Qualitative Theory of Differential Equations

by

Sheri Homeyer

Dr. Rodine, Math 499H

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The problem of stability is of primary concern in the qualitative theory of differential equations, and has occupied mathematicians for the past century. The problem appears when considering solutions to the differential equation $\dot{x}=f(t,x)$ where $x=(x_1(t),...,x_n(t))^T$ and $f(t,x)$ is a nonlinear function of $x_1,...,x_n$. While no known method of solving this equation explicitly exists even for the case n=2, it is possible to discuss the qualitative properties of $x_1(t)$ and $x_2(t)$ where $x_1(t)$ and $x_2(t)$ denote, for example, the populations, at time t, of two competing species. The qualitative attributes under consideration include points of equilibrium and the question of the stability of solutions in neighborhoods of these points. Note that an equilibrium point occurs when the values z_1 and z_2 exist at which the two species can coexist together in a steady state; that is, if there are z_1 and z_2 such that $x_1(t)=z_1$ and $x_2(t)=z_2$ is a solution of the differential equation (above), then (z_1,z_2) is a point of equilibrium. Further note that the problem of stability arises when considering what will happen when members of species 1 are added to the picture. The question is whether the values $x_1(t)$ and $x_2(t)$ will remain near their equilibrium values as time t tends to infinity, or if species 1 will gain a significant advantage from the added members and commence eliminating species 2. While points of equilibrium can be determined directly by observing that $\dot{x}(t)\equiv 0$ if $x(t)\equiv x^0$ and therefore, x^0 is an equilibrium value of the differential equation, if, and only if, $f(t,x^0)\equiv 0$, the question of stability is not as easily resolved. That is, because it is not possible to solve the diffential equation explicitly, the only case of stability it is feasible to consider solving, is the situation where $\mathbf{f}(t,\mathbf{x})$ does not depend on t explicitly, but is a function of $\mathbf{x}$ alone; differential equations such as these are labeled "autonomous." Complete resolution of the stability question is only possible (generally) for two cases of autonomous differential equations, namely: linear systems and equilibrium solutions.

Every solution of the linear differential equation $\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}$ can be identified as either stable or unstable; that is, the question of stability can be answered conclusively. To clarify stability, let $\mathbf{x}=\phi(t)$ be a solution to the differential equation $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})$, and consider the following formal definition of stability:

> The solution $\mathbf{x}=\phi(t)$ of $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})$ is stable if every solution $\psi(t)$ of $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})$ which starts sufficiently close to $\phi(t)$ at $t=0$ must remain close to $\phi(t)$ for all future time t. The solution $\phi(t)$ is unstable if there exists at least one solution $\psi(t)$ of $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})$ which starts near $\phi(t)$ at $t=0$ but which does not remain close to $\phi(t)$ for all future time. More precisely, the solution $\phi(t)$ is stable if for every $\epsilon>0$ there exists $\delta=\delta(\epsilon)$ such that $|\psi_j(t)-\phi_j(t)|<\epsilon$ if $|\psi_j(0)-\phi_j(0)|<\delta(\epsilon)$, $j=1,\ldots,n$ for every solution $\psi(t)$ of $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})$.[1]

It is now possible to settle the stability question for $\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}$, using the following important theorem:

> Theorem 1. (a) Every solution $\mathbf{x}=\phi(t)$ of $\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}$ is stable if all the eigenvalues of $\mathbf{A}$ have negative real part.
> (b) Every solution $\mathbf{x}=\phi(t)$ of $\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}$ is unstable if at least one eigenvalue of $\mathbf{A}$ has positive real part.
> (c) Suppose that all the eigenvalues of $\mathbf{A}$ have real part ≤ 0 and $\lambda_1=i\sigma_1,\ldots,\lambda_l=i\sigma_l$

have zero real part. Let $\lambda_j = i\sigma_j$ have multiplicity k_j. This means that the characteristic polynomial of **A** can be factored into the form

$$p(\lambda) = (\lambda - i\sigma_1)^{k_1} \cdots (\lambda - i\sigma_l)^{k_l} q(\lambda)$$

where all the roots of $q(\lambda)$ have negative real part. Then, every solution $\mathbf{x} = \phi(t)$ of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is stable if **A** has k_j linearly independent eigenvectors for each eigenvalue $\lambda_j = i\sigma_j$. Otherwise, every solution $\phi(t)$ is unstable.[2]

While proving part (a) of Theorem 1, a phenomenon entitled asymptotic stability can be noted. That is, if all the eigenvalues of **A** have negative real part, then every solution $\mathbf{x}(t)$ of $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ tends to zero as t tends to infinity; hence, not only is the equilibrium solution $\mathbf{x}(t) \equiv 0$ stable, but each solution $\phi(t)$ of $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ nears it as t tends to infinity; label this phenomenon asymptotic stability and note further that it is a very strong type of stability. Applying Theorem 1 to answer the question of stability for linear systems requires employing knowledge gained in MATH 240 (Linear Algebra) with respect to finding characteristic polynomials and eigenvalues, as demonstrated by the worked problems in Appendix I: Problem Portfolio.

The problem of stability can also be resolved for other solutions, where the equation to be considered is: $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{g}(\mathbf{x})$ where $\mathbf{g}(\mathbf{x}) = (g_1(\mathbf{x}), \ldots, g_n(\mathbf{x}))$ is very small when compared with **x**. Determining stability for these solutions is accomplished by employing a parallel version of Theorem 1 as follows:

Theorem 2. Suppose that the vector-valued function $\mathbf{g}(\mathbf{x})/\|\mathbf{x}\| \equiv \mathbf{g}(\mathbf{x})/\max\{|x_1|, \ldots$ [illegible] is a continuous function of $x_1, \ldots, x_n$ which vanishes for $\mathbf{x} = 0$. Then,
(a) The equilibrium solution $\mathbf{x}(t) \equiv 0$ of [illegible]

> $\dot{x}=Ax+g(x)$ is asymptotically stable if the equilibrium solution $x(t)\equiv 0$ of the "linearized" equation $\dot{x}=Ax$ is asymptotically stable. Equivalently, the solution $x(t)\equiv 0$ of $\dot{x}=Ax+g(x)$ is asymptotically stable if all the eigenvalues of A have negative real part.
> (b) The equilibrium solution $x(t)\equiv 0$ of $\dot{x}=Ax+g(x)$ is unstable if at least one eigenvalue of A has positive real part.
> (c) The stability of the equilibrium solution $x(t)\equiv 0$ of $\dot{x}=Ax+g(x)$ cannot be determined from the stability of the equilibrium solution $x(t)\equiv 0$ of $\dot{x}=Ax$ if all the eigenvalues of A have real part ≤ 0 but at least one eigenvalue of A has zero real part.[3]

The importance of Theorem 2 becomes apparent, since it can be used to determine the stability of equilibrium solutions of arbitrary autonomous differential equations. Consider x^0, an equilibrium value of the differential equation $\dot{x}=f(x)$, and set $z(t)=x(t)-x^0$, then $\dot{z}=\dot{x}=f(x^0+z)$. Furthermore, $z(t)\equiv 0$ is an equilibrium solution of $\dot{z}=\dot{x}=f(x^0+z)$, the stability of $z(t)\equiv 0$ is equivalent to the stability of $x(t)\equiv x^0$, and finally, $f(x^0+z)$ can also be written as, $f(x^0+z)=Az+g(z)$ where $g(z)$ is small when compared to z. Based upon these facts, the following lemma can be presented:

> Lemma 1. Let $f(x)$ have two continuous partial derivatives with respect to each of its variables $x_1,\ldots,x_n$. Then, $f(x^0+z)$ can be written in the form
> $$f(x^0+z)=f(x^0)+Az+g(z)$$
> where $g(z)/\max\{|z_1|,\ldots,|z_n|\}$ is a continuous function of z which vanishes for $z=0$.[4]

Theorem 2 and Lemma 1 yield the three-step algorithm for deciding whether an equilibrium solution $x(t)\equiv x^0$ of $\dot{x}=f(x)$ is stable or unstable:

1. Set $z=x-x^0$.
2. Write $f(x^0+z)$ in the form $Az+g(z)$ is a vector-valued polynomial in $z_1,\ldots,z_n$ beginning with terms of order two or more.

3. Compute the eigenvalues of A. If all the eigenvalues of A have negative real part, then $\mathbf{x}(t)\equiv\mathbf{x}^0$ is asymptotically stable. If one eigenvalue of A has positive real part, then $\mathbf{x}(t)\equiv\mathbf{x}^0$ is unstable.[5]

Examples of finding all the equilibrium solutions of a system of equations and of determining the stability or instability of these solutions, can be found in Appendix I.

Another facet of qualitative theory is the geometric study of differential equations. Since the intent of pursuing qualitative theory is to acquire the most complete description possible of all solutions of the system of differential equations $dx/dt=f(x,y)$, $dy/dt=g(x,y)$, the graphs of the solutions of the system would assist the qualitative study. Observe that each solution $x=x(t)$, $y=y(t)$ of $dx/dt=f(x,y)$, $dy/dt=g(x,y)$ traces out a curve in 3-space (t,x,y). Note that each of the solutions $x=x(t)$, $y=y(t)$ where $t_0\le t\le t_1$ of $dx/dt=f(x,y)$, $dy/dt=g(x,y)$ also traces out a curve in 2-space (x,y), or the x-y plane. That is, the set of points $(x(t),y(t))$ define a curve C in the x-y plane, as t traverses the interval (t_0,t_1). The curve thus defined is called the orbit or trajectory of the solution $x=x(t)$, $y=y(t)$ and the x-y plane is called the phase-plane of the solutions. While graphing the solutions in 3-space is possible (ex. helix), it is not as manageable as considering the phase-plane. Defining the curve C in 2-space allows for enough diversity to satisfy the intent of qualitative theory, yet does not involve the complexity of three-dimensional plotting. Some examples of 2-dimensional orbits are: spirals, lines, parabolas, sets of curves, and families of elipses. Note that the orbits of the solutions $x=x(t)$, $y=y(t)$ of

$dx/dt=f(x,y), dy/dt=g(x,y)$ are the solution curves of the first-order scalar equation $dy/dx=g(x,y)/f(x,y)$, which, unfortunately, cannot (in general) be solved explicitly. However, securing a precise representation of all the orbits of $dx/dt=g(x,y), dy/dt=g(x,y)$ can still be accomplished, since the system of differential equations $dx/dt=f(x,y)$, $dy/dt=g(x,y)$ sets up a direction field in the x-y plane. That is, the system of differential equations $dx/dt=f(x,y)$, $dy/dt=g(x,y)$ indicates the velocity at which a solution moves along its orbit, as well as which direction the solution is moving. Furthermore, the concept of an orbit can be extended to include curves in three-dimensional space, which is labeled the phase-spaces of the solutions. One of the advantages of considering the orbit or trajectory of the solution, rather than the solution itself, is that it is often feasible to produce the orbit of a solution, without prior knowledge of the solution.

