in implementing this lesson was that the area concept is strongly linked with multiplication and it creates a "mind-set" that prevents the student from perceiving the additive nature of the problem involving the sum of the two areas. The results of Herscovics' study imply that in early algebra, teachers cannot take a student's answer at face value. Further, we must be certain that the student is well aware of the frame of reference (arithmetic or algebra) in which he or she is to respond. (Herscovics, 42)

When I worked on my task analysis of the concept of variable, I relied heavily upon Algebra I: A Process Approach written by Sidney Rachlin for the Curriculum Research and Development Group at the University of Hawaii. As I stated earlier, the task analysis was useful in assisting my understanding of what prerequisite knowledge would be necessary for a student to have before he was introduced to the concept of variable. The first chapter of this book is devoted to problem-solving strategies which I feel all students should be exposed to. One of the first things that is recommended is that the student understands the problem which involves identifying the relevant information that the problem gives, what one is to find, how the information is related, if there is enough information, is the information given relevant, and will determine whether or not rereading the problem will help. In order to

In order to facilitate the task, the book advocates drawing diagrams, making tables, looking for patterns, simplifying the problem, or working the problem backwards. This approach to learning algebra builds on past knowledge and experience by stressing the value of looking back at solution processes. The carrying out of a plan of attack involves keeping the problem continuously in mind, making sure that each step makes sense, keep refining guesses until a solution is derived, trying to solve this problem the same way you solved a similar problem or solving a simpler problem and then adapting the solution to the original problem. The looking back phase of problem solving includes checking whether the question is answered, checking if the answer makes sense, posing new problems like the one just solved to expand understanding of the solution process, extending the problem to search for patterns of solutions, and looking for alternative ways to solve the same problem. I found that when I conducted my interviews, if I was not sure if the student understood the problem or was thoroughly convinced of their answer, this "looking back phase" proved beneficial.

I felt the chapter just prior to the introduction of the symbols of algebra to be quite interesting in that it provided some structure for the conception of the concept of variable. As alluded to earlier, properties seem to provide a vehicle through which one can introduce the concept of variable. This chapter deals with properties in a meaningful way. The chapter begins by discussing fact teams or equations which express addition/subtraction or multiplication/division relationships. In order to exhibit this concept, the process approach involves the students in developing fact teams by filling in missing numbers. The way this approach handles unknowns ranges from blanks, to question marks, to geometric shapes. For example, fact teams are used to show students how they

can be used to restate problems such as "What number subtracted from $2\frac{1}{4}$ equals $1\frac{1}{2}$?"

Fact teams allow the easy transition into the introduction of the properties, such as the commutative, associative and distributive. At this point, the book states the properties using variables but does not refer to them as such. It should be noted that the process approach takes special care to mention that the properties hold for any numbers, whether fractions, decimals or whole numbers. Another important concept which the book devotes time to prior to treatment of "Algebra topics" is positive and negative numbers. I have seen Algebra books that develop this topic long after the introduction of variables and it seems that this topic is challenging enough in and of itself. At this point, the approach also goes into a discussion of absolute value which will lend more credence when this topic is discussed using the symbols of algebra. Throughout this chapter, students are continually exposed to the various representations of unknowns.

I was surprised to see , at this point in the book, such a complete treatment of addition, subtraction, multiplication and division. Yet, no one can argue that there needs to exist, in the minds of Algebra students, a firm understanding of arithmetic fundamentals. The process approach employs a model using pebbles to review the operations in addition to the traditional employment of the number line. White pebbles represent positive numbers and black pebbles represent negative numbers. It is explained that in the model, whenever a pile of pebbles contains both black and white ones, each white one can cancel a black one. Also, the value of each pebble is set at one unit. This approach is especially noteworthy due to the fact that the students come to realize that every number can be represented in an infinite number of ways. Hence, students also come to the conclusion that some ways are easier to evaluate than others. Thus, students are encouraged to solve problems from a number of different view-

points. Although the book does not go into much detail at this point, I felt that the introduction of a balance beam with the pebbles would prove very helpful when students begin to solve equations.

Other topics that were discussed before the "real" Algebra was introduced included powers. In this section, the approach points out that mathematicians create symbols to help them organize their thinking(i.e. exponents). This leads to a discussion of the advantages of generalizations. Throughout this chapter, the students are prompted to formulate their own generalizations or discovery lessons as well as counterexamples. The order of operations and grouping symbols facilitated in the introduction of formulas and hence, indirectly, variables. The students are instructed to view the letters in formulas as representing numbers. Again, this provides another mode in which understanding for variables is enhanced. In conclusion, the symbols of Algebra are introduced. The approach explains the intrinsic nature of symbolism by such examples as the varied ways we represent multiplication, division and even the number zero. A review of the properties helps tie in the symbolism a bit more concretely with Algebra. Finally, the approach devotes a lesson to translating phrases into algebraic expressions. It is at this point that the process approach introduces the concept of variable. It explains how letters are used in several different situations in mathematics. For example, in some cases, letters are used to represent some specific unknown number or numbers that we are trying to find. Here, there is a brief discussion that one may go about finding the unknown either by "guess and test" or "undoing" (working backward). Letters are also used for situations to represent some generalized numbers as explained by the process approach. It is at this time, that the process approach begins referring to these letters as variables. As a novice in this field, this approach was the only one I studied which seemed to manifest meaning for the concept of variable.

After I received approval from the Graduate School, I proceeded to contact the principal of Shepard High School in Palos Heights where I did my student teaching. Please see the letter I have enclosed in the Appendix. I explained my Honors Capstone Project to him and he was very supportive in helping me to organize my student interviews. I decided that I wanted to interview three students: an Honors student, a regular-track student, and a survey student to assess and enhance their understanding of the concept of variable. The interviews lasted for a period of two and one-half weeks. I tried to meet with the three students every other day for thirty minutes. Between the first and second weeks of the interviewing process there was one week of Easter vacation. I was able to conduct between five and six interviews with each student.

As I was working with the Algebra Honors classes as part of my student teaching assignment, I was readily able to obtain volunteers from the two classes to participate in my project after I explained to them what would be entailed. Then I went to the regular Algebra class and the Algebra Survey class to explain my project and was very pleased at the overwhelming number of volunteers I received. I sent out permission letters, forms and sample questions to the parents of those students who volunteered. Please see the letters, forms, and sample questions enclosed in the Appendix. I explained to the students and parents that the student could withdraw from this project anytime that he/she so desired and that the interviews would be audio-taped; however, the students names would be changed to protect their anonymity.

After I received the permission letters back, I consulted with the teachers of those students ^{who} would epitomize a typical honors, regular and survey student respectively. After consultation I chose three girls. For the purposes of this study, the HOnors' student's name is Andrea, the regular student's name is Diane,

and the Survey student's name is Charlene. All three students were receiving A's in their Algebra classes. Andrea is classified as one of the brightest students in her class, Diane was in the HOnors track but at semester time transferred to the regular Algebra class and Charlene is also classified as one of the brightest students in her Survey class.

In addition to the audio-taping, I also had each student, during the interview, write down as much as possible so as to ensure another reference for when I analyzed the interviews. I based my initial assessment questions on Dietmar Kuchemann's article, "Understanding of Numerical Values." My first couple of interviews cannot be classified as true assessments because I intervened with instruction. I tried to build each interview based upon the previous interview in that I picked up each day with where the students' understanding broke down from the previous day. I accumulated questions for my interviews from a variety of different sources including a Graduate Record Exam review book I was studying. My last interview with each student was devoted to posing a recap of similar questions throughout the two and one-half week interviewing period.

At this time, I would like to discuss Kuchemann's understanding of numerical variables which formed the framework of my interviews. Collis, in 1975, helped to construct the following six levels for describing the different ways letters can be used: letter EVALUATED, letter IGNORED, letter as OBJECT, letter as SPECIFIC UNKNOWN, letter as GENERALISED NUMBER, and letter as VARIABLE.

