NORTHERN ILLINOIS UNIVERSITY

What is the Square Root of 2?

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Department of Mathematics
by
Kristine A. Beernink
Dekalb, Illinois
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Approved: Henry S. Leonard Jr.

Department of: Mathematical Sciences

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In mathematics, we are always told that when there are two methods for doing some problem, they will always yield the same results. In this paper, I will see if this is true when using two methods for constructing the real numbers. Why the reals? Well, the reals have always been difficult for me to grasp. You can ask your mother for 1/2 a piece of bread or any other rational piece of bread. But try asking her for $\sqrt{2}$ pieces of bread and she won't know what to give you.

The real numbers are defined by the following 12 axioms. There are functions $+: R \times R \to R$ and $\cdot: R \times R \to R$ and a relation $<$ on R such that $\forall\ x,y,z \in R$, where R is the set of real numbers, and the following are true:

1) $(x+y)+z = x+(y+z)$; $(x\cdot y)\cdot z = x\cdot(y\cdot z)$;
2) $x+y = y+x$; $x\cdot y = y\cdot x$;
3) $x\cdot(y+z) = (x\cdot y) + (x\cdot z)$;
4) $\exists!\ 0 \in R$ such that $0+x=x\ \forall x \in R$;
5) For each $x \in R$, $\exists! y \in R$ such that $x+y=0$, and we write $y=-x$
6) $\exists! 1 \in R$ such that $x\cdot 1=x\ \forall x \in R$; and $1 \neq 0$;
7) For each $x \in R$ with $x \neq 0$, $\exists!\ y \in R$ such that $x\cdot y=1$, and we write $y=x^{-1}$ or $y=1/x$;
8) $x<y$ implies $x+z<y+z$
9) $x<y$ and $y<z$ implies that $x<z$
10) for $x,y \in R$ exactly one of the following is true: $x<y$, $y<x$, $x=y$;
11) $x<y$ and $z>0$ implies that $xz<yz$.

12) Every nonempty set of real numbers that is bounded from above has a least upper bound.

So now we shall construct the real numbers based on these 12 axioms. We will do this first by using Dedekind cuts, which we must first define.

Definition: A Dedekind cut is a set $\alpha \subset Q$ such that

(i) α is nonempty

(ii) $\alpha \neq Q$

(iii) if $a \in \alpha$ and $b \in Q$ and $b<a$ then $b \in \alpha$.

(iv) α does not contain a largest member.

To visualize what a cut is look at the following example:

Example 1: $\alpha = \{p \in Q \mid p<2\}$ is a Dedekind cut.

(i) α is nonempty because $1<2$ so $1 \in \alpha$.

(ii) $\alpha \neq Q$ because $2 \in Q$ but $2 \not< 2$ so $2 \notin \alpha$.

(iii) Let $a \in \alpha$, then $a<2$, now let $b \in Q$ with $b<a$, then $b<2$ so $b \in \alpha$.

(iv) Let $a \in \alpha$ then $a<2$ so $2a<a+2$, $a<(a+2)/2$ and $a+2<4$, $(a+2)/2<2$ so $a<(a+2)/2<2$, obviously $(a+2)/2 \in Q$ so $(a+2)/2 \in \alpha$. So α has no largest member.

I now claim that the set of all Dedekind cuts is R, the set of real numbers. In order to show this we will prove the 12 axioms are true for the set of all Dedekind cuts. For the next few pages the sumbol R will represent the set of all Dedekind cuts.

First we define + in R. Let $\alpha, \beta \in R$ then define $\alpha + \beta = \{a+b \mid a \in \alpha, b \in \beta\}$. But is $\alpha+\beta \in R$? Let's see. (i) $\alpha+\beta$ is nonempty since α and β are nonempty. (ii) Since α and β are cuts neither equal Q. So $\exists$ a, b$\in$Q with $a \notin \alpha$ and $b \notin \beta$. Then if $c \in \alpha$ and $d \in \beta$, c<a and d<b otherwise $a \in \alpha$ and $b \in \beta$ which we know is not true. Therefore, for $\forall$ $c \in \alpha$ and $d \in \beta$ c+d<a+b so $a+b \notin \alpha+\beta$ so $\alpha+\beta \neq Q$. (iii) Let $c \in \alpha+\beta$ then c=a+b for some $a \in \alpha$, $b \in \beta$. Now suppose d<c then d<a+b so d-b<a so $d-b \in \alpha$. Then $d=(d-b)+b \in \alpha+\beta$. (iv) Let $a \in \alpha$ and $b \in \beta$ then since α and β are cuts there is $c \in \alpha$ and $d \in \beta$ with a<c and b<d so a+b<c+d and $c+d \in \alpha+\beta$ so $\alpha+\beta$ has no largest member. Therefore we have shown that $\alpha+\beta$ is indeed an element of R.