Several important applications of the qualitative theory of differential equations are collectively entitled "mathematical theories of war," and include, L. F. Richardson's theory of conflict, as well as Langhester's combat models and the battle of Iwo Jima. Beginning with Richardson's theory, a mathematical model is constructed which delineates the relationship between two nations, each of which is resolved to defend itself from possible attack by the other. Each of the two nations believes the danger of attack to be quite real, and bases its apprehensions upon the preparedness of the other to engage in war. The model under consideration

(Richardson's) is $dx/dt=ky-ax+g$, $dy/dt=lx-by+h$, where $x(t)$ denotes the war potential (armaments) of nation one, $y(t)$ specifies the war potential of nation two, and the rate of change of each depends upon the constants a,b,g,h,k, and l. Note that k indicates the war readiness of nation two, l denotes the war readiness of nation one, a represents the cost of armaments for nation one, b depicts expense of arming nation two, and, finally, g and h designate the grievances that nation one feels toward nation two, and that nation two holds against nation one, respectively. By comparing the European arms race of 1909-1914 (World War I), with Richardson's model, and by incorporationg data such as the defense budgets of each alliance, a high correllation is found between the two, with regard to Richardson's choice of constant values. Understanding of the arms race models as a group was expanded and deepened through this study, as well as through studious endeavors in MATH 360 (Model Building) which included work on the competitive hunter's model, the predator-prey model, and Richardson's model. Lanchestrian models extended those studies to include models for conventional guerilla combat, for example, $dx/dt=-cxy+f(t)$, $dy/dt=-dx+g(t)$, where x denotes guerilla forces, c indicates the combat effectiveness coefficient of the opponent y, $cx(t)y(t)$ designates the combat loss rate for the guerilla force (x) and $dx(t)$ characterizes the loss rate of the non-guerilla force, and finally, $f(t)$ and $g(t)$ express the re-inforcement rates of x and y, respectively. Lanchester's work also produced a mathematical model for a conventional force versus a conventional force, where the model is $dx/dt=-ay+f(t)$, $dy/dt=-by+g(t)$, the form of which is very similar to Lanchester's conventional-guerilla combat model. Through data available from the battle of Iwo Jima (World War II), such as, casualty reports broken down by military unit for each nation, it is possible to test the effectiveness of Lanchester's work. Final conclusions, based upon the Lanchestrian model for Iwo Jima, $dx/dt=-ay+f(t)$, $dy/dt=-bx$ where $x(t)$ and $y(t)$ designate respectively, active American and Japanese forces on Iwo Jima t days after the battle began, yield that it is frequently possible to calculate the values for constants a and b, once the data for a given battle is known; this was the case for Iwo Jima.

By incorporating matrix theory learned in Math 240 (Linear Algebra) with differential equation theory introduced and applied in Math 336 (Differential Equations), and utilizing the foundations gained in Math 229H (with 297), Math 230H (with 297), and Math 232, the study of the qualitative theory of differential equations provided both an excellent summation as well as a furthering of mathematical study in the applied vein. The applicability of these studies is far-reaching and of an ever-growing importance today, particularily when considering mathematical theories of was, such as L.F. Richardson's theory of conflict, as well as Lanchester's combat models and the battle of Iwo Jima. Through these models it is possible to consider various outcomes of a war, given sufficient data. The question/problem of stability has yielded many useful applications, in the realms of physics and war, and further, in the areas of biology, ecology, and medical research.

Endnotes

[1]Martin Braun, Differential equations and their applications, Springer-Verlag New York, Inc., New York, NY,1983,p.376.

[2]Martin Braun,pp.376-377.

[3]Martin Braun,p.384.

[4]Martin Braun,p.388.

[5]Martin Braun,p.389.

Appendix I:

Problem Portfolio

MATH 499H
Prof. Rodine (F 1:30-2 pm)

①

CHAPTER FOUR: Qualitative theory of differential equations

→ Differential Equations and Their Applications
(An Introduction to Applied Mathematics)
3rd edition
by M. Braun

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4.1 Introduction

p. 373: "This quantity is at most $\sqrt{2}\,10^{-4}$"

$$c_1 \le 10^{-4}$$
$$c_1^2 \le 10^{-8}$$
$$c_1^2 \le 2(10^{-8}) - (10^{-8})$$
$$c_1^2 \le [2(10^{-8})^{\frac{1}{2}}]^2 - [(10^{-8})^{\frac{1}{2}}]^2$$
$$c_1^2 \le (\sqrt{2}\cdot 10^{-4})^2 - (10^{-4})^2$$
$$c_1^2 \le (\sqrt{2}\cdot 10^{-4})^2 - (\tfrac{4}{3} - c_2)^2$$

since $\frac{4}{3} - 10^{-4} \le c_2 \le \frac{4}{3} + 10^{-4}$

$$c_1^2 + (\tfrac{4}{3} - c_2)^2 \le (\sqrt{2}\cdot 10^{-4})^2$$

$\frac{4}{3} - c_2 \le 10^{-4}$

$$[c_1^2 + (\tfrac{4}{3} - c_2)^2]^{\frac{1}{2}} \le \sqrt{2}\,10^{-4}$$

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4.1 Excercises

For probs 1-8, find all equilibrium values of the given system of differential equations.

1. $\frac{dx}{dt} = x - x^2 - 2xy$
$\frac{dy}{dt} = 2y - 2y^2 - 3xy$

Solution

$\binom{x_0}{y_0}$ is an equilibrium value of this system iff:

$x_0 - x_0^2 - 2x_0 y_0 = 0$ and $2y_0 - 2y_0^2 - 3x_0 y_0 = 0$

$x_0(1 - x_0 - 2y_0) = 0$ $\quad$ $y_0(2 - 2y_0 - 3x_0) = 0$

$x_0 = 0,\ x_0 = 1 - 2y_0 \Rightarrow y_0 = 0,\ y_0 = \frac{2-3x_0}{2}$

so

1) $x_0 = 0,\ y_0 = 0$ $\quad$ $(0,0)$

2) $x_0 = 0,\ y_0 = \frac{2-0}{2} = 1$ $\quad$ $(0,1)$

3) $y_0 = 0,\ x_0 = 1 - 0 = 1$ $\quad$ $(1,0)$

4) $y_0 = \frac{2-3x_0}{2},\ x_0 = 1 - 2y_0$ $\quad$ $(\frac{1}{2}, \frac{1}{4})$

$\hookrightarrow \frac{2-3x_0}{2} = \frac{1-x_0}{2}$

$2 - 3x_0 = 1 - x_0$

$1 = 2x_0$

$x_0 = \frac{1}{2} \rightarrow y_0 = \frac{1}{4}$

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2. $\frac{dx}{dt} = -\beta xy + \mu$

$\frac{dy}{dt} = \beta xy - \gamma y$

Solution

$\binom{x_0}{y_0}$ is an equilibrium value of this system iff:

$-\beta x_0 y_0 + \mu = 0$ and $\beta xy - \gamma y = 0$

$\beta x_0 y_0 = \mu$ $\qquad y_0(\beta x_0 - \gamma) = 0$

$x_0 = \frac{\mu}{\beta y_0}$ $\qquad \leftarrow y_0 = 0,\ \beta x_0 = \gamma$

so

$y_0 = 0,\ x_0 = \frac{\mu}{\beta y_0} = \frac{\mu}{0} =$ undefined

$\beta x_0 = \gamma,\ y_0 = \frac{\mu}{\beta x_0}$ $\qquad \left(\frac{\gamma}{\beta}, \frac{\mu}{\gamma}\right)$

$\hookrightarrow x_0 = \frac{\gamma}{\beta},\ y_0 = \frac{\mu}{\beta x_0}$

$y_0 = \frac{\mu}{\gamma},\ x_0 = \frac{\gamma}{\beta}$

3. $\frac{dx}{dt} = ax - bxy$

$\frac{dy}{dt} = -cy + dxy$

$\frac{dz}{dt} = z + x^2 + y^2$

Solution

$\binom{x_0}{y_0}$ is an equilibrium value of this system if

$ax_0 - bx_0y_0 = 0$ and $-cy_0 + dx_0y_0 = 0$ and $z_0 + x_0^2 + y_0^2 = 0$

$x_0(a - by_0) = 0$ $\qquad y_0(dx_0 - c) = 0$ $\qquad x_0^2 + y_0^2 = -z_0$

$x_0 = 0,\ y_0 = \frac{a}{b}$ $\qquad y_0 = 0,\ x_0 = \frac{c}{d}$ $\qquad z_0 = -x_0^2 - y_0^2$

so

① $x_0 = 0,\ y_0 = 0,\ z_0 = 0$ $\qquad (0,0,0)$

② $x_0 = \frac{c}{d},\ y_0 = \frac{a}{b},\ z_0 = -\frac{c^2}{d^2} - \frac{a^2}{b^2}$ $\qquad \left(\frac{c}{d}, \frac{a}{b}, -\left(\frac{a^2}{b^2} + \frac{c^2}{d^2}\right)\right)$

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4. $\frac{dx}{dt} = -x - xy^2$