The first level applies to responses where the letter is assigned a numerical value from the outset. In the second level, the student ignores the letter, or at best acknowledges its existence but without giving it meaning. At the third level, the letter is regarded as a shorthand for an object or as an object in its own right. When students reach the fourth level, they regard a letter as a specific but unknown number and can operate upon it directly. At the fifth level, the letter is seen as representing, or at least being able to take

several values rather than just one. Finally, at the sixth level, the letter is seen as representing a range of unspecified values, and a systematic relationship is seen to exist between two such sets of values. Generally, the first three categories indicate a low level of response and it can be argued that for students to have any real understanding of even the beginnings of Algebra they need to be able to deal with problems that require the use of a letter as a specific unknowns, at least when the structure of such items is simple. (Kuchemann, 104)

When we refer to the first level, letter evaluated, this is one of three interpretations by which students avoid having to operate on a specific unknowns in this case by giving the unknown a value. The category also refers to problems where students are asked to find a specific value for an unknown, but again without first having to operate on the unknown. The second category refers to solving problems by "not using" the letters. In this instance, there is no need to handle, transform, or even remember the expression. In the third category, letters can be thought of as objects in and of themselves. In a way, this is easier than if the letters had to be treated as specific unknowns. Evidently, using a letter as an object, which amounts to reducing the letter's meaning from something quite abstract to something far more concrete, allows many students to answer certain problems successfully which they would normally not if they had to use the intended meaning of the letter. We must be careful to distinguish between the objects themselves and their number (specific unknown). The previous three categories all describe ways of avoiding generalised arithmetic by not using the letters as genuine unknowns. When we refer to a letter as a specific unknown, the opposite is true, even though the idea of a specific number is still a rather primitive notion. As opposed to a letter as a specific unknown, where the letter is thought of as having a particular (but unknown) value, a letter

used as a generalised number is able to take more than one value. " A distinction can be made between the idea of a letter taking on several values in turn and a letter representing a set of values simultaneously. However, this is not done here, although it seems to be the second idea rather than the first that later forms part of the concept of variable." (Kuchemann, 109) Apparently, it seems likely that during the course of many algebra problems students flip from one interpretation to the other, depending on which is momentarily more convenient. (Kuchemann, 104-110)

The generalised use of the term 'variable' in arithmetic is a common practice which has obscured both the meaning of the term itself and the differences in meaning that can be given to letters. The concept of a variable clearly implies some kind of understanding of an unknown as its value changes, and if this is to go beyond the ideas already present in seeing a letter as a specific unknown and generalised number, it would seem reasonable to argue that the concept implies some understanding of how the values of an unknown change. One reason why the concept is so elusive is because many items that might be thought to involve variables can be solved at a lower level of interpretation. Perhaps a useful, operational definition of variables may be: "letters are used as variables when a second or higher order relationship is established between them."

(Kuchemann 111)

INTERVIEWS

The way I decided to document my interviews was to recap the dialogue that transpired between the student and myself. The "I" represents the question or statement I posed to the student. The method I used to indicate the student's response was to put the first letter of their name in front of their response. After I share the dialogue, I will respond with the conclusions I drew from that particular session.

Diane - March 25

Diane felt that Algebra is a game which is fun. The aspect of Algebra that Diane liked least is polynomials because they take too long to solve. When I asked Diane what a variable is, she responded that it is a number with a letter. Further, when I asked her why we study variables, she responded by saying that she felt there was no use for variables.

I: If $a + 5 = 8$, $a = ?$

D: $a=3$

I: Why?

D: Subtract 5 from 8.

I: What was your ultimate strategy?

D: To find the missing number.

I: Did you only subtract the 5 from the 8?

D: No, we want to subtract from both sides to get the "a" alone.

I: Perhaps we want to maintain a balance.

I: If $m = 3n + 1$ and $n = 4$, $m = ?$

D: 13

I: Why?

D: All we need to do is multiply 3 by 4 and add the 1.

I: Why did you multiply 3 by 4?

D: Because we're substituting the 4 for the n.

I: If $a + b = 43$, $a + b + 2 = ?$

D: 45

I: Why?

D: Good question. I really don't know. I just figured it out.

I: You must have thought of something when you were solving the problem?

D: If $a+b=43$ and you have the same a and b , just add the 2 to get 45.

I: Would that work for any "a" and "b"?

D: No, probably not.

I: Can you find me an example where it wouldn't work?

D: 100 and 50, we need numbers that will add to 43.

I: If $n - 246 = 762$, $n - 247 = ?$

D: 761

I: Why?

D: If you add one to 246, then you take one off 762.

I: Why?

D: Because we're actually taking one from one side of the equation and putting it one the other side.

I: We do this to help maintain the balance.

I: If $e + f = 8$, $e + f + g = ?$

D: $8 + g$

I: Why?

D: Because we have no number given for g , so we just add the g .

I: We have a specific number and we're adding g onto it and since we do not have a specific value for g , we can write the algebraic expression as $8 + g$.

I: Given this figure(an equilateral triange with sides labelled e) can you tell me what the perimeter of this figure would be?

D: e to the third power

I: Why?

D: You said all 3 sides were equal and you said to get the perimeter you add all 3 sides.

I: If we add $e + e + e$, we get e to the third power?

D: No, because e to the third power means $e \times e \times e$. We would get $e + e + e$.

I: Can that be simplified at all?

D: No.

- I: Are you sure?
- D: Yes.
- I: Can you tell me what $e + e$ is?
- D: No because there's no numbers.
- I: What if I said we can represent this using understood coefficients as in $1e + 1e + 1e$?
- D: $3e$.
- I: There are n sides to a given figure, all of length 2. Can you give me the perimeter of this figure?
- D: It would be $2n$. You don't know how many sides there are and since each side has length 2, the perimeter would be $2n$.
- I: Cabbages cost 8¢ each and turnips cost 6¢ each. If c stands for the number of cabbages bought and t stands for the number of turnips bought, what does $.08c + .06t$ stand for?
- D: It stands for the total number of cabbages and turnips we bought or how much we paid for each.
- I: However, we know that the cabbages are 8¢ each, if c stands for the number of cabbages what does $.08c$ stand for?
- D: If you work the problem out later, say if $c=4$, then you're going to find out how much it will cost to buy 4 cabbages.
- I: If I buy 4 cabbages, how much will it cost?
- D: 32¢
- I: If I have " c " number of cabbages, what does $.08c$ stand for?
- D: I don't know. This is confusing.
- I: Can you tell me how we would algebraically represent the total number of vegetables bought?
- D: $c + t$, So $.08c$ is what we'll pay for c cabbages and $.06t$ stands for how much we'll pay for t turnips.
- I: 4 added to n can be written as $n + 4$. Add 4 onto 8.
- D: $4 + 8 = 12$
- I: Add 4 onto $n + 5$.
- D: $n + 5 + 4 = n + 9$
- I: Add 4 onto $3m$.
- D: $3m + 4$
- I: Can that be simplified any further?

D: No.

I: n multiplied by 4 can be written as $4n$. Multiply 4 by 8.

D: $8 \times 4 = 32$

I: Multiply $n + 5$ by 4.

D: $4(n + 5) = 4n + 20$

I: Multiply $3n$ by 4.

D: $12n$

I: If $r = s + t$ and $r + s + t = 30$, what can you tell me about r ?

D: We can't say anything. We don't know what the letters stand for.

I: We know that r and $s+t$ stand for the same number. Can we represent that in the second equation in any way or can you give me a value for r ?

D: 15

I: Why? How can we write the second equation using only the variable r ?

D: $2r = 30$. So... $r=15$

Of all three students I interviewed, Diane seemed to have the least sophisticated grasp of what the concept of variable meant. It also appeared that Diane was not able to verbalize as well as the others what she was thinking. She responds rather quickly and has a noticeably difficult time explaining her justifications. Obviously, Diane had no difficulty with the level of variable evaluated. Further, she convinced me of a fairly secure grasp of the letter ignored level. I especially noted the way she responded to: if $n-246=762$,,, Her reasoning behind her thinking led me to believe that she understood what was really being asked of her in this problem. Difficulty arose at the level where the letter is viewed as an object. I do not think there was any flaw in her prerequisite knowledge of the concept of perimeter; however, as evidenced by the dialogue, it was not until I prompted her that $e+e+e$ can be thought of as $le+le+le$ did she arrive at the correct response. Perhaps the difficulty was not at this level as evidenced by the fact that she answered the following perimeter question correctly. The error may be attributed to a lapse of pre-

requisite knowledge concerning adding like terms. As we move into the more advanced levels such as the level that views variables as specific unknowns that naturally more difficulty arose. I was surprised that given the fact that I defined what the variables stood for in the problem with cabbages and turnips, Diane was still unable to translate $.08c$ and $.06t$ into English. After I put the problem into a real world situation, greater insight was gained. When I asked her how we would represent the total number of vegetables, this adequately facilitated the correct response. Diane had no problems with the other questions pertaining to variables as specific unknowns until we came to the last problem in this interview. Again, I feel that Diane was caught up in solving for the s and t as well as r . Although she was able to give me the correct answer, I was not convinced she truly understood she could write the equation in one unknown and then solve for r . It should be noted; however, that I had to suggest that approach.

Diane - March 29

I: What can you say about c if $c + d = 10$ and $c < d$?

D: c is going to be a lower number than d

I: Please give me two numbers that when I add them together I'll get 10.

D: 4 and 6

I: Let's let $c=4$, $d=6$. Is $c < d$?

D: Yes.

I: Is that the only case where that will work?

D: No..

I: Can you give me more examples?

D: 2 and 8, 3 and 7, 1 and 9, 0 and 10

I: Can you give me a general statement of what needs to be true of c ?

D: c has to be less than 5

I: Could c be negative?

D: D: Yes, $-2 + 12$, so any number less than 5 will work.

I: Which is larger $2n$ or $n + 2$?

D: $2n$

I: Why?

D: Because $2n$ is 2 times n .

I: Will that always work?

D: No, it could be equal.

I: When will it be equal?

D: When $n=2$. Is there an answer for this? n doesn't have to be the same number on both sides of the equation?

I: If $n + 2 = 2n$...

D: Yes it does.