Now since we know addition in Q is associative we can easily see that $(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma)$. Because $(\alpha+\beta)+\gamma=\{(a+b)+c \mid a \in \alpha, b \in \beta, c \in \gamma\}$ and $\alpha+(\beta+\gamma)=\{a+(b+c) \mid a \in \alpha, b \in \beta, c \in \gamma\}$. So now we have that R satisifes half of axiom 1. To get the other half we need to define $\cdot$ in R, but before doing so we need to establish the < relationship in R.

First $\alpha=\beta$ iff every element of α is an element of β, and every element of β is an element of α. Then define $\alpha<\beta$ iff $\alpha \subset \beta$ and $\alpha \neq \beta$. Now let $\Theta=\{r \in Q \mid r<0\}$ which is obviously an element of R. Then also define $-\alpha$. $-\alpha=\{r \in Q \mid \exists s$ such that $s \notin \alpha$ and $r<-s\}$. It can be shown fairly easily that $-\alpha$ is also a cut. So now we can define $\cdot$ in R in the following manner with $\alpha, \beta \in R$:

$\alpha \cdot \beta = \{r \cdot s \mid r \geqslant 0,\ s \geqslant 0,\ r \in \alpha,\ s \in \beta\} \cup \Theta$, if $\alpha \geqslant \Theta, \beta \geqslant \Theta$

$\alpha \cdot \beta = -[\alpha \cdot (-\beta)]$ if $\alpha \geqslant \Theta, \beta < \Theta$

$\alpha \cdot \beta = -[(-\alpha) \cdot \beta]$ if $\alpha < \Theta, \beta \geqslant \Theta$

$\alpha\cdot\beta=[(-\alpha)\cdot(-\beta)]$ if $\alpha<\Theta, \beta<\Theta$

In order to show that $\alpha\cdot\beta$ is a cut, we need only show it for the case when $\alpha>\Theta$ and $\beta>\Theta$. (i) Since we know that Θ is nonempty, $\alpha\cdot\beta$ is nonempty. (ii) Now since α and β are cuts there are $a\notin\alpha$ and $b\notin\beta$ with $a>0$ and $b>0$ since $\alpha>\Theta$ and $\beta>\Theta$. Then for $c\in\alpha$ and $d\in\beta$ with $c\geqslant0$ and $d\geqslant0$ then $c<a$ and $d<b$ which means $cd<ab$ and $ab\notin\alpha\cdot\beta$ so $\alpha\cdot\beta\neq Q$ (iii) Now let $a\cdot b\in\alpha\cdot\beta$ and $c\in Q$ with $c<a\cdot b$ and with $a>0$ and $b>0$. Then if $c<0$ $c\in\Theta$ so $c\in\alpha\cdot\beta$. If $c=0$ then $c\in\alpha\cdot\beta$ because $\alpha>\Theta$ and $\beta>\Theta$. And if $c>0$ then $c/a<b$ and obviously $c/a\in Q$ and is therefore an element of β. Then $c=(c/a)\cdot a\in\alpha\cdot\beta$. (iv) Let $a\cdot b\in\alpha\cdot\beta$ with $a\in\alpha$, $b\in\beta$ then since α and β are cuts there is $c\in\alpha$ and $d\in\beta$ such that $a<c$ and $b<d$ so $ab<cd$ and $cd\in\alpha\cdot\beta$. So $\alpha\cdot\beta$ has no largest member. Therefore, $\alpha\cdot\beta$ is an element of R.

We also just need to show the associative law is true for the case where $\alpha>\Theta$ and $\beta>\Theta$. Then for $\alpha,\beta,\gamma\in R$ $(\alpha\cdot\beta)\cdot\gamma=$ $\{(a\cdot b)\cdot c \mid a,b,c\geqslant0,\ a\in\alpha, b\in\beta, c\in\gamma\}\cup\Theta$ and $\alpha\cdot(\beta\cdot\gamma)=\{a\cdot(b\cdot c)\mid a,b,c,\geqslant0,$ $a\in\alpha, b\in\beta, c\in\gamma\}\cup\Theta$ which are obviously equal since $\cdot$ is associative in Q. Therefore R satisfies the first axiom. Also from our definitions of + and $\cdot$ it is clear that $\alpha+\beta=\beta+\alpha$ and $\alpha\cdot\beta=\beta\cdot\alpha$ since + and $\cdot$ are commutative in Q. So R also satisfies axiom 2.