$\frac{dy}{dt} = -y - yx^2$

$\frac{dz}{dt} = 1 - z + x^2$

Solution

$\binom{x_0}{y_0}$ is an equilibrium value of this system iff:

$-x_0 - x_0 y_0^2 = 0$ and $-y_0 - y_0 x_0^2 = 0$ and $1 - z_0 + x_0^2 = 0$

$-x_0(1 + y_0^2) = 0$ $\quad$ $-y_0(1 + x_0^2) = 0$ $\quad$ $z_0 = 1 + x_0^2$

$x_0 = 0,\ y_0 = i$ $\quad$ $y_0 = 0,\ x_0 = i$

so

① $x_0 = 0,\ y_0 = 0,\ z_0 = 1$ $\quad$ $(0, 0, 1)$

② $x_0 = i,\ y_0 = i,\ z_0 = 0$ $\quad$ $(i, i, 0)$

5. $\frac{dx}{dt} = xy^2 - x$

$\frac{dy}{dt} = x \sin \pi y$

Solution

$\binom{x_0}{y_0}$ is an equilibrium value of this system iff:

$x_0 y_0^2 - x_0 = 0$ and $x_0 \sin \pi y_0 = 0$

$x_0(y_0^2 - 1) = 0$ $\quad$ $-1 \le \sin \pi y_0 \le 1$

$x_0 = 0,\ y_0 = \pm 1$ $\quad$ $-x_0 \le x_0 \sin \pi y_0 \le x_0$

$\sin^{-1}(-1) \le \pi y_0 \le \sin^{-1}(1)$

$\frac{\sin^{-1}(-1)}{\pi} \le y_0 \le \frac{\sin^{-1}(1)}{\pi}$

$\frac{1}{\pi}\left(\frac{3\pi}{2}\right) \le y_0 \le \frac{1}{\pi}(0)$

$y_0 \ge \frac{3}{2},\ y_0 \le 0$

so

① $x_0 = 0,\ y_0 \ge \frac{3}{2}$
$x_0 = 0,\ y_0 \le 0$
$\Big\}$ $(0, y_0)$ where y_0 is arbitrary

② $y_0 = 1,\ x_0 = x_0$ $\quad$ $(1, x_0)$ where x_0 is arbitrary

③ $y_0 = -1,\ x_0 = x_0$ $\quad$ $(-1, x_0)$ where x_0 is arbitrary

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6. $\frac{dx}{dt} = \cos y$

$\frac{dy}{dt} = \sin x - 1$

Solution

$\binom{x_0}{y_0}$ is an equilibrium value of the system iff:

$\cos y_0 = 0$ and $\sin x_0 - 1 = 0$

$y_0 = \frac{\pi}{2} + 2\pi n$ $n\geq 0$, n integer $\sin x_0 = 1$

$x_0 = \frac{\pi}{2} + 2\pi n$ $n\geq 0$ n integer

So

① $(\frac{\pi}{2}, \frac{\pi}{2})$ $\left(\frac{\pi(4n+1)}{2}, \frac{\pi(4n+1)}{2}\right)$ $n\geq 0$ n positive integer ($n\geq 0$, n integer)

$(\frac{\pi}{2} + 2n\pi, \frac{\pi}{2} + 2n\pi)$

7. $\frac{dx}{dt} = -1 - y - e^x$

$\frac{dy}{dt} = x^2 + y(e^x - 1)$

$\frac{dz}{dt} = x + \sin z$

Solution

$\binom{x_0}{y_0}$ is an equilibrium value of the system iff:

$-1-y-e^x = 0$ and $x^2 + y(e^x-1) = 0$ and $x + \sin z = 0$

$y = -(1+e^x)$ $x^2 = -y(e^x-1)$ $\sin z = -x$

$e^x = -(1+y)$ $x = \pm\sqrt{-y(e^x-1)}$ $z = \sin^{-1}(-x)$

$x = \ln -(1+y)$ $y = \frac{-x^2}{e^x-1}$

thus

$\frac{-x^2}{e^x-1} = -(1+e^x)$, $z = \sin^{-1}(-x)$

$\Rightarrow x^2 = (1+e^x)(e^x-1)$

$x^2 = e^x + e^{2x} - e^x - 1$

$x = \pm\sqrt{e^{2x}-1}$, $y = -1-e^x$, $z = \sin^{-1}(-x)$

so

$x=0$, $y=-2$, $z=n\pi$

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8. $\frac{dx}{dt} = x - y^2$

$\frac{dy}{dt} = x^2 - y$

$\frac{dz}{dt} = e^z - x$

Solution

$\binom{x_0}{y_0}$ is an equilibrium value of this system if:

$x - y^2 = 0$ and $x^2 - y = 0$ and $e^z - x = 0$

$x = y^2$ $\quad$ $x^2 = y$ $\quad$ $e^z = x$

Thus $y^4 = y$ $\quad$ $z = \ln x$

$y^4 - y = 0$

$y(y^3 - 1) = 0$

$y = 0,\ y = 1$ $\quad$ $x = 0,\ x =$ $\quad$ z undefined, $z = 0$

SO

① $x = 1,\ y = 1,\ z = 0$ $\quad$ $(1,1,0)$

9. Consider the system of differential equations

$$\frac{dx}{dt} = ax + by, \qquad \frac{dy}{dt} = cx + dy. \qquad (*)$$

(i) Show that $x=0,\ y=0$ is the only equilibrium point of $(*)$ if $ad - bc \neq 0$

Solution

$\binom{x_0}{y_0}$ is an equilibrium value of this system if:

$ax + by = 0$ and $cx + dy = 0$

$x = \frac{-b}{a} y$ $\quad$ $y = \frac{-c}{d} x$

$\frac{y}{x} = \frac{-a}{b}$ (‡) $\quad$ $\frac{y}{x} = -\frac{c}{d}$ (‡) $\quad$ (‡) provided $x \neq y \neq 0$

$-\frac{a}{b} = -\frac{c}{d} \Rightarrow ad = -bc \Rightarrow ad - bc = 0$

so

① $x = y = 0$ *provided $ad - bc \neq 0$ $\quad$ $(1,1)$*

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9. (ii) Show that (*) has a line of equilibrium pts. if $ad-bc=0$.

Solution

since $x \neq y \neq 0$ for $ad-bc=0$, the only restrictions on x & y are that they not equal 0. since $x = \frac{-a}{b} y_0$ & $y = \frac{-c}{d} x_0$ are all possible solutions (where $ad-bc=0$). Now, since x and y have slope $\frac{-a}{b}$ and $\frac{-c}{d}$ respectively, each describes the graph of a line which does not cross the y-axis (no y-intercept). Hence, if $ad-bc=0$, (*) has a line of equilibrium pts.

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11. (a) Verify that $\underset{\sim}{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-t} = \begin{pmatrix} e^{-t} \\ 0 \\ 0 \end{pmatrix}$

is the solution of the initial value problem

$$\dot{\underline{x}} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix} \underline{x} - \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix} e^{-t}, \quad x(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Solution

$$\dot{\underline{x}} = \frac{d}{dt} \underline{x} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \frac{d}{dt} \begin{pmatrix} e^{-t} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -e^{-t} \\ 0 \\ 0 \end{pmatrix} \checkmark$$

$$\dot{\underline{x}} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix} \begin{pmatrix} e^{-t} \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 2e^{-t} \\ 2e^{-t} \\ -3e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t} \\ 2e^{-t} \\ -3e^{-t} \end{pmatrix} + \begin{pmatrix} -2e^{-t} \\ -2e^{-t} \\ 3e^{-t} \end{pmatrix} = \begin{pmatrix} -e^{-t} \\ 0 \\ 0 \end{pmatrix} \checkmark$$

Further note that $\underline{x}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \checkmark$

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4.2 Stability of Linear Systems

1. $\dot{\underline{x}} = \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix} \underline{x}$

Solution

The characteristic polynomial of the matrix $\underline{A} = \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}$ is . . .

$$p(\lambda) = \det(\underline{A} - \lambda \underline{I}) = \det \begin{pmatrix} 1-\lambda & 1 \\ -2 & -2-\lambda \end{pmatrix}$$
$$= (1-\lambda)(-2-\lambda) - (-2)$$
$$= \lambda^2 + \lambda - 2 + 2$$
$$= \lambda(\lambda+1)$$

So $\lambda=0$, $\lambda=-1$ are eigenvalues of $\underline{A}$.

Then by Thm 1, every eigenvector $\underline{v}$ of $\underline{A}$ with eigenvalue 0 must satisfy the equation

$$\underline{A}\,\underline{v} = \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which means that $v_1 + v_2 = 0$ & $-2v_1 - 2v_2 = 0$ yields $v_2 = -v_1$ so that every eigenvector $\underline{v}$ of $\underline{A}$ w/ eigenvector 0 must be of form $\underline{v} = c \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. * Consequently every soln $\underline{x} = \underline{\phi}(t)$ of $\dot{\underline{x}} = \underline{A}\underline{x}$ is stable, since $\lambda = 0$ is an eigenvalue of multiplicity one and $\underline{A}$ has exactly one linearly independent eigenvector w/ eigenvalue 0.