I: Originally, you said $2n$ would always be greater. Now you just found me an example where $2n = n + 2$. Do you think $2n$ could be less than $n + 2$?

D: I guess so.

I: When?

D: When, ... no.

I: So $2n$ is always greater than or equal to $n + 2$?

D: Correct,

I: What would happen if n were negative?

D: $2n < n + 2$

I: Can you illustrate this?

D: IF $n = -1$.

I: What if $n = -10$?

D: $2n$ would be larger, no $n+2$ would be.

I: So can you find some way to summarize what you just discovered?

D: The answer depends on whether n is positive or negative. If n is positive then the $2n$ is greater, if n is negative $n + 2$ is greater.

I: What if $n=1$?

D: That changes my answer. $n + 2$ is greater. So that means $2n$ is greater as

long as n is equal to or greater than 2.

I: Which is larger b or $-b$?

D: b because positives are always greater than negatives

I: What happens if $b=-3$? What is the opposite of b ?

D: 3

I: Which is bigger?

D: $-b$

I: When will $b > -b$?

D: When b is a positive number.

I: When will $b < -b$?

D: When b is a negative number.

I: What happens when b is 0?

D: We can't say anything.

I concluded that Diane had not yet reached the level of variable as a generalized letter. She was obviously searching for some specific value of c . It was not until I prompted her to formulate several examples of the problem that she could finally generalize. It also seems to me that unless I suggested that c could be negative, she would have neglected to allow c to take on those values as well. As I expected, Diane had a great deal of difficulty with the remaining questions in this interview that were geared towards the highest level of understanding variables. It seems apparent from her responses, that Diane automatically limited the domain of values for variables with the set of positive integers. That perhaps accounts for her immediate reply that $2n$ is larger than $n + 2$, which, granted is a natural first reaction. I was surprised that Diane so quickly realized that the two expressions could be equal given the fact that she had difficulty finding the value for n which would make those two expressions equal. Again, I needed to suggest what would happen if n had the value of a negative number. Diane was quick to respond what one would intuitively expect to happen.

However, she seemed convinced without testing a few examples. Afterwards, she was able to summarize her findings to correlate with the cognitive aspects of a variable as a generalized number. Obviously, she did not transfer or retain what we did in the prior problem given her response to the last question. It seemed that she truly did not comprehend the nature of this level of understanding.

Diane - March 31

I: $L + M + N = L + P + N$ Is this statement always true, sometimes true or never true?

D: Never true.

I: Why?

D: Well, sometimes, because the M and P could equal the same number. Most likely this won't happen,

I: When will it be true?

D: When the M and P stand for the same number

I: Can two variables be used to represent the same number?

D: You're right they can't, no, sure they can.

I: What algebraic expression added to $x + 9y$ equals $-2x + 7y$?

D: Would it be $-4x + 9y$?

I: If I have 4 and I want to add something to it to get 9, what would I add?

D: 5

I: How did you get that?

D: I subtracted $9 - 4$.

I: If I have a quantity which represents a number and I wanted to find out what to add onto it to equal another quantity which also represents a number,...

D: Oh, I could subtract. $-2x - x = -3x$ and $7y - 9y = -2y$. I should add $2y$ and $-3x$.

I: Let's check it.

D: $x + 9y + -2y + -3x = -2x + 7y$

I: What should I add onto $3(y + 3)$ to get $6y + 10$?

D: I think I should simplify to get $3y + 9$, so I should add $(3y + 1)$.

I: Convince me that will work.

D: $3y + 3y = 6y$ and $9 + 1 = 10$; $3y + 3y + 9 + 1 = 6y + 10$

- I: Can you give me an algebraic expression that when we simplify it we get $8c + 12$?
- D: Could this be anything?
- I: Sure.
- D: $2(4c + 6) = 8c + 12$
- I: Is that the only one?
- D: Probably not. $4c + 4c + 6 + 6$
- I: How many more expressions are there?
- D: Well, I found two. There are probably more... alot more. There's about ten more.
- I: NO more than 10?
- D: No more than 10.
- I: What if I asked you to find as many equations as you could which have a solution of $x=7$, how many could you find?
- D: Alot. NOT more than 10.
- I: $4 + 3 = 7$, $7 \times 1 = 7$, ...
- D: So there's about 100.
- I: Could there be an infinite number?
- D: Yes.
- I: Let's go back to our original problem.
- D: There's probably an infinite number.
- I: Convince me.
- D: How much time do we have?
- I: Why?
- D: Because $8c + 12$ represents one number. So just as we could find an infinite number of equations with a solution of 7 we could do the same here.
- I: If $c=1$, $8c + 12 = 20$ and if $c=2$, $8c + 12 = 28$. Those are two numbers.
- D: Well, it can only represent one number at a time, depending on what c is.
- I: $s/8 - 3 = 14$. Solve for s .
- D: $s/8 (8) - 3 = 14(8)$ so $s - 3 = 112$ so $s = 115$
- I: Can you convince me that's correct?
- D: YEs . $115/8 - 3 = 14$ No it doesn't check. How do I do this? This isn't

making sense. I'm not right.

I: You're thinking about clearing fractions?

D: AM I doing this right? It looks right but it's not checking. Actually I should get rid of the 3 first, but it should come out the same way.
 $s/8 = 17$ so $s = 136$.

It checks. $136/8 - 3 = 14$. I didn't think that would make a difference. But it sure does.

I: Why can't we clear the fractions first? I thought we could do that.

D: So did I.

I: What if I have $3 - 4/8 = x$?

D: $2 \frac{8}{8} - 4/8 = 2 \frac{4}{8}$.

I: Could I solve this problem by not clearing the fractions?

D: No because there's no letter.

I: Why don't you multiply the 3 by 8?

D: Because you're supposed to do all multiplications and divisions last.

I: It will work regardless of which way you choose to solve it. HINT: The way you cleared the fractions was wrong. Can you find your mistake?

D: You probably have to multiply the 3 by 8 but I don't know why.

I: Earlier, you told me that we have to make sure we maintain a balance when we solve equations.

D: Oh, I see. $8(s/8) - 3(8) = 14(8)$ so $s - 24 = 112$ so $s = 136$

I: What if I switched this problem to $t/8 - 3 = 14$?

D: It should come out the same. (She goes through the same process to solve for t)

I: Why does it come out the same?

D: Does this go back to the first problem? We just switched the variables and two variables could represent the same number.

I did not plan for this interviewing session to turn out as cohesive as it did. Judging from the quickness of the response I got from the first question, it seems like Diane would have been satisfied with that answer. It was only when she was asked to justify her response that her thinking became clearer. I was

pleased that she realized M and P could represent the same number. However, I thought it odd that she felt that this most likely would not happen. When I asked her if indeed two variables could be used to represent the same number, I think she felt I was trying to trick her, but she stood ~~her~~ firm in her convictions and convinced me she truly understood the problem. With respect to the next problem, I admired her "guess and check" method first. Obviously, there were a few approaches one could take to this problem. After she was stuck for quite a while, it was evident that the easier arithmetic facilitated her thinking. I once heard that Algebra is just arithmetic in disguise and perhaps teachers could benefit their students by correlating their understanding with knowledge they are comfortable and familiar with. I thought I would try to utilize one of the methods found in the Algebra process approach book I studied; namely, that of fact teams. I was surprised to learn that Diane felt there were Only 10 expressions which could be simplified to $8c + 12$. I had to make her see that the entire quantity represents a specific unknown which was facilitated by explaining it in terms of a simple example. The last question we worked on was placed in this interview in light of the article I read where a Wagner interview was conducted with a seventh grade Algebra student. Several misconceptions surfaced which I found quite surprising. Due to the fact that Diane obviously forgot that she had to multiply every term in the equation through by 8, although, I did not analyze this topic in depth in this project, I feel that this is a situation exemplifying why the "unwrapping" approach to solving equations is beneficial. I was happy that she realized that either approach she used should yield the correct answer. When I tried to explain the concept of "clearing fractions" to her, I could not believe that she felt we could not do so on a problem that contained no variables. Since, we were running short on time, I suggested how we clear fractions which served as a gentle reminder for Diane who was then able to solve the problem and convince

herself that she could use either approach to solve the problem. I intended for the last question I posed to be easy. Although she realized the result was the same, I was surprised that she went through the solution process to derive the solution again. Perhaps she was just checking. I admired the way she made the analogy back to the first question in the interview.

Diane - April 14

- I: Please give me an algebraic equation that represents the fact that at a certain university, there are six times as many students as professors.
- D: $6p$
- I: What does p represent?
- D: Professors.
- I: Is that an equation?
- D: NO.
- I: What do we need for an equation?
- D: A variable, an operation sign, and a equal sign.
- I: Do you always need a variable in an equation?
- D: Yes,
- I: Is $4 + 3 = 7$ an equation?
- D: Yes, you don't need variables... $6p = ?$
- I: Do you think we need to introduce something else on the other side of the equal sign?
- D: Yes.
- I: THink about what the problem is telling you.
- D: $6s = p$
- I: What does s stand for?
- D: NO it should be $6p = s$. I'm confused.
- I: What does s stand for?
- D: Students.