Now we need to show that $\cdot$ is distributive in R. Let α, $\beta,\gamma\in R$, and take the case $\alpha\geqslant\Theta$, $\beta\geqslant\Theta$, and $\gamma\geqslant\Theta$. Now let's look at $\alpha\cdot(\beta+\gamma)$ and $(\alpha\cdot\beta)+(\alpha\cdot\gamma)$. $\alpha\cdot(\beta+\gamma)=\alpha\cdot\{b+c\mid b\in\beta, c\in\gamma\}=\{a\cdot(b+c)\mid$ $a\geqslant0, b+c\geqslant0, a\in\alpha, b\in\beta, c\in\gamma\}\cup\Theta$. Then $(\alpha\cdot\beta)+(\alpha\cdot\gamma)=\{a\cdot b \mid a\geqslant0, b\geqslant0,$ $a\in\alpha, b\in\beta\}\cup\Theta+\{a\cdot c \mid a\geqslant0, c\geqslant0, a\in\alpha, c\in\gamma\}\cup\Theta=\{(a\cdot b)+(a\cdot c) \mid a\geqslant0,\ b\geqslant0,$

$c \geqslant 0, a \in \alpha, b \in \beta, c \in \gamma \} \cup \Theta$. With a little thought it is obvious that these two are indeed equal and therefore R satisfies axiom 3.

Earlier we defined $\Theta = \{r \in Q \mid r < 0\}$, we can see that $\Theta + \alpha = \{a + r \mid a \in \alpha, r \in \Theta\} = \{a \mid a \in \alpha\} = \alpha$. This is intuitively clear since Θ cannot add a positive number to the members of a. But we need to know that Θ is unique. Assume $\phi \in R$ with $\phi + \alpha = \alpha$ then $\phi + \alpha = \{a + r \mid a \in \alpha, r \in \phi\}$. If $\phi + \alpha = \alpha$ then every r must be less than zero so that $\phi + \alpha \ngtr \alpha$, but if the r's do not approach 0 $\phi + \alpha < \alpha$. Therefore $\phi = \{r \in Q \mid r < 0\}$ so $\phi = \Theta$ and Θ is unique and R satisfies axiom 4.

Previously we also defined $-\alpha$ in terms of α with $\alpha + (-\alpha) = \{r + (-s) \mid r \in \alpha, -s \in -\alpha\}$ where $-\alpha = \{r \in Q \mid \exists s$ such that $s \notin \alpha$ and $r < -s\}$. Now let $r \in \alpha$ and $s \in -\alpha$. Then $r < -s$ so $r + s < 0$. Therefore $\alpha + (-\alpha) \leqslant \Theta$. But we need to know that this is equal, not just less than or equal. So now suppose $u < 0$. And since α is a cut we can find $t \notin \alpha$, so then for $r \in \alpha$ $r < t$. Let $v = t - r$. Then there $\exists n \in J$ such that $n(-u/2) \geqslant v$. Now since α is a cut and $r \in \alpha$ and $r + n(-u/2) \notin \alpha$ (this is true since $r + n(-u/2) \geqslant t$ and $t \notin \alpha$) we can find $m \in J$ such that $m \leqslant n-1$ and $r + m(-u/2) \in \alpha$ and $r + (m+1)(-u/2) \notin \alpha$. Then obviously $-(r + (m+2)(-u/2)) \in -\alpha$. Then $r + m(-u/2) + (-(r + (m+2)(-u/2)) = u \in \alpha + (-\alpha)$. Since $u < 0$, $u \in \Theta$, thus $\Theta \leqslant \alpha + (-\alpha)$, and therefore $\Theta = \alpha + (-\alpha)$. It is also clear that this $-\alpha$ must be unique so R satisfies axiom 5.

Now let $\omega \in R$ defined by $\omega = \{r \in Q \mid r < 1\}$ and $\alpha > \Theta$, then obviously if $a \in \alpha$, with $a \geqslant 0$ and $w \in \omega$ $a \cdot w \leqslant a$ since $w < 1$. So $a \cdot w \in \alpha$. Then since the definition of $\cdot$ includes the union of Θ we know that $\alpha \cdot \omega \leqslant \alpha$. Now let $a \in \alpha$ with $a \geqslant 0$. Then there is $b \in \alpha$ such that $a < b$, so $0 < a/b < 1$ and so $a/b \in \omega$. Then $a = b \cdot (a/b) \in \alpha \cdot \omega$ so $\alpha \leqslant \alpha \cdot \omega$, therefore $\alpha \cdot \omega = \alpha$ for

$\alpha > \Theta$. By definition of $\cdot$, if $\alpha = \Theta$, $\alpha \cdot \omega = \Theta \cdot \omega = \Theta = \alpha$. Then when $\alpha < \Theta$ we have $\alpha \cdot \omega = -(-\alpha \cdot \omega) = -(-\alpha) = \alpha$, by the definition of $\cdot$. It is also clear that ω is unique and therefore R satisfies axiom 6.