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2. $\dot{\underline{x}} = \begin{pmatrix} -3 & -4 \\ 2 & 1 \end{pmatrix} \underline{x}$

Solution

The characteristic polynomial of the matrix $\underline{A} = \begin{pmatrix} -3 & -4 \\ 2 & 1 \end{pmatrix}$ is ...

$$p(\lambda) = \det(\underline{A} - \lambda \underline{\underline{I}}) = \det \begin{pmatrix} -3-\lambda & -4 \\ 2 & 1-\lambda \end{pmatrix}$$

$$= (-3-\lambda)(1-\lambda) + 8$$

$$= \lambda^2 + 2\lambda + 5$$

Hence $\lambda = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$ OR $\lambda = -1-2i$, $\lambda = -1+2i$ are the eigenvalues of $\underline{A}$. Then by Theorem 1, part b, every solution $\underline{x} = \underline{\phi}(t)$ of $\dot{\underline{x}} = \underline{A}\,\underline{x}$ (where $A = \begin{pmatrix} -3 & -4 \\ 2 & 1 \end{pmatrix}$) is unstable

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3. $\dot{\underline{x}} = \begin{pmatrix} -5 & 3 \\ -1 & 1 \end{pmatrix} \underline{x}$

Solution

The characteristic polynomial of the matrix $\underline{A} = \begin{pmatrix} -5 & 3 \\ -1 & 1 \end{pmatrix}$ is:

$$p(\lambda) = \det(\underline{A} - \lambda \underline{I}) = \det \begin{pmatrix} -5-\lambda & 3 \\ -1 & 1-\lambda \end{pmatrix}$$
$$= (-5-\lambda)(1-\lambda) + 3$$
$$= \lambda^2 + 4\lambda + 3$$
$$= (\lambda+3)(\lambda+1)$$

Hence $\lambda = -1$, $\lambda = -3$ are the eigenvalues of $\underline{A}$.

Then by Thm 1, part A, every soln $\underline{x} = \underline{\phi}(t)$ of $\dot{\underline{x}} = \begin{pmatrix} -5 & 3 \\ -1 & 1 \end{pmatrix} \underline{x}$ is stable, since all eigenvalues of $\underline{A}$ have negative real part.

4. $\dot{\underline{x}} = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \underline{x}$

Solution

The characteristic polynomial of the matrix $\underline{A} = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}$ is:

$$p(\lambda) = \det(\underline{A} - \lambda \underline{I}) = \det \begin{pmatrix} 1-\lambda & -4 \\ 4 & -7-\lambda \end{pmatrix}$$
$$= (1-\lambda)(-7-\lambda) + 16$$
$$= \lambda^2 + 6\lambda + 16$$

Hence $\lambda = \frac{-6 \pm \sqrt{36-64}}{2} = -3 \pm i\sqrt{7}$ or $\lambda = -3 + i\sqrt{7}$, $\lambda = -3 - i\sqrt{7}$ are the eigenvalues of $\underline{A}$.

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5. $\dot{\underline{x}} = \begin{pmatrix} -7 & 1 & -6 \\ 10 & -4 & 12 \\ 2 & -1 & 1 \end{pmatrix} \underline{x}$

Solution

The characteristic polynomial of the matrix $\underline{A} = \begin{pmatrix} -7 & -1 & -6 \\ 10 & -4 & 12 \\ 2 & -1 & 1 \end{pmatrix}$ is

$$p(\lambda) = \det(\underline{A} - \lambda \underline{I}) = \det \begin{pmatrix} -7-\lambda & -1 & -6 \\ 10 & -4-\lambda & 12 \\ 2 & -1 & 1-\lambda \end{pmatrix}$$

$$= -7-\lambda \begin{vmatrix} -4-\lambda & 12 \\ -1 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 10 & 12 \\ 2 & 1-\lambda \end{vmatrix} - 6 \begin{vmatrix} 10 \\ 2 \end{vmatrix}$$

$$= (-7-\lambda)\left[\lambda^2 + 3\lambda - 4 + 12\right] - \left[10 - 10\lambda - 24\right]$$

$$- 6\left[-10 - (-8 - 2\lambda)\right]$$

$$= -\lambda^3 - 3\lambda^2 - 8\lambda$$

$$- 7\lambda^2 - 21\lambda - 56$$

$$+ 10\lambda + 14$$

$$60$$

$$- 12\lambda - 48$$

$$= -\lambda^3 - 10\lambda^2 - 31\lambda - 30$$

$$= -(\lambda+2)(\lambda+3)(\lambda+5).$$

Hence, $\lambda = -2$, $\lambda = -3$, $\lambda = -5$ are the eigenvalues of A.

Then by Thm 1, part A, every soln $\underline{x} = \underline{\phi}(t)$ of $\dot{\underline{x}} = \begin{pmatrix} -7 & 1 & -6 \\ 10 & -4 & 12 \\ 2 & -1 & 1 \end{pmatrix} \underline{x}$ is asymptotically stable, since all eigenvalues of A have negative real part.

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7. $\dot{\underline{x}} = \begin{pmatrix} 0 & 2 & 1 \\ -1 & -3 & -1 \\ 1 & 1 & -1 \end{pmatrix} \underline{x}$

Solution

The characteristic polynomial of the matrix $A = \begin{pmatrix} 0 & 2 & 1 \\ -1 & -3 & -1 \\ 1 & 1 & -1 \end{pmatrix}$ is:

$$p\lambda = \det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 2 & 1 \\ -1 & -3-\lambda & -1 \\ 1 & 1 & -1-\lambda \end{pmatrix}$$

$$= (-\lambda)\begin{vmatrix} -3-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} - 2\begin{vmatrix} -1 & -1 \\ 1 & -1-\lambda \end{vmatrix} + \begin{vmatrix} -1 & -3-\lambda \\ 1 & 1 \end{vmatrix}$$

$$= -\lambda\left[\lambda^2+4\lambda+3-(-1)\right] - 2\left[\lambda+1-(-1)\right] + \left[-1-(-3-\lambda)\right]$$

$$= -\lambda^3-4\lambda^2-3\lambda-\cancel{\lambda}-2\lambda-2-2-1+3+\cancel{\lambda}$$

$$= -\lambda^3-4\lambda^2-5\lambda-2$$

$$= -(\lambda^3+4\lambda^2+5\lambda+2)$$

$$= -(\lambda+1)^2(\lambda+2)$$

Hence, $\lambda = -1$, $\lambda = -2$ are the eigenvalues of A.

Then by Thm 1, part A, every soln. $\underline{x} = \underline{\phi}(t)$ of $\dot{\underline{x}} = \begin{pmatrix} 0 & 2 & 1 \\ -1 & -3 & -1 \\ 1 & 1 & -1 \end{pmatrix} \underline{x}$ is asymptotically stable, since all eigenvalues of A have negative real part.

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9 $\dot{\underline{x}} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix} \underline{x}$

Solution

The char. polynomial of matrix $A = \begin{pmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix}$ is:

$$p(\lambda) = \det(\underline{A} - \lambda \underline{I}) = \det\begin{pmatrix} -\lambda & 2 & 0 & 0 \\ -2 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 2 \\ 0 & 0 & -2 & -\lambda \end{pmatrix}$$

$$= -\lambda \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 2 \\ 0 & -2 & -\lambda \end{vmatrix} - 2 \begin{vmatrix} -2 & 0 & 0 \\ 0 & -\lambda & 2 \\ 0 & -2 & -\lambda \end{vmatrix}$$

$$= -\lambda \left[-\lambda \begin{vmatrix} -\lambda & 2 \\ -2 & -\lambda \end{vmatrix} \right] - 2 \left[-2 \begin{vmatrix} -\lambda & 2 \\ -2 & -\lambda \end{vmatrix} \right]$$

$$= -\lambda [-\lambda (\lambda^2 + 4)] - 2 [-2 (\lambda^2 + 4)]$$

$$= \lambda^2 (\lambda^2 + 4) + 4 (\lambda^2 + 4)$$

$$= (\lambda^2 + 4)(\lambda^2 + 4)$$

Hence $\lambda^2 = -4$, or $\lambda = 2i$ and $\lambda = -2i$ are the eigenvalues of A. Therefore, by part (c) of Theorem 2, every solution $x = \phi(t)$ of $\dot{\underline{x}} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix} \underline{x}$ is stable.

However, no solution is asymptotically stable; this follows immediately from the fact that the general soln. of $\underline{\dot{x}} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix} \underline{x}$ is

$$x(t) = c_1 \begin{pmatrix} -2 \sin 2t \\ 2 \cos 2t \end{pmatrix} + c_2 \begin{pmatrix} 2 \cos 2t \\ 2 \sin 2t \end{pmatrix}.$$

Hence, the solution is periodic, with period π, and no solution $x(t)$ (except $x(t) \equiv 0$) approaches 0 as t approaches infinity.

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11. Determine whether the solutions $x(t) \equiv 0$ and $x(t) \equiv 1$ of the single scalar equation $\dot{x} = x(1-x)$ are stable or unstable.

Solution

Consider $x(t) \equiv 1$. The solution $x(t) \equiv 1$ is stable, since every solution $\psi(t)$ of $\dot{x} = x(1-x)$ which starts sufficiently close to 1 at $t=0$ must remain close to 1 for all future time t.

Consider $x(t) \equiv 0$. The solution $x(t) \equiv 0$ is unstable, since there exists at least one solution $\psi(t)$ of $\dot{x} = x(1-x)$ which starts near 0 at $t=0$, but which does not remain close to 0 for all future time.

15. Show that the stability of any solution $x(t)$ of the nonhomogeneous equation $\dot{\underline{x}} = \underline{A}\underline{x} + \underline{f}(t)$ is equivalent to the stability of the equilibrium solution $\underline{x} \equiv \underline{0}$ of the homogeneous equation $\dot{\underline{x}} = \underline{A}\,\underline{x}$.

Solution

Consider $\dot{\underline{x}} = A\underline{x} + \underline{f}$

Let $\underline{x}_1 = \underline{x} - \underline{f}$

and $\dot{\underline{x}}_1 = \dot{\underline{x}} + \dot{\underline{f}}$

Then $A\underline{x}_1 = A\underline{x} - A\underline{f}$

$= \dot{\underline{x}}_1$

Note: since $\underline{x}_1$ satisfies $\dot{\underline{x}}_1 = A\underline{x}_1$ (homogeneous), thus, $\underline{x}$ satisfies $\dot{\underline{x}} = A\underline{x} + \underline{f}$

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Chapter 4.3

Find all equilibrium solutions of each of the following systems of equations and determine, if possible, whether they are stable or unstable.