I: Why is $6p = s$ correct?

D: There are 6 times as many students as professors so you multiply 6 times the number of professors and you come out with the number of students.

I: Perhaps you can give me a practical example to explain your reasoning.

D: No.

I: Let's say there were 6 professors at this school. How many students would there be?

D: 36

I: 36 represents...

D: 6 times the amount of professors

I: Why isn't the equation $6s = p$?

D: Because you gave me the amount of professors

I: AMount?

D: NUmber

I: So p stands for...

D: A number and that's what variables are.
p stands for the number of professors

I: THat's an important distinction, p stands for the number of professors not professors. What does s stand for?

D: The number of students.

I: Are there more students or professors according to $6p = s$?

D: More students.

I: Is there one student for every six professors or are there six students for every one professor?

D: There are six students for every one professor.

I: Convince me that your answer is right.

D: There are more students than professors so there cannot be one student for every six professors. We're multiplying 6 times the number of professors.

I: I told you there were six times as many students. Doesn't that mean $6s$?

D: No.

I: How can we check your answer?

D: We could make up an example. If there are 11 professors there would be 66 students. This tells me there are more students and for every one professor there are six students.

I: Can you tell me in words what your equation tells us?

D: 6 times the number of students equal the number of professors

I wanted to be sure to include in my interviews the classic "Student and Professor" problem due to the fact that so much of the literature included this problem. Diane's initial response was surprising because I asked for an equation. As I expected she was unable to correctly define her variables. I was also surprised to learn that she felt that variables needed to be present to constitute an equation. Evidently, she has misconceptions concerning the concept of equation as well. As was commonly documented in the literature, Diane made the reversal error. I could not determine whether or not asking her what the variable s stood for made her change her answer to $6p = s$. I feel that she was thinking about the problem the whole time and arrived at the correct answer. Although I kept asking her to justify throughout the interview, I think she had a fairly good understanding of this problem in comparison to the literature I reviewed. I can't be certain, but I think Diane knew that, for example, p stood for the number of professors even though she said that p stood for professors. There may be an implicit understanding for some students; however, I firmly believe that we cannot afford to take this for granted. She told me, in the context of explaining the problem, that the problem meant to multiply 6 times the number of professors to obtain the number of students. I was also surprised that Diane could not initially relate a practical example of this problem. The practical example allowed her to make the distinction that the ~~the~~ variables stood for numbers. I was convinced that she finally understood the problem in light of her final check and the fact that she could correctly translate her equation into English.

Andrea - March 28

Equations and word problems were Andrea's favorite part of Algebra. The aspect of Algebra that Andrea liked least was graphing linear equations. When I asked Andrea what a variable is she responded by saying that it was a letter that stands for an unknown number.

I: If $a + 5 = 8$, $a = ?$

A: Subtract 5 from both sides, so $a = 3$.

I: Why do we subtract 5 from both sides?

A: To get the a alone, to maintain a balance.

I: If $m = 3n + 1$ and $n = 4$, $m = ?$

A: $m = 13$; because you substitute 4 in for the n and 3 times 4 equals 12 plus 1 is 13.

I: Why do we substitute 4 for n ?

A: Because n equals 4.

I: If $a + b = 43$, $a + b + 2 = ?$

A: 45, If $a + b = 43$, when we look at the second equation if we put a parenthesis around the $a + b$ and substitute 43, then if we add 2 we'll get 45

I: So you're looking at $(a + b)$ as standing for one number.

A: We could say $(a + b) = m$, so the second equation becomes $m + 2$, where $m = 43$.

I: If $n - 246 = 762$, $n - 247 = ?$

A: 1509

I: Let's check it.

A: $1509 - 246 = 1263$, so it doesn't check.

I: Let's say I have 5 apples and I took 4 apples away we'd have one apple left. Let's say I take 5 apples away this time. So I'm taking one more apple away. I would have 0 apples left. What happened when I took one more apple away to the answer?

A: I got one less, so you'd get 763.

I: Why?

A: The answer must be more.

I: Why does my answer increase if I take more away?

A: It doesn't work? The answer should be 761. I understand this problem now especially when you asked me the apple problem.

I: If $e + f = g$, $e + f + g = ?$

A: It would be $8 + g$ because $(e + f)$ could be expressed as one variable.

I: Given an equilateral triangle with length of each side as e , what would be the perimeter of the figure?

A: $e + e + e = 3e$

I: Given a figure with n sides each of length 2, find the perimeter.

A: $2n$, like with the previous problem, we already know the number of sides but since we don't know the exact number of sides we can express it like this

I: Cabbages cost 8¢ each and turnips cost 6¢ each. If c stands for the number of cabbages bought and t stands for the number of turnips bought, what does $.08c + .06t$ stand for?

A: It equals the total number of cabbages and turnips bought. $.08c$ stands for 8¢ times the total number of cabbages bought and $.06t$ stands for 6¢ times the total number of turnips bought.

I: What does $c + t$ stand for?

A: It would just be the number of cabbages and turnips bought, but we wouldn't know how much it is.

I: But you told me that $.08c + .06t$ stood for the total number of vegetables bought?

A: No, it would stand for the total price. So $c + t$ is just the total number and not the price.

I: 4 added to n can be written as $n + 4$. Add 4 onto 8.

A: $4 + 8 = 12$

I: Add 4 onto $n + 5$.

A: $n + 4 + 5 = n + 9$

I: Add 4 onto $3n$.

A: $3n + 4$ and this cannot be simplified

I: What can you tell me about r if $r = s + t$ and $r + s + t = 30$?

A: $30 - r = s + t$

I: That's true. Could we find a value for r ?

A: $r = s + t + 30$, no $r = s + t - 30$

I: Let's let r equal the total number of points Shepard and Richards gained at the last track meet. Let s equal the total number of points Shepard received and let t equal the total number of points Richards received. NOW, the total number of points equals 30, so if Shepard scored 15 points, how many points would Richards have scored?

A: 15

I: In algebraic terms how could we represent the procedure you went through to obtain this solution.

A: $r = -s - t + 30$, I think I know what you mean. If $r = s + t$, then you could substitute this in the second equation so $r + r = 30$ or $2r = 30$ or $r = 15$

I: What can you tell me about c if $c + d = 10$ and $c < d$?

A: We know that $c = 10 - d$, $d = 5$, No this can't be. $c < d$ so c can't be 5

I: Can c take on more than one value?

A: Yes. c can be 2, 4, or 1 so c can be 1, 2, 3, or 4. Therefore, $c < 5$.

I: Which is larger b or $-b$?

A: b because if we have 2, $2 > -2$

I: What if $b = -3$? What is the opposite of b ?

A: 3

I: Which is larger b or $-b$?

A: They're equal.

I: $3 = -3$?

A: So, -3 is bigger? So $-b$ is larger than b ?

I: What if $b = 2$?

A: b , so it depends on what b is

I: When will b be larger than $-b$?

A: When b is positive

I: When will b be smaller than $-b$?

A: When b is negative

I: What happens when $b = 0$?

A: b is not positive or negative so it's nothing

I: We cannot say anything.

Since Andrea was the HOnors student, as I expected she progressed the farthest in her first interview. Obviously, Andrea had no problem with the letter evaluated stage. Also, the letter ignored stage seemed to pose no problem for her. I especially found noteworthy the way she looked at the following problem: if $a + b = 43$,...Clear evidence of the fundamentals of this level were displayed as Andrea viewed the $a + b$ as simply standing for one specific number in the second equation. As with the other students; however, problems arose when we got to the problem: $n - 246 = 762$,...She seemed to want to solve for n as opposed to looking at the problem from the standpoint of ignoring the letters. She obviously did not transfer what she did in the preceding problem, which leads me to believe that perhaps she really did not understand the underlying concept of variable at this level. When I put the problem into a real world context, then she was able to understand the relationship. With respect to the level that deals with variables as objects, she seemed to have no apparent difficulty with the whole exception of explaining what $.08c + .06t$ stood for. It was not until after I asked her what $c + t$ stood for was she able to correct her mistake. Since all three students had difficulty with this problem, perhaps it is further evidence that as teachers, we need to be careful about how we define our variables. The specific unknown level posed no problems until we got to the problem concerning what she could tell me about r . I thought Andrea would have no misunderstanding in dealing with this question in light of her thinking in prior problems. I think she got side-tracked when she incorrectly solved for r in the second equation. Perhaps she was trying to apply what we had covered just prior to this due to the fact that she initially was not as adamant about solving for r . I do not know how effective the practical example was except to facilitate a solution for the variable. In the end she was able to conclude without any assistance, that we could substitute the r into the second equation to get $2r = 30$. I think this allowed Andrea to feel convinced that her original

conjecture was indeed correct. Moving to the next level of variable as a generalized number, I was especially impressed with the way Andrea reasoned her way through the solution. It should be noted that I probably gave too convenient a hint when I suggested that c could take on more than one value. In light of the difficulty Andrea had with the last problem, we may conclude that she has not yet reached the highest level of understanding with regard to variables. She restricted almost immediately the domain of the value of b . I was convinced after my questioning that Andrea understood the problem; however, I was not assured that she understood the nature of the problem.