The proof of axiom 7 is quite long but can be done. We will just state how to define α^{-1} when $\alpha > \Theta$ and leave it up to the reader to prove if he wishes. Define $\alpha^{-1} = \{r \in Q \mid r > 0$ and $\exists s \in Q$, $s \notin \alpha$ such that $s < 1/r\} \cup \Theta \cup \{0\}$.

Earlier we defined $<$ in R. Let $\alpha, \beta, \gamma \in R$ with $\alpha < \beta$, then there exists $b \in \beta$ such that $\forall\ a \in \alpha$, $a < b$ then let $c \in \gamma$, since these are elements of Q $a+c < c+b$ thus $\alpha + \gamma < \beta + \gamma$, and axiom 8 is satisfied.

Now let $\alpha, \beta, \gamma \in R$ with $\alpha < \beta$, $\beta < \gamma$. Then from the definition of $<$ in R $\alpha \subset \beta$ and $\beta \subset \gamma$. By the laws of sets this means that $\alpha \subset \gamma$, but since $<$ in R says that $\alpha \neq \beta$, we also have as sets $\alpha \neq \beta$ and therefore $\alpha \neq \gamma$. So $\alpha < \gamma$ when $\alpha < \beta$ and $\beta < \gamma$ in R. So R also satisfies axiom 9.

Now if $\alpha < \beta$ then by definition $\alpha \neq \beta$. If $\alpha < \beta$ and $\beta < \alpha$, then $\beta \subset \alpha$ and $\alpha \subset \beta$, so $\beta = \alpha$ by the laws of sets, but this contradicts $\alpha < \beta$ and $\beta < \alpha$, finally if $\alpha = \beta$ then by definition $\beta \not< \alpha$. So we have shown that for two cuts α and β only one of the following can be true $\alpha < \beta$, $\beta < \alpha$ or $\alpha = \beta$. But for axiom 10 we need to know that for $\alpha, \beta \in R$ one of these is true. Assume that $\alpha \not< \beta$ and $\alpha \neq \beta$. Then $\alpha \not\subseteq \beta$ as sets. Therefore $\exists a \in Q$ with $a \in \alpha$ but $a \notin \beta$, let $b \in \beta$. So $b < a$ since $a \notin \beta$. So $b \in \alpha$, and therefore $\beta \subseteq \alpha$ as sets so $\beta < \alpha$. Thus R satisfies axiom 10.

Now we need to look at multiplication using inequalities. Let $\alpha,\beta,\gamma\in R$ with $\alpha<\beta$. Then there exists $b\in\beta$ such that $\forall a\in\alpha$, $a<b$. Then let $\gamma>\theta$ then $\exists c\in\gamma$ such that $c>0$, so since we are in Q again we see that $ac<bc$. Now obviously if any $a\in\alpha$ is positive then the b we have chosen is also postive so these terms do show up in $\alpha\cdot\gamma$ and $\beta\cdot\gamma$. Since $ac\in\alpha\cdot\gamma$ and $bc\in\beta\cdot\gamma$ with $ac<bc$, $\alpha\cdot\gamma\subset\beta\cdot\gamma$ and $\alpha\cdot\gamma\neq\beta\cdot\gamma$ as sets, so $\alpha\cdot\gamma<\beta\cdot\gamma$. Therefore R satisfies axiom 11.

Now we have the final axiom to prove. Let $S=\{\alpha_\lambda \mid \lambda\in\Lambda\}$ be a non-empty set of cuts, with an upper bound β. Then β is also a cut and $\alpha_\lambda\leqslant\beta$ $\forall\lambda\in\Lambda$. Now let $\bar{\alpha}=\bigcup_{\lambda\in\Lambda}\alpha_\lambda$. We must show first that $\bar{\alpha}$ is a cut. i) since S is non-empty $\exists\alpha_\lambda\in S$ and $\alpha_\lambda\subseteq\bar{\alpha}$ and since α_λ is a cut α_λ is not empty and therefore $\bar{\alpha}$ is not empty. ii) Since β is a cut $\beta\neq Q$ and since β is an upper bound of S $\bar{\alpha}\subseteq\beta$ so $\bar{\alpha}\neq Q$. iii) Given $p\in\bar{\alpha}$ and $q\in Q$ with $q<p$ we know since $p\in\bar{\alpha}$ and $\bar{\alpha}=\bigcup_{\lambda\in\Lambda}\alpha_\lambda$, that $p\in\alpha_\lambda$ for some $\lambda\in\Lambda$. Then since α_λ is a cut $q<p$ imiplies that $q\in\alpha_\lambda$ so $q\in\bar{\alpha}$. iv) If $p\in\bar{\alpha}$ then $p\in\alpha_\lambda$, for some $\lambda\in\Lambda$. Since α_λ is a cut $\exists q\in\alpha_\lambda$ with $p<q$. Since $q\in\alpha_\lambda$, $q\in\bar{\alpha}$, so $\bar{\alpha}$ has no largest member. Therefore $\bar{\alpha}$ is indeed a cut, and since $\bar{\alpha}=\bigcup_{\lambda\in\Lambda}\alpha_\lambda$, $\bar{\alpha}$ is an upper bound of S.