1. $\dot{x} = x - x^3 - xy^2$

   $\dot{y} = 2y - y^5 - yx^4$

<u>Solution</u>

$x - x^3 - xy^2 = 0$ iff $x(1 - x^2 - y^2) = 0$

iff $x = 0$ or $x^2 + y^2 = 1$

$2y - y^5 - yx^4 = 0$ iff $y(2 - y^4 - x^4) = 0$

iff $y = 0$ or $x^4 + y^4 = 2$

	x	y	Stability
i)	0	0	unstable
ii)	1	0	unstable
iii)	-1	0	unstable
iv)	0	$2^{1/4}$	stable
v)	0	$-2^{1/4}$	stable

$\underline{f}(\underline{x}) = \begin{pmatrix} x_1 - x_1^3 - x_1x_2^2 \\ 2x_2 - x_2^5 - x_2x_1^4 \end{pmatrix}$

$\underline{f}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\underline{f}(\underline{x}^0 + \underline{z}) = \underline{f}\begin{pmatrix} 0 + z_1 \\ 0 + z_2 \end{pmatrix} = \underline{f}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1 - z_1^3 - z \\ 2z_2 - z_2^5 - z \end{pmatrix}$

i) <u>$x(t) \equiv 0,\ y(t) \equiv 0$</u>

$\underline{z} = \underline{x} - \underline{x}^0 = \begin{pmatrix} x_1 - 0 \\ x_2 - 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} -z_1^3 - z \\ -z_2^5 - z \end{pmatrix}$

$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

$A\underline{v} = \lambda\underline{v}$

or $|A - \lambda I| = 0$

$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix}$

$\begin{vmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = 0$

$v_1 = \lambda v_1 \Rightarrow \lambda_1 = 1$

$(1-\lambda)(2-\lambda) = 0$

$2v_2 = \lambda v_2 \Rightarrow \lambda_2 = 2$

$\lambda_1 = 1,\ \lambda_2 = 2$

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1. i) continued

$\lambda = 1$: $(A - I)\underline{v} = \underline{0}$ $\qquad \left(\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix} - \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)\underline{v} = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$

λ vector: $\begin{pmatrix}c_1\\ 0\end{pmatrix}$ $\qquad v_1$ arbitrary, $v_2 = 0$

$\lambda = 2$: $(A - 2I)\underline{v} = \underline{0}$

λ vector: $\begin{pmatrix}0\\ c_2\end{pmatrix}$ $\qquad \left(\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix} - \begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}\right)\underline{v} = \begin{pmatrix}-1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$

$v_1 = 0$, v_2 arbitrary

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ii) $x(t) \equiv 1$, $y(t) \equiv 0$: $\qquad \underline{f}(\underline{v}^c + \underline{z}) = \underline{f}\begin{pmatrix}1+z_1\\ 0+z_2\end{pmatrix} = \begin{pmatrix}(1+z_1) - (1+z_1)^3 - (1+z_1)z_2^2\\ 2z_2 - z_2^5 - z_2(1+z_1)^4\end{pmatrix}$

set $u = x-1$, $v = y$ $\rightarrow \frac{du}{dt} = \frac{dx}{dt} = (u+1) - (u+1)^3 - (u+1)v^2$

$\rightarrow \frac{dv}{dt} = \frac{dy}{dt} = 2v - v^5 - v(u+1)^4$

$$\frac{d}{dt}\begin{pmatrix}u\\ v\end{pmatrix} = \begin{pmatrix}-2 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} + \begin{pmatrix}-u^3 - 3u^2 - uv^2 - v^2\\ -u^4v - 4u^3v - 6u^2v - 4uv - v^5\end{pmatrix}$$

$A = \begin{pmatrix}-2 & 0\\ 0 & 1\end{pmatrix}$

$|A - \lambda I| = 0$: $\begin{vmatrix}-2-\lambda & 0\\ 0 & 1-\lambda\end{vmatrix} = -(2+\lambda)(1-\lambda)$

$\lambda_1 = 1$, $\lambda_2 = -2$

{UNSTABLE}

iii) $x(t) \equiv -1$, $y(t) \equiv 0$:

set $u = x+1$, $v = y$ $\rightarrow \frac{du}{dt} = \frac{dx}{dt} = (u-1) - (u-1)^3 - (u-1)v^2$

$\rightarrow \frac{dv}{dt} = \frac{dy}{dt} = 2v - v^5 - v(u-1)^4$

$$\frac{d}{dt}\begin{pmatrix}u\\ v\end{pmatrix} = \begin{pmatrix}-2 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} + \begin{pmatrix}3u^2 - u^3 + v^2 - uv^2\\ -u^4v + 4u^3v - 6u^2v + 4uv - v^5\end{pmatrix}$$

$A = \begin{pmatrix}-2 & 0\\ 0 & 1\end{pmatrix}$

As above: $\lambda_1 = 1$, $\lambda_2 = -2$

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1. continued

iv) $x(t) = 0, \quad y(t) = \sqrt[4]{2}$:

set $u = x, \quad v = y - 2^{1/4} \quad \rightarrow \frac{du}{dt} = \frac{dx}{dt} = u - u^3 - u(v + 2^{\frac{1}{4}})^2$

$\rightarrow \frac{dv}{dt} = \frac{dy}{dt} = 2(v + 2^{\frac{1}{4}}) - (v + 2^{\frac{1}{4}})^5 - (v + 2^{\frac{1}{4}})u$

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1-\sqrt{2} & 0 \\ 0 & 2(1-2^{\frac{9}{4}}) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \text{---} \\ \text{---} \end{pmatrix}$$

$|A - \lambda I| = 0$

$$\begin{vmatrix} (1-\sqrt{2})-\lambda & 0 \\ 0 & 2(1-2^{\frac{9}{4}})-\lambda \end{vmatrix} = 0$$

$(1 - \sqrt{2} - \lambda)(2(1 - 2^{9/4}) - \lambda) = 0$

eigenvalues: $\lambda_1 = 1 - \sqrt{2}, \quad \lambda_2 = 2(1 - 2^{9/4})$

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v) $x(t) = 0, \quad y(t) = -\sqrt[4]{2}$:

set $u = x, \quad v = y + 2^{\frac{1}{4}} \quad \rightarrow \frac{du}{dt} = \frac{dx}{dt} = u - u^3 - u(v + 2^{\frac{1}{4}})^2$

$\rightarrow \frac{dv}{dt} = \frac{dy}{dt} = 2(v - 2^{\frac{1}{4}}) - (v - 2^{\frac{1}{4}})^5 - (v - 2^{\frac{1}{4}})u$

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1-\sqrt{2} & 0 \\ 0 & 2(1-2^{9/4})-\lambda \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \text{---} \\ \text{---} \end{pmatrix}$$

$|A - \lambda I| = 0$

... eigenvalues: $\lambda_1 = 1 - \sqrt{2}, \quad \lambda_2 = 2(1 - 2^{9/4})$ (same as abov

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3. $\dot{x} = x^2 + y^2 - 1$

$\dot{y} = 2xy$

Solution

$x^2 + y^2 - 1 = 0$ iff $x^2 + y^2 = 1$

$2xy = 0$ iff $x = 0$ or $y = 0$

	x	y	stability
i)	0	1	unstable
ii)	0	-1	unstable
iii)	1	0	unstable
iv)	-1	0	stable

$f(x) = \begin{pmatrix} x_1^2 + x_2^2 - 1 \\ 2x_1x_2 \end{pmatrix}$

$f\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

i) $\underline{x(t) \equiv 0,\ y(t) \equiv 1}$: $f(x^0 + z) = f\begin{pmatrix} 0 + z_1 \\ 1 + z_2 \end{pmatrix} = \begin{pmatrix} z_1^2 + (1+z_2)^2 - 1 \\ 2z_1(1+z_2) \end{pmatrix}$

$= \begin{pmatrix} z_1^2 + 2z_2 + z_2^2 \\ 2z_1 + 2z_1z_2 \end{pmatrix}$

Set $u = x,\ v = y - 1$ $\rightarrow \frac{du}{dt} = \frac{dx}{dt} = u^2 + (v+1)^2 - 1 = u^2 + v^2 + 2v$

$\rightarrow \frac{dv}{dt} = \frac{dy}{dt} = 2u(v+1) = 2uv + 2u$

$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} u^2 + v^2 \\ 2uv \end{pmatrix}$

$A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$

$\det|A - \lambda I| = \det\begin{vmatrix} -\lambda & 2 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 - 4$

$\lambda_1 = 2,\ \lambda_2 = -2$

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3. continued

ii) $\underline{x(t)\equiv 0,\ y(t)\equiv -1}$: $\underline{f}(\underline{x}^0+\underline{z}) = \underline{f}\begin{pmatrix}0+z_1\\-1+z_2\end{pmatrix} = \begin{pmatrix}z_1^2+(z_2-1)^2-1\\2z_1(z_2-1)\end{pmatrix}$

$$\Rightarrow \begin{pmatrix}z_1^2+z_2^2-2z_2\\2z_1z_2-2z_1\end{pmatrix}$$

set $u=x,\ v=y+1 \rightarrow \frac{du}{dt}=\frac{dx}{dt} = u^2+(v-1)^2-1 = u^2+v^2-2v$

$\rightarrow \frac{dv}{dt}=\frac{dy}{dt} = 2u(v-1) \quad = 2uv-2u$

$$\frac{d}{dt}\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}0&-2\\-2&0\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix} + \begin{pmatrix}u^2+v^2\\2uv\end{pmatrix}$$

$$A = \begin{pmatrix}0&-2\\-2&0\end{pmatrix}$$

$\det|A-\lambda I| = \det\begin{vmatrix}-\lambda&-2\\-2&-\lambda\end{vmatrix} = \lambda^2-4$

$\lambda_1=2,\ \lambda_2=-2$

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iii) $\underline{x(t)\equiv 1,\ y(t)\equiv 0}$: $\underline{f}(\underline{x}^0+\underline{z}) = \underline{f}\begin{pmatrix}1+z_1\\0+z_2\end{pmatrix} = \begin{pmatrix}(1+z_1)^2+z_2^2-1\\2(1+z_1)z_2\end{pmatrix}$

$$= \begin{pmatrix}z_1^2+2z_1+z_2^2\\2z_1z_2+2z_2\end{pmatrix}$$

set $u=x-1,\ y=v \rightarrow \frac{du}{dt}=\frac{dx}{dt} = (u+1)^2+v^2-1 = u^2+2u+v^2$