Andrea - April 13

- I: Please write an algebraic equation to express the following: at a certain school, there are six times as many students as professors.
- A: n =professors, $6n$ =number of students, $n=6n$
- I: In order to have an equation we need...
- A: A balance .
- I: Can we represent the number of students as $6n$?
- A: No, maybe we'll let n =professors or the number of professors
- I: Why is it the number?
- A: Because a variable represents a number--we can't have $6n$ =number of students--you don't have a total
- I: No.
- A: $n=6n$
- I: In words, what does that tell me?
- A: The number of professors is equal to 6 times the number of professors and that doesn't make sense. Why don't we let n =number of students-- n =the number of students--we need two variables--you didn't tell me we could use two variables
- I: Can I have an equation with two variables?
- A: Yes, so let's let $6x$ =number of professors--so $n=6x$ which means the number of students equals six times the number of professors--that will work--but you couldn't really solve for it

- I: Since you told me $6x$ = the number of professors I read this equation as the number of students equals the number of professors.
- A: No the number of students is 6 times the number of professors
- I: But you're letting $6x$ be the variable that represents the number of professors
- A: So x equals the number of professors--so $n=6x$ -This works
- I: Convince me.
- A: Let's say $x=2$ so there are 2 professors --then $n=12$ (number of students) and 12 is 6×2 . The number of students is greater than the number of professors. There are 6 students for every professor.
- I: Joanne purchased p pencils and Steve purchased 3 more than $\frac{1}{2}$ as many as Joanne. Could you tell me what is the relationship between the number of pencils Steve purchased and $(p + 6)/2$?
- A: p =number of pencils Joanne has and $p/2 + 3$ =number of pencils Steve has--
 $p/2 + 3 = (p + 6)/2$ This is the same. Isn't it?
- I: So if I add $\frac{1}{2} + \frac{1}{2} = \frac{1}{4}$?
- YOU told me all I needed to do was to add the numerators and then the denominators.
- A: $p/2 + 3 < (p + 6)/2$ Let's say Joanne purchased 2 pencils-- Oh, we would get $4 = 4$
- I: What happened?
- A: They are the same.
- I: Why? How do we add fractions?
- A: Oh, I see. $p/2 + 3 = (P + 6)/2$.

I was very pleased by the way Andrea approached the classic "Student and Professor" problem. She was able to correctly define her variables rather quickly. As found with the other two students, her first response was not an equation and she did not see any apparent need to introduce two variables. Once she told me in words what her original equation stated, she detected she was wrong and immediately defined another variable. After that she was able to come up with the correct equation even though she defined her second variable incorrectly. After dwelling a bit more on the English that corresponded to her algebraic expression, she concluded that simply x would stand for the number

of professors. When she convinced me it worked, I knew she truly understood the problem. She even initiated using a real world situation. As I would expect from an Honors student she completed this problem at a much higher rate and level of sophistication. The second problem I used in this interview was one I found in the Graduate Record Exam book. Difficulty with fractions arose a couple of times throughout this interviewing process which should be noted as an area that definitely needs to be stressed more stringently in the Algebra classroom. For example, Andrea felt $p/2 + 3 = (p+3)/2$. She temporarily ignored the question I posed to her pertaining to the validity of that statement and dealt with the task at hand. After she determined the original quantities were equal after a practical example, then we were able to work on the prerequisite knowledge. A gentle reminder about adding fractions was all that seemed to be needed to correct the error. I especially liked the way she demonstrated her understanding by setting up the problem just as one would in setting up an addition of fractions in arithmetic.

Andrea - April 15

I: If $x < 0$, What is the relationship between $1-x$ and $x-1$?

A: $1-x < x-1$

I: Why?

A: Let's say $x=2$ $-1 < 1$

I: But our initial condition is $x < 0$, so what values can x take on?

A: So x is negative. So $1-x < x-1$

I: Why?

A: Let's say $x=-1$, $2 > -2$

I: Will that work for any x ?

A: Only if x is less than 0.

I: What if $x=0$?

A: $1-x$ is greater than or equal to $x-1$ when x is less than or equal to 0

I: The average of 2 numbers is $2x$. If one of the numbers is y , what does the other number have to be?

A: $2y$?

I: How did you get $2y$?

A: To find the average, you add then multiply --so then $y + y$ times that by 2 you get $2y$ --

I: What would I do if I wanted to find the average of two of your test scores?

A: I would add the two test scores and divide by 2-- $2x/y$ would have to be the other number.

I: How could we check that?

A: Let's try a number--Let $y=2$ -- If $2x/2$ would give you x --Let us try $y=2$, $x=3$ -- This would give us 3. If $y=2$, the average is $2x$, then we would get 2..The other number is y also.

I: If one number is y , and the other number is y , how can we check that?

A: $(y + y)/2 = 2y/2 = y$ but the average is $2x$

I: Let's say I wanted to find the average of two of your test scores from last semester. I remember that one of your scores was 98 and the average was 97. How would I find your other test score?

A: 96 because $98 + 96 = 194/2 = 97$ because 96 is in between

I: Let us say one of your scores was an 88 instead?

A: $88 + x / 2 = 97$

I: You could have received extra credit on the test.

A: $97 \times 2 = 194$ Our answer would be 106

I: How did you get 106?

A: I sort of undid what I would normally do.

I: Can you relate this to our problem?

A: $4x - y$ is that correct?

I: How can we check this?

A: $4x - y + y / 2 = 4x / 2 = 2x$ The other number is $4x - y$.

I: A phone call from Duluth to Chicago cost \$1 for the first 3 minutes and 20¢ for each additional minute. If $r > 3$, a phone call r minutes long will cost how many dollars?

A: r =number of minutes--It would be $20r$ --You know $3r = \$1$.-- $\$1 + 20r > 3$ --

A: All we know is if $r > 3$, ... I don't know or understand the problem.

I: Which one of these is the correct answer? $3r/5$; $r-10/5$; $r-3/5$; $r+2/5$; $r + 15/5$

A: Maybe we should try a practical example. Let's say they talked for 5 minutes. We already know that it costs \$1 for the first 3 minutes. 2 extra minutes, so that would be 40¢ So it would cost \$1.40 for 5 minutes.

I: Think about the steps that you just went through. Remember that we want to know the cost in dollars. Maybe we could represent cents in terms of dollars. 20¢ is what part of a dollar?

A: $1/5$, I don't understand how any of those choices could be correct.

Andrea - April 20

I: Let us continue with our problem from last time.

A: 20¢ is $1/5$ of a dollar-- $r + 1 + 1/5 = r + 2/5$ --This is the correct answer--
\$1 can be represented as $5/5$ or 1 so now that you know it's r minutes long, and $r > 3$ so we know it will be more than \$1, we can put $r + 1$ which is the number of minutes over 3 and that would be $r + 1 + 1/5$ for the extra minutes -- If it's $r + 1$ --number of minutes over 3--this is the number of minutes plus one

A: This works because say I talked for 3 minutes, if I plug this into my equations I would get $3 + 2/5 = 5/5 = \$1$ which we know is right.--Let us say I talked for 7 minutes we would have $7 + 2/5 = 9/5 = \$1.80$ --\$1 for the first 3 minutes --we have 4 minutes over at 20¢ each minute--which comes out to 80¢-- so we are right

I: I'm confused how $r + 1 + 1/5 = (r + 2)/5$

A: We have the number of minutes plus one which is the total if we talked for 3 minutes for each additional it's $1/5$ of a dollar

I: $1 + 1/5 =$

A: $1 \frac{1}{5}$

I: $1 \frac{1}{5} + r =$

A: I just put everything over 5

I: Remember how we add fractions...

A: It would be $(5r + 5 + 1)/5 = 5r + 6/5$ That is wrong because if I talk for 3 minutes I won't get \$1.

I: Let us work with an example. Let's say I talked for 10 minutes. What is the first thing to do?

A: We know it is \$1 for the first 3 minutes--Next we subtract $10-3=7$. Then $.20(7)=\$1.40$... $\$1.40 + \$1 = \$2.40$

I: Remember $.20 = 1/5$

A: $7(1/5) = 7/5 = 1.40$

I: In other words, explain what you did.

A: We subtracted 3 minutes--Then we took $(r-3)1/5$...Then we have $r-3 / 5$

I: Let us check for 3 minutes.

A: No

I: What does $r-3$ tell us?

A: number of minutes left

I: We haven't taken into account...