Now we will show that $\bar{\alpha}$ is the least upper bound of S. Let β be an upper bound of S, then assume $\beta<\bar{\alpha}$. Then $\exists p\in\bar{\alpha}$ with $p\notin\beta$, but since $p\in\bar{\alpha}$ then by definition of $\bar{\alpha}$ $p\in\alpha_\lambda$ for some $\lambda\in\Lambda$ so for $\forall b\in\beta$ $b<p$, and therefore β is not an upper bound of S. Therefore $\bar{\alpha}\leqslant\beta$ for all β an upper bound of S, thus $\bar{\alpha}$ is the least upper bound of S. And finally we have shown that indeed the set R of all

Dedekind cuts satisfies all 12 axioms, and thus defines the real numbers.

Now that we have found the real numbers as defined by Dedekind cuts, we shall derive the reals again, but this time by completing the rational number using Cauchy sequences. To do this we need some definitions.

Definition: A valuation of rank 1 of a field k is a mapping, $|\,|$, from k into an ordered field such that for all $a, b \in k$:

i) $|a| \geqslant 0'$ and $=0'$ iff $a=0$, where zero is the identity element of k and $0'$ is the identity element of the ordered field.

ii) $|ab| = |a||b|$

iii) $|a+b| \leqslant |a| + |b|$

Obviously the absolute value function satisfies the three requirements, so we will use the absolute value function as our valuation on Q. We also need to know exactly what a Cauchy sequence is.

Definition: The sequence $\{a_n\}$ is called a Cauchy sequence in a field k with respect to the valuation $|\,|$, if for any $\varepsilon > 0$, there exists an integer N such that $|a_n - a_m| < \varepsilon$ for all $n, m \geqslant N$.

When using these Cauchy sequences we will need to speak of their limits.

Definition: Let k be a field and $|\,|$ a valuation on k. Let $\{a_n\}$ be a sequence of elements of k. The sequence $\{a_n\}$ is said to converge to the element $a \in k$ (and a is said to be a limit of $\{a_n\}$, denoted by $\lim a_n = a$, or $a_n \to a$) if, for every real number $\varepsilon > 0$, there exists an integer N such that $|a_n - a| < \varepsilon$ for all $n \geq N$.

Finally, before we complete Q we need to know what exactly it means for a field to be complete.

Definition: The field k is called complete with respect to the valuation $|\,|$ if every Cauchy sequence of k with respect to $|\,|$ has a limit in k.

From the definition of complete it is clear that Q is indeed not complete. For example, the sequence 1, 1.4, 1.41, 1.414, ... in Q, this sequence is Cauchy but it is not convergent in Q, therefore Q is not a complete field.

Now we will find it useful to define some algebra of Cauchy sequences. Let $\{a_n\}$ and $\{b_n\}$ be Cauchy sequences of Q with respect to $|\,|$. Then define addition and multiplication as: $\{a_n\} + \{b_n\} = \{a_n + b_n\}$ and $\{a_n\} \cdot \{b_n\} = \{a_n b_n\}$.

With one more definition we may begin completing Q.

Definition: A null sequence with respect to the valuation $|\,|$, is a sequence, $\{a_n\}$, which satisfies $\forall \varepsilon > 0$, $\exists N \in J$ such that $|a_n| < \varepsilon$ $\forall n \geq N$.

An example of a null sequence in Q is $\{1/n\}_{n=1}^{\infty}$.

For the remainder of the completion we will let A represent the set of Cauchy sequences of Q. Within A it is obvious that multiplication is associative and distributive and that $\langle A,+\rangle$ is an abelian group, therefore A is a ring. The zero element of A is the Cauchy sequence $\{0\}_{n=1}^{\infty}$.