$\rightarrow \frac{dv}{dt}=\frac{dy}{dt} = 2(u+1)v \quad = 2uv+2v$

$$\frac{d}{dt}\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}2&0\\0&2\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix} + \begin{pmatrix}u^2+v^2\\2uv\end{pmatrix}$$

$$A = \begin{pmatrix}2&0\\0&2\end{pmatrix}$$

$\det|A-\lambda I| = \det\begin{vmatrix}2-\lambda&0\\0&2-\lambda\end{vmatrix} = (2-\lambda)^2 = 0$

$\lambda_1=2,\ \lambda_2=2$

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3. continued

(iv) $\underline{x}^0 = $ [illegible] $\qquad \underline{f}(\underline{x}^0+\underline{z}) = \underline{f}\begin{pmatrix} -1+z_1 \\ 0+z_2 \end{pmatrix} = \begin{pmatrix} (z_1-1)^2+z_2^2-1 \\ 2(z_1-1)z_2 \end{pmatrix}$

$$= \begin{pmatrix} z_1^2-2z_1+z_2^2 \\ 2z_1z_2-2z_2 \end{pmatrix}$$

Set $u = x+1,\ v = y \rightarrow \frac{du}{dt} = \frac{dx}{dt} = (u-1)^2+v^2-1 = u^2-2u+v^2$

$\rightarrow \frac{dv}{dt} = \frac{dy}{dt} = 2(u-1)v = 2uv-2v$

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} u^2+v^2 \\ 2uv \end{pmatrix}$$

$$A = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

$$\det|A-\lambda I| = \det\begin{vmatrix} -2-\lambda & 0 \\ 0 & -2-\lambda \end{vmatrix} = (-2-\lambda)^2$$

$\lambda_1 = -2,\ \lambda_2 = -2$

{STABLE}

5. $\dot{x} = \tan(x+y)$

$\dot{y} = x+x^3$

Solution

$\tan(x+y) = 0$ iff $x+y = n\pi$ where $n = 0, 1, 2, 3, \ldots$

$x+x^3 = 0$ iff $x(1+x^2) = 0$

iff $x = 0$ or $x = \pm i$

$$\underline{f}(\underline{x}) = \begin{pmatrix} \tan(x_1+x_2) \\ x_1+x_2^3 \end{pmatrix}$$

$$\underline{f}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\underline{f}(\underline{x}^0+\underline{z}) = \underline{f}\begin{pmatrix} 0+z_1 \\ n\pi+z_2 \end{pmatrix} = \underline{f}\begin{pmatrix} z_1 \\ n\pi+z_2 \end{pmatrix} = \begin{pmatrix} \tan(z_1+z_2+n\pi) \\ z_1+z_1^3 \end{pmatrix}$$

Set $u = x,\ v = y-n\pi \rightarrow \frac{du}{dt} = \frac{dx}{dt} = \tan(u+v+n\pi)$

$\rightarrow \frac{dv}{dt} = \frac{dy}{dt} = u+u^3$

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \tan(u+v+n\pi) \\ u^3 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\det|A-\lambda I| = \det\begin{vmatrix} -\lambda & 0 \\ 1 & -\lambda \end{vmatrix} = \lambda^2-1 = (\lambda-1)(\lambda+1)$$

$\lambda_1 = 1,\ \lambda_2 = -1$

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Verify that the origin is an equilibrium point of each of the following systems of equations and determine, if possible, whether it is stable or unstable.

7. $\dot{x} = y + 3x^2$

$\dot{y} = x - 3y^2$

Solution

$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ is an equilibrium value of this system iff:

$$y_0 + 3x_0^2 = 0 \quad \text{and} \quad x_0 + 3y_0^2 = 0$$

$$y_0 = -3x_0^2 \qquad x_0 = -3y_0^2$$

$$y_0^2 = \frac{-1}{3}y_0$$

$$x_0 = \frac{\pm i\sqrt{3y_0}}{3}$$

$$-3y_0^2 = \frac{\pm i\sqrt{3y_0}}{3}$$

$$-9y_0^2 = \pm i\sqrt{3y_0}$$

$$-9y_0^2 = 3y_0$$

$$-9y_0^2 - 3y_0 = 0$$

$$-3y_0(y_0+1) = 0$$

$$y_0 = 0 \text{ or } y = -1$$

$x_0 = 0$

$\underline{x(t) \equiv 0, \ y(t) \equiv 0}$: $\quad f(\underline{x_0} + z) = f\begin{pmatrix} 0+z_1 \\ 0+z_2 \end{pmatrix} = \begin{pmatrix} z_2 + 3z_1^2 \\ z_1 - 3z_2^2 \end{pmatrix}$

Set $u = x, \ v = y \rightarrow \frac{du}{dt} = \frac{dx}{dt} = v + 3u^2$

$\rightarrow \frac{dv}{dt} = \frac{dy}{dt} = u - 3v^2$

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 3u^2 \\ -3v^2 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\det |A - \lambda I| = \det \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1$$

$\lambda_1 = 1, \ \lambda_2 = -1$

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9. $\dot{x} = e^{x+y} - 1$

$\dot{y} = \sin(x+y)$

Solution

$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ is an equilibrium value of the system if

$e^{x+y} - 1 = 0$ and $\sin(x_0+y_0) = 0$

$e^{x_0+y_0} = 1$ $x_0+y_0 = n\pi$ where $n = 0, 1, 2, 3, \ldots$

$x_0 + y_0 = 0$

SO $(x_0, y_0) = (0,0)$

$x(t) \equiv 0,\ y(t) \equiv 0$: $f(x_0 + z) = f\begin{pmatrix} 0+z_1 \\ 0+z_2 \end{pmatrix} = \begin{pmatrix} e^{z_1+z_2} - 1 \\ \sin(z_1+z_2) \end{pmatrix}$

set $u = x,\ v = y$ $\rightarrow \frac{du}{dt} = \frac{dx}{dt} = e^{u+v} - 1$

$\rightarrow \frac{dv}{dt} = \frac{dy}{dt} = \sin(u+v)$

$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} e^{u+v} - 1 \\ \sin(u+v) \end{pmatrix}$

$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\det|A - \lambda I| = \det\begin{vmatrix} -\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = \lambda^2$

$\lambda_1 = 0,\ \lambda_2 = 0$

$\boxed{\text{STABLE}}$

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11. $\dot{x} = \cos y - \sin x - 1$

$\dot{y} = x - y - y^2$

Solution

$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ is an equilibrium value of this system if

$\cos y_0 - \sin x_0 - 1 = 0$ and $x_0 - y_0 - y_0^2 = 0$

$\cos y_0 - \sin x_0 = 1$ $\qquad x_0 = y_0(1 + y_0)$

$\cos y_0 = \sin x_0 + 1$

Letting $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ solves both equations above, and hence, we have a point of equilibrium

$x(t) \equiv 0,\ y(t) \equiv 0$: $\quad \underline{f}(\underline{x}_0 + \underline{z}) = \underline{f}\begin{pmatrix} 0 + z_1 \\ 0 + z_2 \end{pmatrix} = \begin{pmatrix} \cos z_2 - \sin z_1 - 1 \\ z_1 - z_2 - z_2^2 \end{pmatrix}$

Let $u = x,\ v = y \Rightarrow \frac{du}{dt} = \frac{dx}{dt} = \cos v - \sin u - 1$

$\Rightarrow \frac{dv}{dt} = \frac{dy}{dt} = u - v - v^2$

$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \cos v - \sin u - 1 \\ -v^2 \end{pmatrix}$

$A = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$

$\det|A - \lambda I| = \det\begin{vmatrix} -\lambda & 0 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 + \lambda - 1$

$\lambda = \frac{-1 \pm \sqrt{1+4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$

$\lambda_1 = \frac{\sqrt{5}-1}{2}, \quad \lambda_2 = \frac{-\sqrt{5}-1}{2}$

{STABLE}

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13. $\dot{x} = -x - y - (x^2+y^2)^{3/2}$

$\dot{y} = x - y + (x^2+y^2)^{3/2}$

Solution

$\binom{x_0}{y_0}$ is an equilibrium value of the system if

$-x_0 - y_0 - (x_0^2+y_0^2)^{3/2} = 0$ and $x_0 - y_0 + (x_0^2+y_0^2)^{3/2} = 0$

$x_0 + y_0 = -(x_0^2+y_0^2)^{3/2}$ $\qquad$ $x_0 - y_0 = -(x_0^2+y_0^2)^{3/2}$

$x_0 + y_0 = x_0 - y_0$

$y_0 = -y_0 \rightarrow x_0 = -(x_0^2)^{\frac{3}{2}} = -x_0^3 = 0$

So $x_0 = 0,\ y_0 = 0$

$x(t) \equiv 0,\ y(t) \equiv 0$: $\quad f(x_0+z) = f\begin{pmatrix} 0+z_1 \\ 0+z_2 \end{pmatrix} = \begin{pmatrix} -z_1 - z_2 - (z_1^2+z_2^2)^{3/2} \\ z_1 - z_2 + (z_1^2+z_2^2)^{3/2} \end{pmatrix}$

Set $u = x,\ v = y \rightarrow \frac{du}{dt} = \frac{dx}{dt} = -u - v - (u^2+v^2)^{3/2}$

$\rightarrow \frac{dv}{dt} = \frac{dy}{dt} = u - v + (u^2+v^2)^{3/2}$

$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} -(u^2+v^2)^{3/2} \\ (u^2+v^2)^{3/2} \end{pmatrix}$

$A = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$

$\det|A - \lambda I| = \det\begin{vmatrix} -1-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} = (\lambda+1)^2 - (-1) = \lambda^2 + 2\lambda + 2$

$\lambda = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$

$\lambda_1 = -1 + i, \quad \lambda_2 = -1 - i$

$\boxed{\text{STABLE}}$

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15. $\dot{x} = e^{x+y+z} - 1$