A: The first 3 minutes,,,s o, $r-3/5 + 1 = r-3/5 + 5/5 = r + 2/5$ and that checks

I decided to combine two interviewing sessions into this analysis because we did not finish a problem in the first session. All of the questions I used in these interviewing sessions were found in the Graduate Record Exam book. The first question demonstrated to me that Andrea still had not yet reached the highest level of variable understanding due to the fact that she, as the other students did, restricted the domain of the variable. It should be noted that understanding of the problem only emerged after I prompted her to consider the different cases. The next problem seemed to incorporate several of Kuchemann's levels of understanding. Again, as detected with the other students there appears to be a tendency to initially guess at a solution and seem content until I persuade them to check the answer. I interpret this as an indication that there should be a greater emphasis on making students prove their answers. A real world situation was employed again. This seems to offer a tremendous help to the students whereby they are able to translate the steps or processes they go through into words and in turn into Algebra. Andrea appropriately admitted that she "undoes" what she would normally do in a given arithmetic problem. The last question was intended to be challenging in that it assimilates some of the intrinsic concepts behind the highest level of understanding variables.

It should be noted that I only worked on this problem with Andrea. She did a fantastic job of thinking aloud in this problem which I know was the key to her final success. Since she worked on the problem between the two interviewing sessions, I know that giving her options for the final answer probably guided her thinking and perhaps it was too significant a hint. I think she may have tried to derive the correct answer as a process of elimination. One of the keys to this problem was understanding that the solution needed to be expressed in terms of dollars. After she selected a solution, then she tried to justify it. Perhaps that is why I could not follow her reasoning. As detected with the other students, further difficulty was discovered with fractions. Again, the key element was working through a real world situation as described earlier.

Charlene - March 25

When I asked Charlene what she liked most about Algebra, she said graphing. When I asked her what she liked least, she responded with Geometry. Then I asked her what a variable is and she answered by saying that it was a letter that represented a number or a specific unknown. Charlene, when asked why we even study variables, responded by saying they were useful when we studied word problems in that the variables could represent the unknowns.

I: If $a + 5 = 8$, $a = ?$

C: $a + 5 - 5 = 8 - 5$ so $a = 3$

I: Why do you subtract 5 from both sides of the equation?

C: We need to keep the equation balanced and get a by itself.

I: If $m = 3n + 1$ and $n = 4$, $m = ?$

C: $m=13$

I: Why?

C: You need to put the 4 in for the n, so you can write $m = 3(4) + 1$. Then you can multiply 3 by 4 to get 12 plus 1 will give you 13.

I: Why should we substitute 4 for n?

C: Because $n=4$

I: If $a + b = 43$, $a + b + 2 = ?$

C: I'm going to subtract 2 from 43 which will equal 41 and then you find two numbers so that $a + b$ will give you 41.

I: If I had 6 apples and then I added 2 more apples, how many apples would I have?

C: 8

I: $a + b$ represents two numbers when added together we will get 43. Now if I should take those two numbers which equal 43 and add 2 what should I get?

C: Can you give me two numbers that when added together will give me 43?

C: 22 and 21

I: Let a equal the variable that stands for the number 22 and let b equal the variable that stands for the number 21. What happens if I add 2?

C: 45, I see. I understand the Problem,

I: together will give me 45?

C: Yes.

I: If $n - 246 = 762$, $n - 247 = ?$

C: $n - 246 + 246 = 762 + 246$ --I'm doing the same thing as I did in the first problem. If I add $762 + 246$ I'm going to get what n is-- $n=1008$

I: What is $n-247$?

C: 1009--but wait, is that right--If I take $1008 - 247$ I get 761.

I: What is the relationship between $n - 247$ and $n - 246$?

C: It's going to make the 762 lower. As long as you add 1 to 246 you're going to be subtracting one from 762.

I: Why?

C: I don't know.

I: Let's think about another example. If I have 5 apples and I take 3 away, we'll have 2 left. If I have 5 apples and I take 4 away, or one more away, I have 1 apple left.
If $e + f = 8$, $e + f + g = ?$

C: Let $e=5$, and $f=3$, so $5 + 3 = 8$. Should I plug any number in for g ? Does g also need to equal 8?

I: Not necessarily

C: If $g=2$, then we would get 10

I: Let's not substitute any values in for e and f . We know $e + f$ could be any number. So if $e + f$ are any two numbers that when added together will give me 8, what happens if we add g to that sum?

C: The number's going to be more than 8.

I: How much more?

C: Whatever g is equal to.

I: How can we represent that in terms of an algebraic expression?

C: $e + f = 8 + g$

I: What is $e + f + g$?

C: $8 + g$

I had no problem with concluding that Charlene was at least at the letter evaluated stage. She appeared to have no difficulty responding to the first two questions. Her understanding seemed to break down at the letter ignored

stage. Charlene tried to apply equation solving heuristics to the questions pertaining to this level. When I prompted her with a simpler version of the questions at this level as well as reminded her what variables really were, she seemed to gain a greater understanding. However, I was not convinced that she could apply what we had analyzed during the simpler problem to the more complex one. I knew that we would definitely have to review the problem: $n-246 = 762...$ It seems that she was determined to solve for n as opposed to ignoring the specific value for n and looking at the problem as a whole. Perhaps she viewed the two equations as separate and unrelated. I believe that by the time we got through the last problem, she was beginning to gain a deeper understanding of the letter ignored stage as evidenced by the markedly shorter time she arrived at the correct answer.

Charlene - March 29

- I: Given an equilateral triangle with length of each side as e , find the perimeter.
- C: e to the third power or $e + e + e$
- I: So $e + e + e = e$ to the third power?
- C: no, I don't know.
- I: What is $4x + 3x$?
- C: $7x$
- I: What is $e + e$?
- C: $2e$
- I: What is $e + e + e$?
- C: $3e$
- I: Given a figure of n sides, each of length 2, what is the perimeter?
- C: I can't think of the equation.
- I: Let's say we knew there were 5 sides to this figure. Each side was 2 units long. What is the perimeter?
- C: We could multiply 2 by 5.
- I: What does the 2 and the 5 stand for?

C: 2 is the length of each side and 5 is the number of sides

I: Let's go back to our original problem,

C: 2 times n or $2n$

I: Cabbages cost 8¢ each and turnips cost 6¢ each. If c stands for the number of cabbages bought and t stands for the number of turnips bought, how could we represent the total number of vegetables bought?

C: Couldn't we just write $8c + 6t = n$ by dropping the decimal point?

I: Are you sure?

C: Yes

I: Why don't we write down what our variables stand for (c=number of cabbages bought and t=number of turnips bought) How can we write the total number of cabbages and turnips bought?

C: $c + t$

I: What would $.08c + .06t$ stand for?

C: The total amount that they cost--It wouldn't tell you how many you bought just how many you bought or how much you spent

I: On what?

C: Both of them together.

I: So $.08c$ tell us what?

C: The cost of cabbages

I: How many cabbages?

C: One

I: If c stands for the number of cabbages bought and we don't know what that number is, if we bought 4 cabbages how much would we pay?

C: 32¢, so it is the cost of c cabbages

I: So what does $.08c + .06t$ tell us?

C: The cost of c cabbages and t turnips

I: 4 added to n can be written as $n + 4$ Add 4 onto 8.

C: 12

I: Add 4 onto $n + 5$

C: $n + 5 + 4 = n + 9$

I: Add 4 onto $3n$.

I: But you wrote $3n + 4$

C: We can't add those two together because they are not like terms.

I: How many numbers does $3n$ represent?

C: As many as you want, no , only 3

I: What if I put a value in for n , say $n=3$?

C: $3n = 9$

I: HOW many numbers does $3n$ represent?

C: one

As was observed when interviewing the Algebra student, Charlene exhibited difficulty in combining like terms. This shows that Charlene's understanding of variables broke down at the level of variable as an object. Rather than pointing out that the coefficient in front of the e 's is one, I thought I would see if she could add $4x + 3x$. When she readily answered that question, she seemed to make the correlation with the problem at hand. The ambiguity of n sides in the following problem seemed to cause problems for Charlene. As soon as we introduced a similar problem and identified what the numbers stood for she was able to generalize the solution to the more complex problem. To ensure that she understood I posed another perimeter problem for her which she was readily able to answer. Due to the fact that the other two students had difficulty with the "cabbage and turnip" problem, I decided to ask her an "easier" question to start off with. In light of her response, I feel that what she and the other students were determined to do was to set up an equation to be solved as one would normally do when faced with a word problem. It was not until I had Charlene actually write down what the variables stood for that she was able to solve the problem. Then when I asked her what the $.08c + .06t$ stood for she was able to give me the correct answer. This gave me an indication of how important correctly defining variables is and how that can facilitate understanding of letters as objects. With respect to the level that assesses a variable as a specific unknown, Charlene did not

appear to have any misconceptions on the surface. I was glad I inquired as to how many numbers $3n$ represented. I was surprised to see that Charlene did not understand the juxtaposition nature of variables. It was helpful to show an example of substituting a value in for n , because it was only at this time that she was able to arrive at the correct answer.

Charlene - March 31

I: Is the following statement always true, sometimes true or never true:
 $L + M + N = L + P + N$? If so, when?

C: I have never seen a problem like this before.

I: When do we have a true equation or what has to happen?

C: There needs to be a balance

I: Does $L = L$?

C: As long as they both represent the same number.

I: Does $L + N = L + N$?

C: Yes

I: We have two other variables, M and P

C: If we add M and P to one side of the equation we need to add them to the other

I: We don't have a M and a P on both sides..Why isn't $L + M + N = L + P + N$ not true?