Let M represent the set of all null sequences. Then $\langle M,+\rangle$ is an additive subgroup of A because + is associative, M contains the zero element $\{0\}_{n=1}^{\infty}$, and if $\{a_n\}_{n=1}^{\infty}$ is a null sequence then $\{-a_n\}_{n=1}^{\infty}$ obviously is a null sequence, so all the elements inverses are present. Now consider $\{a_n\}\in M$ and $\{b_n\}\in A$. Since $\{b_n\}$ is a Cauchy sequence, $|b_n| \leqslant k$ $\forall n\in J$ for some k. Then $|a_n b_n| \leqslant |a_n|k$ and $|b_n a_n| \leqslant k|a_n|$. But $\{|a_n|k\}$, $\{k|a_n|\}\in M$, so $\langle M,+\rangle$ is an ideal of A.

Next we need to show that $\langle M,+\rangle$ (to be called M from now on) is a maximal ideal of A. Choose $\{a_n\}\in A$, with $\{a_n\}\notin M$. Then $\exists\varepsilon>0$ and $\exists N\in J$ such that $|a_n| \geqslant\varepsilon$ $\forall n>N$. If this were not true then $\forall\varepsilon>0$ and $\forall N\in J$ $\exists n>N$ such that $|a_n| <\varepsilon$. But since $\{a_n\}\in A$ $\exists N\in J$ such that $|a_m-a_n| <\varepsilon$ $\forall$ $n,m>N$. Then $|a_m| \leqslant |a_n| + |a_m-a_n| <2\varepsilon$ $\forall\varepsilon>0$ and $\forall$ $m>N$ which implies that $\{a_n\}$ is a null sequence which is the opposite of what we assumed. Therefore if $\{a_n\}\in A$ but $\{a_n\}\notin M$ $\exists\varepsilon>0$ and $\exists N\in J$ such that $|a_n| \geqslant\varepsilon$ $\forall n>N$. Since $\{a_n\}\in A$, $\forall\varepsilon*>0$ $\exists M\in J$ such that $|a_m-a_n| <\varepsilon*\varepsilon^2$ for all $m,n>M$. Define $\{b_n\}$ as

$$\{b_n\} = \begin{cases} 0 \text{ for } n \leqslant N \\ 1/a \quad \text{for } n > N \end{cases}$$

Then $\{b_n\}\in A$ because if $m,n>\max\{M,N\}$ then $|b_n-b_m| = |1/a_n - 1/a_m|$ $= |(a_m - a_n)/a_m a_n| \leqslant |(a_m - a_n)|/\varepsilon^2 < \varepsilon*$.

Now suppose I is an ideal of A and $I \supset M$, with $\{a_n\} \in I$. Then $\{a_n\}\{b_n\}=\{0,0,\ldots,0,1,1,\ldots\}=\{1,1,\ldots\}-\{1,1,\ldots,1,0,0,\ldots\}\in I$. But $\{c_n\}=\{1,1,\ldots,1,0,0,\ldots\}\in I$ because it is a null sequence. Then $\{a_n\}\{b_n\}+\{c_n\}=\{1,1,\ldots\}\in I$ because $\langle I,+\rangle$ is a subgroup of A. Therefore I=A because I contains the multiplicative identity of A. Therefore M is a maximal ideal of A, and that means A/M is a field by a Theorem from Abstract Algebra.

Now we need to show that A/M is the completion of Q and consequently the set of Real Numbers. We will begin by showing that A/M satisfies the axioms for the real numbers. Since A/M is a field, we already know that it satisfies axioms 1 through 7.

An element in A/M looks like $\{a_n\}+M$, where $\{a_n\}\in A$. We define + in A/M as if $\alpha,\beta\in A/M$ then $\alpha+\beta=\{a_n\}+M+\{b_n\}+M=\{a_n+b_n\}+M$ and $\cdot$ in A/M as $\alpha\cdot\beta=(\{a_n\}+M)\cdot(\{b_n\}+M)=\{a_n\cdot b_n\}+M$.

Now we must look into the ordering of the field A/M. Let $\alpha,\beta\in A/M$, then we say $\alpha=\beta$ if and only if $|a_n - b_n| \to 0$. Then define $\alpha<\beta$ by $\alpha<\beta$ if and only if $\exists N\in J$ such that $a_n<b_n\ \forall n \geqslant N$ and $\alpha\neq\beta$. Now let $\alpha,\beta,\gamma\in A/M$ with $\alpha<\beta$. Then we know $\exists N\in J$ such that $a_n<b_n\ \forall n \geqslant N$. Now look at $\alpha+\gamma$ and $\beta+\gamma$, clearly the terms with $n > N$ look like $a_n+c_n < b_n+c_n$ so that $\alpha+\gamma<\beta+\gamma$. So axiom 8 is satisfied by A/M.