$\dot{y} = \sin(x+y+z)$

$\dot{z} = x - y - z^2$

Solution

$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ is an equilibrium value of this system iff:

$e^{x_0+y_0+z_0} - 1 = 0$ and $\sin(x_0+y_0+z_0) = 0$ and $x_0 - y_0 - z_0^2 = 0$

$e^{x_0+y_0+z_0} = 1$ $\qquad$ $x_0 + y_0 + z_0 = n\pi$ where $n = 0, \ldots$

$x_0 + y_0 + z_0 = 0$

$n\pi = 0 \Rightarrow n = 0$

$x_0 = -y_0 - z_0 = y_0 + z_0^2$

$2y_0 = -z_0^2 - z_0$

$y_0 = \frac{1}{2} z_0 (z_0 - 1)$

Suppose $y_0 = 0$, then $z_0 = 0$ or $z_0 = 1$

But if $z_0 = 1$, then $x_0 = -1$ & the eqs are not solvable

So $y_0 = 0$, $z_0 = 0$ must be true, and hence, $x_0 = 0$

$x(t) \equiv 0,\ y(t) \equiv 0,\ z(t) \equiv 0$: $f(x_0 + z) = f\begin{pmatrix} 0 + z_1 \\ 0 + z_2 \\ 0 + z_3 \end{pmatrix} = \begin{pmatrix} e^{z_1+z_2+z_3} - 1 \\ \sin(z_1+z_2+z_3) \\ z_1 - z_2 - z_3^2 \end{pmatrix}$

Set $u = x$, $v = y$, $w = z$ $\rightarrow \frac{du}{dt} = \frac{dx}{dt} = e^{u+v+w} - 1$

$\rightarrow \frac{dv}{dt} = \frac{dy}{dt} = \sin(u+v+w)$

$\rightarrow \frac{dw}{dt} = \frac{dz}{dt} = u - v - w^2$

$\frac{d}{dt}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} e^{u+v+w} - 1 \\ \sin(u+v+w) \\ -w^2 \end{pmatrix}$

$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}$

$\det|A - \lambda I| = \det\begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 1 & -1 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 0 \\ -1 & -\lambda \end{vmatrix} = -\lambda(\lambda^2) = -\lambda^3$

$\lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = 0$

$\boxed{\text{UNSTABLE}}$

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17. $\dot{x} = x - \cos y - z + 1$

$\dot{y} = y - \cos z - x + 1$

$\dot{z} = z - \cos x - y + 1$

Solution

$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ is an equilibrium solution of this system iff.

$x_0 - \cos y_0 - z_0 + 1 = 0$ and $y_0 - \cos z_0 - x_0 + 1 = 0$ and $z_0 - \cos x_0 - y_0 + 1 = 0$

$x_0 - z_0 - \cos y_0 = -1$ $\quad y_0 - x_0 - \cos z_0 = -1$ $\quad z_0 - y_0 - \cos x_0 = -1$

Letting $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ solves all three above equations, yielding an equilibrium solution for this system.

$x(t) \equiv 0,\ y(t) \equiv (0),\ z(t) \equiv (0)$: $\quad f(x_0 + z) = f\begin{pmatrix} 0+z_1 \\ 0+z_2 \\ 0+z_3 \end{pmatrix} = \begin{pmatrix} z_1 - \cos z_2 - z_3 + 1 \\ z_2 - \cos z_3 - z_1 + 1 \\ z_3 - \cos z_1 - z_2 + 1 \end{pmatrix}$

set $u = x,\ v = y,\ w = z \longrightarrow \frac{du}{dt} = \frac{dx}{dt} = u - \cos v - w + 1$

$\longrightarrow \frac{dv}{dt} = \frac{dy}{dt} = v - \cos w - u + 1$

$\longrightarrow \frac{dw}{dt} = \frac{dz}{dt} = w - \cos u - v + 1$

$$\frac{d}{dt}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} 1-\cos v \\ 1-\cos w \\ 1-\cos u \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\det|A - \lambda I| = \det\begin{vmatrix} 1-\lambda & 0 & -1 \\ -1 & 1-\lambda & 0 \\ 0 & -1 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & 0 \\ -1 & 1-\lambda \end{vmatrix} - \begin{vmatrix} -1 & 1-\lambda \\ 0 & -1 \end{vmatrix}$$

$$= (1-\lambda)[(1-\lambda)^2] - [1]$$

$$= (\lambda^2 - 2\lambda + 1)(1-\lambda) - 1$$

$$= -\lambda^3 + 2\lambda^2 - \lambda + \lambda^2 - 2\lambda + 1 - 1$$

$$= -\lambda^3 + 3\lambda^2 - 3\lambda$$

$$= -\lambda(\lambda^2 - 3\lambda + 3)$$

$\lambda_1 = 0,\ \lambda_2 = \frac{3 \pm \sqrt{9-12}}{2} = \frac{3}{2} + \frac{i\sqrt{3}}{2},\ \lambda_3 = \frac{3}{2} - \frac{i\sqrt{3}}{2}$

$\boxed{\text{UNSTABLE}}$

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18 continued.

EXACTLY $e=0 \Rightarrow \frac{dx}{dt} = gz - hx, \ \frac{dy}{dt} = \frac{c}{a+bx} - ky, \ \frac{dz}{dt} = -fz$

$z=0; \ \frac{c}{a+bx} - ky = 0 \qquad x=0$ (if $h \neq 0$)

Hence $(0, \frac{c}{ak}, 0)$ is a point of equilibrium.

Set $u=x, \ v=y-\frac{c}{ak}, \ w=z \ \rightarrow \ \frac{du}{dt} = \frac{dx}{dt} = gw - hu$

$\rightarrow \ \frac{dv}{dt} = \frac{dy}{dt} = \frac{c}{a+bu} - kv - \frac{ck}{ak}$

$\rightarrow \ \frac{dw}{dt} = \frac{dz}{dt} = -fw$

Now expand $(1+\frac{b}{a}u)^{-1}$ in binomial series --- (see prev. page)

$$\frac{d}{dt}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -h & 0 & g \\ -cb/a^2 & -k & 0 \\ 0 & 0 & -f \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} 0 \\ - \\ 0 \end{pmatrix}$$

$$A = \begin{vmatrix} -h & 0 & g \\ -cb/a^2 & -k & 0 \\ 0 & 0 & -f \end{vmatrix}$$

$$\det|A - \lambda I| = \det\begin{vmatrix} -h-\lambda & 0 & g \\ -cb/a^2 & -k-\lambda & 0 \\ 0 & 0 & -f-\lambda \end{vmatrix} = (-h-\lambda)\begin{vmatrix} -k-\lambda & 0 \\ 0 & -f-\lambda \end{vmatrix} + g\begin{vmatrix} -cb/a^2 & -k-\lambda \\ 0 & 0 \end{vmatrix}$$

$$= (-h-\lambda)(-k-\lambda)(-f-\lambda)$$

$$= -(h+\lambda)(k+\lambda)(f+\lambda)$$

$\lambda_1 = -h, \ \lambda_2 = -k, \ \lambda_3 = -f$

{STABLE}

EXACTLY $c=0 \Rightarrow \frac{dx}{dt} = gz - hx, \ \frac{dy}{dt} = -ky, \ \frac{dz}{dt} = ey - fz$

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In each of probs 1 and 3, verify that $x(t)$, $y(t)$ is a solution of the given system of equations, and find its orbit.

1. $\dot{x} = 1$ / $\dot{y} = 2(1-x)\sin(1-x)^2$ / $x(t) = 1+t$ / $y(t) = \cos t^2$

Solution

It is easily verified that $x(t) = 1+t$ and $y(t) = \cos t^2$ is a solution of the system of differential equations $\dot{x} = 1$ and $\dot{y} = 2(1-x)\sin(1-x)^2$ since $\frac{d}{dt}x(t) = \dot{x} = 1$ and since $\frac{d}{dt}y(t) = \dot{y} = -2t\sin t^2$ where $t = x-1$ so that $\dot{y} = -2(x-1)\sin(x-1)^2 = 2(1-x)\sin(1-x)^2$.

As t runs from $-\infty$ to ∞, the set of points $(1+t, \cos t^2)$ traces out the graph of $y = \cos(x-1)^2 = \cos(1-x)^2$ in the x-y plane. Hence, $y = \cos(x-1)^2$ is the orbit of the solution $x = 1+t$, $y = \cos t^2$, $-\infty \le t \le \infty$

3. $\dot{x} = 1+x^2$ / $\dot{y} = (1+x^2)\sec^2 x$ / $x(t) = \tan t$ / $y(t) = \tan(\tan t)$

Solution

It is easily verified that $x(t) = \tan t$ and $y(t) = \tan(\tan t)$ is a solution of the system of differential equations $\dot{x} = 1+x^2$ and $\dot{y} = (1+x^2)\sec^2 x$, since $\frac{d}{dt}x(t) = \sec^2 t = 1+\tan^2 t = 1+x^2$ and since $\frac{d}{dt}y(t) = \sec^2(\tan t)\sec^2(t) = \sec^2 x\,(1+x^2) = (1+x^2)\sec^2 x$.

As t runs from $-\infty$ to ∞, the set of points $(\tan t, \tan(\tan t))$ traces the graph of $y = \tan x$ in the x-y plane. Hence, $y = \tan x$ is the orbit of the solution $x = \tan t$, $y = \tan(\tan t)$, $-\infty < t < \infty$.

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Find the orbits of the following systems.

5. $\dot{x} = y \,/\, \dot{y} = -x$

Solution

The orbits of the system of differential equations above are the solution curves of the scalar equation $\frac{dy}{dx} = \frac{-x}{y}$. This equation is separable as follows: $y\,dy = -x\,dx$ so that $\int y\,dy = -\int x\,dx$ or $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + c'$ or $y^2 = -x^2 + 2c'$, c constant. Thus, the orbits of $\dot{x} = y$, $\dot{y} = -x$ are the set of all curves $y = \pm\sqrt{-x^2+2c'}$ (c constant) or $x^2 + y^2 = 2c'$ or $x^2 + y^2 = c^2$ (c constant).