I: Because it's not balanced. There's a M on one side and a P on the other

I: Could M ever equal P ?

C: Yes, when we substitute the same value for M and P . So M and P could represent the same number.

I: Could $L + M + N = L + P + N$ even though we have different variables on either side of the equal sign?

C: Yes

I: When might this happen?

C: If we put the same number in for M and P we would have a balanced equation.

I: Is our statement always true, sometimes true, or never true?

C: Sometimes, no always, or only when we put the same number in for M and P

C; Sometimes

I: What number divided by 24 equals $3/4$?

C: $n/24 = 3/4$ This is the equation. I don't know.

I: What do we need to do in order to solve this problem, or what is our strategy?

C: It is just reducing a fraction, correct? We have to reduce $n/24$. So $n=6$.

I; So, $6/24 = 3/4$? How can we be sure?

C: $6/24 = 1/4$ So I'm wrong.

I: What are you thinking about doing here?

C: I'm trying to find a number that 24 divides into.

I: Could we cross-multiply to solve?

C: So $4n/4 = 72/4$ or $n = 18$

C: When we reduce $18/24$ we get $3/4$.

In the last interviewing session, Charlene had particular difficulty with the level of letter ignored. It is here that I initially detected a collapse of her understanding. I found another problem that assessed this level of understanding in the Algebra Learning Project discussed earlier. After I initially posed the question to Charlene, I was pleased that she responded by saying that she had never seen a problem like this before. Throughout the interviewing process, I stressed to the students to analyze each problem and try to apply approaches they have learned in the past on the task at hand. Evidently, Charlene knew when an equation would be balanced, hence, true. I wasn't sure why. I asked her if $L = L$ but her response convinced me that she retained something from the previous session; namely, $L = L$ only if the variable represented the same number on both sides of the equation. Charlene naturally had a notion that in order for an equation to be balanced the same letters had to be present on both sides of the equation. When I pointed out to her that $4 + 4 = 8$ I think she began to realize that a balanced equation did not necessarily indicate a mirror image

on both sides of the equal sign. I had to prompt her to consider whether or not M could ever equal P and she quickly responded correctly; yet that did not assist her with the original problem. After a while, she concluded that the statement was sometimes true. It should be noted that Charlene did this in her own mind. Another question from the Algebra Learning Project was the second question I used in this interview. She wrote the equation that corresponded to my verbal formulation of the question. When I asked her what the ultimate goal would be in solving the problem, I was interested that she approached this problem in a "non-traditional" way in that I believe most Algebra students would attack this problem as solving the equation. It would be interesting to determine whether or not this had anything to do with the fact that she was an Algebra Survey student. After her "guess and check" method was not working (15 minutes) I suggested cross-multiplying and she immediately solved the problem and checked it. She obviously exhibited a difficulty with techniques used in solving equations. Finally, we should take note of the fact that she checked the problem from her original viewpoint of the problem as determining what fraction could be reduced to $\frac{3}{4}$.

Charlene - April 14

- I: Please give me an algebraic equation to represent the following: There are 6 times as many students as professors.
- C: $6s$, Six times as more means $6s - 6s +$ (as many more means plus)--I think it should be $6s + p$. I know $6s$ is right.
- I: What is the first thing you do whenever you work on a word problem?
- C: I read it maybe twice and then find the variable which represents the unknown. I see if there's a known and I also look for key words (is means =)
- I: So you first try to figure out which variables represent the unknowns?
- C: Yes
- I: You introduced a variable when you wrote $6s$, what was the variable?
- C: s
- I: What does that variable stand for?
- C: Students

- I: What does 6s mean?
- C: It means 6 more than students, no wait... it means 6 times more than professors
- I: Are there more students or professors?
- C: MOre students
- I: In an equation can we have more than one variable?
- C: Yes, so we need another variable for professors-- it would be $6s + p = n$.
- I: What does n stand for?
- C: I don't know. The amount of students--No, the point is we don't even need n
- I: That is a possibility. HOw many unknowns do we have in our original problem?
- C: Two, so we don't need n.
- I: So we need an equation that utilizes two variables. What is true of all Equations?
- C: It is the same on both sides. It is balanced.
- I: Can we relate one variable in terms of the other?
- C: No
- I: In English, what does $6s + p$ mean?
- C: 6 times as many students as professors
- I: What if we had 36 professors, how many students would we have?
- C: Souldn't we multiply 36 by 6? That equals 216.
- I: Does that make sense or are there six times as many students as professors? Are there more students?
- C: Yes
- I: PLease refresh my memory, what is a variable?
- C: A letter that stands for an unknown or number.
- I: So what does s really represent?
- C: The total number of students.
- I: This is an important distinction. We need to be careful in defining our variables. What does p stand for?
- C: The total number of professors.
- I: In this problem what is there a relationship between?

C: s and p

I: If an equation is a relationship and the two things we're relating are s and p , what does that tell you about how the equation might look?

C: $s + p = 6s$

I: In English, tell me what you have written.

C: The total number of students plus the total number of professors equals 6 times as many students as professors.

I: What does $6s$ mean?

C: 6 times the total number of students

I: Does that make sense according to our problem?

C: Yes, No it doesn't sound right

I: If there were four professors how many students are there and why?

C: 24 because $6 \times 4 = 24$

I: I gave you the total number of professors so what did you do to get the total number of students?

C: I multiplied the total number of professors by 6

I: NOW, in Algebra translate what you just told me.

C: $6s \times p = s$

I: ^eTell me what that says.

C: Six times the number of students times the number of professors equals the total number of students

I: Let's go back to our practical example.

C: $6p = s$

I wanted to make sure I included the classic "Student and Professor" problem in all of my interviews. The type of errors Charlene made were not the common misconceptions found in the literature. As with Diane, Charlene began by not giving me an equation. In spite of the problem solving heuristics Charlene stated she employed, specifically, picking out key words which indicate operations, she came up with $6s + p$. Yet she could not pick out the key word that indicated addition. As I expected, Charlene was unable to correctly define her variables.

She was able to interpret the problem in that she saw there were ^{more} students than professors. It seems that further insight into the problem arose when I asked Charlene to translate the equation into English, I think the key to understanding came when we resorted to a practical example. Reminding Charlene what a variable really represents enabled her to eventually define her variables correctly. As opposed to the other students, I really do not feel that she had an intuitive understanding. In other words, the other girls said that p stood for professor but given the context of translating, they were able to correctly define their variables. It was not until we utilized a practical example, analyzed the steps, translated the steps into English and then put the words in terms of an algebraic expression that Charlene was finally able to derive the correct answer.

FINAL REMARKS

The concluding remarks for each student are taken from the last interview session.

DIANE - APRIL 27

I asked Diane if she had learned anything the last few sessions or if she was thinking about certain concepts differently. She responded by saying that she really tries to think about what the variable means and when she encounters difficult problems she tries to recall what we discussed. Also, she was able to correct her original misconception that variables are not always represented by letters. One of the final questions I asked was what happens to the value of $1/x$ as x gets larger. She immediately began to approach this problem by trying out different values of x . She was then able to conclude behind the highest level of understanding. Since Diane exhibited difficulty with the "cabbage and turnip" problem I asked her another question similar to it: Carol earned d dollars during the week. She spends e dollars for clothes and f dollars for food. Can you write an algebraic expression that shows how many dollars she had by the end of the week? She correctly defined her variables and arrived at the correct solution. A number of articles I read alluded to the fact that many students have difficulty interpreting the meaning of algebraic definition of absolute value. Diane seemed to have no problems with this as evidenced by the fact that she related it to similar problems we had discussed before. I concluded the interview by asking her to write an equation that represented the fact that in a cafeteria there are $1/2$ as many coffee cups as juice glasses. Although we spent quite a while, Diane employed several of the aspects of variable understanding we had utilized the past couple of weeks.

In conclusion, by the end of our interviewing she was definitely able to verbalize her thoughts more accurately.

ANDREA - APRIL 27

With respect to Andrea's last session I basically asked her the same questions as I did with Diane. She impressed me by the way she reasoned through the problem. What happens to $1/x$ as x gets larger? Further, she had no trouble with the problem assessing her understanding of defining variables (i.e. clothes, food, etc) and took special care in defining her variables. As opposed to Diane, Andrea also had no trouble interpreting the Algebraic definition of absolute value. Since she had prior difficulty with the always, sometimes or never problem, I asked her a similar problem which she was able to correctly define the solution. Further, I was still concerned with her understanding of fractions so I asked another problem dealing with that. Through the results of that problem, I saw that cleared up her original misconceptions. Finally, I asked her the problem: find an Algebraic equation to represent that there are $7 \frac{1}{2}$ times as many coffee cups as juice glasses in the cafeteria.

In conclusion, it was evident that Andrea had retained much of what we discussed in the last few weeks and her conception of variables was definitely enhanced.