Now suppose that $\alpha<\beta$ and $\beta<\gamma$, then we have that for some $N\in J$ $a_n<b_n$ and $b_n<c_n$ and since these are rational numbers it is obvious that $a_n<c_n$ for $n > N$ and $\alpha\neq\gamma$. Therefore $\alpha<\gamma$ which is axiom 9.

Now if $\alpha < \beta$ then by definition $\alpha \neq \beta$. If $\alpha < \beta$ and $\beta < \delta$, then $\exists$ $N \in J$ such that for $n \geq N$ $a_n < b_n$ and $\exists M \in J$ such that for $m \geq M$ $b_n < a_n$. Let $N' = \max\{N, M\}$ then for $n \geq N'$ we have that $a_n < b_n$ and $b_n < a_n$. Since these are rational numbers we know that these two inequalities can not be true at the same time. So this contradicts our assumption of $\alpha < \beta$ and $\beta < \alpha$. Finally if $\alpha = \beta$ by definition we know that $\beta \nless \alpha$ so we have just shown that for two elements of A/M only one of the following can be true: $\alpha < \beta$, $\beta < \alpha$, or $\alpha = \beta$. It can be shown that for $\alpha, \beta \in A/M$ exactly one of the previous conditions is always true. Therefore A/M also satisfies axiom 10.

Now let $\alpha < \beta$ and $\gamma > M$ then we know that $\exists N \in J$ such that for $n \geq N$ $a_n < b_n$. Now we also know that $\exists \varepsilon > 0$ and $\exists N' \in J$ such that $c_n \geq \varepsilon$ for $\forall n \geq N'$ from an earlier proof and the fact that $\gamma > M$. Then $\alpha \cdot \gamma = \{a_n \cdot c_n\} + M$ and $\beta \cdot \gamma = \{b_n \cdot c_n\} + M$, then we can look at the elements of the sets using $N'' = \max\{N, N'\}$. Then obviously for $n \geq N''$ we know that $a_n c_n < b_n c_n$. Now we see that $|a_n c_n - b_n c_n| = |c_n(a_n - b_n)| \geq \varepsilon |a_n - b_n|$. And since we know that $\alpha \neq \beta$, we know that $|a_n - b_n| \not\to 0$. Then since $\varepsilon > 0$ we know that $|a_n c_n - b_n c_n| \not\to 0$ so by definition $\alpha\gamma \neq \beta\gamma$. Therefore $\alpha\gamma < \beta\gamma$ and A/M satisfies axiom 11.

Now all we need to do is prove axiom 12. To do this we must first show that Q is dense in A/M up to a ring isomorphism. Let $q \in Q$ and define $f: Q \dashrightarrow A/M$ by $f(q) = \{q\} + M$ where $\{q\}$ is the Cauchy sequence q, q, ... Obviously f is one to one and onto, but it must also satisfy the following: if q, r Q then $f(q+r) = f(q) + f(r)$

and $f(qr)=f(q)f(r)$. Let $q,r\in Q$ then $f(q+r) = \{q+r\}+M = (\{q\}+M) + (\{r\}+M) = f(q)+f(r)$ and $f(qr)=\{qr\}+M=(\{q\}+M)(\{r\}+M)=f(q)f(r)$. Furthermore if $q<r$ then $f(q) = \{q\} + M < \{r\} + M = f(r)$ by the definition of $<$ in A/M. Therefore f is an order preserving isomorphism of Q onto $Q'=\{\{q\}+M \mid q\in Q\}$, and $Q' \subset A/M$.

Now let $\alpha = \{a_n\}+M \in A/M$, choose $\varepsilon>0$, then $|a_m-a_n| < \varepsilon$ for $n,m>N$. For a fixed m satisfying this let $\beta=\{a_m,a_m,a_m,\ldots\}+M\in Q'$. Then $\beta+(-\alpha)=\{a_m,a_m,\ldots\}+M+\{-a_n\}+M=\{a_m+-a_n\}+M$. Therefore in A/M we have $|\{a_m+-a_n\}+M| < \varepsilon$ for $n>N$. Therefore Q' is dense in A/M.

The final axiom says that every nonempty set of real numbers that is bounded from above has a least upper bound. If we could show that A/M is complete with respect to a valuation $|\,|$, this final axiom would be satisfied. First we define the valuation $|\,|$ on A/M by , if $\alpha\in A/M$ then $|\alpha| = \{|a_n|\}+M$. Since $||a_n| - |a_m|| \leqslant |a_n - a_m|$ and $\{a_n\}$ is a Cauchy sequence, then $\{|a_n|\}$ is a Cauchy sequence and therefore $|\alpha|\in A/M$. Since we know A/M is an ordered field, the proposed valuation is into an ordered field.