7. $\dot{x} = y(1+x+y) \,/\, \dot{y} = -x(1+x+y)$

Solution

The orbits of the system of differential equations above are the partially described by the solution curves of $\frac{dy}{dx} = \frac{-x(1+x+y)}{y(1+x+y)} = -\frac{x}{y}$. As seen above, the equation is separable yielding $x^2 + y^2 = c^2$ as an orbit. Consider $1+x+y=0$ or $x+y=-1$. Let $x = x_0$, $y = y_0$, and hence $x_0 + y_0 = -1$. Thus, the orbits of $\dot{x} = y(1+x+y)$, $\dot{y} = x(1+x+y)$ are the set of points described by $x = x_0$, $y = y_0$, $x_0 + y_0 = -1$ and the set of all curves $x^2+y^2=c^2$ (circles), minus the previously described points. (c constant).

9. $\dot{x} = xye^{-3x} \,/\, \dot{y} = -2xy^2$

Solution

The orbits of the system of differential equations above are the solution curves of the scalar equation $\frac{dy}{dx} = \frac{-2xy^2}{xye^{-3x}} = \frac{-2y}{e^{-3x}}$. Solve as follows: $\frac{dy}{-2y} = \frac{dx}{e^{-3x}}$ so that $\int \frac{dy}{-2y} = \int \frac{dx}{e^{-3x}}$ or $-\frac{1}{2}\int \frac{dy}{y} = \int \frac{dx}{e^{-3x}}$ or $-\frac{1}{2}\ln|y| = \int e^{3x}dx$ or $-\frac{1}{2}\ln|y| = \frac{1}{3}e^{3x} + c'$ or $\ln|y| = -\frac{2}{3}e^{3x} - 2c'$ or $|y| = e^{-\frac{2}{3}e^{3x}}e^{-2c'}$ or $y = e^{-\frac{2}{3}e^{3x}}e^{-2c'}$ or $y = ce^{-\frac{2}{3}e^{3x}}$ (c constant). Please note, only holds for $x=0$. Now, let $x = x_0$ and $y = y_0$. Consider $x = x_0$, $y = y_0$ and $y = y_0 = 0$, $x = x_0$. Hence, the orbits of $\dot{x} = xye^{-3x}$, $\dot{y} = -2xy^2$ are the set of points described by $x = 0$, $y = y_0$, also $x = x_0$, $y = 0$, and finally the set of all curves $y = ce^{-\frac{2}{3}e^{3x}}$ (c c

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11. $\dot{x} = ax - bxy$ / $\dot{y} = cx - dxy$ (a, b, c, d positive)

<u>Solution</u>

Assume a, b, c, d positive <u>constants</u>.

Then the orbits of the system of differential equations above are partially described by the solution curves of the scalar equation $\frac{dy}{dx} = \frac{cx - dxy}{ax - bxy} = \frac{c - dy}{a - by}$. This equation is separable as follows:

$\frac{a - by}{c - dy} dy = dx$, yielding... $\int \frac{a - by}{c - dy} dy = \int dx$ <u>or</u> $a \int \frac{dy}{c - dy} - b \int \frac{y\,dy}{c - dy} = x + K$ (K constant)

<u>or</u> $a \cdot \frac{-1}{d} \int \frac{du}{u} - b \int \frac{y\,dy}{c - dy} = x + K$ where $u = c - dy$ & $du = -d\,dy$

<u>or</u> $-\frac{a}{d} \ln|c - dy| - b \int \frac{y\,dy}{c - dy} = x + K$, K constant

<u>or</u> $-\frac{a}{d} \ln|c - dy| - b\left[\frac{-y}{d} - \frac{c}{d^2} \ln|c - dy|\right] = x + K$ (K constant)

<u>or</u> $\frac{b}{d} y - x - \frac{a}{d} \ln|c - dy| + \frac{bc}{d^2} \ln|c - dy|] = K$ (K constant)

<u>or</u> $by - dx - a \ln|c - dy| + \frac{bc}{c} \ln|c - dy|] = K$ (K=K' constant)

<u>or</u> $(by - dx) - \frac{(ad - bc)}{d} \ln|c - dy| = K$ (K constant)

Thus, the orbits of $\dot{x} = ax - bxy$ and $\dot{y} = cx - dxy$ are the set of points described by $x = 0$, $y = y_0$, and the set of all curves $(by - dx) - \frac{(ad - bc)}{d} \ln|c - dy| = K$ (K constant), $x > 0$. (K constant), $x < 0$ and $(by - dx) - \frac{(ad - bc)}{d} \ln|c - dy| = K$

13. $\dot{x} = 2xy$ / $\dot{y} = x^2 - y^2$

<u>Solution</u>

The orbits of the above system of differential equations are partially described by the solution curves of the scalar equation $\frac{dy}{dx} = \frac{x^2 - y^2}{2xy} = \frac{x}{2y} - \frac{y}{2x}$. This equation is solved as follows:

$\frac{dy}{dx} - \frac{x}{2y} + \frac{y}{2x} = 0$, yielding... $\frac{dy}{dx} - \frac{1}{2}\frac{x}{y} + \frac{1}{2}\frac{y}{x} = 0$

Final solution (by exactness) is $xy^2 = \frac{1}{3}x^3 + K$ (K constant)

<u>or</u> $xy^2 - \frac{1}{3}x^3 = K$ (K constant)

Thus, the orbits of $\dot{x} = 2xy$ and $\dot{y} = x^2 - y^2$ are the sets of points described by $xy^2 - \frac{1}{3}x^3 = K$, $x = 0$, $y = 0$, and the all of the curves described by. (K constant).

Chapter 4.5.1 3/24/86

1. Suppose that what moves a government to arm is not the magnitude of the other nation's armaments, but the difference between its own and theirs. Then,

$$\frac{dx}{dt} = k(y-x) - \alpha x + g, \qquad \frac{dy}{dt} = l(x-y) - \beta y + h.$$

Show that every solution of this system of equations is stable if $k_1 l_1 < (\alpha_1 + k_1)(\beta_1 + l_1)$ and unstable if $k_1 l_1 > (\alpha_1 + k_1)(\beta_1 + l_1)$.

Solution

Consider $\frac{dx}{dt} = k(y-x) - \alpha x + g$ or $\frac{dx}{dt} = ky - (k+\alpha)x + g$

and $\frac{dy}{dt} = l(x-y) - \beta y + h$ or $\frac{dy}{dt} = -(l+\beta)y + lx + h$

Now have that

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -(k+\alpha) & k \\ l & -(l+\beta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} g \\ h \end{pmatrix}$$

The characteristic polynomial of the matrix $A = \begin{pmatrix} -(k+\alpha) & k \\ l & -(l+\beta) \end{pmatrix}$

above is

$$p(\lambda) = \det(A - \lambda I)$$
$$= \det \begin{pmatrix} -(k+\alpha)-\lambda & k \\ l & -(l+\beta)-\lambda \end{pmatrix}$$
$$= [(k+\alpha)-\lambda][(l+\beta)-\lambda] - kl$$
$$= \lambda^2 - [(k+\alpha)+(l+\beta)]\lambda + (k+\alpha)(l+\beta) - kl.$$

Consider,

$$\lambda^2 - (\alpha+\beta+k+l)\lambda + \alpha l + \beta k + \alpha\beta = 0$$

or $\lambda[\lambda - (\alpha+\beta+k+l)] = -\alpha l - \beta k - \alpha\beta$

Then since desire stability, want $\lambda < 0$, so want

$-\alpha l - \beta k - \alpha\beta < 0$ or $\lambda < \alpha + \beta + k + l$

Look at $-\alpha_1 l_1 - \beta_1 k_1 - \alpha_1\beta_1 < 0$ or $k_1 l_1 - \alpha_1\beta_1 - \alpha_1 l_1 - \beta_1 k_1 - l_1 k_1 < 0$

or $k_1 l_1 - (\alpha_1 + k_1)(\beta_1 + l_1) < 0$

or $k_1 l_1 < (\alpha_1 + k_1)(\beta_1 + l_1)$

Similarly, instability would be present where

$k_1 l_1 > (\alpha_1 + k_1)(\beta_1 + l_1)$

Chapter 4.5.1

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3. Suppose in Problem 2 that the z nation is a pacifist nation, while x and y are pugnacious nations. Then,

$$\frac{dx}{dt} = -\alpha x + ky + kz + g_1$$

$$\frac{dy}{dt} = kx - \alpha y - kz + g_2$$

$$\frac{dz}{dt} = 0 \cdot x + 0 \cdot y - \alpha z + g_3$$

Show that every solution $x(t), y(t), z(t)$ of the above system of equations is stable if $k < \alpha$, and unstable if $k > \alpha$.

Solution

Consider
$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} -\alpha & k & k \\ k & -\alpha & k \\ 0 & 0 & -\alpha \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}$$

The characteristic polynomial of matrix A (above) is determined by:

$$p(\lambda) = \det(\underline{A} - \lambda \underline{I}) = \det \begin{pmatrix} -\alpha-\lambda & k & k \\ k & -\alpha-\lambda & k \\ 0 & 0 & -\alpha-\lambda \end{pmatrix} = -\alpha-\lambda \begin{vmatrix} -\alpha-\lambda & k \\ k & -\alpha-\lambda \end{vmatrix}$$

$$= (-\alpha-\lambda)[(-\alpha-\lambda)^2 - k^2]$$

Consider $(-\alpha-\lambda)[(-\alpha-\lambda)^2 - k^2] = 0$

so that $\lambda = -\alpha$ or $(\lambda+\alpha)^2 = k^2$

$$\lambda + \alpha = k$$

$$\lambda = -\alpha + k$$

Now since want $\lambda < 0$, want $\lambda = -\alpha + k < 0$ or want $k < \alpha$ (if want to prove stability for every solution $x(t), y(t), z(t)$ of the above system), where α, k are positive constants.

Similarly, instability would exist for $k > \alpha$.