CHARLENE - April 27th

Charlene's last interview did not go as well as I anticipated. She expressed anxiety for about a week concerning being taped. I do feel that it may have affected her responses; even though she insisted on continuing her sessions. As with the other students I asked her the same questions. She appeared to have difficulty with all of them and I found myself reviewing and hinting quite often. I did not have to do this with the other students. We spent more of the time dealing with the problem which was similar to the "Student/Professor" problem. I concluded that she really did not understand that problem at all. After going through this other problem situation, I feel she gained some insight but was still confused. I was not discouraged because I know Charlene's understanding of variables was enhanced and I was content to accept that due to other cognitive processes. Some students will remain at a level of certain understanding.

APPENDICES

Kathleen Gavin
Northern Illinois University

De [REDACTED] s [REDACTED]
February 19, 1988

Dear Dr. McAllister,

I would like to request your permission to interview and tutor three students in Algebra I: one from an honors course, one from a regular course and one from a survey course during my mathematics student teaching with Mrs. Reid. Along with student teaching, I will be working on my Honors Capstone Project which concerns itself with exploring the concept of variable, assessing Algebra I students' understanding of the concept, determining how students' misconceptions concerning variables lead to problems in solving equations, and finally, to assess how individualized tutoring in this area will enhance the students' understanding.

Many students have difficulty making the transition from junior high to high school mathematics and in particular, they have difficulty understanding what these things are that we begin referring to as variables when students begin Algebra I. Therefore, I would like to work with three students to help them understand this very important concept. I will work with each student over a two week period for thirty minutes every other day. Each time I work with the student I will assess his or her understanding of variables and provide the appropriate instruction to enhance this understanding.

All of the tutoring sessions will be audio-taped; however, the students' anonymity will be protected by changing their name in reporting the data. Further, the interview questions, the tutoring techniques, and the assessment will be under the direct supervision of my Honors Advisor, Dr. Nancy Mack, Assistant Professor, Northern Illinois University.

I will contact you within the week to see if this proposal meets with your approval. If so, I will also be asking you for the procedures you have concerning sending permission letters home to parents. If you have any questions regarding this study, please feel free to contact me at [REDACTED] or Dr. Mack (815) 753-6748. I am enclosing a sample permission letter to the parents as well as sample interview questions. Thank you for your help.

Sincerely,

Kathleen Gavin
Honors Student

Dr. Nancy Mack
Assistant Professor

Kathleen Gavin
Northern Illinois University

«Parents»

«Address»

Dear «Parents»,

I would like to request your permission to interview and tutor «Child's Name» in Algebra. My name is Kathleen Gavin and I am going to be student teaching at Shepard High School this semester. Along with my student teaching, I will be working on completing my Honors degree from Northern Illinois University. My Honors Capstone Project consists of exploring the concept of variable, assessing Algebra I students' understanding of the concept, determining how students' misconceptions concerning variables lead to problems solving equations, and finally, to assess how individualized tutoring in this area will enhance the students' understanding.

Many students have difficulty making the transition from junior high school to high school mathematics, and in particular they have difficulty understanding what these things are that we begin referring to as variables when students begin Algebra I. Therefore, I would like to work with three students to help them understand this very important concept. I will work with each student individually over a two week period. Each time I meet with a student I will interview him or her to assess understanding and provide appropriate instruction to enhance this understanding. All of the tutoring sessions will be audio-taped; however, the student's anonymity will be protected by changing his or her name in reporting the data. Further, the interview questions, the tutoring techniques and the assessment will be under the direct supervision of my Honors Advisor, Dr. Nancy Mack, Assistant Professor of Mathematics, Northern Illinois University.

If you would be willing for me to interview and tutor «Child's Name», please sign and return the attached permission form. Please be aware that I may receive permission to work with several students; however, I regret that I will only be able to work with three of the students I receive permission for. Additionally, I give «Child's Name» permission to withdraw from this project before its completion if «he/she» desires. If you have any questions regarding my project, please feel free to contact me ([REDACTED]) Dr. Mack ((815)753-6748). Thank you for your help.

[REDACTED] sincerely,

Kathleen Gavin
Honors Student

Nancy K. Mack
Assistant Professor

Kathleen Gavin

Honors Capstone Project
Variables: What Are They and Why Are They Important?

Please respond to both parts.

Yes. I am willing to allow you to interview and tutor my son/daughter.

I am sorry, but I would prefer that my son/daughter does not participate.

Signature of Parent or
Guardian

Date

Yes. I am willing to allow you to audio-tape your tutoring sessions with my son/daughter.

I am sorry, but I would prefer that you do not audio-tape your tutoring sessions with my son/daughter.

Signature of Parent or
Guardian

Date

Kathleen Gavin
Honors Capstone

Sample Questions

1. In your own words can you tell me what a variable is?
2. At Windy's restaurant, for every four people who order cheesecake, there are five people who order strudel. Can you represent this as an algebraic expression? (Hint: let C represent the number of cheesecakes and S represent the number of strudels)
3. How may we represent the algebraic expression: the sum of three consecutive even integers? If unable to answer, can you give me an example or tell me what is the relationship between three consecutive even integers?
4. Solve for v: $30 = 2/5v + 10$
5. Carol earned D dollars during the week. She spent C dollars for clothes and F dollars for food. Write an expression using D, C, and F that shows the number of dollars she had left.

REQUEST FOR UNIVERSITY HONORS INDEPENDENT STUDY

PLEASE TYPE

Kathleen Ann Gavin
Student Name

[REDACTED], [REDACTED]
[REDACTED] Address

Math - 499H
Department and Course Number

0971
Computer Reference Number

Spring 1988
Semester of Registration

February 1, 1988
Date of Request

Attach additional pages as needed.

1. Describe below, in detail, the goal(s) of the work you propose. What will you study?

The goal of my Honors Capstone project is to explore the concept of the variable as it relates to an Algebra I course. I believe this to be a significant endeavor in that many of the problems with solving algebraic equations arise from students' misconceptions concerning variables. Initially, I will define the concept of variable. Next, I will examine studies concerning students' understanding of variables for the types of questions asked as well as the students' misconceptions and how these misconceptions lead to difficulties in solving equations. During student teaching this semester, I will conduct student interviews with an honors student, an average student, and a student from an Algebra Survey class to examine their understanding of the concept of variable. After the initial interviews, I will devote two weeks of individualized instruction for each of these students. The purpose of the instruction is to enhance the students' understanding and to examine the changes in their understanding.

2. Please list major works dealing with this topic (published materials relevant to your project):

Dynamics of Teaching Secondary School Mathematics, Cooney, Davis, Henderson
Mathematics Teacher, Volume X, JULY 1986
Research Within Reach, Driscoll
School Science and Mathematics, Volume LXXXII Number 6, October 1982
Mathematics Teacher, NOVember 1980
Journal for Research in Mathematics Education, Volume XII, 1981
Focus on Learning Problems in Mathematics, Volume 5, 1983
Children's Understanding of Mathematics
Students' Performance in Algebra: Results from the National Assessment
of Educational Progress

3. Describe the methodology of your proposed study. For example, if you plan regular conferences with your advisor, how often? What written work will your study produce? If you will be working in a laboratory, what equipment will be needed and what procedures will you follow?

I plan to meet with my advisor, Dr. Nancy Mack, once or twice a week prior to the beginning of my student teaching. Meetings during student teaching will be decided at a later time. I will research studies and materials concerning variables. Further, it will be important to learn how to conduct interviews which my advisor will assist me with. For example, I plan to tape all my interviews so that it will be easier to evaluate and discuss them with my advisor. Finally, in order to develop lessons to enhance students' conceptions of variables, it will be necessary to research Algebra texts as well as results of studies which include teaching suggestions.

4. What tangible evidence of the project's completion will you submit?

I will submit, a research paper that will include a summary of critical words already done in this field, the findings of my student interviews, and the lessons I used to enrich the students' understanding of variables.

5. How will the project/thesis be presented and to whom?

I plan to present this project in its written form to my advisor, the math department and to the Honors Director.

6. List the courses you have taken (Honors and non-Honors) which provide a background for this study.

MATH 412

MATH 421

MATH 420

LDEP 313

- . Please indicate the proposed title of your project/thesis.

Variables: What Are They and Why Are They Important?

Kathleen A. Harin
Student Signature



Request approved:

Nancy S. Mack
(Instructor)



I hereby certify that the above mentioned independent study does not duplicate in content and/or method similar material offered in a regular course in this, the semester of enrollment or the immediately preceding or immediately subsequent semesters.

(Department Chairperson)

(Date)

(Director, University Honors Program)

(Date)

REFERENCES

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- GRE General Test - No. 5 (1987). Educational Testing Services.
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- Rachlin, Sidney. (1987). Algebra I: A Process Approach. Honolulu, Hawaii. Curriculum Research and Development Group. Chapters 1 and 2.
- Rosnick, Peter. "Some Misconceptions Concerning the Concept of Variable" Mathematics Teacher. September, 1981 (418-420).
- Wagner, Sigrid. "What Are These Things Called Variables." Mathematics Teacher. October, 1983. (47-L - 479).
- Wagner, Sigrid. (1981). Algebra Learning Project.