Next we must show that $|\,|$ is well defined. Let $\{a_n\} + M = \{b_n\}+M$, then from the definition of a residue class we know that $|a_n-b_n| \to 0$ and therefore $||a_n| - |b_n|| \to 0$ so $\{|a_n|\} + M = \{|b_n|\}+M$. Therefore $|\,|$ is well-defined. Lastly we must show that $|\,|$ is indeed a valuation. i) $|\alpha| \geqslant M$ and $= M$ iff $\alpha = M$. This is true because we showed earlier that if $\{a_n\} + M \neq M$ then $|a_n| \geqslant \varepsilon$ for some $\varepsilon > 0$, $\forall n \geqslant N$ for some $N\in J$, which means the only time $|\alpha| = M$ is when $\alpha = M$. ii) Let $\beta=\{b_n\}+M$ then $|\beta| =\{|b_n|\}+M$ then $|\alpha\beta| = \{|a_nb_n|\}+M=\{|a_n||b_n|\}+M=(\{|a_n|\}+M)\cdot(\{|b_n|\} + M) =$

$|\alpha||\beta|$, so $|\alpha\beta| = |\alpha||\beta|$. iii) $|\alpha + \beta| = \{|a_n + b_n|\} + M \leqslant \{|a_n| + |b_n|\} + M = (\{|a_n|\} + M) + (\{|b_n|\} + M = |\alpha| + |\beta|$, so $|\alpha+\beta| \leqslant |\alpha| + |\beta|$. Therefore $||$ as defined is a valuation of A/M. Finally we must show that A/M is complete.

Let $\alpha = \{\alpha_1, \alpha_2, \ldots\}$ be a Cauchy sequence in A/M where each $\alpha_i \in Q'$. In other words $\alpha_i = \{q_i, q_i, \ldots\} + M$ with $q_i \in Q$. Then $|\alpha_i| = \{|q_i|, |q_i|, \ldots\} + M$ and $\{q_i\}_{i=1}^{\infty}$ must be a Cauchy sequence in Q. Let $\beta = \{q_i\}_{i=1}^{\infty} + M$, then since $\forall\varepsilon>0\ \exists N \in J$ such that for $n,m>N$ $|q_m - q_n| < \varepsilon$, $|\alpha_m - \beta| = \{|q_m - q_i|\}_{i=1}^{\infty} + M < \{\varepsilon\} + M$ where $\{\varepsilon\}+M>M$, for $\forall i > N$. Therefore $\lim \alpha = \beta$ and $\beta \in A/M$.

Now let $\alpha = \{\alpha_1, \alpha_2, \ldots\}$ be an arbitrary Cauchy sequence in A/M, with $|\alpha_n - \alpha_{n+1}| = \varepsilon_n$. Then since $\{\alpha_n\}$ is a Cauchy sequence the sequence $\{\varepsilon_n\}$ must be a null sequence. Then since we know that Q' is dense in A/M $\exists q_n' \in Q' \subset A/M$ such that $|q_n' - \alpha_n| < \varepsilon_n$. Also since α is a Cauchy sequence, for $\forall\varepsilon>M$ $\exists N'$ such that $|\alpha_m - \alpha_n| < 1/3\varepsilon$ for $n,m>N'$ and an N" such that $\varepsilon_p < 1/3\varepsilon$ for $p>N''$. Then let $N = \max(N', N'')$ so that for n, m, $p > N$, $|q_m' - \alpha_m|$, $|\alpha_m - \alpha_n|$, and $|\alpha_n - q_n'|$ are all $<1/3\varepsilon$. Then $|q_m' - q_n'| \leqslant |q_m' - \alpha_m| + |\alpha_m - \alpha_n| + |\alpha_n - q_n'| < 1/3\varepsilon + 1/3\varepsilon + 1/3\varepsilon = \varepsilon$. Thus $\{q_i'\}_{i=1}^{\infty}$ is a Cauchy sequence in Q' and there is $\beta = \{q_n\}_{n=1}^{\infty} + M$ as above with $\lim \{q_i'\} = \beta$. Therefore since $\{q_n' - \alpha_n\}$ was shown to be a null sequence for each α_n in α, $\lim \alpha = \beta \in A/M$. Therefore A/M is complete and satisfies all 12 axioms of the real numbers.

So both of the methods say they have found the real numbers. And since it has been shown that if a set satisfies the 12 axioms given at the beginning of this paper, it is the set of real numbers, I conclude that both methods must work. This does not, however, solve the dilemma of $\sqrt{2}$ pieces of bread. But my mother does a lot of other things for me, so if she can't give me $\sqrt{2}$ pieces of bread, I guess I will just have to get it for myself.

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